Bogachev V.I.*, Röckner M.,** Shaposhnikov S.V.*

Estimates of densities of stationary distributions and transition probabilities of diffusion processes[1)]

Abstract

We obtain lower bounds for solutions to second order elliptic and parabolic equations on the whole space. Our method is based on the study of the dependence of a constant in Harnack's inequality on the coefficients of the equation. As an application we obtain lower bounds for densities of stationary distributions and transition probabilities of diffusion processes with unbounded drift coefficients.

Keywords: Harnack inequality, transition probabilities, stationary distribution, lower bounds for solutions to parabolic equations.

AMS Subject Classification: 35K10, 35K12, 60J35, 60J60, 47D07

1. Introduction

The goal of this work is to obtain lower bounds for densities of stationary distributions and transition probabilities of diffusion processes with unbounded (possibly rapidly increasing) drift coefficients. To this end, we obtain lower bounds on densities of solutions to elliptic and parabolic equations of the form

$$\mathcal{L}^*\mu = 0 \tag{1.1}$$

for Borel measures μ on $\mathbb{R}^d$ or on $\mathbb{R}^d\times(0,1)$, respectively. Here $\mathcal{L}$ is an elliptic or parabolic second order operator of the form

$$\mathcal{L}u(x) := \partial_{x_i}(a^{ij}(x)\partial_{x_j}u(x)) + b^i(x)\partial_{x_i}u(x)$$

or

$$\mathcal{L}u(x,t) := \partial_t u(x,t) + \partial_{x_i}(a^{ij}(x,t)\partial_{x_j}u(x,t)) + b^i(x,t)\partial_{x_i}u(x,t),$$

where the summation over repeated indices is taken, and the interpretation of our equation is as follows.

We shall say that a Borel measure μ on $\mathbb{R}^d$ satisfies the weak elliptic equation (1.1) if the functions a^{ij} and b^i are integrable on every compact set in $\mathbb{R}^d$ with respect to the measure μ and, for every $u\in C_0^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d}\mathcal{L}u\,d\mu = 0. \tag{1.2}$$

A Borel measure μ on $(0,1)\times\mathbb{R}^d$ satisfies the weak parabolic equation (1.1) if the functions a^{ij} and b^i are integrable on every compact set in $\mathbb{R}^d\times(0,1)$ with respect to μ and, for every function $u\in C_0^\infty(\mathbb{R}^d\times(0,1))$, we have

$$\int_{\mathbb{R}^d\times(0,1)}\mathcal{L}u\,d\mu = 0. \tag{1.3}$$

* Department of Mechanics and Mathematics, Moscow State University, 119992 Moscow, Russia.

** Department of Mathematics, Purdue University, 150 N. University Str., West Lafayette, IN 47907-2067, USA.

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In this paper we assume that the matrices $A(x) = (a^{ij}(x))$ are nondegenerate, which guarantees the absolute continuity of μ. For this reason, throughout we consider measures μ represented in the form $\mu(dt\,dx) = \mu_t(dx)\,dt$ by means of a family of Borel measures $(\mu_t)_{t\in(0,1)}$ on $\mathbb{R}^d$. In this case (1.3) can be written as

$$\int_0^1 \int_{\mathbb{R}^d} \mathcal{L}u(x,t)\mu_t(dx)\,dt = 0.$$

We say that the measure $\mu = \mu_t\,dt$ satisfies the initial condition $\mu_0 = \nu$ at $t = 0$ if ν is a Borel measure on $\mathbb{R}^d$ and

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \zeta(x)\,\mu_t(dx) = \int_{\mathbb{R}^d} \zeta(x)\,\nu(dx) \tag{1.4}$$

for all $\zeta \in C_0^\infty(\mathbb{R}^d)$.

In the last decade, such equations attracted considerable attention; papers [1]–[12] contain diverse comments and surveys on the principal results, some of which will be mentioned below. The main feature of the estimates obtained below as compared to the previously known results is that no boundedness, dissipativity or coercivity of the drift coefficient b is assumed. For example, in the elliptic case, we obtain the estimate

$$\varrho(x) \ge C_1 \exp(-C_2|x|^{\beta+1})$$

under the assumption that the duffusion coefficient A is uniformly bounded and uniformly invertible and the function $|b(x)|$ is majorized by $C(1+|x|^\beta)$. This estimate substantionally reinforces the important recent result from [12], where considerably stronger restrictions on the coefficients are imposed (in particular, the two-fold differentiability of b and the three-fold differentiability of A is assumed). For the parabolic equation, we obtain a similar, but somewhat weaker estimate

$$\varrho(x,t) \ge \exp\Bigl\{-K_t(1+|x|^{2\beta}+|x|^2)\Bigr\}.$$

Along with the upper bounds of the same form obtained in [10] and [11] this provides a sufficiently precise description of the decay of solutions at infinity. Our proofs are based on a study of the dependence of constants in the Harnack inequality on the coefficients of the equation. As an application we give simple sufficient conditions for the inclusion $|\nabla\varrho/\varrho| \in L^p(\mu)$, which reinforces a result from [12]. Finally, one more result of our work gives broad sufficient conditions for the existence of finite entropy of the solution to the parabolic equation for probability measures at any time $t_0 > 0$ with an arbitrary initial distribution. This result is important for applying the estimates of solutions obtained in our paper [10], since they heavily use the existence of entropy of a solution at the moment starting from which we estimate the solution.

Let us remark that the indicated divergence form equations arise as equations for stationary distributions and transition probabilities of stochastic differential equations in the Stratonovich form. For a stochastic differential equation in the Ito form

$$d\xi_t = A_0(\xi_t)dw_t + \frac{1}{2}b(\xi_t)dt$$

analogous equations are written with a non-divergence operator

$$Lu := a^{ij}\partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u,$$

where $A = A_0^T A_0$. We consider operators of divergence form, which simplifies some formulations, but indicate the corresponding analogs for non-divergence form operators.

For an arbitrary domain $\Omega \subset \mathbb{R}^d$ let $W^{q,1}(\Omega)$ denote the Sobolev space of functions belonging to $L^q(\Omega)$ along with their generalized first order partial derivatives. This space is equipped with the standard norm

$$\|f\|_{W^{q,1}(\Omega)} := \|f\|_{L^q(\Omega)} + \|\nabla f\|_{L^q(\Omega)},$$

where $\|\cdot\|_{L^q(\Omega)}$ denotes the $L^q(\Omega)$-norm of scalar or vector functions. Let $C^{0,\delta}(\Omega)$ denote the space of all functions (possibly vector-valued or matrix-valued) on Ω that are Hölder continuous of order δ. Given a matrix-valued mapping A on Ω, we set $\|A\|_{C^{0,\delta}} := \sup_{x,y\in\Omega}\Big(\|A(x)\| + \|A(x)-A(y)\|/|x-y|^\delta\Big)$.

Let $J\subset(0,1)$ be an interval and let $\|\cdot\|_{q,\Omega\times J}$ denote the $L^q(\Omega\times J)$-norm on scalar or vector functions. Finally, let $\mathbb{H}^{q,1}(\Omega\times J)$ denote the space of all measurable functions f on $\Omega\times J$ with finite norm

$$\|f\|_{\mathbb{H}^{q,1}(\Omega\times J)} := \Big(\int_J \big(\|f(\cdot,t)\|_{W^{q,1}(\Omega)}\big)^q\,dt\Big)^{1/q}.$$

For notational simplicity the gradient of a function u on $\mathbb{R}^d\times(0,1)$ with respect to the argument from $\mathbb{R}^d$ is denoted by

$$\nabla u := \nabla_x u = (\partial_{x_1}u,\dots,\partial_{x_d}u).$$

Let $B(x,R)\subset\mathbb{R}^d$ denote the open ball of radius R centered at a point x.

2. Investigation of constants in the Harnack inequality

In this section we investigate the dependence of a constant in the Harnack inequality for elliptic and parabolic equations on the coefficients of the equation. We begin with the elliptic case. First we recall certain known a priori estimates of solutions to elliptic equations, which will be employed in the proof of the p-integrability of the logarithmic gradient of the measure μ.

Let Ω be a bounded domain in $\mathbb{R}^d$, let functions $h^1,\dots,h^d$ be locally integrable in Ω, and let $x\mapsto A(x)=(a^{i,j}(x))_{1\le i,j\le d}$ be a measurable locally bounded matrix-valued mapping on Ω such that the matrices $A(x)=(a^{ij}(x))$ are positive symmetric. We shall say that a function u from the class $W^{q,1}(\Omega)$, where $q>1$, is a solution to the equation

$$\partial_{x_i}(a^{ij}\partial_{x_j}u + h^i) = 0 \tag{2.1}$$

in Ω if, for every function $\varphi\in C_0^1(\Omega)$, the equality

$$\int_\Omega \partial_{x_i}\varphi(a^{ij}\partial_{x_j}u + h^i)\,dx = 0$$

holds.

Theorem 2.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^d$ with a boundary of the class C^1, $A\in C^{0,\delta}(\Omega)$, where $\delta>0$, and that there exists a constant $\alpha>0$ such that $A(x)\ge\alpha I$ for all $x\in\Omega$. Let $h^i\in L^q(\Omega)$, where $q>1$. If a function u from $W^{q,1}(\Omega)$ satisfies equation (2.1) in Ω, then*

$$\|u\|_{W^{q,1}(\Omega)} \le C\big(\|h\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\big),$$

where the number C depends only on d,q,α,Ω, and $\|A\|_{C^{0,\delta}}$, and, for any fixed d,q, and Ω, the quantity C is a locally bounded function of $\alpha>0$ and $\|A\|_{C^{0,\delta}}$.

This theorem is a partial case of the result formulated by Morrey in his book [13, p. 156], where only the idea of the proof was outlined. A complete proof with an investigation of the dependence of the constant on the coefficients has been given in [14] (see the theorem and corollary in [14]). Let us observe that here we need only that $q>1$, although it is assumed in [14] that $q\ge d/(d-2)$. The fact that the latter condition can be dropped in the situation under consideration follows at once from the reasoning in [14] since the coefficients due to which this condition was necessary there are absent in equation (2.1) (see the H_q^1-conditions in [14]). In order to achieve a complete correspondence with the results in [14] and Theorem 2.1 one has to use also the inequality $\|u\|_{L^1(\Omega)}\le |\Omega|^{(q-1)/q}\|u\|_{L^q(\Omega)}$.

Corollary 2.1. *If $\Omega=B(z,R)$, where $R<1$, then the estimate from Theorem 2.1 holds with a constant $C=C(d,q,\alpha,\|A\|_{C^{0,\delta}})R^{-1}$.*

Proof. Let us consider the change of variables $x = z + Ry$. The function $v(y) = u(z + Ry)$ satisfies in $B(0,1)$ the equation

$$\partial_{y_j}\big(a^{ij}(z+Ry)\partial_{y_i}v(y) + Rh^i(z+Ry)\big) = 0.$$

By assumption, the matrix A is Hölder continuous, and we may assume that the Hölder constant does not change since $R < 1$. Then the following estimate holds for the function v:

$$\|v\|_{W^{q,1}(B(0,1))} \le C(d,q,\alpha)\big(\|v\|_{L^q(B(0,1))} + R\|h\|_{L^q(B(0,1))}\big).$$

Returning to the initial coordinates and taking into account that $R < 1$ we obtain

$$\|v\|_{W^{q,1}(B(0,1))} = R^{-d/q}\|u\|_{L^q(B(z,R))} + R^{1-d/q}\|\nabla u\|_{L^q(B(z,R))} \ge R^{1-d/q}\|u\|_{W^{q,1}(B(z,R))}.$$

Similarly, we have

$$\begin{aligned}\|v\|_{L^q(B(0,1))} + R\|h\|_{L^q(B(0,1))} &= R^{-d/q}\|u\|_{L^q(B(z,R))} + R^{1-d/q}\|h\|_{L^q(B(z,R))}\\ &\le R^{-d/q}\big(\|u\|_{L^q(B(z,R))} + \|h\|_{L^q(B(z,R))}\big).\end{aligned}$$

On account of these estimates we obtain

$$\|u\|_{W^{q,1}(B(z,R))} \le C(d,q,\alpha,\|A\|_{C^{0,\delta}})R^{-1}\big(\|u\|_{L^q(B(z,R))} + \|h\|_{L^q(B(z,R))}\big),$$

as required. ☐

We shall assume that there exist constants $\gamma \ge 0$ and $\alpha > 0$ such that

$$\sum_{i,j}|a^{i,j}(x)|^2 \le \gamma^2 \quad \text{and} \quad A(x) \ge \alpha\cdot I \quad \text{for all } x\in\Omega. \tag{2.2}$$

Let a mapping $b\colon\ \Omega \to \mathbb{R}^d$ be measurable and satisfy the condition

$$\sup_{x\in\Omega}|b(x)| \le B < \infty.$$

Suppose that a nonnegative function $u \in W^{2,1}(\Omega)$ satisfies the equation

$$\partial_{x_i}(a^{ij}\partial_{x_j}u - b^i u) = 0, \tag{2.3}$$

i.e., for every function $\varphi \in C_0^1(\Omega)$, one has the equality

$$\int_\Omega \partial_{x_i}\varphi(a^{ij}\partial_{x_j}u - b^i u)\,dx = 0. \tag{2.4}$$

Under the stated assumptions, the function u has a locally Hölder continuous version, in particular, this version is locally bounded on Ω (see Theorem 8.22 in [16]).

For every integrable function v we set

$$v_\Omega = \frac{1}{|\Omega|}\int_\Omega v\,dx.$$

In our study of constants in the Harnack inequality for solutions to equation (2.3) we have to reproduce the main steps of the proof of Theorem 8.18 in [16]. Let us note that in the remark following Theorem 8.20 in [16], the desired dependence of the constant on the coefficients of the equation is indicated without proof. Since the exact form of this dependence is important for the sequel we give a complete justification.

The following lemma is true (see Lemma 7.21 in [16]).

Lemma 2.1. *Let Ω be a convex domain and let $v \in W^{1,1}(\Omega)$. Suppose that there exists a constant K such that, for every ball $B(x_0,R)$, the following inequality holds:*

$$\int_{\Omega\cap B(x_0,R)}|\nabla v|\,dx \le KR^{d-1}.$$

Then there exist positive constants σ_0 and C, which depend only on d, such that

$$\int_\Omega \exp\Big(\frac{\sigma}{K}|v - v_\Omega|\Big)\,dx \le C(\operatorname{diam}\Omega)^d,$$

where $\sigma = \sigma_0|\Omega|(\operatorname{diam}\Omega)^{-d}$ *and* $\operatorname{diam}\Omega = \sup_{x,y\in\Omega} |x-y|$.

Theorem 2.2. *Let* $\theta = 1 + 4\delta$, *where* $0 < \delta \le 3$. *If* $B(y, \theta R) \subset \Omega$, *then the following inequality holds for the continuous version of the function* u*:*

$$\sup_{x\in B(y,R)} u(x) \le C \inf_{x\in B(y,R)} u(x), \tag{2.5}$$

where

$$C = \exp\bigl\{c(d)\delta^{-1}\bigl(\gamma\alpha^{-1} + B\alpha^{-1}R\bigr)\bigr\},$$

and the number $c(d)$ *depends only on* d.

Proof. We shall assume that $d \ge 3$. The case $d \le 2$ reduces to the case $d = 3$ by passing to the function $u(x_1, x_2)\exp(-|x_3|)$.

1. Let $R = 1, y = 0$. Let $\eta \in C_0^1(\Omega)$ and $u_k = u + k, k > 0$. Let $\beta \in (-\infty, +\infty)$. Plugging the function $\varphi = \eta^2 u_k^\beta$ into the integral identity (2.4), which can be easily justified, we obtain

$$\beta \int_\Omega (A\nabla u, \nabla u_k) u_k^{\beta-1}\eta^2\,dx = -2\int_\Omega (A\nabla u, \nabla\eta) u_k^\beta \eta\,dx + \beta\int_\Omega (bu, \nabla u_k) u_k^{\beta-1}\eta^2\,dx + 2\int_\Omega (bu, \nabla\eta) u_k^\beta\eta\,dx.$$

Applying the Cauchy inequality and taking into account our assumptions on the coefficients we obtain

$$\int_\Omega |\nabla u_k| u_k^{\beta-1}\eta^2\,dx \le C\int_\Omega (\eta + |\nabla\eta|)^2 u_k^{\beta+1}\,dx, \tag{2.6}$$

where

$$C = 4\Bigl(\Bigl(\frac{\gamma}{|\beta|\alpha} + \frac{B}{\alpha}\Bigr)^2 + \frac{B}{|\beta|\alpha}\Bigr).$$

Since $\alpha \le d\gamma$, one has

$$C \le 16d\Bigl(\frac{\gamma}{|\beta|\alpha} + \frac{B}{\alpha}\Bigr)^2.$$

So we shall use estimate (2.6) with $C = 16d\Bigl(\gamma|\beta|^{-1}\alpha^{-1} + B\alpha^{-1}\Bigr)^2$.

Let $w = u_k^{(\beta+1)/2}$ if $\beta \ne -1$ and $w = \ln u_k$ if $\beta = -1$. If $\beta = -1$, we have

$$\int_\Omega |\nabla w|^2\eta^2\,dx \le C_1\int_\Omega (\eta + |\nabla\eta|)^2\,dx, \tag{2.7}$$

where

$$C_1 = 16d\Bigl(\frac{\gamma}{\alpha} + \frac{B}{\alpha}\Bigr)^2.$$

If $\beta \ne -1$, we have

$$\int_\Omega |\nabla w|^2\eta^2\,dx \le C_2\int_\Omega (\eta + |\nabla\eta|)^2 w^2\,dx, \tag{2.8}$$

where

$$C_2 = 4d(\beta+1)^2\Bigl(\frac{\gamma}{|\beta|\alpha} + \frac{B}{\alpha}\Bigr)^2.$$

2. Set

$$F(p, r) = \Bigl(\int_{B(0,r)} u_k^p\,dx\Bigr)^{1/p}, \quad p \in (-\infty, +\infty).$$

We shall find numbers $p_0 = p_0(d, \delta, \alpha, \gamma, B) > 0$ and $C_3 = C_3(d, \delta, \alpha, \gamma, B) > 0$ such that

$$F(p_0, 1 + 3\delta) \le C_3 F(-p_0, 1 + 3\delta).$$

Let $r \le 1$, let $B_{r+\delta r}$ be an arbitrary ball of radius $r + \delta r$ contained in $B(0, \theta)$, and let B_r be the ball of radius r with the same center. Let us take a smooth function η such that

$\eta(x) = 1$ if $x \in B_r$, $\eta(x) = 0$ if $x \notin B_{r+\delta r}$, and $0 \le \eta \le 1$ and $|\nabla\eta| \le 2(\delta r)^{-1}$. According to estimate (2.7) with $\beta = -1$ we obtain

$$\int_{B_r} |\nabla w|\,dx \le (2r)^{d/2}\Big(\int_{B_r} |\nabla w|^2\,dx\Big)^{1/2} \le K_0(d)\delta^{-1}C_1^{1/2}r^{d-1}.$$

By Lemma 2.1 there exist numbers $C(d)$ and $K(d)$, which depend only on d, such that the following inequality holds:

$$\int_{B(0,1+3\delta)} \exp\big(p_0|w - w_{B(0,1+3\delta)}|\big)\,dx \le K(d), \quad p_0 = C(d)\delta C_1^{-1/2}.$$

Therefore, one has

$$\int_{B(0,1+3\delta)} e^{p_0 w}\,dx \int_{B(0,1+3\delta)} e^{-p_0 w}\,dx \\ \le \int_{B(0,1+3\delta)} e^{p_0 w - w_{B(0,1+3\delta)}}\,dx \int_{B(0,1+3\delta)} e^{w_{B(0,1+3\delta)} - p_0 w}\,dx \le K(d)^2.$$

Recalling that $w = \ln u_k$ we obtain the estimate

$$F(p_0, 1+3\delta) \le C_3 F(-p_0, 1+3\delta),$$

where

$$C_3 = K(d)^{2/p_0} = \exp\big(c_1(d)\delta^{-1}\alpha^{-1}(\gamma + B)\big),$$

and the number $c_1(d)$ depends only on d.

3. It is known that

$$F(+\infty, 1) := \sup_{B(0,1)} u_k = \lim_{p\to+\infty} F(p,1), \quad F(-\infty,1) := \inf_{B(0,1)} u_k = \lim_{p\to-\infty} F(p,1).$$

We shall find numbers $C_4 = C_4(d,\delta,\alpha,\gamma,B) > 0$ and $C_5 = C_5(d,\delta,\alpha,\gamma,B) > 0$ such that

$$F(+\infty,1) \le C_4 F(p_0, 1+3\delta) \le C_3 F(-p_0, 1+3\delta) \le C_5 F(-\infty, 1).$$

By the triangle inequality we have

$$\Big(\int_{\mathbb{R}^d} |\nabla(\eta w)|^2\,dx\Big)^{1/2} \le \Big(\int_{\mathbb{R}^d} |\eta\nabla w|^2\,dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^d} |w\nabla\eta|^2\,dx\Big)^{1/2}.$$

According to estimate (2.8) with $\beta \ne -1$ we obtain

$$\Big(\int_\Omega |\nabla(\eta w)|^2\,dx\Big)^{1/2} \le 2(C_2+1)^{1/2}\Big(\int_\Omega (\eta + |\nabla\eta|)^2 w^2\,dx\Big)^{1/2}. \tag{2.9}$$

Let a smooth function η be such that $\eta(x) = 1$ if $|x| \le r_1$, $\eta(x) = 0$ if $|x| \ge r_2$, where $1 < r_1 < r_2 \le 1 + 3\delta$, and $0 \le \eta \le 1$ and $|\nabla\eta| \le 2/(r_2 - r_1)$. By the Sobolev embedding theorem, inequality (2.9) yields

$$\Big(\int_{B_{r_1}(0)} w^{2d/(d-2)}\,dx\Big)^{(d-2)/2d} \le \frac{C_0(d)(C_2+1)^{1/2}}{r_2 - r_1}\Big(\int_{B_{r_2}(0)} w^2\,dx\Big)^{1/2},$$

where the number $C_0(d)$ depends only on d. Let $\beta + 1 > 0$. Since $w = u_k^{(\beta+1)/2}$, we have

$$F\Big(\frac{d}{d-2}(\beta+1), r_1\Big) \le \Big(\frac{C_0(d)(C_2+1)^{1/2}}{r_2 - r_1}\Big)^{2/(\beta+1)} F(\beta+1, r_2).$$

Set $q = d/(d-2) > 1$. We shall apply the obtained estimate to the numbers r_1 and r_2 of the form $r_1 = r_m = 1 + \delta 2^{-m}$, $r_2 = r_{m-1} = 1 + \delta 2^{-m+1}$, and we shall take for $\beta + 1$ the numbers $q^m p$, where $m = 0, 1, 2, \ldots$, and the number p is given by the equality

$$p = d(d-1)^{-1}((d-2)/d)^k,$$

in which k is a natural number such that

$$\frac{\ln p_0 - \ln d + \ln(d-1)}{\ln(d-2) - \ln d} < k < \frac{\ln p_0 - \ln d + \ln(d-1)}{\ln(d-2) - \ln d} + 2.$$

A natural number in the indicated interval of length 2 exists because its left end is positive. We observe that

$$(d-2)^2 d^{-2} p_0 < p < p_0.$$

In addition, for all nonnegative integers m one has the estimate

$$|q^m p - 1| = |d(d-1)^{-1}(1-2/d)^{k-m} - 1| \geq (d-1)^{-1},$$

which is obvious from the consideration of the cases $k \geq m$ and $k < m$. It follows from this estimate that $C_2 \leq 4d(d-1)^2(\beta+1)^2\alpha^{-2}(\gamma+B)^2$, whence by the relationship $p < p_0$ we obtain the inequality

$$C_2 \leq 4d(d-1)^2 q^{2m} p_0^2 \Big(\frac{\gamma}{\alpha} + \frac{B}{\alpha}\Big)^2 = 4\delta^2 C(d)^2 d(d-1)^2 q^{2m}.$$

Thus, we arrive at the relationship

$$F(q^{m+1}p, r_m) \leq \big(Q(d)\delta^2\big)^{2m/(q^m p)} F(q^m p, r_{m-1}),$$

where the number $Q(d)$ depends only on d. This relationship yields the estimate

$$F(q^m p, r_m) \leq \big(Q(d)\delta^2\big)^{S/p} F(p, 1+\delta), \quad S := S(d) := 2\sum_{m=1}^{\infty} m q^{-m}.$$

By Hölder's inequality $F(p, 1+\delta) \leq \exp(c_2(d)/p_0) F(p_0, 1+3\delta)$. Finally, we obtain the inequality $F(+\infty, 1) \leq C_4 F(p_0, 1+3\delta)$, where

$$C_4 = \exp\Big\{c_3(d)\delta^{-1}\alpha^{-1}(\gamma+B)\Big\}.$$

If $\beta+1 < 0$, we have

$$F(\beta+1, r_2) \leq \Big(\frac{4C_0(d)(C_2+1)^{1/2}}{r_2 - r_1}\Big)^{2/(|\beta+1|)} F\Big(\frac{d}{d-2}(\beta+1), r_1\Big).$$

Repeating the previous reasoning, we obtain the estimate

$$F(-p_0, 1+3\delta) \leq C_5 F(-\infty, 1), \quad C_5 = \exp\Big\{c_4(d)\delta^{-1}\alpha^{-1}(\gamma+B)\Big\}.$$

Along with our previous estimates this yields the assertion of the theorem with a constant of the required form in the case $R = 1$, $y = 0$. The case $R \neq 1$ reduces to the considered case by the change of variable $x = y + Rz$. The function $v(z) = u(y + zR)$ satisfies the equation

$$\partial_{z_i}\big(a^{ij}\partial_{z_j} v - Rb^i v\big) = 0.$$

Hence B must be replaced by BR and the constants γ and α remain unchanged. □

It should be noted that it is mistakenly claimed in [16, Problem 8.3, p. 217] that in the case when A is symmetric (which we assume), the constant can be refined as follows: $C = \exp\Big(c(d)(\gamma/\alpha + BR)^{1/2}\Big)$. However, this is not true in the case where $A = I$ and $b(x) = -x$ because the solution is Gaussian and cannot be estimated from below by $C_1 \exp(-C_2|x|)$.

Remark 2.1. In the case where our elliptic operator is written in the non-divergence form $L = a^{ij}\partial_{x_i}\partial_{x_j} + b^i\partial_{x_i}$ and A is locally Lipschitzian, the same lower bound holds if we replace b^i by $b^i - \partial_{x_j} a^{ij}$.

We now consider the Harnack inequality for the parabolic equation. Let Ω be a bounded domain in $\mathbb{R}^d$, let $Q = \Omega \times (0,1)$, and let $A = (a^{ij})_{1\leq i,j\leq d}$ be a measurable matrix-value mapping on Q such that there exist constants $\gamma \geq 0$ and $\alpha > 0$ such that

$$\sum_{i,j} |a^{i,j}(x,t)|^2 \leq \gamma^2 \quad \text{and} \quad A(x,t) \geq \alpha \cdot I \quad \text{for all } (x,t) \in Q. \tag{2.10}$$

In addition, let $b\colon\ Q \to \mathbb{R}^d$ be a measurable vector field such that

$$\sup_{(x,t)\in Q} |b(x,t)| \le B < \infty.$$

Suppose that a nonnegative function $u \in \mathbb{H}^{2,1}(Q)$ satisfies the equation

$$\partial_t u = \partial_{x_i}(a^{ij}\partial_{x_j}u - b^i u), \tag{2.11}$$

i.e., for every function $\varphi \in C_0^1(Q)$, one has the equality

$$\int\!\!\int_Q \Big[-\varphi_t u + \partial_{x_i}\varphi\big(a^{ij}\partial_{x_j}u - b^i u\big)\Big]\,dx\,dt = 0.$$

It follows from the general theory of parabolic equations (see, e.g., Theorem 8.1 in §8 and Theorem 10.1 in §10 in Chapter 3 of [15]) that under our assumptions any solution u has a version that is locally Hölder continuous.

Let us fix a point $(\bar{x},\bar{t}) \in Q$. Let $R(\bar{x},r)$ be the open cube with the edge length r centered at the point $\bar{x}$. Set

$$Q(r) = R(\bar{x},r)\times(\bar{t}-r^2,\bar{t}), \quad Q^*(r) = R(\bar{x},r)\times(\bar{t}-8r^2,\bar{t}-7r^2).$$

The following classical theorem is true (see Theorem 3 in [17] generalizing a result from [18]).

Theorem 2.3. *Let $Q(3r)\subset Q$. Then, for the continuous version of the function u satisfying equation* (2.11)*, we have*

$$\sup_{(x,t)\in Q^*(r)} u(x,t) \le C \inf_{(x,t)\in Q(r)} u(x,t),$$

where the number $C = C(d,\alpha,\gamma,Br)$ depends only on d,α,γ, and Br.

As in the elliptic case, we are interested in a more precise form of dependence of C on the indicated parameters.

Let $Q' = \Omega'\times J'$, where $\Omega'\subset\Omega$ and $J'\subset[0,1]$ is an interval. If $p,q>1$, we set

$$\|f\|_{p,q,Q'} := \Big(\int_{J'}\Big(\int_{\Omega'}|f(x,t)|^p\,dx\Big)^{\frac{q}{p}}dt\Big)^{\frac{1}{q}}.$$

If $q=\infty$, we set

$$\|f\|_{p,\infty,Q'} := \sup_{J'}\|f(\cdot,t)\|_{L^p(\Omega')}.$$

The proof of the following lemma can be found in §3 of Chapter 2 in [15].

Lemma 2.2. *Let $d>2$. For every function $v\in\mathbb{H}^{2,1}(Q')\bigcap L^{2,\infty}(Q')$ such that for almost all $t\in J'$ the function $x\to v(x,t)$ has compact support in Ω', one has the inequality*

$$\|v\|_{p,q,Q'} \le c(d,p,|\Omega'|)\big(\|\nabla v\|_{L^2(Q')} + \|v\|_{2,\infty,Q'}\big),$$

where $2\le q$, $2\le p\le 2d/(d-2)$ and $1/q + d/(2p) = d/4$.

Remark 2.2. Let us single out an important partial case of this lemma, which will be used below. If $p=q=2(d+2)/d$, then the above inequality takes the form

$$\|u\|_{L^{2(d+2)/d}(Q')} \le C\Big(\|\nabla u\|_{L^2(Q')} + \|u\|_{2,\infty,Q'}\Big),$$

where C depends only on d and $|\Omega|$.

Set $u_\varepsilon = u+\varepsilon$, $\varepsilon>0$, and

$$\mathcal{H} = \begin{cases} \frac{1}{\beta+1}u_\varepsilon^{\beta+1}, & \beta\ne -1\\ \ln u_\varepsilon, & \beta=-1.\end{cases}$$

Lemma 2.3. *Let $\eta \in C_0^1(Q)$. For almost all $\tau_1, \tau_2 \in (0,1)$ one has*

$$\operatorname{sign}\beta\Big(\int_\Omega [\eta(x,\tau_2)^2\mathcal{H}(x,\tau_2) - \eta(x,\tau_1)^2\mathcal{H}(x,\tau_1)]\,dx + \frac{\alpha\beta}{2}\int_{\tau_1}^{\tau_2}\int_\Omega \eta^2 u_\varepsilon^{\beta-1}|\nabla u_\varepsilon|^2\,dx\,dt\Big)$$
$$\le C(\alpha,\gamma,\beta)\int_{\tau_1}^{\tau_2}\int_\Omega (|\nabla\eta| + \eta)^2 u_\varepsilon^{\beta+1}\,dx\,dt + 2\int_{\tau_1}^{\tau_2}\int_\Omega \eta|\eta_t||\mathcal{H}|\,dx\,dt,$$

where

$$C(\alpha,\gamma,\beta) = \frac{16d}{\alpha|\beta|}(|\beta|B + \gamma)^2.$$

Proof. It suffices to repeat the reasoning from steps 2 and 9 in [17], where one has to set $f = g = h = 0$ and take into account that $d\gamma > \alpha$. □

Let us fix a point $(\bar{x}, \bar{t}) \in Q$. Let $R(r)$ be the open cube of the edge length r centered at the point $\bar{x}$. Let $D = R(1/2) \times (0,1)$ and let $Q^+(l)$ and $Q^-(l)$ be the rectangles contained in D and obtained from $R(1/2) \times (0,1/2)$ and $R(1/2) \times (1/2,1)$ respectively by means of the transformations of the form

$$t \mapsto l^2 t + c_1, x \mapsto lx + c_2$$

with some numbers c_1, c_2, l. Set $\psi(w) = \sqrt{w}$ if $w > 0$ and $\psi(w) = 0$ if $w \le 0$.

Lemma 2.4. *Let $D \subset Q$. For all $Q^+(l), Q^-(l)$ we have*

$$\iint_{Q^-(l)}\iint_{Q^+(l)} \psi\Big(\log\frac{u_\varepsilon(y,s)}{u_\varepsilon(x,t)}\Big)\,dy\,ds\,dx\,dt \le B_1 l^{2d+4},$$

where

$$B_1 = C(d)\Big(1 + \frac{1}{\alpha} + \frac{(\alpha+\sqrt{\alpha})}{\alpha\sqrt{\alpha}}(B+\gamma)\Big).$$

Proof. A detailed proof is given in Section 6 of [17]. For complete correspondence it suffices to note that the constant C_6 indicated there has the form $C_6 = 4C(\alpha,\gamma) = 64d\alpha^{-1}(B+\gamma)^2$ in our case. □

Let $D^+ = R(1/2) \times (0,1/4)$ and $D^- = R(1/2) \times (3/4,1)$. Then Lemma 2.4 and the well-known Moser's lemma [18] yield the following assertion.

Corollary 2.2. *Let $D \subset Q$. Then the following inequality holds:*

$$\iint_{D^-} u_\varepsilon^{\lambda/B_1^2}\,dy\,ds \iint_{D^+} u_\varepsilon^{-\lambda/B_1^2}\,dx\,dt \le K,$$

where B_1 is the constant from Lemma 2.4 and the numbers $\lambda = \lambda(d)$ and $K = K(d)$ depend only on d.

Theorem 2.4. *Let $Q(3r) \subset Q$. Then the following inequality holds for the continuous version of the function u:*

$$\sup_{(x,t)\in Q^*(r)} u(x,t) \le C \inf_{(x,t)\in Q(r)} u(x,t),$$

where

$$C := C(d,\alpha,\gamma,B,r) := \exp\Big\{c(d)\Big(1 + \frac{1}{\alpha} + \frac{(\alpha+\sqrt{\alpha})}{\alpha\sqrt{\alpha}}(Br+\gamma)\Big)^2\Big\}.$$

Proof. We shall follow the proof of Theorem 3 in [17].

1. Let $\bar{x} = 0, \bar{t} = 1$, $r = 1/3$, and $Q(3r) = R(1) \times (0,1) \subset Q$. As above, we set $u_\varepsilon = u + \varepsilon$, $\varepsilon > 0$. It suffices to obtain the estimate

$$\sup_{(x,t)\in Q^*(1/3)} u_\varepsilon(x,t) \le C \inf_{(x,t)\in Q(1/3)} u_\varepsilon(x,t), \quad C = C(d,\alpha,\gamma,B,1/3).$$

If $1/3 \le s \le 1/2$, we set

$$S(s) = R(s) \times \big((1-s)/6, (1+s)/6\big).$$

Let l and l' be two numbers such that $1/3 < l' < l < 1/2$ and let a smooth function η be such that $\eta(x,t) = 1$ if $(x,t) \in S(l')$, $\eta(x,t) = 0$ outside $S(l)$, and

$$0 \le \eta \le 1, \quad |\nabla\eta| \le 2(l-l')^{-1}, \quad |\partial_t \eta| \le 6(l-l')^{-1}.$$

Let $\beta > -1$ and $v = u_\varepsilon^{(\beta+1)/2}$. According to Lemma 2.3 we have

$$\|\eta\nabla v\|^2_{L^2(S(l))} \le C_1 \|v\|^2_{L^2(S(l))}, \quad \|\eta v\|^2_{2,\infty} \le C_2 \|v\|^2_{L^2(S(l))},$$

where

$$C_1 = C(d)\frac{(\beta+1)^2}{(l-l')^2}\Big(\frac{B}{\alpha} + \frac{\gamma}{|\beta|\alpha}\Big)^2, \quad C_2 = \frac{\alpha|\beta|C_1}{(\beta+1)}.$$

Applying Lemma 2.2 we obtain

$$\|v\|_{L^{2(d+2)/d}(S(l'))} \le (C_3+1)^{1/2}\|v\|_{L^2(S(l))},$$

where

$$C_3 = C(d)\frac{(\beta+1)(\beta+1+\alpha|\beta|)}{(l-l')^2}\Big(\frac{B}{\alpha} + \frac{\gamma}{|\beta|\alpha}\Big)^2.$$

As in the elliptic case, if $p > 0$, we set

$$F(p,l) = \Big(\int\int_{S(l)} u_\varepsilon^p \, dx\, dt\Big)^{1/p}.$$

We have proved that if $\beta + 1 > 0$ and $q = (d+2)/d$, then the following inequality holds:

$$F(q(\beta+1), l') \le (C_3+1)^{1/(\beta+1)} F(\beta+1, l). \tag{2.12}$$

Set $l = l_m = 3^{-1}(1+2^{-m-1}), l' = l'_m = 3^{-1}(1+2^{-m-2})$ and $p_0 = \lambda/B_1^2$, where λ is the constant from Corollary 2.2 and the number B_1 is defined in Lemma 2.4. There exists a number p satisfying the condition $(d/(d+2))^2 p_0 < p < p_0$ such that for some constant $C(d) > 1$, which depends only on the dimension d, for all m we have

$$C_3 \le C(d)\frac{q^{2m}p_0(1+\alpha)}{(l-l')^2}\Big(\frac{B}{\alpha}+\frac{\gamma}{\alpha}\Big)^2 \le C(d)\frac{q^{2m}p_0 B_1^2}{(l-l')^2} \le C(d)(2q)^{2m}.$$

Therefore, applying inequality (2.12) with $\beta + 1 = pq^m$, we obtain the estimate

$$F(q^{m+1}p, l'_m) \le \big(C(d)q+1\big)^{2m/(pq^m)} F(q^m p, l_m).$$

Thus, we have

$$F(q^m p, r_m) \le \big(C(d)(q+1)\big)^{2p^{-1}S} F(p, 1/2), \quad S := S(d) := \sum_{m=1}^\infty mq^{-m}.$$

By Hölder's inequality

$$F(p,1/2) \le |S(1/2)|^{1/p-1/p_0} F(p_0, 1/2).$$

Finally, we obtain

$$F(+\infty, 1/3) \le C_4 F(p_0, 1/2), \quad C_4 = \exp\{c(d)B_1^2\}.$$

2. Let $Q(l) = R(l) \times (1-l^2, 1)$ if $1/3 \le l \le 1/2$. Whenever $p < 0$ we set

$$F(p,l) = \Big(\int\int_{Q(l)} u_\varepsilon^{-p}\, dx\, dt\Big)^{-1/p}.$$

Similarly to Step 1, whenever $\beta + 1 < 0$ we have

$$F(\beta+1, r_2) \le \big(C_3+1\big)^{1/(|\beta+1|)} F\Big(\frac{d}{d-2}(\beta+1), r_1\Big).$$

Repeating the previous reasoning we obtain

$$F(-p_0, 1/2) \le C_5 F(-\infty, 1/2), \quad C_5 = \exp\big\{C(d) B_1^2\big\}.$$

3. We observe that $S(1/2) \subset D^{-1}$ and $Q(1/2) \subset D^+$. According to Corollary 2.2 we have

$$F(p_0, 1/2) \le C_6 F(-p_0, 1/2), \quad C_6 = \exp\big\{C(d) B_1^2\big\}.$$

On account of the established inequalities we obtain the required estimate. The general case, where $r \ne 1$ and $(\bar{x}, \bar{t}) \ne (0, 1)$, reduces to the considered case by the change of coordinates

$$(x, t) \to ((x - \bar{x})/r, 1 + (t - \bar{t})/r^2).$$

The theorem is proven. □

Our next result further refines the obtained estimate with respect to depends on r. Its advantage as compared to the previous theorem is that now B^2 enters without factor r.

Theorem 2.5. *Suppose that $B(z_0, \theta r) \subset \Omega$ for some $\theta > 1$ and $r > 0$. Then, whenever $0 < s < t < 1$ and $x, y \in B(z_0, r)$, the following inequality holds for the continuous version of u:*

$$u(y, s) \le u(x, t) \exp\Big\{K\Big(\frac{|x - y|^2}{t - s} + (B + 1)^2 \frac{t - s}{\delta^2} + 1\Big)\Big\},$$

where $\delta = \min\{(\theta - 1)r, \sqrt{s}\}$ and the number K depends only on $d, \alpha,$ and γ as follows:

$$K := c(d)\Big|1 + \alpha^{-1} + (\alpha^{-1} + \alpha^{-1/2})\gamma\Big|^2,$$

and $c(d)$ depends only on d.

Proof. Let us fix $x \in B(z_0, r)$ and $y \in B(z_0, r)$. Let

$$\delta = \min\{(\theta - 1)r, \sqrt{s}\} \quad \text{and} \quad q_0 = \frac{\delta}{9d(B + 1)}.$$

Then $\delta < 1$, $q_0 B \le 1$, and for all $z \in B(z_0, r)$ we have $R(z, 3q_0) \subset B(z_0, \theta r) \subset \Omega$. Indeed, for every $z_1 \in R(z, 3q_0)$ we have

$$|z_1 - z_0| \le |z_1 - z| + |z - z_0| \le 3dq_0 + R \le \theta R.$$

We observe that whenever $s < \tau < t$ one has the inequality $\tau - 9q_0^2 \ge s - 9q_0^2 \ge 0$, hence $(\tau - 9q_0^2, \tau) \subset [0, 1]$. Therefore, $Q(3q_0) = R(z, 3q_0) \times (\tau - 9q_0^2, \tau) \subset Q$.

Let $x_n = y + n(x - y)/N$ and $t_n = s + n(t - s)/N$, where $n = 0, 2, \dots, N$. Then $y = x_0$, $s = t_0$, $x = x_N$, $t = t_N$, and

$$|x_n - x_{n-1}| = \frac{|x - y|}{N}, \quad |t_n - t_{n-1}| = \frac{t - s}{N}.$$

Set

$$q = \frac{1}{2}\Big(\frac{1}{q_0^2} + \frac{56}{t - s} + \frac{256|x - y|^2}{(t - s)^2}\Big)^{-1/2}.$$

We have

$$q \le \min\Big\{q_0, \sqrt{\frac{t - s}{56}}, \frac{t - s}{16|x - y|}\Big\}.$$

Let

$$N = \Big[\frac{t - s}{8q^2}\Big] + 1,$$

where $[r]$ denotes the integer part of r. We observe that with this choice of N the following inequalities hold:

$$\frac{2|x - y|}{q} \le \frac{t - s}{8q^2} \le N, \quad \frac{t - s}{7q^2} - \frac{t - s}{8q^2} = \frac{t - s}{56q^2} \ge 1.$$

It follows at once that
$$|x_n - x_{n-1}| = \frac{|x-y|}{N} \le \frac{q}{2} \le \frac{q_0}{2}.$$
Since N is the minimal natural number greater than $(t-s)/8q^2$ and one has
$$\frac{t-s}{7q^2} - \frac{t-s}{8q^2} \ge 1,$$
we obtain
$$\frac{t-s}{8q^2} \le N \le \frac{t-s}{7q^2}.$$
Therefore,
$$7q^2 \le \frac{t-s}{N} \le 8q^2.$$
Taking into account that $x_{n-1} \in R(x_n, q)$, $7q^2 \le t_n - t_{n-1} \le 8q^2$ and
$$R(3q, x_n) \times (t_n - 9q^2, t_n) \subset R(3q_0, x_n) \times (t_n - 9q_0^2, t_n) \subset Q,$$
we apply Theorem 2.4 with $r = q$ and obtain
$$u(x_{n-1}, t_{n-1}) \le \sup_{(z,\tau)\in Q^*(q)} u(z,\tau) \le C_0 \inf_{(z,\tau)\in Q(q)} u(z,\tau) \le C_0 u(x_n, t_n),$$
where
$$Q^*(q) = R(x_n, q) \times (t_n - 8q^2, t_n - 7q^2), \quad Q(q) = R(x_n, q) \times (t_n - q^2, t_n),$$
$$C_0 := \exp\Bigl(c(d)\Bigl|1 + \alpha^{-1} + (\alpha^{-1} + \alpha^{-1/2})\gamma\Bigr|^2\Bigr).$$
Indeed, we have $qB \le q_0 B \le 1$. We obtain the following recurrent relationship:
$$u(x_{n-1}, t_{n-1}) \le C_0 u(x_n, t_n), \quad 0 \le n \le N.$$
Therefore, one has the inequality
$$u(y,s) = u(x_0, t_0) \le C_0^N u(x_N, t_N) = C_0^N u(x,t).$$
Plugging in the value of N indicated above, we obtain
$$u(y,s) \le u(x,t) \exp\Bigl\{K_0\Bigl(\frac{t-s}{8q^2} + 1\Bigr)\Bigr\},$$
where $K_0 = \ln C_0$. Finally, we arrive at the estimate
$$u(y,s) \le u(x,t) \exp\Bigl\{K\Bigl(\frac{|x-y|^2}{t-s} + (B+1)^2\frac{t-s}{\delta^2} + 1\Bigr)\Bigr\},$$
where $K = K(d,\alpha,\gamma)$ has the desired form and $\delta = \min\{(\theta-1)r, \sqrt{s}\}$. □

Remark 2.3. In the case where our parabolic operator is written in the non-divergence form $L = \partial_t + a^{ij}\partial_{x_i}\partial_{x_j} + b^i\partial_{x_i}$ and the mappings $x \mapsto A(x,t)$ are locally Lipschitzian uniformly in t, the same lower bound holds if we replace b^i by $b^i - \partial_{x_j} a^{ij}$.

3. Lower estimates of densities

Here we obtain pointwise lower bounds for densities of measures satisfying weak elliptic and parabolic equations. To this end, we employ the results on constants in the Harnack inequality obtained above. In addition, in the elliptic case, we obtain sufficient conditions for the inclusion $\nabla \ln \varrho \in L^p(\mu)$ for all $p > 1$, which reinforces analogous results obtained in [12].

It is interesting to compare lower bounds obtained here with upper bounds from [10] and [11], which we recall for the reader's convenience. Suppose that a probability measure μ with a density ϱ satisfies equation (1.1), where the mappings A and A^{-1} are uniformly bounded and satisfy the conditions $a^{ij} \in W^{\beta,1}_{loc}(\mathbb{R}^d)$ and $|b| \in L^\beta(\mu)$ with some $\beta > d$.

Suppose we are given a positive function $\Phi \in W^{1,1}_{loc}(\mathbb{R}^d)$. Then, for the validity of the estimate
$$\varrho(x) \le C\Phi(x)^{-1}$$
the following conditions are sufficient: $\Phi, |\nabla\Phi|^\beta, |\nabla a^{ij}|^d \in L^1(\mu)$ (see Theorem 3.1 in [11]). For example, if the mapping A is uniformly bounded, uniformly invertible and uniformly Lipschitzian, then in order to have the estimate
$$\varrho(x) \le C\exp(-\kappa|x|^\beta) \tag{3.1}$$
with some $C > 0$, $\beta > 0$, and $\kappa > 0$, the following conditions are sufficient:
$$\exp(M|x|^\beta) \in L^1(\mu), \quad |b(x)| \le C_0 + C_1\exp(M_0|x|^\beta), \quad 0 \le M_0 < d^{-1}M.$$
Among the listed conditions only one, the inclusion $\exp(M|x|^\beta) \in L^1(\mu)$, is not explicitly expressed via A and b. In order to ensure also this condition in terms of the coefficients, it suffices to have the following estimate:
$$\limsup_{|x|\to\infty} |x|^{-\beta}(b(x), x) < -\beta d^2 M \sup_{i,j,x} |a^{ij}(x)|.$$
This follows by considering the Lyapunov function $V(x) = \exp(M|x|^\beta)$; see details in [3], [11].

Suppose that a probability measure μ on $\mathbb{R}^d\times[0,1)$ satisfies the parabolic equation (1.3), (1.4), where the mappings A and A^{-1} are uniformly bounded, the mappings $x \mapsto A(x,t)$ are uniformly Lipschitzian with a common constant, $|b| \in L^\beta(\mu)$ with some $\beta > d+2$ and $\sup_{t\in(0,1)} \|b(\,\cdot\,,t)\|_{L^d(\mu_t)} < \infty$. Suppose also that we are given a function $\Phi \ge c > 0$ on $\mathbb{R}^d$ with locally bounded second order derivatives such that $\varrho(x,0) \le C\Phi(x)^{-1}$, $\Phi \in L^1(\mu_0)$, and
$$\Phi^{1+\varepsilon},\ |L\Phi|^{\beta/2}\Phi^{1-\beta/2},\ |A\nabla\Phi|^\beta\Phi^{1-\beta} \in L^1(\mu), \quad \sup_{t\in[0,1)} \int_{\mathbb{R}^d} \Phi(x)\varrho(x,t)\,dx < \infty$$
with some $\varepsilon > 0$. Then, according to Theorem 3.3 in [10], which can be used by passing to the non-divergence form of our equation, for every $\tau < 1$ there exists a number C_τ such that
$$\varrho(x,t) \le C_\tau\Phi(x)^{-1} \quad \text{for almost all } (x,t) \in \mathbb{R}^d \times [0,\tau].$$
For example, let A and A^{-1} be uniformly bounded, let A be uniformly Lipschitzian in x, and let $\beta > d+2$, $r > 0$, $\varepsilon > 0$, and $K > 0$ be such that
$$|b| \in L^\beta(\mu),\ \exp\Bigl((2K+\varepsilon)|x|^r\Bigr) \in L^1(\mu),\ \sup_{t\in[0,1)} \int_{\mathbb{R}^d} \exp(K|x|^r)\varrho(x,t)\,dx < \infty.$$
Suppose that $\sup_{t\in(0,1)} \|b(\,\cdot\,,t)\|_{L^d(\mu_t)} < \infty$ and that the function $\exp(K|x|^r)\varrho(x,0)$ is bounded and integrable over $\mathbb{R}^d$. Then, for every $\tau < 1$, there exists a number $C(\tau)$ such that
$$\varrho(x,t) \le C(\tau)\exp(-K|x|^r), \quad (x,t) \in \mathbb{R}^d \times [0,\tau].$$
We obtain similar, although not of exactly the same order, lower bounds. For example, in the elliptic case, in place of κ in estimate (3.1) we get some other constant in the lower bound (but the exponent β does not change).

Suppose that a nonnegative locally bounded measure μ on $\mathbb{R}^d$ has a density ϱ such that $\varrho \in W^{2,1}(B)$ for every ball $B \subset \mathbb{R}^d$. Let the measure μ satisfy equation (1.1) on $\mathbb{R}^d$, where
$$\mathcal{L} = \partial_{x_i}(a^{ij}\partial_{x_j}) + b^i\partial_{x_i},$$
the matrix-valued mapping $A = (a^{ij})_{1\le i,j\le d}$ is measurable, the functions $\|A(x)\|$ and $\|A(x)^{-1}\|$ are locally bounded, and the coefficient $b = (b^i)$ is a measurable locally bounded vector field.

Let V be a continuous increasing function on $[0,\infty)$ with $V(0) > 0$.

Theorem 3.1. *Let $|b(x)| \le V(|x|/\theta)$, where $\theta > 1$. Set*

$$\alpha(r) := \sup_{|x|\le r} \|A(x)^{-1}\|, \quad \gamma(r) := \sup_{|x|\le r} \|A(x)\|.$$

Then there exists a positive number $K(d)$ depending only on d such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x) \ge \varrho(0) \exp\bigl\{-K(d)(\theta-1)^{-1}\alpha(\theta|x|)^{-1}\bigl(\gamma(\theta|x|) + V(|x|)|x|\bigr)\bigr\}.$$

In particular, if $\|A(x)\| \le \gamma$ and $\|A(x)^{-1}\| \le \alpha$, then Then there exists a positive number $K = K(d, \alpha, \gamma, \theta)$ such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x) \ge \varrho(0) \exp\bigl\{-K\bigl(1 + V(|x|)|x|\bigr)\bigr\}.$$

Proof. Let us fix x. Let us consider the ball $B(0, \theta|x|)$. Theorem 2.2 gives the desired estimate since $\sup\limits_{z\in B_{\theta|x|}(0)} |b(z)| \le V(|x|)$. $\square$

Example 3.1. Suppose that condition (2.2) is satisfied with $\Omega = \mathbb{R}^d$. If, for some numbers $c_1, c_2 > 0$ for almost all x one has the estimate

$$|b(x)| \le c_1|x|^\beta + c_2,$$

then there exists a constant K such that the following inequality holds:

$$\varrho(x) \ge \varrho(0) \exp\bigl\{-K\bigl(1 + |x|^{\beta+1}\bigr)\bigr\}.$$

If

$$\sup_{x,i,j}\Bigl[\|A(x)\| + \|A(x)^{-1}\| + |\nabla a^{ij}(x)|\Bigr] < \infty,$$

$$|b(x)| \le c_1|x|^\beta + c_2, \quad \limsup_{|x|\to\infty} |x|^{-\beta-1}(b(x), x) < 0,$$

then according to what we have proved we obtain the following two-sided estimate:

$$\exp\bigl\{-K_1\bigl(1 + |x|^{\beta+1}\bigr)\bigr\} \le \varrho(x) \le \exp\bigl\{-K_2\bigl(1 + |x|^{\beta+1}\bigr)\bigr\}.$$

For example, if $A = I$ and $b^i(x) = x_i$, then the measure with density $\varrho(x) = \exp(-|x|^2/2)$ is a solution. The results obtained above ensure the estimate

$$\exp(-K_1(1 + |x|^2)) \le \varrho(x) \le \exp(-K_2(1 + |x|^2))$$

with some numbers $K_1, K_2 > 0$, which gives a sufficiently adequate description of the decay at infinity, although does not yield a precise asymptotic.

It should be noted that the hypothesis that $\limsup_{|x|\to\infty} |x|^{-\beta-1}(b(x), x) < 0$ is only needed to ensure the integrability of $\exp(M|x|^\beta)$ and can be replaced by the latter.

By using the obtained estimates we can give an effectively verified condition for the membership in $L^p(\mu)$ for the logarithmic gradient $\nabla\varrho/\varrho$ of the measure μ. In the case $p = 2$ simple sufficient conditions were obtained in [1], [2]. The first general result for $p > 2$ has recently been established in [12]. The condition found below improves this result since we do not require the differentiability of the drift coefficient and assume lower regularity of the diffusion coefficient (it is assumed in [12] that $a^{ij} \in C^3(\mathbb{R}^d)$ and $b \in C^2(\mathbb{R}^d)$). This weakening of the conditions on the coefficients has become possible due to the fact that, unlike [12], we do not use methods of the theory of nonlinear equations. Let $L^p(\mu)$ denote the space of all measurable functions that are integrable of order p with respect to the measure μ on $\mathbb{R}^d$. Let $W^{p,1}(\mu)$ be the Sobolev space of functions that belong to the space $L^p(\mu)$ along with their generalized first order partial derivatives.

Suppose that μ is a probability measure on $\mathbb{R}^d$ satisfying the elliptic equation (1.1).

Theorem 3.2. *Let $a^{ij} \in C^{0,\delta}(\mathbb{R}^d) \cap W^{p_0,1}_{loc}(\mathbb{R}^d)$, where $\delta > 0$ and $p_0 > d$. Suppose that condition (2.2) is satisfied with $\Omega = \mathbb{R}^d$. Let a positive function $\Phi \in W^{1,1}_{loc}(\mathbb{R}^1)$ increase on $[0, +\infty)$ such that $\Phi(N+1) \le C\Phi(N)^{1+\varepsilon}$ with some $C, \varepsilon > 0$, and let the functions $\Phi(|x|)$ and $\Phi'(|x|)^{p_1}$ with some $p_1 > d$ be integrable against the measure μ on $\mathbb{R}^d$. Suppose also that there exist numbers $p > 1$, $\theta > 1$, and $\gamma \in [0, 1/d)$ such that*

$$|b(x)| \le C_0\Phi(|x| - \theta)^{\gamma}, \quad |\nabla a^{ij}(x)|^d \le C_0\Phi(|x|),$$

$$\sum_{N=1}^{\infty} N^{d-1}\Phi(N)^{-q} < \infty, \quad \text{where } q := 1 - \gamma(2p + \varepsilon d).$$

Then $\ln \varrho \in W^{p,1}(\mu)$.

Proof. As shown in the above mentioned Theorem 3.1 of paper [11], under our assumptions the density ϱ is estimated as follows:

$$\varrho(x) \le C_1\Phi(|x|)^{-1}.$$

For any fixed $x \in \mathbb{R}^d$ with $|x| > 2$ we have

$$B := \sup_{z \in B(x,\theta)} |b(z)| \le C_0\Phi(|x|)^{\gamma}.$$

Let $0 < r < \min\{1/B, 1\}$. Then $rB < 1$. Hence by Theorem 2.2 there exists a constant $K = K(\alpha, \gamma, d, \theta, \delta)$ such that for every $y \in B(x, r)$ one has the inequality

$$K^{-1}\varrho(x) \le \varrho(y) \le K\varrho(x). \tag{3.2}$$

Let us estimate the following integral:

$$I := \int_{B(x,r)} \frac{|\nabla\varrho(y)|^p}{\varrho(y)^{p-1}}\, dy.$$

According to inequality (3.2) we have

$$I \le \varrho(x)^{1-p}K^{p-1} \int_{B(x,r)} |\nabla\varrho(y)|^p\, dy.$$

By using Corollary 2.1 and the estimate

$$\int_{B(x,r)} |b|^p \varrho^p\, dy \le B^p \int_{B(x,r)} \varrho^p\, dy$$

we obtain that there exist constants C_2 and C_3 independent of r and x such that

$$\int_{B(x,r)} |\nabla\varrho(y)|^p\, dy \le C_2 r^{-p}(B^p + 1) \int_{B(x,r)} |\varrho(y)|^p\, dy \le C_3 r^{d-p} B^p \varrho(x)^p.$$

Therefore,

$$I \le C_4 r^{d-p} B^p \varrho(x),$$

where the number C_4 is independent of x and r. By the above estimates for B and ϱ, letting $r = 2^{-1}\Phi(x)^{-\gamma}$, we obtain the inequality

$$\int_{B(x,r)} \frac{|\nabla\varrho(y)|^p}{\varrho(y)^{p-1}}\, dy \le C_5\Phi(x)^{2\gamma p - \gamma d - 1},$$

where C_5 does not depend on x and r. Let $Q(x, r)$ be the open cube centered at the point x with the edge length r. Then this estimate holds, of course, for $Q(x, r) \subset B(x, r)$. Set

$$Q_N = Q(0, N+1) \backslash Q(0, N), \quad N > 1.$$

There exists a constant C_6, which depends only on the dimension d, such that Q_N can be covered by cubes $Q(x, r)$ with pairwise disjoint interiors and $r = 2^{-1}(C_0+1)^{-1}\Phi(N+1)^{-\gamma}$, whose total number does not exceed

$$C_6 N^{d-1}\Phi(N+1)^{d\gamma}.$$

According to what has been proved above we obtain

$$I_N := \int_{Q_N} \frac{|\nabla \varrho(y)|^p}{\varrho(y)^{p-1}}\, dy \le C_5 C_6 N^{d-1} \Phi(N+1)^{d\gamma} \Phi(N)^{2\gamma p - \gamma d - 1}.$$

Since $\Phi(N+1) \le C\Phi(N)^{1+\varepsilon}$, one has the inequality

$$N^{d-1}\Phi(N+1)^{d\gamma}\Phi(N)^{2\gamma p-\gamma d-1} \le CN^{d-1}\Phi^{-q}.$$

Hence the series $\sum_{N=1}^\infty I_N$ converges. Therefore, $|\nabla \ln \varrho|^p \varrho \in L^1(\mathbb{R}^d)$. $\square$

Corollary 3.1. *Let $a^{ij} \in C^{0,\delta}(\mathbb{R}^d) \cap W^{p_0,1}_{loc}(\mathbb{R}^d)$, where $\delta > 0$ and $p_0 > d > 1$. Suppose that condition (2.2) is satisfied with $\Omega = \mathbb{R}^d$. Let $p > 1$. Suppose that for some $M > 0$ and $\beta > 0$ the function $\exp(M|x|^\beta)$ is integrable with respect to the measure μ on $\mathbb{R}^d$ and that*

$$|b(x)| \le C_0 \exp\bigl\{\kappa |x|^\beta\bigr\}, \quad |\nabla a^{ij}(x)| \le C_0 \exp\bigl\{\kappa|x|^\beta\bigr\}, \quad \textit{where } 0 < 2\kappa d \max(p,d) < M.$$

Then $\ln \varrho \in W^{p,1}(\mu)$. In particular, if for every $\kappa > 0$ there is a number $C(\kappa)$ such that

$$|b(x)| + |\nabla a^{ij}(x)| \le C(\kappa) \exp\bigl\{\kappa |x|^\beta\bigr\},$$

then $\ln\varrho \in W^{p,1}(\mu)$ for all $p \in [1, +\infty)$.

Proof. There is a sufficiently small number $\varepsilon_0 > 0$ such that for $\gamma := (2\max(p,d))^{-1} - \varepsilon_0$ one has $\gamma > 0$ and $\kappa\gamma^{-1} < Md^{-1}$. Let us take for Φ the function $\Phi(r) = \exp(M_0|r|^\beta)$, where M_0 is chosen in the interval $(\kappa\gamma^{-1}, Md^{-1})$. Since $2p\gamma < 1$, there is $\varepsilon > 0$ such that $q = 1 - \gamma(2p + \varepsilon d) > 0$. Set $\theta = 2$. We observe that

$$\exp\bigl(\kappa r^\beta\bigr) \le C_1 \exp\bigl(\gamma M_0 |r-2|^\beta\bigr)$$

since $\kappa < \gamma M_0$. Hence $|b(x)| \le C_0 C_1 \Phi(|x| - 2)^\gamma$. It is also clear that for some $p_1 > d$ the function $\Phi'(|x|)^{p_1}$ is integrable with respect to μ since $M_0 d < M$. It is easily seen that all other assumptions of Theorem 3.2 are fulfilled. $\square$

Similarly we prove the following result.

Corollary 3.2. *Let $a^{ij} \in C^{0,\delta}(\mathbb{R}^d) \cap W^{p_0,1}_{loc}(\mathbb{R}^d)$, where $\delta > 0$ and $p_0 > d > 1$. Suppose that condition (2.2) is satisfied with $\Omega = \mathbb{R}^d$. Let $p > 1$. Suppose that for some $\beta > 0$ the function $|x|^\beta$ is integrable with respect to the measure μ on $\mathbb{R}^d$ and that*

$$|b(x)| \le C_0 + C_0|x|^\sigma, \quad |\nabla a^{ij}(x)| \le C_0 + C_0 |x|^{\beta/d}, \quad \textit{where } 0 < \sigma d < \beta.$$

Then $\ln\varrho \in W^{p,1}(\mu)$ for every $p \in [1, (1+\beta d^{-1})(2\sigma)^{-1})$.

Let us proceed to the parabolic equation. Let a measurable matrix-valued mapping $A = (a^{ij})_{1\le i,j\le d}$ on $\mathbb{R}^d \times (0,1)$ satisfy condition (2.10) with $\Omega = \mathbb{R}^d$ and let b be a measurable vector field on $\mathbb{R}^d \times (0,1)$.

Suppose that a nonnegative measure μ with a density ϱ on $\mathbb{R}^d \times (0,1)$ such that $\varrho \in \mathbb{H}^{2,1}(B \times J)$ for any ball $B \subset \mathbb{R}^d$ and any interval J with compact closure in $(0,1)$ satisfies equation (1.1).

Let V be a continuous increasing function on $[0,\infty)$ with $V(0) > 0$.

Theorem 3.3. *Let $\sup_{t\in(0,1)} |b(x,t)| \le V(|x|/\theta)$ for almost all $x \in \mathbb{R}^d$, where $\theta > 1$. Let*

$$\alpha(r) := \sup_{t\in(0,1),|x|\le r} \|A(x,t)^{-1}\|, \quad \gamma(r) := \sup_{t\in(0,1),|x|\le r} \|A(x,t)\|.$$

Then, there exists a positive number $K = K(d)$ such that the continuous version of the function ϱ satisfies the inequality

$$\varrho(x,t) \ge \varrho(0,s) \exp\Bigl\{-K(d)\bigl|1 + \alpha(\theta|x|)^{-1} + (\alpha(\theta|x|)^{-1} + \alpha(\theta|x|)^{-1/2})\gamma(\theta|x|)\bigr|^2$$
$$\times \Bigl(1 + \frac{t-s}{s} V(|x|)^2 + \frac{t}{t-s}|x|^2\Bigr)\Bigr\}, \quad 0 < s < t < 1,\ x \in \mathbb{R}^d.$$

In particular, if $\|A(x,t)\| \le \gamma$ *and* $\|A(x,t)^{-1}\| \le \alpha$*, then there exists a positive number* $K = K(d,\alpha,\gamma,\theta)$ *such that the continuous version of the function* ϱ *satisfies the inequality*

$$\varrho(x,t) \ge \varrho(0,s)\exp\Bigl\{-K\Bigl(1+\frac{t-s}{s}V(|x|)^2+\frac{t}{t-s}|x|^2\Bigr)\Bigr\}, \quad 0<s<t<1,\ x\in\mathbb{R}^d.$$

Proof. We take $\Omega = B(0,\theta|x|)$ and $y=0$ in Theorem 2.5. This gives the required estimate. ☐

Corollary 3.3. *Suppose that under the assumptions of Theorem* 3.3 *one has* $\|A(x,t)\| \le \gamma$ *and* $\|A(x,t)^{-1} \le \alpha$ *and that for almost all* $t \in (0,1)$ *the function* $x \mapsto \varrho(x,t)$ *does not vanish identically, then for every closed interval* $[\tau_1,\tau_2]$ *in* $(0,1)$ *there exists a number* $K = K(d,\alpha,\gamma,\theta,\tau_1,\tau_2) \ge 0$ *such that for all* $t \in [\tau_1,\tau_2]$ *and* $x \in \mathbb{R}^d$ *the following inequality holds:*

$$\exp\Bigl(-K\bigl(1+V(|x|)^2+|x|^2\bigr)\Bigr) \le \varrho(x,t) \le \exp\Bigl(K\bigl(1+V(|x|)^2+|x|^2\bigr)\Bigr).$$

Proof. It follows from Theorem 2.5 and our hypothesis that the function $t \mapsto \varrho(0,t)$ has no zeros on $(0,1)$. Let us consider the interval $[\tau',\tau'']$, where $\tau' = \tau_1/2$, $\tau'' = 1-(1-\tau_2)/2$. The first of the estimates we are proving follows by Theorem 3.3, in which one should take $s=\tau'$. The second estimate is clear from Theorem 2.5. ☐

Example 3.2. If $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded and for some constants $c_1>0$ and $c_2>0$ the inequality

$$\sup_{t\in(0,1)} |b(x,t)| \le c_1(1+|x|^\beta)$$

holds for almost all x, then there exists a positive number K such that

$$\varrho(x,t) \ge \varrho(0,s)\exp\Bigl\{-K\Bigl(1+\frac{t-s}{s}|x|^{2\beta}+\frac{1}{t-s}|x|^2\Bigr)\Bigr\}.$$

For example, if

$$L = \partial_t + \frac{1}{2}\Delta,$$

then the measure $(2\pi t)^{-1/2}e^{-|x|^2/2t}\,dx\,dt$ is a solution. Our results yield a number $K>0$ such that $\varrho \ge e^{-K(\delta)|x|^2/t}$ in every strip $\mathbb{R}^d\times(\delta,1)$, where $\delta>0$. Similarly, our lower estimate is exact in the case of a linear drift coefficient, but it becomes less precise in the case of a quadratic growth of $|b|$; e.g., if $\varrho(x,t) = C\exp(-|x|^3)$, then $\exp(-K|x|^4)$ appears in our lower bound.

Let us give conditions on the coefficients A and b ensuring two-sided exponential estimates of the density of the solution in the parabolic case.

Example 3.3. Suppose that $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded, the functions $x \mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant, and that for some $r>1$, $\sigma\ge 0$, $K>0$, and $K'>K$ we have

$$|b(x,t)| \le C + C|x|^{r-1+\sigma}, \quad \varrho(x,0) \le C\exp(-K'|x|^r),$$
$$(x,b(x,t)) \le c_1 - c_2|x|^r, \quad c_2 > 2rK\sup_{x,t}\|A(x,t)\|.$$

Then, for every closed interval $[\tau_1,\tau_2]\subset(0,1)$, there exist numbers C_1, C_2, and K_0 such that

$$C_1\exp\Bigl(-K_0|x|^{2r+2\sigma-2}-K_0|x|^2\Bigr) \le \varrho(x,t) \le C_2\exp(-K|x|^r), \quad (x,t)\in\mathbb{R}^d\times[\tau_1,\tau_2].$$

The upper bound follows from Example 3.1 of paper [10], and the lower bound follows by the above results. Unlike the elliptic case, here there is no coincidence of the powers of $|x|$ in the lower and upper bounds. We observe that the indicated conditions also give the existence of a solution $\mu = \mu_t\,dt$ with probability measures μ_t with an arbitrary initial distribution (see [8], [19]).

One more application of our results is concerned with the proof of the existence of finite entropy of any solution with respect to the space variable at any positive t for every initial distribution. The existence of finite entropy, which is useful in many respects, is necessary for applying the results of paper [10], which give the integrability of $|\nabla\varrho(x,t)|^2/\varrho(x,t)$.

Let $V \in C^2(0,\infty)$ be a continuous increasing function on the half-line $[0,\infty)$ such that $V(0)>0$ and $\lim\limits_{r\to\infty} V(r) = +\infty$.

Corollary 3.4. *Let the measures μ_t be probabilistic. Suppose that $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded and the function $|x|^2+V(|x|)$ is integrable with respect to the measure μ and that for some $\theta>1$ the following inequality holds:*

$$\sup_{t\in(0,1)} |b(x,t)|^2 \le V(|x|/\theta), \quad x\in\mathbb{R}^d.$$

Then, for almost every $s\in(0,1)$, we have

$$\int_{\mathbb{R}^d} \varrho(x,s)|\ln\varrho(x,s)|\,dx < \infty.$$

Proof. By Harnack's inequality $\varrho(t,x)>0$ for any $t>0$. Let $[\tau_1,\tau_2]\subset(0,1)$. According to Corollary 3.3, there exists a number $K=K(d,\alpha,\gamma,\theta,\tau_1,\tau_2)>0$ such that

$$\exp\Bigl\{-K\bigl(1+|x|^2+V(|x|)\bigr)\Bigr\} \le \varrho(x,s) \le \exp\Bigl\{K\bigl(1+|x|^2+V(|x|)\bigr)\Bigr\}, \quad s\in[\tau_1,\tau_2], x\in\mathbb{R}^d.$$

By Fubini's theorem, for almost every $s\in(0,1)$ the function $x\mapsto(|x|^2+V(|x|))\varrho(x,s)$ is integrable over $\mathbb{R}^d$. For such s we obtain

$$\int_{\mathbb{R}^d} \varrho(x,s)|\ln\varrho(x,s)|\,dx \le \int_{\mathbb{R}^d} K(1+|x|^2+V(|x|))\varrho(x,s)\,dx.$$

The corollary is proven. $\square$

Corollary 3.5. *Let $A\in H^{2,1}_{loc}(\mathbb{R}^d\times(0,1))$, let the measures μ_t be probabilistic, and let $\nu=\mu_0$. Let $V(r)\ge c_1+c_2r^2$ and let $A(x,t)$ and $A(x,t)^{-1}$ be uniformly bounded. Suppose that the function $V(|x|)$ is integrable with respect to ν and that the function $V_0(x):=V(|x|)$ satisfies the inequality $\mathcal{L}V_0(x)\le C$ with some constant $C>0$. In addition, suppose that for some $\theta>1$ the following inequality holds:*

$$\sup_{t\in(0,1)} |b(x,t)|^2 \le V(|x|/\theta), \quad x\in\mathbb{R}^d.$$

Then we have

$$\sup_{s\in(0,1)} \int_{\mathbb{R}^d} \varrho(x,s)|\ln\varrho(x,s)|\,dx < \infty.$$

Proof. Set $b_0^i := \partial_{x_j}a^{ij}+b^i$. Then

$$\mathcal{L} = \partial_t + a^{ij}\partial_{x_i}\partial_{x_j} + b_0^i\partial_{x_i}.$$

Therefore, we can apply the results of paper [19], where divergence form operators are considered. According to Lemma 2.2 from [19], for almost all $s\in[0,1)$ we have

$$\int_{\mathbb{R}^d} V(|x|)\varrho(x,s)\,dx \le Cs + \int_{\mathbb{R}^d} V(|x|)\,\nu(dx).$$

By the continuity of ϱ and Fatou's theorem this inequality holds for every $s\in(0,1)$, which enables us to apply the same reasoning as in the previous corollary. $\square$

Example 3.4. Suppose that in Corollary 3.5 it is known additionally that the functions $x\mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant. Then, for every closed interval $[\tau_1,\tau_2]\subset(0,1)$, according to Theorem 2.1 of [10] we have

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^d} \frac{|\nabla\varrho(x,t)|^2}{\varrho(x,t)}\,dx\,dt < \infty.$$

Example 3.5. Suppose that $A(x,t)$ and $A(x,t)^{-1}$ are uniformly bounded, the functions $x \mapsto a^{ij}(x,t)$ are uniformly Lipschitzian with a common constant and that

$$|b(x,t)| \le c_0 \exp(c|x|^r), \quad (b(x,t),x) \le c_1 - c_2|x|^r, \quad c_2 > crd^2.$$

Let the function $\exp(c|x|^r)$ be integrable with respect to $\nu = \mu_0$. Then

$$\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^d} \frac{|\nabla \varrho(x,t)|^2}{\varrho(x,t)}\, dx\, dt < \infty$$

for every closed interval $[\tau_1, \tau_2] \subset (0,1)$. In order to justify this example it suffices to take $V(z) = \exp(M|z|^r)$ with $M < c$ sufficiently close to c.

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