Analisi Matematica. —-*Dissipative stochastic equations in Hilbert space with time dependent coefficients.* Nota di GIUSEPPE DA PRATO e MICHAEL RÖCKNER, presentata dal Corrisp. G. Da Prato.

ABSTRACT. —We prove existence and, under an additional assumption, uniqueness of an evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ for a stochastic differential equation with time dependent dissipative coefficients. Then we prove that the corresponding transition evolution operator $P_{s,t}\varphi$ is attracted as $t \to +\infty$ to a limit curve (which is independent of s) for any continuous and bounded "observable" φ.

KEY WORDS. —Dissipative stochastic equations, evolution systems of measures, mixing.

RIASSUNTO. —*Equazioni stocastiche dissipative is spazi di Hilbert aventi coefficienti dipendenti dal tempo.* Proviamo l'esistenza e, sotto un'ipotesi addizionale, l'unicità di un sistema di evoluzione di misure $(\nu_t)_{t\in\mathbb{R}}$ per un'equazione differenziale stocastica con coefficienti dipendenti dal tempo. Inoltre dimostriamo che l'operatore di transizione corrispondente $P_{s,t}\varphi$ è attratto per $t \to +\infty$ a una curva limite (independente da s) per ogni"osservabile" φ continua e limitata.

1 Introduction

We are given a separable Hilbert space H (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$); we denote by $L(H)$ the space of all linear bounded operators in H and by $\mathcal{P}(H)$ the set of all Borel probability measures on H. We are also given a cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ in H.

We are concerned with the following stochastic differential equation

$$dX = (AX + F(t,X))dt + \sqrt{C}\, dW(t), \quad X(s) = x \in H, \tag{1.1}$$

where $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup e^{tA} in H, $C \in L(H)$ and $F\colon D(F) \subset \mathbb{R} \times H \to H$ is such that $F(t,\cdot)$ is dissipative for all $t \in \mathbb{R}$.

When s is negative, in order to give a meaning to equation (1.1), we shall extend $W(t)$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ for all $t < 0$. To do so we take another cylindrical process $W_1(t)$ independent of $W(t)$ and set

$$\overline{W}(t) = \begin{cases} W(t) \text{ if } t \geq 0, \\ \\ W_1(-t) \text{ if } t \leq 0. \end{cases}$$

Moreover, we denote by $\overline{\mathcal{F}}_t$ the σ–algebra generated by $\overline{W}(s), s \leq t,\ t \in \mathbb{R}, k \in \mathbb{N}$.

Concerning A, C, F we shall assume that

Hypothesis 1.1 *(i) There is $\omega > 0$ such that $\langle Ax, x\rangle \leq -\omega|x|^2$ for all $x \in D(A)$.*

(ii) $C \in L(H)$ is symmetric, nonnegative and such that

$$\int_0^{+\infty} \text{Tr}\,[e^{tA}Ce^{tA^*}]dt < +\infty.$$

(iii) $F : \mathbb{R} \times H \to H$ is continuous and there exist $M > 0$ and $K > 0$ such that

$$|F(t, 0)| \leq M, \quad |F(t, x)-F(t, y)| \leq K|x-y|, \quad \textit{for all}\, x, y \in H,\ t \in \mathbb{R}.$$

Moreover,

$$\langle F(t, x) - F(t, y), x - y\rangle \leq 0, \quad \textit{for all}\, x, y \in H,\ t \in \mathbb{R}.$$

A mild solution $X(t, s, x)$ of (1.1) is an adapted stochastic process $X \in C([s, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}))$ such that

$$X(t, s, x) = e^{(t-s)A}x + \int_s^t e^{(t-r)A}F(r, X(r, s, x))dr + W_A(t, s), \quad t \geq s, \tag{1.2}$$

where $W_A(t, s)$ is the *stochastic convolution*

$$W_A(t, s) = \int_s^t e^{(t-r)A}\sqrt{C}\, d\overline{W}(r), \quad t \geq s. \tag{1.3}$$

It is well known that, in view of Hypothesis 1.1–(ii), $W_A(t, s)$ is a Gaussian random variable in H with mean 0 and covariance operator $Q_{t,s}$ given by

$$Q_{t.s}x = \int_s^t e^{rA}Ce^{rA^*}xdr, \quad t \geq s,\ x \in H \tag{1.4}$$

and that thre exists a unique mild solution of (1.1), see e.g. [5]. We define the transition evolution operator

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t,s,x))], \quad t \geq s, \ \varphi \in C_b(H),$$

where $C_b(H)$ is the Banach space of all continuous and bounded mappings $\varphi : H \to \mathbb{R}$ endowed with the sup norm

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$

Remark 1.2 It is easy to check that $P_{s,t}$ is *Feller*, that is $P_{s,t}\varphi \in C_b(H)$ for all $varphi \in C_b(H)$ and any $s < t$.

The aim of the paper is to prove the existence and, under a suitable condition, uniqueness of an evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ indexed by $\mathbb{R}$, see [2]. This means that each ν_t is a probability measure on H and

$$\int_H P_{s,t}\varphi(x)\nu_s(dx) = \int_H \varphi(x)\nu_t(dx) \quad \text{for all } \varphi \in C_b(H), \ s < t. \tag{1.5}$$

This concept is the natural generalization of the notion of an invariant measure to non autonomous systems. We notice that an evolution system of measures indexed by $\mathbb{R}$ is a measure solution of the corresponding (dual) Kolmogorov equation in all the real line. So, it is a generalization of a measure solution of (1.1) on half–lines, see the paper [1].

Using the system $(\nu_t)_{t\in\mathbb{R}}$ we are able to study the asymptotic behaviour of $P_{s,t}\varphi(x)$. We prove that

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_H \varphi(x)\nu_t(dx), \tag{1.6}$$

and

$$\lim_{t\to+\infty} \left[P_{s,t}\varphi(x) - \int_H \varphi(x)\nu_t(dx)\right] = 0. \tag{1.7}$$

The second result implies that $P_{s,t}\varphi(x)$ approaches as $t \to +\infty$ a curve, parametrized by t, which is independent of s and x. This is the natural generalization of the strongly mixing property for an autonomous dissipative sytem.

In a paper in preparation we shall study the case when the coefficient $F(t,x)$ is singular, generalizing the results in [3].

2 Existence and uniqueness of an evolution family of measures indexed by $\mathbb{R}$

It is convenient to write equation (1.1) as a family of deterministic equations indexed by $\omega \in \Omega$. Setting $Y(t) = X(t,s,x) - W_A(t,s)$, we see that $Y(t)$ fulfills the deterministic evolution equation,

$$Y'(t) = AY(t) + F(t, Y(t) + W_A(t,s)), \quad Y(s) = x. \tag{2.1}$$

Lemma 2.1 *For any $m \in \mathbb{N}$ there is $C_m > 0$ such that*

$$\mathbb{E}\left(|X(t,s,x)|^{2m}\right) \le C_m(1 + e^{-m\omega(t-s)}|x|^{2m}). \tag{2.2}$$

Proof. Multiplying (2.1) by $|Y(t)|^{2m-2}Y(t)$ and taking into account Hypothesis 1.1, yields for a suitable constant C_m^1,

$$\begin{aligned}
&\frac{1}{2m}\,\frac{d}{dt}\,|Y(t)|^{2m} \le -\omega|Y(t)|^{2m} + \langle F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&+\langle F(t, Y(t) + W_A(t,s)) - F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&\le -\omega|Y(t)|^{2m} + \langle F(t, W_A(t,s)), Y(t)\rangle |Y(t)|^{2m-2}\\
&\le -\frac{\omega}{2}\,|Y(t)|^{2m} + C_m^1|F(W_A(t,s))|^{2m}.
\end{aligned}$$

By a standard comparison result it follows that

$$|Y(t)|^{2m} \le e^{-m\omega(t-s)}|x|^{2m} + 2mC_m^1\int_s^t e^{-m\omega(t-\sigma)}|F(\sigma, W_A(t,\sigma))|^{2m}d\sigma,$$

and finally we find that, for some constant C_m^2,

$$\begin{aligned}
&|X(t,s,x)|^{2m} \le C_m^2 e^{-m\omega(t-s)}|x|^{2m}\\
&+C_m^2\left(\int_s^t e^{-m\omega(t-\sigma)}|F(\sigma, W_A(t,\sigma))|^{2m}d\sigma + |W_A(t,s)|^{2m}\right).
\end{aligned} \tag{2.3}$$

Now the conclusion follows taking expectation, recalling that in view of Hypothesis 1.1,

$$|F(t,x)| \le |F(t,0)| + |F(t,x) - F(t,0)| \le M + K|x|, \quad t \in \mathbb{R},\ x \in H,$$

and using the fact that

$$\sup_{t\in\mathbb{R}, t\ge s} \mathbb{E}|W_A(t,s)|^{2m} < +\infty.$$

□

The following lemma gives a generalization to the time dependent case of a result proved in [4].

Lemma 2.2 *Assume that Hypothesis* 1.1 *holds. Then for any* $t \in \mathbb{R}$, *there exists* $\eta_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ *(independent of* x*) such that*

$$\lim_{s \to -\infty} X(t, s, x) = \eta_t \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.4}$$

Moreover,

$$\mathbb{E}|X(t, s, x) - \eta_t|^2 \le 2e^{-2\omega(t-s)}(|x|^2 + C_2). \tag{2.5}$$

Proof. Let $h > 0$ and set $Z(t) = X(t, s, x) - X(t, s - h, x)$, $t \ge s$. Then $Z(t)$ is the mild solution of the following problem

$$\begin{cases} Z'(t) = AZ(t) + F(t, X(t, s, x)) - F(t, X(t, s - h, x)) \\ \\ Z(s) = x - X(s, s - h, x). \end{cases} \tag{2.6}$$

Multiplying (2.6) by $Z(t)$ and taking into account Hypothesis 1.1, yields

$$\frac{1}{2}\,\frac{d}{dt}\,|Z(t)|^2 \le -\omega|Z(t)|^2.$$

Therefore

$$|X(t, s, x) - X(t, s - h, x)|^2 = |Z(t)|^2 \le e^{-2\omega(t-s)}|x - X(s, s - h, x)|^2.$$

Now, by Lemma 2.1 it follows that

$$\mathbb{E}|X(t, s, x) - X(t, s - h, x)|^2 \le 2e^{-2\omega(t-s)}(|x|^2 + C_2(1 + e^{-2\omega h}|x|^2). \tag{2.7}$$

Consequently, for any $t \in \mathbb{R}$ and any $x \in H$, there exists the limit

$$\lim_{s \to -\infty} X(t, s, x) := \eta_t(x) \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Moreover, letting $h \to \infty$, yields (2.5) (if we know that $\eta_t(x)$ is independent of x).

It remains to show that $\eta_t(x)$ is independent of x.

Let $x, y \in H$ and set $V(t) = X(t, s, x) - X(t, s, y)$. Then $V(t)$ is the solution of the following problem

$$\begin{cases} V'(t) = AV(t) + F(t, X(t, s, x)) - F(t, X(t, s, y)) \\ \\ V(s) = x - y. \end{cases} \tag{2.8}$$

Multiplying (2.8) by $V(t)$ and taking into account Hypothesis 1.1, yields

$$\frac{1}{2}\,\frac{d}{dt}\,|V(t)|^2 \le -\omega|V(t)|^2,$$

so that

$$|X(t,s,x)-X(t,s,y)|^2 = |V(t)|^2 \le e^{-2\omega(t-s)}|x-y|^2.$$

Letting $s\to-\infty$ we see that $\eta_t(x)=\eta_t(y)$, as required. □

In the following we shall denote by ν_t the law of η_t, $t\in\mathbb{R}$.

Proposition 2.3 *$(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures indexed by $\mathbb{R}$,*

$$\int_H P_{s,t}\varphi(x)\nu_s(dx) = \int_H \varphi(x)\nu_t(dx), \quad s\le t,\ \varphi\in C_b(H). \tag{2.9}$$

Moreover, for all $\varphi\in C_b(H)$ we have

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_H \varphi(y)\nu_t(dy), \quad x\in H \tag{2.10}$$

Proof. Let us first prove (2.10). Let $\varphi\in C_b(H)$. Letting $s\to-\infty$ in the identity

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t,s,x))],$$

and recalling (2.4), yields

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \mathbb{E}[\varphi(\eta_t)] = \int_H \varphi(y)\nu_t(dy),$$

and (2.10) is proved. Let us prove (2.9). Let $s<t<\tau$. Letting $s\to-\infty$ in the identity

$$P_{s,t}P_{t,\tau}\varphi(x) = P_{s,\tau}\varphi(x),$$

recalling Remark 1.2 and taking into account (2.10), yields

$$\int_H P_{t,\tau}\varphi(y)\nu_t(dy) = \int_H \varphi(y)\nu_\tau(dy).$$

□

The following result give informations on the asymptotic behaviour of $P_{s,t}\varphi(x)$ when $t\to\infty$.

Proposition 2.4 *Let $\varphi\in C_b^1(H)$. Then for any $s\in\mathbb{R}$ and $x\in H$, we have*

$$\lim_{t\to+\infty}\left[P_{s,t}\varphi(x) - \int_H \varphi(x)\nu_t(dx)\right] = 0. \tag{2.11}$$

Proof. Fix $s \in \mathbb{R}$ and $x \in H$ and choose $s_1 < s$. Set, for $t > s$

$$X(t) = X(t, s, x), \quad Y(t) = X(t, s_1, x)$$

and $Z(t) = X(t) - Y(t)$. Then we have

$$\frac{d}{dt}\, Z(t) = AZ(t) + F(t, X(t)) - F(t, Z(t)), \quad Z(s) = x - X(s, s_1, x).$$

Multiplying scalarly both sides of this identity by $Z(t)$ and taking into account the dissipativity of $F(t, \cdot)$ yields

$$\frac{d}{dt}\, |Z(t)|^2 \leq 2\omega |Z(t)|^2,$$

so that

$$|X(t, s, x) - X(t, s_1, x)|^2 = |Z(t)|^2 \leq e^{-2\omega(t-s)} |x - X(s, s_1, x)|^2.$$

Letting $s_1 \to -\infty$ yields

$$|X(t, s, x) - \eta_t|^2 = |Z(t)|^2 \leq e^{-2\omega(t-s)} |x - \eta_s|^2.$$

Consequently

$$\left| P_{s,t}\varphi(x) - \int_H \varphi(x)\nu_t(dx) \right|^2 = |\mathbb{E}[\varphi(X(t, s, x))] - \mathbb{E}[\varphi(\eta_t)]|^2$$

$$\leq \|\varphi\|^2_{C_b^1(H)} \mathbb{E}(|X(t, s, x) - \eta_t|^2) \leq \|\varphi\|^2_{C_b^1(H)} e^{-2\omega(t-s)} \mathbb{E}(|x - \eta_s|^2),$$

which yields the conclusion. □

We end the paper with a uniqueness result.

Proposition 2.5 *Assume that $(\zeta_t)_{t \in \mathbb{R}}$ is an evolution system of measures indexed by $\mathbb{R}$ and that there exists $C > 0$ such that*

$$\sup_{t \in \mathbb{R}} \int_H |x|^2 \zeta_t(dx) \leq C.$$

Then $\zeta_t = \nu_t$ for all $t \in \mathbb{R}$.

Proof. Let $\varphi \in C_b^1(H)$. By the assumption we have for $s < t$

$$\int_H P_{s,t}\varphi(x)\zeta_s(dx) = \int_H \varphi(x)\zeta_t(dx).$$

We claim that

$$\lim_{s\to-\infty}\int_H P_{s,t}\varphi(x)\zeta_s(dx)=\int_H \varphi(x)\nu_t(dx). \tag{2.12}$$

By the claim it follows that $\zeta_t=\nu_t$ by the arbitrariness of φ. To prove the claim write

$$\begin{aligned}\int_H P_{s,t}\varphi(x)\zeta_s(dx) &= \int_H\left(P_{s,t}\varphi(x)-\int_H\varphi(y)\nu_t(dy)\right)\zeta_s(dx)\\ &\quad+\int_H\varphi(y)\nu_t(dy).\end{aligned} \tag{2.13}$$

But, since

$$P_{s,t}\varphi(x)-\int_H\varphi(y)\nu_t(dy)=\mathbb{E}(\varphi(X(t,s,x)-\varphi(\eta_t)),$$

we have, taking into account (2.5)

$$\begin{aligned}|P_{s,t}\varphi(x)-\int_H\varphi(y)\nu_t(dy)|^2 &\le \|\varphi\|^2_{C^1_b(H)}\mathbb{E}(|X(t,s,x)-\eta_t)|^2)\\ &\le 2e^{-2\omega(t-s)}(|x|^2+C_2)\|\varphi\|^2_{C^1_b(H)}.\end{aligned}$$

So,

$$\begin{aligned}&\left|\int_H\left(P_{s,t}\varphi(x)-\int_H\varphi(y)\nu_t(dy)\right)\zeta_s(dx)\right|\\ &\le 2\|\varphi\|^2_{C^1_b(H)}e^{-2\omega(t-s)}\left(C_2+\int_H|x|^2\zeta_s(dx)\right),\end{aligned}$$

and the conclusion follows. □

References

[1] V. Bogachev, G. Da Prato and M. Röckner, *Existence of solutions to weak parabolic equations for measures*, Proc. London Math. Soc., (3), 88, 753-774, 2004.

[2] G. Da Prato and M. Röckner, *Singular dissipative stochastic equations in Hilbert spaces*, Probab. Theory Relat. Fields, **124**, 2, 261–303, 2002.

[3] G. Da Prato and M. Röckner, *A note on non autonomous stochastic differential equations*, Proceedings of the Ascona Conference 2005 on Stochastic Analysis, to appear.

[4] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[5] P. Guiotto, *Cauchy problem for nonautonomous Kolmogorov equations.* Stochastic Anal. Appl. 14 no. 4, 393–410, 1996.