STOCHASTIC NONLINEAR PERRON-FROBENIUS THEOREM*

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We establish a stochastic nonlinear analogue of the Perron-Frobenius theorem on eigenvalues and eigenvectors of positive matrices. The result is formulated in terms of an automorphism T of a probability space $(\Omega, \mathcal{F}, P)$ and a random mapping $D(\omega, \cdot) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. Under assumptions of monotonicity and homogeneity of $D(\omega, \cdot)$, we prove the existence of scalar and vector measurable functions $\alpha(\omega) > 0$ and $x(\omega) > 0$ satisfying the equation $\alpha(\omega)x(T\omega) = D(\omega, x(\omega))$ almost surely.

Key words and Phrases: Random dynamical systems, Perron-Frobenius theory, Nonlinear cocycles, Stochastic equations, Random monotone mappings, Hilbert-Birkhoff metric.
2000 Mathematics Subject Classifications: 37H10, 37H15.

*S. Pirogov gratefully acknowledges financial support from the German Science Foundation research project SFB-701 (Bielefeld University). He is thankful to Yu. Kondratiev and other colleagues at the University of Bielefeld for their kind hospitality during his visits in 2005-2006.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $T : \Omega \to \Omega$ its *automorphism*, i.e., a one-to-one mapping such that T and T^{-1} are measurable and preserve the measure P. For each $\omega \in \Omega$, let $D(\omega) = D(\omega, x)$ be a mapping of the set $\mathbb{R}^n_+$ of non-negative n-dimensional vectors into itself, continuous and positively homogeneous (of degree one) in x and $\mathcal{F}$-measurable in ω. Define

$$C(t, \omega) = D(T^{t-1}\omega)D(T^{t-2}\omega)...D(\omega),\ t = 1, 2, ..., \tag{1}$$

where the product means the composition of maps, and $C(0, \omega) = Id$ (the identity map). Then we have

$$C(t, T^s\omega)C(s, \omega) = C(t + s, \omega),\ t, s \geq 0, \tag{2}$$

i.e., the mapping $C(t, \omega)$ is a *cocycle* over the dynamical system $(\Omega, \mathcal{F}, P, T)$ (see, e.g., Arnold [1]). In what follows, it will be convenient to write $C(t, \omega)x$ and $D(\omega)x$ for the result of application of the corresponding map to the point x.

For two vectors $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$, we write $x \leq y$ (resp. $x < y$) if $x^i \leq y^i$ (resp. $x^i < y^i$) for all i. The notation $x \prec y$ means that $x \leq y$ and $x \neq y$. We write $|x|$ for $|x^1| + ... + |x^n|$. A mapping $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called *completely monotone* if it preserves each of the relations $x \leq y$, $x \prec y$ and $x < y$ between two vectors $x, y \in \mathbb{R}^n_+$ (clearly, if A preserves the second relation, it preserves the first). A mapping A is termed *strictly monotone* if the relation $x \prec y$ implies $A(x) < A(y)$.

We will assume that the mappings $D(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ ($\omega \in \Omega$) are completely monotone and the cocycle $C(t, \omega)$ satisfies the following condition.

(**C**) For almost all $\omega \in \Omega$, there is a natural number l (depending on ω) such that the mapping $C(l, \omega)$ is strictly monotone.

The main result of this paper is as follows.

Theorem 1. *(a) There exists a measurable vector function $x(\omega) > 0$ and a measurable scalar function $\alpha(\omega) > 0$ such that*

$$\alpha(\omega)x(T\omega) = D(\omega)x(\omega),\ \ |x(\omega)| = 1\ \ (a.s.). \tag{3}$$

(b) The pair of functions $(\alpha(\cdot), x(\cdot)) \geq 0$ satisfying (3) is determined uniquely up to the equivalence with respect to the measure P.

(c) If $t \to \infty$, then

$$\frac{C(t, T^{-t}\omega)a}{|C(t, T^{-t}\omega)a|} \to x(\omega) \ (a.s.), \tag{4}$$

where convergence is uniform in $a \succ 0$.

(d) Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be sub-σ-algebras of $\mathcal{F}$ such that the random maps $D(T^{-1}\omega)x, D(T^{-2}\omega)x, \ldots$ are $\mathcal{F}_0$-measurable and the random maps $D(\omega)x, D(T^{-1}\omega)x, \ldots$ are $\mathcal{F}_1$-measurable (for each x). Then one can select versions of $x(\cdot)$ and $\alpha(\cdot)$ satisfying (3) and (4), which are $\mathcal{F}_0$- and $\mathcal{F}_1$-measurable, respectively.

The above result may be regarded as a stochastic non-linear generalization of the Perron–Frobenius theorem: $x(\cdot)$ and $\alpha(\cdot)$ play the roles of an "eigenvector" and an "eigenvalue" of the random mapping $D(\omega)$ with respect to the dynamical system $T : \Omega \to \Omega$. For linear maps $D(\omega)$ (non-negative random matrices), the first result of this kind was obtained by Evstigneev [6], see also Arnold, Gundlach and Demetrius [2] and references therein. In the papers cited, conditions somewhat stronger than (**C**) were imposed. Infinite-dimensional analogues, especially pertaining to random linear operators in spaces of functions and measures, were considered by many authors; a comprehensive treatment with various applications was provided by Kifer [10]. There exists a vast literature on non-linear (deterministic) versions of the Perron-Frobenius theorem; for a review of it see e.g. Gaubert and Gunawardena [9]. An important role in this literature has been played by the paper by Kohlberg [11], the ideas of which we use in the present study.

Problems related to the stochastic Perron-Frobenius theory arise in various areas of applied mathematics—in particular, in models of evolutionary biology (see Arnold, Gundlach and Demetrius [2]) and in mathematical finance (Dempster, Evstigneev and Schenk-Hoppé [4]). The results obtained in this work are motivated primarily by applications in mathematical economics. They make it possible to establish the existence of equilibrium in some stochastic versions of the von Neumann-Gale model of economic dynamics (see e.g. Evstigneev and Taksar [8]). These applications, involving concepts and techniques that are beyond the scope of this paper, will be discussed in detail in a separate publication.

Several comments about the assumptions imposed are in order. Consider a *concave* mapping $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, i.e. a mapping satisfying $A(\theta x + (1 - \theta)y) \geq \theta A(x) + (1 - \theta)A(y)$ for all $x, y \in \mathbb{R}^n_+$ and $\theta \in [0, 1]$. Clearly, if A is homogeneous, then A is concave if and only if it is *superadditive*:

$$A(x + y) \geq A(x) + A(y). \tag{5}$$

A superadditive mapping $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ preserves the relation $\succ$ if and only if

$$A(h) \succ 0 \text{ for } h \succ 0; \tag{6}$$

it preserves the relation $>$ if and only if

$$A(h) > 0 \text{ for } h > 0, \tag{7}$$

and it is strictly monotone if and only if

$$A(h) > 0 \text{ for } h \succ 0. \tag{8}$$

(These assertions are immediate from (5).) Thus, for a concave homogeneous mapping, its complete monotonicity is equivalent to the validity of (6) and (7), and its strict monotonicity is equivalent to (8). If the mapping under consideration is linear, i.e., defined by a non-negative matrix A, then property (6) (resp. (7)) means that A does not have zero columns (resp. zero rows). Consequently, completely monotone linear maps are defined by non-negative matrices A without zero rows and columns. Such mappings are strictly monotone when $A > 0$. (For matrices the symbol "$>$" means that all the matrix elements are strictly positive.)

2 A stochastic contraction principle

The proof of Theorem 1 is based on a stochastic generalization of the following known result regarding contraction mappings (see, e.g., Eisenack and Fenske [5]).

Let X be a compact space with a metric ρ and let $f : X \to X$ be a mapping satisfying $\rho(f(x), f(y)) < \rho(x, y)$ for all $x \neq y$. Then f has a unique fixed point $\bar{x}$, and $f^k(x) \to \bar{x}$ for each $x \in X$.

This result was used by Kohlberg [11] in connection with a nonlinear (deterministic) analogue of the Perron-Frobenius theorem.

We formulate a stochastic version of the above contraction principle. Let $(X, \mathcal{X})$ be a measurable space and $f(\omega, x)$ a jointly measurable mapping of $\Omega \times X$ into X. Let Y be a measurable subset of X equipped with a metric ρ such that Y is separable with respect to this metric and the Borel measurable structure on Y coincides with the measurable structure induced from X. Define

$$f_k(\omega, x) := f(T^{k-1}\omega, x) \ (k = 0, \pm 1, \pm 2, ...), \tag{9}$$

$$f^{(k)}(\omega, x) := f_0(\omega) f_{-1}(\omega) ... f_{-k}(\omega)(x) \ (k = 0, 1, 2, ...).$$

Assume that the measurable space $(X, \mathcal{X})$ is standard and the mapping f satisfies the following requirements.

(f.1) For each $\omega \in \Omega$, the map $f(\omega, x)$ transforms Y into itself and is continuous on Y with respect to the metric ρ.

(f.2) There is a sequence of $\mathcal{F}$-measurable sets $\Omega_0 \subseteq \Omega_1 \subseteq ... \subseteq \Omega$ such that $P(\Omega_m) \to 1$ and for each $m = 0, 1, ...,$ and $\omega \in \Omega_m$ the following conditions hold:

the set $X^{(m)}(\omega) := f^{(m)}(\omega, X)$ is contained in Y and is compact with respect to the metric ρ;

for all $x, y \in Y$ with $x \neq y$, we have

$$\rho(f^{(m)}(\omega, x), f^{(m)}(\omega, y)) < \rho(x, y). \tag{10}$$

Since the sequence of sets Ω_m is increasing, there exists a measurable function $m(\omega)$ with non-negative integer values such that for each $\omega \in \Omega_0 \cup \Omega_1 \cup ...$ (and hence for almost all ω), we have $\omega \in \Omega_k$ for all $k \geq m(\omega)$.

The stochastic generalization of the above contraction principle is as follows.

Theorem 2. *(i) There exists a measurable mapping $\xi : \Omega \to Y$ for which equation*

$$\xi(T\omega) = f(\omega, \xi(\omega)) \ (a.s.) \tag{11}$$

holds and

$$\lim_{m(\omega) \leq k \to \infty} \sup_{x \in X} \rho(\xi(\omega), f_0(\omega) ... f_{-k}(\omega)(x)) = 0 \tag{12}$$

with probability one.

(ii) If $\eta : \Omega \to X$ is any (not necessarily measurable) solution to (11), then $\eta = \xi$ (a.s.).

(iii) Let $\mathcal{G}_0 \subseteq \mathcal{F}$ be a σ-algebra such that the mappings $f_{-k}(\omega, x)$, $k = 0, 1, ...,$ of the space $\Omega \times X$ into X are $\mathcal{G}_0 \times \mathcal{X}$-measurable and $\Omega_m \in \mathcal{G}_0$ for all $m \geq 0$. Then there exists an $\mathcal{G}_0$-measurable mapping ξ possessing the properties described in (i) and (ii).

According to (12), the random sequence $f_0 ... f_{-k}(x)$ converges to $\xi(\omega)$ uniformly in x with probability one. Note that the distance ρ between $f_0 ... f_{-k}(x)$ and $\xi(\omega)$ involved in (12) is defined only if $f_0 ... f_{-k}(x) \in Y$. By virtue of (f.2), this inclusion holds for almost all ω and all $k \geq m(\omega)$, therefore the limit in (12) is taken over $k \geq m(\omega)$.

For a proof of Theorem 2 see Evstigneev and Pirogov [7].

3 Proof of the main result

We begin with a lemma. For each $t = 0, \pm 1, ...$, denote by $\mathcal{G}_t$ the smallest σ-algebra with respect to which the mappings $\omega \mapsto D(T^j \omega)x$, $j \leq t-1$, are measurable for every $x \in \mathbb{R}^n_+$.

Lemma 1. *There exists a sequence of $\mathcal{G}_0$-measurable sets $\Gamma_1 \subseteq \Gamma_2 \subseteq ... \subseteq \Omega$ such that $P(\Gamma_m) \to 1$ and, for each $m = 0, 1, 2, ...,$ and $\omega \in \Gamma_m$, the mapping $C(m, T^{-m}\omega)$ is strictly monotone.*

Proof. For each $t \geq 1$, consider the set Δ_m of those ω for which the mapping $C(m, \omega)x = C(m, \omega, x)$ is strictly monotone in x. Let us show that Δ_m is $\mathcal{G}_m$-measurable. Denote by B_N the set of points $x \in \mathbb{R}^n_+$ with rational coordinates satisfying $|x| \leq N$ and by e_i the vector in $\mathbb{R}^n_+$ whose ith coordinate is 1 and the others are 0. We have $\Delta_m \in \mathcal{G}_m$ because $\omega \in \Delta_m$ if and only if

$$\inf_{x \in B_N} [C^j(m, \omega, x + \frac{1}{i} e_k) - C^j(m, \omega, x)] > 0$$

for all $N, i = 1, 2, ...$ and $k, j = 1, 2, ...n$, where $C^j(\cdot)$ stands for the jth coordinate of $C(\cdot)$.

Observe that $\Delta_m \subseteq \Delta_{m+1}$. This is a consequence of the relation $C(m + 1, \omega) = D(T^m \omega) C(m, \omega)$ and the following assertion (the proof of which is straightforward):

$(*)$ If a there are two mappings $A, B : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ one of which is strictly monotone and the other is completely monotone, the product AB is strictly monotone.

By virtue of assumption (**C**), we have $P(\Delta_1 \cup \Delta_2 \cup ...) = 1$. Therefore the inclusion $\Delta_m \subseteq \Delta_{m+1}$ $(m = 1, 2, ...)$, which we have established, implies $P(\Delta_m) \to 1$.

Define $\Gamma_m = T^m \Delta_m$. Then $C(m, T^{-m}\omega)$ is strictly monotone if and only if $\omega \in \Gamma_m$, the set Γ_m belongs to $\mathcal{G}_0$, and $P(\Gamma_m) = P(\Delta_m) \to 1$. Furthermore, we have

$$C(m+1, T^{-m-1}\omega) = C(m, T^{-m}\omega)D(T^{-m-1}\omega)$$

(see (2)), and so if $\omega \in \Gamma_m$, then $\omega \in \Gamma_{m+1}$ by virtue of $(*)$. Consequently, $\Gamma_1 \subseteq \Gamma_2 \subseteq ...$, which completes the proof. □

Proof of Theorem 1. We will apply Theorem 2 to the mapping

$$f(\omega, x) = \frac{D(\omega, x)}{|D(\omega, x)|}, \; x \in X, \tag{13}$$

where $X = \{x \in \mathbb{R}^n_+ : |x| = 1\}$. The mapping f is well-defined because $D(\omega, x) \neq 0$ for $x \neq 0$. Denote by $\mathcal{X}$ the Borel σ-algebra on X. Then the measurable space $(X, \mathcal{X})$ is standard and the function $f(\omega, x)$ is $\mathcal{F} \times \mathcal{X}$-measurable because it is $\mathcal{F}$-measurable in ω and continuous in $x \in X$.

We wish to verify the assumptions of Theorem 2 for f. To this end we define $Y = \{x \in X : x > 0\}$ and consider the Hilbert-Birkhoff metric $\rho(x, y)$ on Y (see the Appendix). Condition (f.1) follows from the fact that $D(\omega, x) > 0$ for $x > 0$ and from the continuity of $f(\omega, \cdot)$ in the Euclidean metric and hence in the metric ρ on Y.

Consider the sets $\Gamma_1 \subseteq \Gamma_2 \subseteq ...$ constructed in Lemma 1. Let us show that condition (f.2) holds for the sets $\Omega_m := \Gamma_{m+1} \in \mathcal{F}$ $(m = 0, 1, ...)$. Define $f_m(\omega)$ by (9) and observe that

$$f^{(m)}(\omega, x) = f_0(\omega)f_{-1}(\omega)...f_{-m}(\omega)(x) = \frac{C(m+1, T^{-m-1}\omega)x}{|C(m+1, T^{-m-1}\omega)x|} \tag{14}$$

$(m = 0, 1, 2, ...)$ by virtue of homogeneity of the mappings at hand. Let m be any non-negative integer and let $\omega \in \Omega_m = \Gamma_{m+1}$. Then, according to Lemma 1, the mapping $C(m+1, T^{-m-1}\omega)$ is strictly monotone. Consequently (see (14)), $f^{(m)}(\omega, x) > 0$ for all $x \in X$, and so $f^{(m)}(\omega, X)$ is a subset in Y compact with respect to the Euclidean metric, and hence with respect to the metric ρ, as a continuous image of the compactum X. The contraction property (10) follows from Proposition A.1 in the Appendix. Thus all the conditions sufficient for the validity of Theorem 2 are verified.

Consider the mapping ξ described in assertion (i) of Theorem 2 and put

$$x(\omega) = \xi(\omega),\ \alpha(\omega) = |D(\omega)x(\omega)|. \tag{15}$$

Then (11) implies (3), which proves assertion (a) of Theorem 1. To prove (b), take any $(\alpha'(\cdot), x'(\cdot)) \geq 0$ satisfying (3), fix any $x_0 \in X$ and define $\eta(\omega) = x'(\omega)$ if $|x'(\omega)| = 1$ and $\eta(\omega) = x_0$ otherwise. Then $\eta(\omega) = x'(\omega)$ (a.s.), $\eta(T\omega) = f(\omega, \eta(\omega))$ (a.s.) and $|\eta(\omega)| = 1$ for all ω. Consequently, $x' = \eta = \xi$ (a.s.) by virtue of part (ii) of Theorem 2. Furthermore, $\alpha'(\omega) = |D(\omega)x'(\omega)| = |D(\omega)x(\omega)| = \alpha(\omega)$ (a.s.). Assertion (c) of Theorem 1 follows from (12), (18) and formula (14) applied to $x := a/|a|$.

To verify (d), assume that the maps $D(T^{-1}\omega)x$, $D(T^{-2}\omega)x$ are $\mathcal{F}_0$-measurable and the maps $D(\omega)x$, $D(T^{-1}\omega)x$, $D(T^{-2}\omega)x, ...$ are $\mathcal{F}_1$-measurable for each $x \in X$. For every $m \geq 0$, the mapping $D(T^{-m-1}\omega, x)$ is continuous in x and $\mathcal{G}_0$-measurable in ω, consequently, $f_{-m}(\omega, x)$ is $\mathcal{G}_0 \times \mathcal{X}$-measurable. The sets Γ_m constructed in Lemma 1 belong to $\mathcal{G}_0$, and so $\Omega_m = \Gamma_{m+1} \in \mathcal{G}_0$ $(m = 0, 1, ...)$. By virtue of assertion (iii) of Theorem 2, there exist an $\mathcal{G}_0$-measurable mapping $\xi : \Omega \to Y$ satisfying (11). It remains to define $x(\omega)$ and $\alpha(\omega)$ by (15). Then $x(\omega)$ is $\mathcal{F}_0$-measurable and $\alpha(\omega)$ is $\mathcal{F}_1$-measurable (because $\mathcal{G}_0 \subseteq \mathcal{F}_0$, $\mathcal{G}_1 \subseteq \mathcal{F}_1$), and so $x(\cdot)$ and $\alpha(\cdot)$ satisfy all the conditions in (d). $\square$

4 A counterexample

It should be noted that the main content of Theorem 1 lies in the construction of a *measurable* solution to equations (3). The existence of solutions to (3) in the class of all, not necessarily measurable, functions immediately follows from the Schauder-Tichonoff fixed point principle in the linear space $\mathcal{L}$ of all functions $z : \Omega \to \mathbb{R}^n$ with the topology of pointwise convergence. Indeed, the mapping defined by

$$z(\cdot) \longmapsto \frac{D(T^{-1}\omega, z(T^{-1}\omega))}{|D(T^{-1}\omega, z(T^{-1}\omega))|}$$

is continuous and transforms the convex compact set $\{z(\cdot) : z(\omega) \in \mathbb{R}^n_+, |z(\omega)| = 1 \text{ for all } \omega\}$ into itself; hence it has a fixed point, which yields a solution to (3).

It turns out that condition (**C**) is essential for the existence of measurable solutions to (3)—even in the case of linear mappings $D(\omega, \cdot)$, i.e. non-negative random matrices. Suppose that

$$D(\omega) = \begin{pmatrix} 0 & \gamma(\omega) \\ 1 & 0 \end{pmatrix},$$

where $\gamma(\omega) \geq 1$ is a measurable function. Further, assume that the square $\Theta := T^2$ of the given automorphism $T : \Omega \to \Omega$ is ergodic and there is no measurable function $\beta(\omega) > 0$ such that $\gamma(\omega) = \beta(T\omega)\beta(\omega)$ (a.s.). (For example, let T be the Bernoulli shift associated with a sequence $\omega = (..., s_{-1}, s_0, s_1, ...)$ of independent random variables taking values 1 and 2 with probability 1/2, and $\gamma(\omega) = s_0$.) Under these assumptions, *equations (3) do not have solutions in the class of measurable functions* $x(\cdot) \geq 0$ *and* $\alpha(\cdot) \geq 0$.

To prove the above assertion, assume the contrary: such solutions, $x(\omega) = (u(\omega), v(\omega))$ and $\alpha(\omega)$, exist. Then we have

$$\alpha(\omega)u(T\omega) = \gamma(\omega)v(\omega),\ \alpha(\omega)v(T\omega) = u(\omega),\ u(\omega) + v(\omega) = 1 \text{ (a.s.)}, \quad (16)$$

and so $\alpha(\omega) = \gamma(\omega)v(\omega)+u(\omega) \geq 1$ and $u(T^2\omega) = \alpha^{-1}(T\omega)\alpha^{-1}(\omega)\gamma(T\omega)u(\omega)$. Define $\Gamma = \{\omega : u(\omega) > 0\}$. Clearly $P(\Gamma) > 0$, and the last equation implies $\Gamma \subseteq \Theta^{-1}(\Gamma)$. Consequently, Γ is an invariant measurable set for Θ, and so $\Gamma = \Omega$ (mod0), which yields $u(\omega) > 0$ (a.s.) and $v(\omega) > 0$ (a.s.). From the first two equations in (16), we get

$$u(T\omega)v(T\omega)^{-1} = \gamma(\omega)u(\omega)^{-1}v(\omega) \text{ (a.s.)},$$

and so $\gamma(\omega) = \beta(T\omega)\beta(\omega)$, where $\beta(\omega) = u(\omega)v(\omega)^{-1}$. A contradiction.

Appendix

We present for the reader's convenience some general facts about the Hilbert-Birkhoff metric. Let Y denote the set of $y = (y_1, ..., y_n) > 0$ with $\sum y_i = 1$. For $x, y \in Y$ put

$$\rho(x, y) = \ln[\max_i \frac{x_i}{y_i} \cdot \max_j \frac{y_j}{x_j}]. \quad (17)$$

It is known that formula (17) defines a complete metric on Y (the Hilbert-Birkhoff metric), and the topology induced by ρ on Y coincides with the

Euclidean topology on Y—see Birkhoff [3] and Seneta [12]. A key role in this work is played by the following fact. Let F be a mapping $\mathbb{R}^n_+ \to \mathbb{R}^n_+$ such that $F(x) \neq 0$ for $x \in Y$. Define $f(x) = F(x)/|F(x)|$, $x \in Y$.

Proposition A.1. *If $F(x)$ is homogeneous and strictly monotone, then $f(x)$ is contracting on Y in the metric ρ, i.e. $\rho(f(x), f(y)) < \rho(x, y)$ for $x, y \in Y$ with $x \neq y$.*

Proof (cf. Kohlberg [11, Lemma 1]). Consider some $x, y \in Y$ with $x \neq y$ and define $\lambda = \max_i(x_i/y_i)$ and $\mu = \max_j(y_j/x_j)$. Then we have $\rho(x, y) = \ln(\lambda\mu)$, $\lambda y \geq x$ and $\mu x \geq y$. Furthermore, $\lambda y \neq x$ and $\mu x \neq y$ because otherwise $x = y$ (recall that $|x| = |y| = 1$). Consequently,

$$\lambda F(y) = F(\lambda y) > F(x),\ \mu F(x) = F(\mu x) > F(y)$$

by virtue of strict monotonicity and homogeneity. Therefore

$$\max_i \frac{F^i(x)}{F^i(y)} < \lambda,\ \max_j \frac{F^j(y)}{F^j(x)} < \mu,$$

and so

$$\max_i \frac{f^i(x)}{f^i(y)} \max_j \frac{f^j(y)}{f^j(x)} = \max_i \frac{F^i(x)/|F(x)|}{F^i(y)/|F(y)|} \max_j \frac{F^j(y)/|F(y)|}{F^j(x)/|F(x)|} < \lambda\mu,$$

which yields $\rho(f(x), f(y)) < \rho(x, y)$. □

Proposition A.2. *We have*

$$|x - y| \leq n(e^{\rho(x,y)} - 1),\ x, y \in Y. \tag{18}$$

Proof. Inequality (18) is a consequence of the following one:

$$1 + |x_k - y_k| \leq \max_i \frac{x_i}{y_i} \cdot \max_j \frac{y_j}{x_j}\ (k = 1, ..., n). \tag{19}$$

To prove (19) we may assume without loss of generality that $k = 1$ and $y_1 > x_1$. Then we have

$$\max_i \frac{x_i}{y_i} = \max\{\frac{x_1}{y_1}, \max_{i>1} \frac{x_i}{y_i}\} \geq \max\{\frac{x_1}{y_1}, \frac{1 - x_1}{1 - y_1}\} = \frac{1 - x_1}{1 - y_1}$$

because $\sum_{i>1} x_i = 1 - x_1$, $\sum_{i>1} y_i = 1 - y_1$ and $y_1 > x_1$. Analogously, we get

$$\max_j \frac{y_j}{x_j} \geq \frac{y_1}{x_1},$$

which implies (19) for $k = 1$ and $y_1 > x_1$. □

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