Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach

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Dedicated to Professor Masatoshi Fukushima on the occasion of his 70th birthday

Abstract: The main objective of this paper is to prove the essential self-adjointness of Dirichlet operators in $L^2(\mu)$ where μ is a Gibbs measure on an infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$. This operator can be regarded as a perturbation of the Ornstein-Uhlenbeck operator by a nonlinearity and corresponds to a parabolic stochastic partial differential equation (=SPDE, in abbreviation) on $\mathbb{R}$. In view of quantum field theory, the solution of this SPDE is called a $P(\phi)_1$-time evolution.

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1 Introduction

The uniqueness problem for infinite dimensional diffusion operators plays a crucial role in several areas of mathematical physics including Euclidean quantum field theory and

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statistical mechanics. Hence such problems are discussed in many areas of stochastic analysis. However, these problems are still understood very insufficiently in the sense that there are several important types of infinite dimensional diffusion operators for which it is not known whether uniqueness holds or not. The most prominent example in which essential self-adjointness is not known is the stochastic quantization of a $P(\phi)_2$-quantum field in infinite volume. Even in finite volume, this problem was open for many years and only solved in Liskevich-Röckner [17] and then independently in Da Prato-Tubaro [7]. We refer to Eberle [11] and references therein for a detailed review. However, we would like to mention two references here, namely, Shigekawa [21] and Albeverio-Kondratiev-Röckner [1]. In both papers, techniques were developed which work to prove essential self-adjointness for special classes of operators. [21] is based on the Malliavin calculus, while [1] is based on the analysis of stochastic differential equations associated with certain approximating operators. An analytic variant of the latter led to the proof of essential self-adjointness in [17] for the stochastic quantization of $P(\phi)_2$ in finite volume.

All these approaches, however, do not apply to show the main result of the present paper, namely the essential self-adjointness for the diffusion operators of $P(\phi)_1$-quantum fields in infinite volume. The diffusion operators are defined through Dirichlet forms on an infinite volume path space $C(\mathbb{R},\mathbb{R}^d)$ with a Gibbs measure. The Gibbs measure is associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |w'(x)|^2 dx + \int_{\mathbb{R}} U(w(x))dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is an interaction potential function. Our methods are based on quite recent work by Da Prato-Tubaro [8] and Da Prato-Röckner [6] where an L^p-analysis of Kolmogorov operators in infinitely many variables is developed. Their work is based on the theory of SPDE in an essential way and gives a new approach to tackle such uniqueness problems. In this paper we adopt their approach, however, with substantial necessary modifications.

The organization of this paper is as follows: In Section 2, we present the framework and state our main results. In Section 3, we present basic properties of parabolic SPDEs. In Section 4, we give some results about the Ornstein-Uhlenbeck semigroup and its generator. By using these results, we can state the key approximations by cylinder functions. It implies that our Dirichlet operator can be regarded as a perturbation of the Ornstein-Uhlenbeck operator by a nonlinearity. Finally in Section 5, we prove the main theorem and discuss the connection with our SPDE. There is an enormous literature on uniqueness problems for diffusion operators. We only mention here that a weaker type of uniqueness, namely Markov uniqueness, was also studied intensively (see e.g. Takeda [24] and Röckner-Zhang [25]). For the precise connections, we again refer to [11], where non-symmetric operators are also treated and where it is discussed in detail why neither Markov uniqueness nor essential self-adjointness (strong uniqueness) can be deduced from the fact that the associated stochastic (partial) differential equation has a unique solution.

Finally, we would like to emphasize that to the best of our knowledge, this paper is the first where essential self-adjointness for a Dirichlet operator is proved in infinite volume, i.e., where the differential operator, which determines the drift term, is defined on an unbounded domain.

2 Framework and Main Result

Let us introduce some notations and objects we will be working with. First we define a weight function $\rho_r \in C^\infty(\mathbb{R},\mathbb{R}), r \in \mathbb{R}$, by $\rho_r(x) := e^{r\chi(x)}$, $x \in \mathbb{R}$, where $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. We fix a constant $r > 0$ such that $K_1 + 2r^2 > 0$, where the constant K_1 is denoted in condition **(U1)** below. We set $E = L^2_r(\mathbb{R},\mathbb{R}^d) := L^2(\mathbb{R},\mathbb{R}^d : \rho_{-2r}(x)dx)$. This space is a Hilbert space with the inner product defined by

$$(X,Y)_E := \int_{\mathbb{R}} \big(X(x),Y(x)\big)_{\mathbb{R}^d}\rho_{-2r}(x)dx, \quad X,Y \in E.$$

Moreover, we set $H := L^2(\mathbb{R},\mathbb{R}^d)$ and denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ the corresponding norms of E and H, respectively. We regard the dual space E^* of E as $L^2(\mathbb{R},\mathbb{R}^d : e^{2r\chi(x)}dx)$.

We also introduce a suitable subspace of $C(\mathbb{R},\mathbb{R}^d)$. For functions in $C(\mathbb{R},\mathbb{R}^d)$, we set

$$\|w\|_{r,\infty} := \sup_{x\in\mathbb{R}} |w(x)|\rho_{-r}(x) \ \text{ for } r \in \mathbb{R},$$

and consider

$$\mathcal{C} := \bigcap_{r>0} \big\{w \in C(\mathbb{R},\mathbb{R}^d) \mid \|w\|_{r,\infty} < \infty\big\}.$$

Then it becomes a Fréchet space with the system of norms $\|\cdot\|_{r,\infty}$. We easily see that the inclusion $\mathcal{C} \subset E \cap C(\mathbb{R},\mathbb{R}^d)$ is dense with respect to the topology of E. We endow $C(\mathbb{R},\mathbb{R}^d)$ with the σ-field $\mathcal{B}$ generated by the point evaluation and denote by $\mathcal{P}(C(\mathbb{R},\mathbb{R}^d))$ the class of all probability measures on the space $(C(\mathbb{R},\mathbb{R}^d),\mathcal{B})$. For $T > 0$, we also denote by $\mathcal{B}_T$ and $\mathcal{B}_{T,c}$ the σ-fields of $C(\mathbb{R},\mathbb{R}^d)$ generated by $\{w(x); -T \leq x \leq T\}$ and $\{w(x); x \leq -T, x \geq T\}$, respectively.

In this paper, we impose the following conditions on the potential function $U \in C(\mathbb{R}^d,\mathbb{R})$:

(U1) There exist a constant $K_1 \in \mathbb{R}$ and a convex function $\widetilde{U} : \mathbb{R}^d \to \mathbb{R}$ such that

$$U(z) = -\frac{K_1}{2}|z|^2 + \widetilde{U}(z), \qquad z \in \mathbb{R}^d.$$

(U2) There exist $K_2 > 0$ and $p > 0$ such that

$$|\widetilde{\nabla} U(z)| \leq K_2(1 + |z|^p), \qquad z \in \mathbb{R}^d,$$

where $\widetilde{\nabla} U(z) := -K_1 z + \partial_0\widetilde{U}(z), z \in \mathbb{R}^d$ and $\partial_0\widetilde{U}$ is the minimal section of the subdifferential $\partial\widetilde{U}$. (The reader is referred to Showalter [22] for definitions of the subdifferential for a convex function and its minimal section.)

(U3) $\lim_{|z|\to\infty} U(z) = \infty$.

As examples of U satisfying the above conditions, we can include the case

$$U(z) = \sum_{j=0}^{2m} a_j |z|^j, \quad a_1 = 0,\ a_{2m} > 0,\ m \in \mathbb{N}.$$

Especially, we are interested in a square potential and a double-well potential. Those are, $U(z) = a|z|^2$ and $U(z) = a(|z|^4 - |z|^2)$, $a > 0$, respectively.

Remark 2.1 *In the case of $U \in C^1(\mathbb{R}^d, \mathbb{R})$, $\widetilde{\nabla} U$ defined in condition* **(U2)** *coincides with the usual gradient ∇U. Moreover condition* **(U1)** *is equivalent to the following one-sided Lipschitz condition:*

(U1)' $\big(\nabla U(z_1) - \nabla U(z_2), z_1 - z_2\big)_{\mathbb{R}^d} \geq -K_1 |z_1 - z_2|^2, \qquad z_1, z_2 \in \mathbb{R}^d.$

Remark 2.2 *For the convex function $\widetilde{U}$, we define the Moreau-Yosida approximation by*

$$\widetilde{U}_n(z) := \inf_{z' \in \mathbb{R}^d} \big\{ \widetilde{U}(z') + n|z - z'|^2 \big\}, \qquad z \in \mathbb{R}^d, n \in \mathbb{N}.$$

Then $\widetilde{U}_n$ is differentiable and

$$\lim_{n\to\infty} \widetilde{U}_n(z) = \widetilde{U}(z), \quad \lim_{n\to\infty} \nabla \widetilde{U}_n(z) = \partial_0 \widetilde{U}(z), \qquad z \in \mathbb{R}^d.$$

Now, we introduce a Gibbs measure. Consider the Schrödinger operator $H_U := -\frac{1}{2}\Delta + U$ on $L^2(\mathbb{R}^d, \mathbb{R})$, where $\Delta := \sum_{i=1}^d \partial^2/\partial z_i^2$ is the d-dimensional Laplacian. Then the condition **(U3)** assures that H_U has purely discrete spectrum and a complete set of eigenfunctions. We denote by $\lambda_0 (> \min U)$ the minimal eigenvalue and by Ω the corresponding normalized eigenfunction in $L^2(\mathbb{R}^d, \mathbb{R})$. It is called ground state and it decays exponentially. See Theorems X. 28, XIII. 47, XIII. 67 and XIII. 70 in Reed-Simon [19] for details.

Let $\mathcal{W}_{-T,z_1;T.z_2}$, $T > 0, z_1, z_2 \in \mathbb{R}^d$, be the path measure of Brownian bridge such that $w(-T) = z_1, w(T) = z_2$. We sometimes regard this measure as a probability measure on the space $(C(\mathbb{R}, \mathbb{R}^d), \mathcal{B})$ by considering $w(x) = z_1$ for $x \leq -T$ and $w(x) = z_2$ for $x \geq T$. We define $\mu(A)$ for $A \in \mathcal{B}_T$, $T > 0$, by

$$\begin{aligned} \mu(A) \quad := \quad & e^{2T\lambda_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Omega(z_1)\Omega(z_2) p(2T, z_1, z_2) \\ & \times \mathbb{E}^{\mathcal{W}_{-T,z_1;T,z_2}} \Big[\exp\big(- \int_{-T}^{T} U(w(x))dx \big); A \Big] dz_1 dz_2, \qquad (2.1) \end{aligned}$$

where $p(t, z_1, z_2)$ is the transition probability of standard Brownian motion on $\mathbb{R}^d$. Then by the Feynman-Kac formula and the Markov property of Brownian motion, we can see that μ is well-defined as an element of $\mathcal{P}(C(\mathbb{R},\mathbb{R}^d))$ and it satisfies the following DLR-equation for every $T>0$ and μ-a.e. $\xi \in C(\mathbb{R},\mathbb{R}^d)$:

$$\mu(dw|\mathcal{B}_{T,c})(\xi) \;=\; Z_{T,\xi}^{-1}\exp\big(-\int_{-T}^{T} U(w(x))dx\big)\mathcal{W}_{-T,\xi(-T);T,\xi(T)}(dw), \tag{2.2}$$

where $Z_{T,\xi}$ is a normalizing constant. See Proposition 2.7 in Iwata [14] for details. Although generally there exist other μ's in $\mathcal{P}(C(\mathbb{R},\mathbb{R}^d))$ satisfying the DLR-equation (2.2), in this paper we only consider the Gibbs measure μ which has been constructed in (2.1).

Remark 2.3 *In* [3], *Betz and Lörinczi prove that if, for some $a>2$, $U(z)$ grows at infinity faster than $|z|^a$ but slower than $|z|^{2a-2}$, then there is a unique Gibbs measure on $C(\mathbb{R},\mathbb{R}^d)$. See Theorem* 3.4 *of* [3] *for details.*

Here we note that the Gibbs measure μ is supported on $\mathcal{C}$ by using the standard moment estimates of Brownian motion. Then by the continuity of the inclusion map of $\mathcal{C}$ into E, we can regard $\mu \in \mathcal{P}(E)$ by identifying it with its image measure under the inclusion map.

By virtue of the DLR-equation (2.2), the Gibbs measure μ is $C_0^\infty(\mathbb{R},\mathbb{R}^d)$-quasi-invariant, i.e., $\mu(\cdot+k)$ and μ are mutually equivalent and

$$\mu(k+dw)=\Lambda(k,w)\mu(dw) \tag{2.3}$$

holds for every $k\in C_0^\infty(\mathbb{R},\mathbb{R}^d)$. The Radon-Nikodym density $\Lambda(k,w)$ is represented by

$$\Lambda(k,w)=\exp\Big\{\int_{\mathbb{R}}\Big(U\big(w(x)\big)-U\big(w(x)+k(x)\big)-\frac{1}{2}|k'(x)|^2+(w(x),\Delta_x k(x))_{\mathbb{R}^d}\Big)dx\Big\}, \tag{2.4}$$

where $\Delta_x := d^2/dx^2$. For details the reader is referred to Theorem 3.21 in Iwata [14] or Lemma 4.1 in Funaki [12]. Moreover, we have μ is translation invariant, i.e., $\tau_x\circ\mu=\mu$, where the shift operator $\{\tau_x\}_{x\in\mathbb{R}}$ on $C(\mathbb{R},\mathbb{R}^d)$ is defined by $\tau_x w(\cdot) := w(\cdot - x), x\in\mathbb{R}$. Hence by combining this with the fact that Ω decays exponentially, we see that

$$\int_E\Big(\int_{\mathbb{R}}|w(x)|^{2m}\rho_{-2r}(x)dx\Big)\mu(dw)\le\frac{1}{r}\int_{\mathbb{R}^d}|z|^{2m}\Omega(z)^2dz<\infty \tag{2.5}$$

holds for any $m\in\mathbb{N}$ and $r>0$. These properties will be used below.

Now we define the space of smooth cylinder functions. Let $K\subset E^*$ be a dense linear subspace of E. We say a function $F:E\longrightarrow\mathbb{R}$ is in a class $\mathcal{FC}_b^\infty(K)$ if there exist $n\in\mathbb{N}$, $\{\varphi_1,\cdots,\varphi_n\}\subset K$ and a function $f\equiv f(\alpha_1,\ \cdots,\alpha_n)\in C_b^\infty(\mathbb{R}^n)$ such that

$$F(w)\equiv f(\langle w,\varphi_1\rangle,\cdots,\langle w,\varphi_n\rangle),\quad w\in E. \tag{2.6}$$

Here we use the notation $\langle w,\varphi\rangle := \int_{\mathbb{R}}(w(x),\varphi(x))_{\mathbb{R}^d}dx$ if the integral is absolutely converging and denote $\mathcal{FC}_b^\infty := \mathcal{FC}_b^\infty\big(C_0^\infty(\mathbb{R},\mathbb{R}^d)\big)$ for simplicity.

Since K is dense in E, we have supp $(\mu)=E$. See Proposition 2.7 in Albeverio-Röckner [2] for the proof. Hence two different functions in $\mathcal{FC}_b^\infty(K)$ represent two different μ-classes. Note that $\mathcal{FC}_b^\infty(K)$ is dense in $L^2(\mu)$.

For $F \in \mathcal{FC}_b^\infty$, we also define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(w)(x) := \sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle)\varphi_j(x), \quad x \in \mathbb{R}. \tag{2.7}$$

We consider a pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \frac{1}{2}\int_E \big(D_H F(w), D_H G(w)\big)_H \mu(dw), \quad F, G \in \mathcal{FC}_b^\infty.$$

Then by virtue of the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance, we have the following integration by parts formula for any $F, G \in \mathcal{FC}_b^\infty$ and $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$:

$$\begin{aligned}
&\int_E \big(D_H F(w), \varphi\big)_H G(w)\mu(dw) \\
&= -\int_E F(w)\big(\varphi, D_H G(w)\big)_H \mu(dw) - \int_E F(w)G(w)\beta_\varphi(w)\mu(dw),
\end{aligned} \tag{2.8}$$

where β_φ is the logarithmic derivative of the Gibbs measure μ in the direction $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ in the sense of

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_E F(w)\big\{\Lambda(\varepsilon\varphi, w) - 1\big\}\mu(dw) = \int_E F(w)\beta_\varphi(w)\mu(dw), \quad F \in \mathcal{FC}_b^\infty.$$

Here by recalling Remark 2.2 and (2.4), we easily see that

$$\beta_\varphi(w) = \langle w, \Delta_x \varphi\rangle - \langle \widetilde{\nabla} U(w(\cdot)), \varphi\rangle, \quad \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^d). \tag{2.9}$$

Next we define a differential operator $\mathcal{L}_0$ with domain $\mathcal{FC}_b^\infty$ by

$$\mathcal{L}_0 F(w) := \frac{1}{2}\mathrm{Tr}(D_H^2 F(w)) - \frac{1}{2}\big\langle \widetilde{\nabla} U(w(\cdot)), D_H F(w)\big\rangle + \frac{1}{2}\big\langle w, \Delta_x D_H F(w(\cdot))\big\rangle, \tag{2.10}$$

that is, if $F(w) = f(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle)$, then

$$\begin{aligned}
\mathcal{L}_0 F(w) &= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\langle \varphi_i, \varphi_j\rangle \\
&\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial f}{\partial \alpha_i}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big) \cdot \big\{\langle w, \Delta_x \varphi_i\rangle - \langle \widetilde{\nabla} U(w(\cdot)), \varphi_i\rangle\big\}.
\end{aligned}$$

Then (2.8) and (2.9) imply the equality

$$\mathcal{E}(F, G) = \big(-\mathcal{L}_0 F, G\big)_{L^2(\mu)}, \quad F, G \in \mathcal{FC}_b^\infty. \tag{2.11}$$

This means the operator $\mathcal{L}_0$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable on $L^2(\mu)$. So we can define

$\mathcal{D}(\mathcal{E})$ as the completion of $\mathcal{FC}_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form.

(2.9) also implies that the operator $\mathcal{L}_0$ is symmetric in $L^2(\mu)$. In many applications, it is an important problem whether one has essential self-adjointness for $\mathcal{L}_0$, i.e., self-adjointness of the closure $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ of $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ in $L^2(\mu)$. The reason is that in general there are many lower bounded self-adjoint extensions $\widetilde{\mathcal{L}}_2$ of $\mathcal{L}_0$ in $L^2(\mu)$ which therefore define symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathcal{L}}_2}\}_{t\geq 0}$ generated by them. In fact, there always exists one such extension called the Freidrichs extension which is the operator corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. If $\mathcal{L}_0$ is essentially self-adjoint, there is hence only one such semigroup. Consequently, only one such dynamics associated with the Gibbs measure μ exists.

The following is the main result of this paper. In Theorem 5.1, we give a more extended statement, i.e., we show that our semigroup is not only unique but also represented by the solution of a parabolic SPDE (3.2) on the infinite interval $\mathbb{R}$ in the case of $U \in C^1(\mathbb{R}^d, \mathbb{R})$.

Theorem 2.4 *The pre-Dirichlet operator* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *is essentially self-adjoint in* $L^2(\mu)$.

As a corollary of this theorem, we obtain the Markov uniqueness. See e.g. Chapter 1 in Eberle [11] for the proof. We recall that a Dirichlet form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ in $L^2(\mu)$ is an extension of $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ if $\mathcal{FC}_b^\infty \subset \mathrm{Dom}(\mathcal{E})$ and $\mathcal{E}(F,G) = \big(-\mathcal{L}_0 F, G\big)_{L^2(\mu)}$ for any $F \in \mathcal{FC}_b^\infty$ and $G \in \mathrm{Dom}(\mathcal{E})$.

Corollary 2.5 *The Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is the unique extension of* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$.

3 Preliminaries from Parabolic SPDEs

In this section, we make some preparations starting from the underlying parabolic SPDE for our later use. Throughout this section, we suppose $U \in C^1(\mathbb{R}^d, \mathbb{R})$.

Let $(\Theta, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete probability space with filtration on which an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted H-cylindrical Brownian motion (white noise process) $\{B_t\}_{t\geq 0}$ is defined. More precisely, for a complete orthonormal system (C.O.N.S.) $\{h_j\}_{j=1}^\infty$ of H,

$$B_t(\cdot) = \big(\beta_j(t) h_j(\cdot)\big)_{j=1}^\infty \quad t \geq 0, \tag{3.1}$$

where $\{\beta_j\}_{j=1}^\infty$ is a sequence of independent one-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motions. See Chapter 4 in Da Prato-Zabczyk [9] for details.

We consider the following parabolic SPDE which is called time dependent Ginzburg-Landau type SPDE:

$$dX_t(x) = \frac{1}{2}\big\{\Delta_x X_t(x) - \nabla U(X_t(x))\big\}dt + dB_t(x), \quad x \in \mathbb{R},\ t > 0. \tag{3.2}$$

Following e.g. [9], Iwata [15] and Shiga [20], we call a $\mathcal{C}$-valued $\{\mathcal{F}_t\}$-adapted continuous stochastic process $X := \{X_t(x)\}$ a mild solution of (3.2) with initial datum $X_0 = w \in \mathcal{C}$

if X satisfies the stochastic integral equation

$$\begin{aligned} X_t(x) &= G_t w(x) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} g(t-s,x,y)\nabla U(X_s(y))dyds \\ &\quad + \int_0^t \int_{\mathbb{R}} g(t-s,x,y)dB_s(y)dy, \qquad x\in\mathbb{R},\ t\ge 0, \end{aligned} \tag{3.3}$$

$\mathbb{P}$-almost surely. Here we denote the heat kernel by

$$g(t,x,y) := \frac{1}{\sqrt{2\pi t}}\exp\Big\{-\frac{(x-y)^2}{2t}\Big\}, \quad t>0, x,y\in\mathbb{R}$$

and the heat semigroup by

$$G_t w(x) := \int_{\mathbb{R}} g(t,x,y)w(y)dy, \quad x\in\mathbb{R}.$$

It is well-known that SPDE (3.2) has a unique solution living in $C([0,\infty),\mathcal{C})$ for every initial datum $w\in\mathcal{C}$ under conditions **(U1)'** and **(U2)**. Hereafter we sometimes consider the solution as an element in $C([0,\infty),E)$. Moreover, we also have that the solution is in $C([0,\infty),E)$ for any initial datum $w\in E$ in the case where ∇U is Lipschitz continuous. See Theorems 3.2, 5.1 and 5.2 in [15] and Theorem 2.1 in Funaki [13] for details. In the sequel, we denote by $X^w := \{X_t^w(\cdot)\}_{t\ge 0}$ the solution of SPDE (3.2) with initial datum $w\in\mathcal{C}$ and by P_w the probability measure on $C([0,\infty),E)$ induced by X^w.

We define the transition semigroup $\{P_t\}_{t\ge 0}$ by

$$P_tF(w) := \mathbb{E}\big[F(X_t^w)\big] = \int_E F(y)P_w(X_t\in dy), \quad w\in\mathcal{C},\ F\in C_b(E,\mathbb{R}). \tag{3.4}$$

Here we recall that the Gibbs measure μ is a reversible measure of our dynamics. That is,

$$\int_E F(w)P_tG(w)\mu(dw) = \int_E P_tF(w)G(w)\mu(dw), \quad t\ge 0, \tag{3.5}$$

holds for $F,G\in C_b(E,\mathbb{R})$. See Lemma 2.9 in Iwata [14] for details. Then $\{P_t\}_{t\ge 0}$ can be extended to an $L^2(\mu)$ -symmetric strongly continuous contraction semigroup. We denote by $(\mathcal{L}_2,\mathrm{Dom}(\mathcal{L}_2))$ its infinitesimal generator.

Now, we set

$$\mathcal{C}_\infty^\infty := \bigcap_{k=0}^\infty \bigcap_{r>0} \Big\{\varphi\in C^\infty(\mathbb{R},\mathbb{R}^d) \mid \|\frac{d^k\varphi}{dx^k}\|_{-r,\infty} < \infty\Big\}.$$

It is obvious that $C_0^\infty(\mathbb{R},\mathbb{R}^d)\subset\mathcal{C}_\infty^\infty$ and $\mathcal{C}_\infty^\infty$ is dense in E. We remark that the differential operators D_H and $\mathcal{L}_0$ can be naturally extended to the domain $\mathcal{F}C_b^\infty(\mathcal{C}_\infty^\infty)$ as (2.7) and (2.10), respectively. To prove our main result, we need

Proposition 3.1 *We have* $(\mathcal{L}_0, \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)) \subset (\mathcal{L}_2, \mathrm{Dom}(\mathcal{L}_2))$*, that is, for* $F \in \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)$*,* $F \in \mathrm{Dom}(\mathcal{L}_2)$ *and* $\mathcal{L}_2 F = \mathcal{L}_0 F$*.*

Proof. Let $F \in \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)$ be given as (2.6). Then by similar arguments as in the proof of Theorem 2.1 in [20], (3.3) implies that $\mathbb{P}$-almost surely,

$$\langle X_t^w, \varphi\rangle = \langle w, \varphi\rangle + \frac{1}{2}\int_0^t \langle X_s^w, \Delta_x \varphi\rangle ds - \frac{1}{2}\int_0^t \big\langle \nabla U(X_s^w(\cdot)), \varphi\big\rangle ds + \langle B_t, \varphi\rangle, \ t \geq 0 \quad (3.6)$$

holds for every $\varphi \in \mathcal{C}_\infty^\infty$. Where $\langle B_t, \varphi\rangle$ is a one-dimensional $\{\mathcal{F}_t\}$-Brownian motion multiplied by $\|\varphi\|_H$. Then the Itô formula implies

$$\begin{aligned} F(X_t^w) &= F(w) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial \alpha_i}\big(\langle X_s, \varphi_1\rangle, \cdots, \langle X_s, \varphi_n\rangle\big) d\langle X_s^w, \varphi_i\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle X_s^w, \varphi_1\rangle, \cdots, \langle X_s^w, \varphi_n\rangle\big) d\big[\langle X_\cdot^w, \varphi_i\rangle, \langle X_\cdot^w, \varphi_j\rangle\big]_s \\ &= F(w) + \int_0^t \mathcal{L}_0 F(X_s^w) ds + \int_0^t \big(D_H F(X_s^w), dB_s\big)_H. \end{aligned} \quad (3.7)$$

Here we note that (2.5) implies $\mathcal{L}_0 F \in L^p(\mu),\ p \geq 1$. Then by taking expectation on both sides of (3.7), we have

$$P_t F(w) = \mathbb{E}[F(X_t^w)] = F(w) + \int_0^t P_s(\mathcal{L}_0 F)(w) ds, \quad w \in \mathcal{C}. \quad (3.8)$$

and thus

$$\lim_{t\to 0} \frac{1}{t}\big(P_t F(w) - F(w)\big) = \lim_{t\to 0}\frac{1}{t}\int_0^t P_s(\mathcal{L}_0 F)(w) ds = \mathcal{L}_0 F(w), \quad w \in \mathcal{C}.$$

Moreover, by taking into account the invariance of the Gibbs measure μ, we have

$$\begin{aligned} \int_E \Big|\frac{1}{t}(P_t F(w) - F(w))\Big|^2 \mu(dw) &= \int_E \Big|\frac{1}{t}\int_0^t P_s(\mathcal{L}_0 F)(w) ds\Big|^2 \mu(dw) \\ &\leq \frac{1}{t}\int_0^t ds\Big\{\int_E |P_s(\mathcal{L}_0 F)(w)|^2 \mu(dw)\Big\} \\ &\leq \int_E |\mathcal{L}_0 F(w)|^2 \mu(dw) < \infty. \end{aligned}$$

Therefore by Lebesgue's dominated convergence theorem,

$$\lim_{t\to 0}\frac{1}{t}(P_t F - F) = \mathcal{L}_0 F \quad \text{in } L^2(\mu).$$

This completes the proof. ∎

Before closing this section, we give another representation of the stochastic integral equation (3.3) for our later use. We fix a constant $\kappa > 0$ with $\kappa > 2r^2$ and set $\omega := \frac{\kappa}{2} - r^2$. We divide the potential function U into

$$U(z) = \frac{\kappa}{2}|z|^2 - V(z), \quad z \in \mathbb{R}^d,$$

and consider

$$S_t w(x) := e^{-\kappa t/2} G_t w(x), \qquad x \in \mathbb{R}.$$

Then we have

Lemma 3.2 *$\{S_t\}_{t\geq 0}$ is a strongly continuous contraction semigroup on E and we have the estimate*

$$\|S_t w\|_E \leq e^{-\omega t}\|w\|_E, \quad w \in E. \tag{3.9}$$

Proof. Since the strong continuity of the semigroup $\{G_t\}_{t\geq 0}$ on E is almost obvious (cf. Lemma 2.2 in Funaki [13]), it is sufficient to show the estimate (3.9). To show this, we need an elementary and useful estimate on $g(t,x,y)$. By $|\chi'| \leq 1$ and the convexity of χ, we easily have

$$\frac{1}{2}\Delta_x \rho_{-2r}(x) \leq 2r^2 \rho_{-2r}(x), \quad x \in \mathbb{R}.$$

Hence by standard potential theory, this leads us to

$$\int_{\mathbb{R}} g(t,x,y)\rho_{-2r}(y)dy \leq e^{2r^2 t}\rho_{-2r}(x), \quad t > 0, x \in \mathbb{R}. \tag{3.10}$$

(cf. e.g. Lemma 9.44 in Da Prato-Zabcyzk [10].)

Then we can proceed as

$$\begin{aligned}
\|S_t w\|_E^2 &\leq e^{-\kappa t}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} g(t,x,y)|w(y)|^2 dy\Big)\rho_{-2r}(x)dx \\
&= e^{-\kappa t}\int_{\mathbb{R}} |w(y)|^2\Big(\int_{\mathbb{R}} g(t,x,y)\rho_{-2r}(x)dx\Big)dy \\
&\leq e^{-\kappa t}\int_{\mathbb{R}} |w(y)|^2\big(e^{2r^2 t}\rho_{-2r}(y)\big)dy \\
&= e^{-2\omega t}\|w\|_E^2,
\end{aligned}$$

where we used (3.10) for the third line. This completes the proof. ∎

Let $A : \mathrm{Dom}(A) \subset E \to E$ be the infinitesimal generator of $\{S_t\}_{t\geq 0}$. By the Hille-Yosida theorem, $(A, \mathrm{Dom}(A))$ is m-dissipative and (3.9) leads us to

$$(Aw, w)_E \leq -\omega\|w\|_E^2, \quad w \in \mathrm{Dom}(A). \tag{3.11}$$

Moreover we note that $\mathcal{C}_\infty^\infty \subset \mathrm{Dom}(A)$ and

$$Aw = \frac{1}{2}\Delta_x w - \frac{\kappa}{2}w, \quad w \in \mathcal{C}_\infty^\infty.$$

Remark 3.3 $(A, \mathrm{Dom}(A))$ *is not a symmetric operator on E. In fact, we obtain the following expression of A^* by an easy calculation:*

$$A^*w = \frac{1}{2}\Delta_x w - 2r\chi' w' + \{2r^2(\chi')^2 - r\Delta_x\chi - \frac{\kappa}{2}\}w, \quad w \in \mathcal{C}_\infty^\infty.$$

The following proposition is more or less obvious, however we include a proof for the reader's convenience.

Proposition 3.4 *Let X^w be the solution of the SPDE* (3.2). *Then it is the solution of the SPDE*

$$dX_t(x) = \frac{1}{2}\big(\Delta_x - \kappa\big)X_t(x)dt + \frac{1}{2}\nabla V(X_t(x))dt + dB_t(x), \quad x \in \mathbb{R}, t > 0,$$

with initial datum w. Namely, it satisfies the stochastic integral equation

$$\begin{aligned} X_t^w(x) &= S_t w(x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}} e^{-\kappa(t-s)/2} g(t-s,x,y)\nabla V(X_s^w(y))dyds \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{-\kappa(t-s)/2} g(t-s,x,y)dB_s(y)dy, \quad x \in \mathbb{R},\ t \geq 0, \end{aligned} \tag{3.12}$$

holds $\mathbb{P}$-almost surely. Moreover, the converse also holds.

Proof. First we note the equality

$$e^{\kappa(t-s)/2} = 1 + \frac{\kappa}{2}e^{-\kappa s/2}\int_s^t e^{\kappa\tau/2}d\tau, \quad 0 \leq s \leq t. \tag{3.13}$$

Then by (3.13) and the semigroup property for $\{G_t\}_{t\geq 0}$, we have the following expansion on the first term of the right hand side of (3.3):

$$\begin{aligned} G_t w(x) &= S_t w(x) + \frac{\kappa}{2}\int_0^t e^{\kappa\tau/2} S_t w(x)d\tau \\ &= S_t w(x) + \frac{\kappa}{2}\int_0^t S_{t-\tau}(G_\tau w)(x)d\tau. \end{aligned} \tag{3.14}$$

Now we give the expansion on the second term of the right hand side of (3.3). By using (3.13), Fubini's theorem and the semigroup property for $\{G_t\}_{t\geq 0}$, it holds that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} g(t-s,x,y)\nabla U(X_s^w(y))dyds \\ &= \int_0^t e^{\kappa(t-s)/2} S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds \\ &= \int_0^t S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds \\ &\quad + \frac{\kappa}{2}\int_0^t e^{-\kappa s/2}\Big(\int_s^t e^{\kappa\tau/2}d\tau\Big) S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds \\
&\quad +\frac{\kappa}{2}\int_0^t e^{\kappa\tau/2}\Big(\int_0^\tau e^{-\kappa s/2} S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds\Big)d\tau \\
&= \int_0^t S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds \\
&\quad +\frac{\kappa}{2}\int_0^t S_{t-\tau}\Big(\int_0^\tau G_{\tau-s}\big\{\nabla U(X_s^w(\cdot))\big\}(\cdot)ds\Big)(x)d\tau. \qquad (3.15)
\end{aligned}
$$

Next we proceed to the expansion on the third term of the right hand side of (3.3). Here we recall (3.1). By using (3.13), stochastic Fubini's theorem and the semigroup property for $\{G_t\}_{t\ge 0}$, we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} g(t-s,x,y)dB_s(y)dy \\
&= \sum_{j=1}^\infty \int_0^t e^{\kappa(t-s)/2} S_{t-s}h_j(x)d\beta_j(s) \\
&= \sum_{j=1}^\infty \int_0^t S_{t-s}h_j(x)d\beta_j(s) + \frac{\kappa}{2}\sum_{j=1}^\infty \int_0^t e^{-\kappa s/2}\Big(\int_s^t e^{\kappa\tau/2}d\tau\Big) S_{t-s}h_j(x)d\beta_j(s) \\
&= \sum_{j=1}^\infty \int_0^t S_{t-s}h_j(x)d\beta_j(s) + \frac{\kappa}{2}\sum_{j=1}^\infty \int_0^t e^{\kappa\tau/2}\Big(\int_0^\tau e^{-\kappa s/2} S_{t-s}h_j(x)d\beta_j(s)\Big)d\tau \\
&= \sum_{i=1}^\infty \int_0^t S_{t-s}e_i(x)d\beta_i(s) + \frac{\kappa}{2}\int_0^t S_{t-\tau}\Big\{\sum_{j=1}^\infty \int_0^\tau G_{t-s}h_j(\cdot)d\beta_j(s)\Big\}(x)d\tau. \qquad (3.16)
\end{aligned}
$$

Finally, we combine (3.14), (3.15) and (3.16). Then by (3.3), we have

$$
\begin{aligned}
X_t(x) &= S_t w(x) - \frac{1}{2}\int_0^t S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds + \sum_{j=1}^\infty \int_0^t S_{t-s}h_j(x)d\beta_j(s) \\
&\quad +\frac{\kappa}{2}\int_0^t S_{t-\tau}\Big\{(G_\tau w) - \frac{1}{2}\Big(\int_0^\tau G_{\tau-s}\big\{\nabla U(X_s^w(\cdot))\big\}ds\Big) \\
&\qquad\qquad +\Big(\sum_{j=1}^\infty \int_0^\tau G_{t-s}h_j(\cdot)d\beta_j(s)\Big)\Big\}(x)d\tau \\
&= S_t w(x) - \frac{1}{2}\int_0^t S_{t-s}\big\{\nabla U(X_s^w(\cdot))\big\}(x)ds + \sum_{j=1}^\infty \int_0^t S_{t-s}h_j(x)d\beta_j(s) \\
&\quad +\frac{\kappa}{2}\int_0^t S_{t-s}\big(X_s^w(\cdot)\big)(x)ds \\
&= S_t w(x) + \frac{1}{2}\int_0^t S_{t-s}\big\{\nabla V(X_s^w(\cdot))\big\}(x)ds + \sum_{j=1}^\infty \int_0^t S_{t-s}h_j(x)d\beta_j(s)
\end{aligned}
$$

$$
\begin{aligned}
&= S_t w(x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}} e^{-\kappa(t-s)/2} g(t-s,x,y) \nabla V(X_s^w(y)) dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}} e^{-\kappa(t-s)/2} g(t-s,x,y) dB_s(y) dy, \qquad x \in \mathbb{R},\ t \geq 0.
\end{aligned}
$$

The converse can be shown in the same manner. This completes the proof. ∎

Remark 3.5 *Here we give an abstract representation of* (3.12) *for our later use. Let Q be a bounded linear operator on E defined by $Qw := \rho_{-2r} w, w \in E$. For the H-cylindrical Brownian motion $\{B_t\}_{t\geq 0}$, we consider*

$$
W_t(\cdot) := \sum_{j=1}^{\infty} \beta_j(t)(Q^{-1/2} h_j)(\cdot), \quad t \geq 0.
$$

Then $\{W_t\}_{t\geq 0}$ is a E-cylindrical Brownian motion because $\{Q^{-1/2}h_j\}_{j=1}^{\infty}$ is a C.O.N.S. of E. Let $b : \mathcal{C} \to \mathcal{C} \subset E$ be a continuous map defined by

$$
b(w)(\cdot) := \frac{1}{2}(\nabla V)(w(\cdot)), \quad w \in \mathcal{C}.
$$

By the proof of Lemma 4.1 *below, we can see $S_{t-s}\sqrt{Q}$ is a Hilbert-Schmidt operator on E. Hence* (3.12) *is interpreted as the E-valued stochastic integral equation*

$$
X_t^w = S_t w + \int_0^t S_{t-s} b(X_s^w) ds + \int_0^t S_{t-s}\sqrt{Q} dW_s, \quad t \geq 0. \tag{3.17}
$$

4 Some Results on the Ornstein-Uhlenbeck Semigroup

In this section, we present some properties of the Ornstein-Uhlenbeck semigroup $\{R_t\}_{t\geq 0}$ and its infinitesimal generator L. In this paper, we consider $\{R_t\}_{t\geq 0}$ on suitable subsets of continuous functions on E so that the domain of its generator L lies between $\mathcal{F}\mathcal{C}_b^{\infty}$ and $\mathrm{Dom}(\overline{\mathcal{L}}_0)$. However, since $\{R_t\}_{t\geq 0}$ is not strongly continuous, we need a more refined treatment based on Da Prato-Röckner [6] and Da Prato-Tubaro [8].

4.1 Characterization of the Ornstein-Uhlenbeck Semigroup and its Infinitesimal Generator

At the beginning of this subsection, we present a lemma which is necessary to define the Ornstein-Uhlenbeck semigroup. Hereafter, we often use the notation e^{tA} instead of S_t.

Lemma 4.1 *We define a bounded linear operator $Q_\infty : E \to E$ by*

$$
Q_\infty w := \int_0^\infty e^{tA} Q e^{tA^*} w dt, \quad w \in E. \tag{4.1}
$$

Then Q_∞ is invertible and $\mathrm{Tr}(Q_\infty) < \infty$.

Proof. For the first assertion, we need to show $\mathrm{Ker}(Q_\infty) = \{0\}$ in E. We recall $\sqrt{Q}w = \rho_{-r}w, w \in E$. Then for $w \in \mathrm{Ker}(Q_\infty)$, we have

$$0 = (Q_\infty w, w)_E = \int_0^\infty \|\sqrt{Q}e^{tA^*}w\|_E^2 dt.$$

Hence for a.e. $t \geq 0$, $\sqrt{Q}e^{tA^*}w = 0$ holds and by the continuity with respect to t, we obtain $\sqrt{Q}w = 0$. This leads us to $w = 0$.

For the second assertion, we consider the natural embedding map $i : H \to E$, i.e., $i(h) := h, h \in H$. Then the adjoint operator $i^* : E \to H$ is represented by $i^*(w) = \rho_{-2r}w = Qw, w \in E$. By noting that $Q = ii^*$, we can see that

$$\mathrm{Tr}(Q_\infty) \leq \int_0^\infty \mathrm{Tr}\{(e^{tA}i)(e^{tA}i)^*\}dt \leq \int_0^\infty \|e^{tA}i\|_{H\otimes E}^2 dt.$$

On the other hand, we have

$$\begin{aligned}
\|e^{tA}i\|_{H\otimes E}^2 &= \sum_{j=1}^\infty \|e^{tA}i(h_j)\|_E^2 \\
&= e^{-\kappa t}\sum_{j=1}^\infty \Big\| \int_{\mathbb{R}} g(t,\cdot,y)h_j(y)dy\Big\|_E^2 \\
&= e^{-\kappa t}\int_{\mathbb{R}}\sum_{j=1}^\infty \big(g(t,x,\cdot),h_j(\cdot)\big)_H^2 \rho_{-2r}(x)dx \\
&= e^{-\kappa t}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} g(t,x,y)^2 dy\Big)\rho_{-2r}(x)dx \\
&= e^{-\kappa t}\int_{\mathbb{R}} dy\Big(\int_{\mathbb{R}} g(t,x,y)^2\rho_{-2r}(x)dx\Big) \\
&\leq \frac{e^{-\kappa t}}{\sqrt{2\pi t}}\int_{\mathbb{R}} dy\Big(\int_{\mathbb{R}} g(t,x,y)\rho_{-2r}(x)dx\Big) \\
&\leq \frac{e^{-\gamma t}}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{2r^2 t}\rho_{-2r}(y)dy \\
&= \frac{e^{-2\omega t}}{\sqrt{2\pi t}}\int_{\mathbb{R}} \rho_{-2r}(y)dy = Ct^{-1/2}e^{-2\omega t},
\end{aligned}$$

where we used (3.10) for the sixth line. Therefore, we can conclude that

$$\mathrm{Tr}(Q_\infty) \leq C\int_0^\infty t^{-1/2}e^{-2\omega t}dt = C\Gamma(1/2) < \infty.$$

This completes the proof. ∎

Now we are in a position to introduce the Ornstein-Uhlenbeck semigroup $\{R_t\}_{t\geq 0}$. Let $Q_t, t \geq 0$, be a bounded linear operator on E defined by

$$Q_t w := \int_0^t e^{sA}Qe^{sA^*}w ds, \quad w \in E.$$

We remark this operator is of trace class by Lemma 4.1. We denote by N_{Q_t} the Gaussian measure on E with mean 0 and covariance operator Q_t.

Next we introduce some function spaces on which the Ornstein-Uhlenbeck semigroup will act. We denote by $UC_{b,2}(E)$ the Banach space of all functions $F : E \to \mathbb{R}$ such that $\frac{F(\cdot)}{1+\|\cdot\|_E^2}$ is uniformly continuous and bounded. Endowed with the norm

$$\|F\|_{b,2} := \sup_{w \in E} \frac{|F(w)|}{1 + \|w\|_E^2} \ ,$$

$UC_{b,2}(E)$ is a Banach space. For E-valued continuous functions, we can also define $UC_{b,2}(E,E)$ in the same manner. Moreover, $C^1_{b,2}(E)$ denotes the subspace of $UC_{b,2}(E)$ of those functions F which are continuously differentiable with

$$\|DF\|_{b,2} := \sup_{w \in E} \frac{\|DF(w)\|_E}{1 + \|w\|_E^2} < \infty \ ,$$

where $DF : E \to E$ means the E-Fréchet derivative of F. We have the relation

$$D_H F = Q^{1/2} DF.$$

Then the Ornstein-Uhlenbeck semigroup $\{R_t\}_{t \geq 0}$ is given by

$$R_t F(w) := \int_E F(e^{tA} w + y) N_{Q_t}(dy), \quad w \in E, F \in UC_{b,2}(E). \tag{4.2}$$

For $F \in UC_{b,2}(E,E)$, $\{R_t\}_{t \geq 0}$ can be defined in the same manner as (4.2). In this case, the integral should be regarded as a Bochner integral. The following result is straightforward. We include a proof for completeness.

Proposition 4.2 *R_t maps $UC_{b,2}(E)$ into itself for all $t \geq 0$ and*

$$\|R_t F\|_{b,2} \leq (1 + \mathrm{Tr}(Q_\infty)) \ \|F\|_{b,2}. \tag{4.3}$$

Moreover R_t maps $C^1_{b,2}(E)$ into itself for all $t \geq 0$ and

$$\|DR_t F\|_{b,2} \leq (1 + \mathrm{Tr}(Q_\infty)) \ \|DF\|_{b,2}. \tag{4.4}$$

Proof. Since the proofs for the first assertion and the second assertion are almost the same, we only show the second assertion. (For the proof of the first assertion, see Proposition 2.1 in Da Prato [5].) For $F \in C^1_{b,2}(E)$, we easily have

$$\big(DR_t F(w), k\big)_E = \int_E \big(DF(e^{tA} w + y), e^{tA} k\big)_E N_{Q_t}(dy), \quad k \in E.$$

This implies the intertwining property of the Ornstein-Uhlenbeck semigroup

$$DR_t F = e^{tA^*} R_t DF, \quad F \in C^1_{b,2}(E).$$

Hence we have

$$\begin{aligned}\frac{\|DR_tF(w)\|_E}{1+\|w\|_E^2} &\le \int_E \frac{\|e^{tA^*}DF(e^{tA}w+y)\|_E}{1+\|w\|_E^2}N_{Q_t}(dy)\\ &\le \int_E \frac{\|DF(e^{tA}w+y)\|_E}{1+\|w\|_E^2}N_{Q_t}(dy)\\ &\le \|DF\|_{b,2}\int_E \frac{1+\|e^{tA}w+y\|_E^2}{1+\|w\|_E^2}N_{Q_t}(dy)\\ &\le \|DF\|_{b,2}\int_E (1+\|y\|_E^2)N_{Q_t}(dy)\\ &\le \big(1+\mathrm{Tr}(Q_\infty)\big)\ \|DF\|_{b,2},\end{aligned}$$

where we used Lemma 3.2 for the second and the fourth lines. This leads us to the desired estimate (4.4). ∎

Lemma 4.3 *The Ornstein-Uhlenbeck semigroup $\{R_t\}_{t\ge0}$ has the following representation:*

$$R_tF(w) := \mathbb{E}[F(Y_t^w)] = \int_E F(y)R_w(Y_t\in dy), \quad w\in E,\ F\in UC_{b,2}(E),$$

where R_w is the probability measure on $C([0,\infty),E)$ induced by the Ornstein-Uhlenbeck process $Y^w=\{Y_t^w(\cdot)\}_{t\ge0}$, i.e., the solution of the SPDE

$$dY_t(x) = \frac{1}{2}\big(\Delta_x-\kappa\big)Y_t(x)dt + dB_t(x), \qquad x\in\mathbb{R},\ t>0, \tag{4.5}$$

with initial datum $Y_0=w\in E$.

Proof. By Proposition 3.4 and Remark 3.5, the solution of (4.5) is given by the following representation:

$$Y_t^w = e^{tA}w + \int_0^t e^{(t-s)A}\sqrt{Q}dW_s, \quad t\ge0.$$

Hence obviously $Y_t^w, t\ge0$, is a Gaussian random variable on E with mean $e^{tA}w$ and the covariance operator is given by

$$\int_0^t (e^{sA}\sqrt{Q})(e^{sA}\sqrt{Q})^*ds = \int_0^t e^{sA}(\sqrt{Q})^2e^{sA^*}ds = Q_t.$$

(See Theorem 5.2 in [9]). This completes the proof. ∎

The Ornstein-Uhlenbeck semigroup $\{R_t\}_{t\ge0}$ is not strongly continuous in $UC_{b,2}(E)$. However, it can be proved that it is a π-semigroup in the sense of Priola [18]. Thus one can define its infinitesimal generator L through its Laplace transform

$$\Psi_\lambda F(w) = \int_0^\infty e^{-\lambda t}R_tF(w)dt, \quad w\in E,\ \lambda>0.$$

By virtue of Proposition 4.2, it is easy to see that every Ψ_λ maps $UC_{b,2}(E)$ and $C^1_{b,2}(E)$ into themselves for all $\lambda > 0$, respectively, and that $\{\Psi_\lambda\}_{\lambda>0}$ is a pseudo-resolvent. Consequently, there exists a unique closed operator L in $UC_{b,2}(E)$ such that

$$R(\lambda, L) = (\lambda - L)^{-1} = \Psi_\lambda, \quad \lambda > 0.$$

We call L the infinitesimal generator of R_t on $UC_{b,2}(E)$.

Since the image of the resolvent is independent of $\lambda > 0$, we can set

$$\mathcal{D}(L, UC_{b,2}(E)) := R(\lambda, L)(UC_{b,2}(E)), \quad \mathcal{D}(L, C^1_{b,2}(E)) := R(\lambda, L)(C^1_{b,2}(E)).$$

Remark 4.4 *It holds that $F \in \mathcal{D}(L, UC_{b,2}(E))$ and $LF = G$ if and only if*

$$\text{(i)} \quad \lim_{t\searrow 0} \frac{1}{t}(R_tF(w) - F(w)) = G(w), \ w \in E. \tag{4.6}$$

$$\text{(ii)} \quad \sup_{t>0} \frac{1}{t}\|R_tF - F\|_{b,2} < \infty. \tag{4.7}$$

The reader is referred to Remark 2.2 *in* [5] *and Proposition* 2.2.8 *in* [18] *for the details.*

Proposition 4.5 *$\mathcal{F}\mathcal{C}^\infty_b(\mathcal{C}^\infty_\infty) \subset \mathcal{D}(L, C^1_{b,2}(E))$ holds and we have*

$$LF(w) = \frac{1}{2}\mathrm{Tr}\big(D^2_H F(w)\big) + \langle w, AD_HF(w)\rangle, \quad F \in \mathcal{F}\mathcal{C}^\infty_b(\mathcal{C}^\infty_\infty). \tag{4.8}$$

Namely, for $F(w) = f(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle)$, $\varphi_i \in \mathcal{C}^\infty_\infty$, $i = 1, \cdots, n$, we obtain

$$\begin{aligned} LF(w) &= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\langle\varphi_i, \varphi_j\rangle \\ &\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial f}{\partial\alpha_i}\big(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle\big)\cdot\big\{\langle w, \Delta_x\varphi_i\rangle - \kappa\langle w, \varphi\rangle\big\}. \end{aligned}$$

Proof. We denote the right-hand side of (4.8) by L_0F. Since $\langle w, AD_HF(w)\rangle$ has a linear growth with respect to $\|w\|_E$ and is smooth in the Fréchet sense, we have $L_0F \in C^1_{b,2}(E)$.

First we show the inclusion $\mathcal{F}\mathcal{C}^\infty_b(\mathcal{C}^\infty_\infty) \subset \mathcal{D}(L, UC_{b,2}(E))$. We only need to check two conditions in Remark 4.4. By repeating the argument in the proof of Proposition 3.1, for $F \in \mathcal{F}\mathcal{C}^\infty_b(\mathcal{C}^\infty_\infty)$, we easily obtain

$$\lim_{t\searrow 0}\frac{1}{t}\big(R_tF(w) - F(w)\big) = L_0F(w), \quad w \in E. \tag{4.9}$$

On the other hand, by noting that $L_0F \in UC_{b,2}(E)$, we also have

$$\begin{aligned} \frac{1}{t}\|R_tF - F\|_{b,2} &\le \frac{1}{t}\int_0^t \|R_sL_0F\|_{b,2}ds \\ &\le \frac{1}{t}\int_0^t \big(1 + \mathrm{Tr}(Q_\infty)\big)\|L_0F\|_{b,2}ds \\ &= \big(1 + \mathrm{Tr}(Q_\infty)\big)\|L_0F\|_{b,2}, \end{aligned} \tag{4.10}$$

where we used Proposition 4.2 for the second line. Hence by (4.9) and (4.10), we have the expression (4.8).

Finally, by combining $L_0 F \in C^1_{b,2}(E)$ and $F = \Psi_\lambda(L_0 F)$, it is easy to see that $\mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty) \subset \mathcal{D}(L, C^1_{b,2}(E))$. This completes the proof. ∎

4.2 Approximations by Cylinder Functions

The main object of this subsection is to show that functions in $\mathcal{D}(L, C^1_{b,2}(E))$ can be approximated point-wise in the graph norm by functions in $\mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty)$ with uniformly bounded norm. These approximations are not possible by using simple sequences, but k-sequences, $k \in \mathbb{N}$, that is sequences $\{F_n\} = \{F_{n_1,\cdots,n_k}\}$ depending on k indices. We say that $\{F_n\}$ is convergent to F if

$$\lim_{n\to\infty} F_n(w) := \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} F_{n_1,\cdots,n_k}(w) = F(w), \quad w \in E.$$

Proposition 4.6 (1) *Let $F \in \mathcal{D}(L, C^1_{b,2}(E))$. Then there exists a 4-sequence $\{F_n\}_{n\in\mathbb{N}^4} = \{F_{n_1,\cdots,n_4}\} \subset \mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty)$ such that for all $w \in E$ we have*

$$\lim_{n\to\infty} F_n(w) = F(w), \quad \lim_{n\to\infty} DF_n(w) = DF(w), \quad \lim_{n\to\infty} LF_n(w) = LF(w) \tag{4.11}$$

and the estimates

$$\|F_n\|_{b,2} \le \frac{2e}{e-1}\big(1 + \mathrm{Tr}(Q_\infty)\big) \cdot \big(\|F\|_{b,2} + \|LF\|_{b,2}\big), \tag{4.12}$$

$$\begin{aligned}\|DF_n\|_{b,2} \le{}& \frac{2e}{e-1}\big(1 + \mathrm{Tr}(Q_\infty)\big) \\ &\times \big(2\|F\|_{b,2} + \|DF\|_{b,2} + 2\|LF\|_{b,2} + \|DLF\|_{b,2}\big),\end{aligned} \tag{4.13}$$

$$\|LF_n\|_{b,2} \le 1 + 2\big(2 + \mathrm{Tr}(Q_\infty)\big) \cdot \big(\|F\|_{b,2} + \|LF\|_{b,2}\big). \tag{4.14}$$

(2) $\mathcal{D}(L, C^1_{b,2}(E)) \subset \mathrm{Dom}(\overline{\mathcal{L}}_0)$ *and the following identity holds:*

$$\overline{\mathcal{L}}_0 F = LF + (b, DF)_E, \quad F \in \mathcal{D}(L, C^1_{b,2}(E)), \tag{4.15}$$

where $b : \mathrm{Dom}(b) \subset E \to E$ is a measurable mapping with $\mathrm{Dom}(b) = \mathcal{C}$ is defined by

$$b(w)(\cdot) := \frac{1}{2}\widetilde{\nabla} V(w(\cdot)), \quad w \in \mathcal{C}.$$

Before giving the proof, we need some preparations about the operator $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$.

Lemma 4.7 *For all $F \in \mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty)$, we have*

$$\int_E \mathcal{L}_0 F(w) F(w) \mu(dw) = -\frac{1}{2}\int_E \|D_H F(w)\|_H^2 \mu(dw).$$

Consequently, $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty))$ is dissipative in $L^2(\mu)$.

Proof. Let $F \in \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)$ be given as (2.6) with $K = \mathcal{C}_\infty^\infty$. We construct an approximating sequence of $\mathcal{FC}_b^\infty$. Let $\eta_k \in C_0^\infty(\mathbb{R},\mathbb{R}), k > 0$, be a cut-off function such that $0 \le \eta \le 1$, $\eta_k(x) = 1$ if $|x| \le k$, $\eta(x) = 0$ if $|x| \ge 2k$, $|\eta_k'| \le 2/k$ and $|\Delta_x \eta_k| \le 8/k^2$. We define $F_k \in \mathcal{FC}_b^\infty, k \in \mathbb{N}$, by $F_k(w) := f\big(\langle w, \eta_k\varphi_1\rangle, \cdots, \langle w, \eta_k\varphi_n\rangle\big)$. Then we have the expressions

$$\begin{aligned} D_H F_k(w) &= \sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}\big(\langle w, \eta_k\varphi_1\rangle, \cdots, \langle w, \eta_k\varphi_n\rangle\big)\cdot \eta_k\varphi_j, \qquad (4.16)\\ \mathcal{L}_0 F_k(w) &= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial\alpha_i\partial\alpha_j}\big(\langle w, \eta_k\varphi_1\rangle, \cdots, \langle w, \varphi_n\eta_k\rangle\big)\langle \eta_k\varphi_i, \eta_k\varphi_j\rangle \\ &\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial f}{\partial \alpha_i}\big(\langle w, \eta_k\varphi_1\rangle, \cdots, \langle w, \eta_k\varphi_n\rangle\big) \\ &\quad \times \big\{\langle w, (\Delta_x\eta_k)\varphi_i + 2\eta_k'\varphi_i' + \eta_k\Delta_x\varphi_i\rangle - \langle \nabla U(w(\cdot)), \eta_k\varphi_i\rangle\big\}. \qquad (4.17)\end{aligned}$$

By noting (4.16), (4.17), $\eta_k \to 1, \eta_k' \to 0, \Delta_x\eta_k \to 0$ as $k \to \infty$ and the integrability (2.5), we can use Lebesgue's dominated convergence theorem, and thus we have

$$\|F - F_k\|_{L^2(\mu)} + \|D_H F - D_H F_k\|_{L^2(\mu;H)} + \|\mathcal{L}_0 F - \mathcal{L}_0 F_k\|_{L^2(\mu)} \to 0 \quad \text{as } k \to \infty. \qquad (4.18)$$

On the other hand, we have the equality $\mathcal{E}(F_k, F_k) = \big(-\mathcal{L}_0 F_k, F_k\big)_{L^2(\mu)}$ for each $k \in \mathbb{N}$ by recalling (2.11). Hence we can complete the proof by combining this with the convergence (4.18). ∎

By this lemma, we see that $(\mathcal{L}_0, \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty))$ is closable in $L^2(\mu)$. Then we have the following lemma:

Lemma 4.8 *The closure of $(\mathcal{L}_0, \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty))$ in $L^2(\mu)$ coincides with $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$.*

Proof. We denote by $(\widetilde{\mathcal{L}}_0, \mathrm{Dom}(\widetilde{\mathcal{L}}_0))$ the closure of $(\mathcal{L}_0, \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty))$ in $L^2(\mu)$. We only need to show that for any $F \in \mathrm{Dom}(\widetilde{\mathcal{L}}_0)$, there exists an approximation sequence of $\mathcal{FC}_b^\infty$ with respect to the graph norm. First, we choose a sequence $\{F_m\}_{m=1}^\infty \subset \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)$ such that

$$\|F - F_m\|_{L^2(\mu)} + \|\widetilde{\mathcal{L}}_0 F - \mathcal{L}_0 F_m\|_{L^2(\mu)} < \frac{1}{m}.$$

We set $F_m(w) = f_m(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_{n(m)}\rangle)$, where $n(m) \in \mathbb{N}, \varphi_1, \cdots, \varphi_{n(m)} \in \mathcal{C}_\infty^\infty$ and $f_m \in C_b^\infty(\mathbb{R}^{n(m)})$. For each F_m, we construct an approximated sequence $\{F_{m,k}\}_{k=1}^\infty \subset \mathcal{FC}_b^\infty$ by defining $F_{m,k} := (F_m)_k$. See the proof of Lemma 4.7 for the meaning of $(F_m)_k$.

By (4.18), for each $m \in \mathbb{N}$, we have

$$\|F_m - F_{m,k}\|_{L^2(\mu)} + \|\mathcal{L}_0 F_m - \mathcal{L}_0 F_{m,k}\|_{L^2(\mu)} \to 0 \quad \text{as } k \to \infty.$$

Hence for each $m \in \mathbb{N}$, there exists a sequence $\{m(k)\}_{k=1}^\infty$ with $\lim_{k\to\infty} m(k) = \infty$ and

$$\|F_m - F_{m,m(k)}\|_{L^2(\mu)} + \|\mathcal{L}_0 F_m - \mathcal{L}_0 F_{m,m(k)}\|_{L^2(\mu)} < \frac{1}{k}.$$

Finally, we consider $\{F_{m,m(m)}\}_{m=1}^\infty \subset \mathcal{FC}_b^\infty$. By the above arguments, we easily see that it is the desired sequence. This completes the proof. ∎

Remark 4.9 *By Proposition* 3.1 *and Lemma* 4.8*, we know that*

$$(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0)) \subset (\mathcal{L}_2, \mathrm{Dom}(\mathcal{L}_2)). \tag{4.19}$$

The hard part will be to prove the dense inclusion (see Subsection 5.2 *below).*

Proof of Proposition 4.6. (1) We mainly follow the argument in [8]. However, since we need some modifications in our situation, we give the proof for the reader's convenience. We proceed in several steps.

Step 1: For $F \in \mathcal{D}(L, C^1_{b,2}(E))$, we construct an approximated sequence of cylinder functions. Take $\{e_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ to be a fixed C.O.N.S. of E throughout the proof.

We define a finite dimensional projection $\Pi_{n_1} : E \to E, n_1 \in \mathbb{N}$, by

$$\Pi_{n_1}(w) := \sum_{j=1}^{n_1} (w, e_j)_E e_j, \quad w \in E,$$

and define $F_{n_1} : E \to \mathbb{R}$ by $F_{n_1}(w) := F(\Pi_{n_1}(w))$. Moreover, we define $f_{n_1} : \mathbb{R}^{n_1} \to \mathbb{R}$ by $f_{n_1}(\alpha_1, \cdots, \alpha_{n_1}) := F(\alpha_1 e_1 + \cdots + \alpha_{n_1} e_{n_1})$ for $\alpha = (\alpha_1, \cdots, \alpha_{n_1}) \in \mathbb{R}^{n_1}$. Then we obviously have

$$F_{n_1}(w) = f_{n_1}\big((w, e_1)_E, \cdots, (w, e_{n_1})_E\big)$$

and since $F \in C^1_{b,2}(E)$, we have that $f_{n_1} \in C^1(\mathbb{R}^{n_1}, \mathbb{R})$ and

$$\sup_{\alpha \in \mathbb{R}^{n_1}} \frac{|f_{n_1}(\alpha)|}{1 + |\alpha|^2} \le \|F\|_{b,2} \ , \qquad \sup_{\alpha \in \mathbb{R}^{n_1}} \frac{|\nabla f_{n_1}(\alpha)|}{1 + |\alpha|^2} \le \|DF\|_{b,2} \ , \tag{4.20}$$

where ∇ stands for the gradient on $\mathbb{R}^{n_1}$. We note that (4.20) means $\|F_{n_1}\|_{b,2} \le \|F\|_{b,2}$ and $\|DF_{n_1}\|_{b,2} \le \|DF\|_{b,2}$. Then by recalling that $\lim_{n_1 \to \infty} \|\Pi_{n_1}(w) - w\|_E = 0$ and $DF_{n_1}(w) = DF(\Pi_{n_1}(w))$ for $w \in E$, we obtain

$$\lim_{n_1 \to \infty} F_{n_1}(w) = F(w), \quad \lim_{n_1 \to \infty} DF_{n_1}(w) = DF(w), \quad w \in E. \tag{4.21}$$

Step 2: Since F_{n_1} is not bounded and smooth, we need next approximations. Let $\psi_{n_1,n_2} \in C_0^\infty(\mathbb{R}^{n_1}, \mathbb{R}), n_2 \in \mathbb{N}$, be a cut-off function defined by $\psi_{n_1,n_2}(\alpha) := \eta_{n_2}\big(|\alpha|\big), \alpha \in \mathbb{R}^{n_1}$, where η_{n_2} is defined as in the proof of Lemma 4.8. We note that $|\nabla \psi_{n_1,n_2}| \le 2/n_2$ for all $n_1 \in \mathbb{N}$.

Now we choose a non-negative symmetric function $\zeta \in C_0^\infty(\mathbb{R}^{n_1}, \mathbb{R})$ satisfying $\zeta(\alpha) = 0$ for $|\alpha| \ge 1$ and $\int_{\mathbb{R}^{n_1}} \zeta(\alpha)^2 d\alpha = 1$. Moreover, we define $\zeta_\varepsilon(\alpha) := \varepsilon^{-n_1} \zeta(\alpha/\varepsilon)$ for $\varepsilon > 0$ and define by $g_\varepsilon := (\zeta_\varepsilon * g)$ the mollification of a function g. Here we consider

$$F^{(\varepsilon)}_{n_1,n_2}(w) := \big(\psi_{n_1,n_2} \cdot f_{n_1}\big)_\varepsilon\big((w, e_1)_E, \cdots, (w, e_{n_1})_E\big), \quad n_2 \in \mathbb{N}, \varepsilon > 0.$$

Then for sufficiently small $\varepsilon > 0$, we have the estimates

$$\begin{aligned} \frac{\big|F^{(\varepsilon)}_{n_1,n_2}(w)\big|}{1 + \|w\|_E^2} &\le \frac{2\big|(\psi_{n_1,n_2} \cdot f_{n_1})\big((w, e_1)_E, \cdots, (w, e_{n_1})_E\big)\big|}{1 + \|w\|_E^2} \\ &\le \frac{2\big|f_{n_1}\big((w, e_1)_E, \cdots, (w, e_{n_1})_E\big)\big|}{1 + \|w\|_E^2} \\ &\le \frac{2|F(\Pi_{n_1}(w))|}{1 + \|\Pi_{n_1}(w)\|_E^2} \le 2\|F\|_{b,2} \ , \end{aligned} \tag{4.22}$$

and

$$
\begin{aligned}
\frac{\|DF^{(\varepsilon)}_{n_1,n_2}(w)\|_E}{1+\|w\|_E^2} &\le \frac{2\big|\nabla(\psi_{n_1,n_2}\cdot f_{n_1})\big((w,e_1)_E,\cdots,(w,e_{n_1})_E\big)\big|}{1+\|w\|_E^2} \\
&\le \frac{2\big|(\psi_{n_1,n_2}\nabla f_{n_1})\big((w,e_1)_E,\cdots,(w,e_{n_1})_E\big)\big|}{1+\|w\|_E^2} \\
&\quad + \frac{2\big|(f_{n_1}\nabla \psi_{n_1,n_2})\big((w,e_1)_E,\cdots,(w,e_{n_1})_E\big)\big|}{1+\|w\|_E^2} \\
&\le 2\cdot\frac{\|DF(\Pi_{n_1}(w))\|_E}{1+\|w\|_E^2}+\frac{4}{n_2}\cdot\frac{|F(\Pi_{n_1}(w))|}{1+\|w\|_E^2} \\
&\le 2\|DF\|_{b,2}+4\|F\|_{b,2}. \qquad (4.23)
\end{aligned}
$$

Therefore, there exists a decreasing sequence $\{\varepsilon(j)\}_{j=1}^{\infty}$ such that $\lim_{\varepsilon\searrow 0}\varepsilon(j)=0$ and (4.22), (4.23) hold for every $F^{(\varepsilon(j))}_{n_1,n_2}$.

Finally, we define by $F_n := F^{(\varepsilon(n_3))}_{n_1,n_2} \in \mathcal{FC}_b^{\infty}$ for $n=(n_1,n_2,n_3)\in\mathbb{N}^3$. Then by noting that

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\big\|\big(\psi_{n_1,n_2}\cdot f_{n_1}\big)_{\varepsilon}-\psi_{n_1,n_2}\cdot f_{n_1}\big\|_{L^{\infty}(\mathbb{R}^{n_1})} \\
&\quad =\lim_{\varepsilon\searrow 0}\big\|\nabla\big(\psi_{n_1,n_2}\cdot f_{n_1}\big)_{\varepsilon}-\nabla(\psi_{n_1,n_2}\cdot f_{n_1})\big\|_{L^{\infty}(\mathbb{R}^{n_1},\mathbb{R}^{n_1})}=0,
\end{aligned}
$$

and recalling (4.21), we easily see that

$$
\lim_{n\to\infty}F_n(w)=F(w), \quad \lim_{n\to\infty}DF_n(w)=DF(w), \quad w\in E. \qquad (4.24)
$$

We also note that (4.22) and (4.23) lead us to the estimates

$$
\|F_n\|_{b,2}\le 2\|F\|_{b,2}, \quad \|DF_n\|_{b,2}\le 2\|DF\|_{b,2}+4\|F\|_{b,2}.
$$

Step 3: We proceed to give an approximation for $LF\in C^1_{b,2}(E)$. We set $G:=F-LF\in C^1_{b,2}(E)$. By the above argument, there exists a 3-sequence $\{G_n\}=\{G_{n_1,n_2,n_3}\}\subset\mathcal{FC}_b^{\infty}$ such that (4.24) (with F replaced by G) holds and

$$
\begin{aligned}
\|G_n\|_{b,2} &\le 2(\|F\|_{b,2}+\|LF\|_{b,2}), \\
\|DG_n\|_{b,2} &\le 2(\|DF\|_{b,2}+\|DLF\|_{b,2}+2\|F\|_{b,2}+2\|LF\|_{b,2}).
\end{aligned}
$$

Next we set $F_n:=R(1,L)G_n$. Then $LF_n=F_n-G_n$ and by recalling Proposition 4.2, we have

$$
\begin{aligned}
\|F_n\|_{b,2} &\le \int_0^{\infty}e^{-t}\|R_tG_n\|_{b,2}dt \\
&\le (1+\mathrm{Tr}(Q_{\infty}))\|G_n\|_{b,2} \\
&\le 2\big(1+\mathrm{Tr}(Q_{\infty})\big)\cdot\big(\|F\|_{b,2}+\|LF\|_{b,2}\big), \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
\|DF_n\|_{b,2} &\leq \int_0^\infty e^{-t}\|DR_tG_n\|_{b,2}dt \\
&\leq \big(1+\mathrm{Tr}(Q_\infty)\big)\cdot\big(\|DF\|_{b,2}+\|DLF\|_{b,2}+2\|F\|_{b,2}+2\|LF\|_{b,2}\big), \quad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
\|LF_n\|_{b,2} &\leq \|G_n\|_{b,2}+\|F_n\|_{b,2} \\
&\leq 2\big(2+\mathrm{Tr}(Q_\infty)\big)\cdot\big(\|F\|_{b,2}+\|LF\|_{b,2}\big). \quad (4.27)
\end{aligned}
$$

Therefore Lebesgue's dominated convergence theorem leads us to the convergence

$$
\lim_{n\to\infty} F_n(w)=F(w), \quad \lim_{n\to\infty} DF_n(w)=DF(w), \quad \lim_{n\to\infty} LF_n(w)=LF(w), \quad w\in E. \quad (4.28)
$$

However, F_n is not a cylinder function in general. Thus we need one more approximation.
Step 4: For any $M,N\in\mathbb{N}$, we set

$$
F_{n,M,N}(w):=\frac{1}{M}\sum_{h=0}^{N}\sum_{k=1}^{M}e^{-(h+k/M)}R_{h+k/M}G_n(w), \quad w\in E,
$$

where $R_{h+k/M}G_n$ is represented as

$$
R_{h+k/M}G_n(w)=f_n^{(R,G)}\big(\langle w,\varphi_1\rangle,\cdots,\langle w,\varphi_{n_1}\rangle\big), \qquad f_n^{(R,G)}\in C_b^\infty(\mathbb{R}^{n_1},\mathbb{R}),
$$

and $\{\varphi_i\}_{i=1}^{n_1}$ is given by

$$
\varphi_i(x)=e^{-(h+k/M)}\int_{\mathbb{R}} g(h+k/M,x,y)\rho_{-2r}(y)e_i(y)dy, \quad x\in\mathbb{R}.
$$

Moreover, we note that each φ_i, $i=1,\cdots,n_1$, does not have compact support. So, $\{\varphi_i\}_{i=1}^{n_1}$ is not included in $C_0^\infty(\mathbb{R},\mathbb{R}^d)$ but in $\mathcal{C}_\infty^\infty$. Then we can see that $F_{n,M,N}\in\mathcal{F}\mathcal{C}_b^\infty(\mathcal{C}_\infty^\infty)$.

We have the following estimates on $F_{n,M,N}$ and $DF_{n,M,N}$:

$$
\begin{aligned}
\|F_{n,M,N}\|_{b,2} &\leq \frac{1}{M}\sum_{h=0}^{N}\sum_{k=1}^{M}e^{-(h+k/M)}\|R_{h+k/M}G_n\|_{b,2} \\
&\leq \frac{2}{M}\Big(\sum_{h=0}^{N}e^{-h}\Big)\Big(\sum_{k=1}^{M}e^{-k/M}\Big)\cdot\big(1+\mathrm{Tr}(Q_\infty)\big)\big(\|F\|_{b,2}+\|LF\|_{b,2}\big) \\
&\leq \frac{2e}{e-1}\big(1+\mathrm{Tr}(Q_\infty)\big)\cdot\big(\|F\|_{b,2}+\|LF\|_{b,2}\big), \quad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
\|DF_{n,M,N}\|_{b,2} &\leq \frac{1}{M}\sum_{h=0}^{N}\sum_{k=1}^{M}e^{-(h+k/M)}\|DR_{h+k/M}G_n\|_{b,2} \\
&\leq \frac{2e}{e-1}\big(1+\mathrm{Tr}(Q_\infty)\big) \\
&\qquad \times\big(\|DF\|_{b,2}+\|DLF\|_{b,2}+2\|F\|_{b,2}+2\|LF\|_{b,2}\big). \quad (4.30)
\end{aligned}
$$

Next we proceed to the term $LF_{n,M,N}$. By using a similar argument as in the proof of Proposition 4.5, we easily see that $LF_{n,M,N} \in \mathcal{D}(L, C^1_{b,2}(E))$ and that

$$LF_{n,M,N} = \frac{1}{M}\sum_{h=0}^{N}\sum_{k=1}^{M} e^{-(h+k/M)} R_{h+k/M}(LG_n)(w), \quad w \in E.$$

On the other hand, since $G_n \in \mathcal{FC}_b^\infty$, both the maps $t \mapsto R_t G_n$ and $t \mapsto R_t LG_n$ are continuous as $UC_{b,2}(E)$-valued maps. See Corollary 2.3 in [8] for the details. Moreover, by the intertwining property $DR_tG_n = e^{tA^*}R_tDG_n$, we obtain that DR_tG_n is also continuous on t in $UC_{b,2}(E,E)$. These properties yield the following convergence for any $n = (n_1, n_2, n_3)$:

$$\lim_{M,N\to\infty} \Big\| \int_0^\infty e^{-t}R_tLG_n dt - \frac{1}{M}\sum_{h=0}^{N}\sum_{k=1}^{M} e^{-(h+k/M)}R_{h+k/M}LG_n\Big\|_{b,2} = 0. \tag{4.31}$$

Therefore for any $n_4 \in \mathbb{N}$, there exist $M(n_4), N(n_4) \in \mathbb{N}$ such that

$$\|LF_n - LF_{n,M(n_4),N(n_4)}\|_{b,2} \le 1/n_4. \tag{4.32}$$

Hereafter we replace $F_{n,M(n_4),N(n_4)}$ by $F_n = F_{(n_1,n_2,n_3,n_4)}$. Then (4.27) and (4.32) imply the estimate

$$\|LF_n\|_{b,2} \le 1/n_4 + \|LF_{n_1,n_2,n_3}\|_{b,2} \le 1 + 2(2+\mathrm{Tr}(Q_\infty))\big(\|F\|_{b,2} + \|LF\|_{b,2}\big).$$

We note that (4.29) and (4.30) still hold for F_n.

Finally, we note that a convergence similar to (4.31) also holds for $R_tG_{n_1,n_2,n_3}$ and $DR_tG_{n_1,n_2,n_3}$. Hence the above-mentioned argument and (4.28) imply the point-wise convergence

$$\lim_{n\to\infty} F_n(w) = F(w), \quad \lim_{n\to\infty} DF_n(w) = DF(w), \quad \lim_{n\to\infty} LF_n(w) = LF(w), \quad w \in E.$$

This completes the proof of the first item.

(2) Let $F \in \mathcal{D}(L, C^1_{b,2}(E))$. Then by assertion (1), there exists a 4-sequence $\{F_n\}_{n\in\mathbb{N}^4} \in \mathcal{FC}_b^\infty(\mathcal{C}_\infty^\infty)$ such that the point-wise convergence (4.11) and

$$|F_n(w)| + \|DF_n(w)\|_E + |LF_n(w)| \le C_*(1 + \|w\|_E^2), \quad w \in E, \tag{4.33}$$

holds. Here the constant $C_* > 0$ is given by the sum of the right hand side of (4.12), (4.13) and (4.14).

By noticing that

$$\begin{aligned}
\big(b(w), DF_n(w)\big)_E &= \frac{1}{2}\int_{\mathbb{R}} \big(\widetilde{\nabla}V(w(x)), DF_n(w)(x)\big)_{\mathbb{R}^d}\rho_{-2r}(x)dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \big(\widetilde{\nabla}V(w(x)), \rho_{2r}(x)D_HF_n(w)(x)\big)_{\mathbb{R}^d}\rho_{-2r}(x)dx \\
&= \frac{1}{2}\big\langle \widetilde{\nabla}V(w(\cdot)), D_HF_n(w)\big\rangle
\end{aligned}$$

and combining this with Proposition 4.5, we have

$$\mathcal{L}_0 F_n(w) = LF_n(w) + \big(b(w), DF_n(w)\big)_E, \quad w \in \mathcal{C}. \tag{4.34}$$

Then by taking limits on both sides of (4.34), we obtain the point-wise convergence

$$\lim_{n\to\infty} \mathcal{L}_0 F_n(w) = LF(w) + \big(b(w), DF(w)\big)_E, \quad w \in \mathcal{C}. \tag{4.35}$$

On the other hand, by (4.33)

$$\begin{aligned} |\mathcal{L}_0 F_n(w)| &\le |LF_n(w)| + \|b(w)\|_E \cdot \|DF(w)\|_E \\ &\le C_*(1 + \|w\|_E^2)(1 + \|b(w)\|_E^2), \quad w \in \mathcal{C}. \end{aligned} \tag{4.36}$$

Hence by recalling that $\mu(\mathcal{C}) = 1$, condition **(U2)** and the integrability (2.5), it follows that the right-hand side of (4.36) is in $L^2(\mu)$. Lebesgue's dominated convergence theorem then leads us to

$$\lim_{n\to\infty} \mathcal{L}_0 F_n = LF + \big(b, DF\big)_E, \text{ in } L^2(\mu).$$

Finally, by remembering Lemma 4.8, we have $F \in \mathrm{Dom}(\overline{\mathcal{L}_0})$ and (4.15). This completes the proof. ∎

5 Proof of the Main Result

In this section, we give a proof of the main result, namely, we show the following theorem.

Theorem 5.1 *The pre-Dirichlet operator* $(\mathcal{L}_0, \mathcal{F}C_b^\infty)$ *is essentially self-adjoint in* $L^2(\mu)$*. Moreover, if we assume* $U \in C^1(\mathbb{R}^d, \mathbb{R})$*, the semigroup* $\{T_t\}_{t\ge0}$ *generated by* $(\overline{\mathcal{L}_0}, \mathrm{Dom}(\overline{\mathcal{L}_0}))$ *satisfies the following identity for each* $F \in L^2(\mu)$*:*

$$T_t F = P_t F, \quad \mu\text{-a.s.},$$

where $\{P_t\}_{t\ge0}$ *is the transition semigroup corresponding to SPDE* (3.2)*.*

At the beginning, we make some preparations for the proof of Theorem 5.1. Let $\widetilde{U} \in C(\mathbb{R}, \mathbb{R}^d)$ be given as in condition **(U1)**. That is,

$$\widetilde{U}(z) := U(z) + \frac{K_1}{2}|z|^2 = -V(z) + \frac{1}{2}(K_1 + \kappa)|z|^2, \quad z \in \mathbb{R}^d.$$

We note that $\partial_0 \widetilde{U} : \mathbb{R}^d \to \mathbb{R}^d$ is monotone, i.e., we have

$$\big(\partial_0 \widetilde{U}(z_1) - \partial_0 \widetilde{U}(z_2), z_1 - z_2\big)_{\mathbb{R}^d} \ge 0, \quad z_1, z_2 \in \mathbb{R}^d. \tag{5.1}$$

In this section, we consider SPDE (3.2) as the following stochastic evolution equation on E given by

$$\begin{aligned} dX_t &= AX_t dt + b(X_t)dt + \sqrt{Q}dW_t, \\ &= AX_t dt + \frac{1}{2}(K_1 + \kappa)X_t dt + \widetilde{b}(X_t)dt + \sqrt{Q}dW_t, \end{aligned} \tag{5.2}$$

where the measurable map $\widetilde{b} : \mathrm{Dom}(\widetilde{b}) \subset E \to E$ with $\mathrm{Dom}(\widetilde{b}) = \mathcal{C}$ is defined by

$$\widetilde{b}(w)(\cdot) := -\frac{1}{2}\partial_0 \widetilde{U}(w(\cdot)), \quad w \in \mathrm{Dom}(\widetilde{b}). \tag{5.3}$$

By (5.1), it is obvious that $\widetilde{b}$ is also dissipative, i.e.,

$$\big(w_1 - w_2, \widetilde{b}(w_1) - \widetilde{b}(w_2)\big)_E \le 0, \quad w_1, w_2 \in \mathrm{Dom}(\widetilde{b}).$$

However $\widetilde{b}$ is not continuous on E in general.

In what follows, we give the proof of Theorem 5.1 based on Da Prato-Röckner [6]. We divide it into two steps. In the first subsection, we solve an infinite dimensional elliptic equation which is essential for the proof. We do this under the condition

(D) $\widetilde{b} : E \to E$ is dissipative, smooth and has bounded derivatives of all orders.

Of course, in this case, $\widetilde{b}$ is Lipschitz continuous. Hence SPDE (5.2) can be treated more easily. In the second subsection, we drop condition **(D)**. By adopting the Yosida approximation and regularizing the drift $\widetilde{b}$, we can use the results in the first subsection.

5.1 The Elliptic Problem on the Hilbert Space

Throughout this subsection, we impose condition **(D)** denoted above. Under this condition, we can give the following proposition. Here $C_b^2(E)$ denotes the space of all functions $F : E \to \mathbb{R}$ that are uniformly continuous and bounded together with their first and second derivatives.

Proposition 5.2 *Let $F \in C_b^2(E)$ and let $\{P_t\}_{t\ge 0}$ be the transition semigroup for X defined in* (3.4). *Then $P_t F \in C_b^2(E)$ and it holds that*

$$\big(DP_tF(w), k\big)_E = \mathbb{E}\big[\big(DF(X_t^w), Z_t(w;k)\big)_E\big], \quad w \in E, \ t \ge 0, \tag{5.4}$$

for $k \in E$, where $Z_t(w;k)$ is the mild solution of the first variation equation

$$\frac{du_t}{dt} = Au_t + Db(X_t^w)[u_t]_E, \quad t > 0,$$

with initial datum $u_0 = k$ and we have

$$\|Z_t(w;k)\|_E \le e^{(K_1+2r^2)t/2}\|k\|_E, \quad \mathbb{P}\text{-}a.s. \tag{5.5}$$

Moreover

$$\begin{aligned} D^2P_tF(w)\big[k_1,k_2\big]_{E\times E} &= \mathbb{E}\big[\big(DF(X_t^w), Z_t(w;k_1,k_2)\big)_E\big] \\ &\quad + \mathbb{E}\big[D^2F(X_t^w)\big[Z_t(w;k_1), Z_t(w;k_2)\big]_{E\times E}\big], \quad w \in E, \ t \ge 0, \end{aligned}$$

holds for $k_1, k_2 \in E$, where $Z_t(w;k_1,k_2)$ is the mild solution of the equation

$$\frac{dv_t}{dt} = Av_t + Db(X_t^w)[v_t]_E + D^2b(X_t^w)\big[Z_t(w;k_1), Z_t(w;k_2)\big]_{E\times E}, \quad t > 0, \tag{5.6}$$

with initial datum $v_0 = 0$. We also have the estimate

$$\|Z_t(w;k_1,k_2)\|_E \le \frac{\|D^2 b\|_\infty}{\sqrt{2K_1+4r^2}} e^{(3K_1+6r^2+1)t/2} \cdot \|k_1\|_E \|k_2\|_E, \quad \mathbb{P}\text{-}a.s. \tag{5.7}$$

Proof. This proposition can be proved in just the same way as Chapter 4 of Cerrai's book [4]. Unfortunately, a complete proof would require several pages and is too long to be repeated. Here we only explain the derivation of the estimate (5.7) for the reader's convenience. (Note that the estimate (5.5) is essentially obtained in Lemma 2.1 of Kawabi [16].) We set $v_t := Z_t(w;k_1,k_2)$ and multiply (5.6) by v_t. Then by taking into account (3.11) and the dissipativity of $\tilde{b}$, we have

$$\begin{aligned}
\frac{d}{dt}\|v_t\|_E^2 &\le -2\omega\|v_t\|_E^2 + 2\big(Db(X_t^w)[v_t]_E, v_t\big)_E \\
&\quad +2\Big(D^2 b(X_s^w)\big[Z_s(w;k_1), Z_s(w;k_2)\big]_{E\times E}, v_t\Big)_E \\
&\le (-2\omega + K_1 + \kappa)\|v_t\|_E^2 + 2\big(D\tilde{b}(X_t^w)[v_t]_E, v_t\big)_E \\
&\quad +2\big(\|D^2 b\|_\infty \|Z_s(w;k_1)\|_E \|Z_s(w;k_2)\|_E\big)\cdot\|v_t\|_E \\
&\le (-2\omega + K_1 + \kappa + 1)\|v_t\|_E^2 + \|D^2 b\|_\infty^2 \|Z_s(w;k_1)\|_E^2 \|Z_s(w;k_2)\|_E^2 \\
&\le (-2\omega + K_1 + \kappa + 1)\|v_t\|_E^2 \\
&\quad +\|D^2 b\|_\infty^2 \big(e^{(K_1+2r^2)t/2}\|k_1\|_E\big)^2 \big(e^{(K_1+2r^2)t/2}\|k_2\|_E\big)^2 \\
&= (K_1 + 2r^2 + 1)\|v_t\|_E^2 + e^{2(K_1+2r^2)t}\|D^2 b\|_\infty^2 \|k_1\|_E^2 \|k_2\|_E^2,
\end{aligned} \tag{5.8}$$

where we used (5.5) for the fourth line. Needless to say, by lack of regularity for v_t, the above computations are formal. However, we can approximate v_t by means of more regular solutions to justify (5.8). (For details of these approximations, see Proposition 6.2.2 of [4] or the mollifier technique in Lemma 2.1 of [16].)

By remembering $K_1 + 2r^2 > 0$, the Gronwall inequality leads us to

$$\begin{aligned}
\|v_t\|_E^2 &\le e^{(K_1+2r^2+1)t}\Big(\int_0^t e^{2(K_1+2r^2)s} ds\Big)\|D^2 b\|_\infty^2 \|k_1\|_E^2 \|k_2\|_E^2 \\
&\le \frac{e^{(3K_1+6r^2+1)t}}{2(K_1+2r^2)}\|D^2 b\|_\infty^2 \|k_1\|_E^2 \|k_2\|_E^2.
\end{aligned}$$

This completes the proof of the estimate (5.7). ∎

Proposition 5.3 *Let $F \in C_b^2(E)$ and we consider the elliptic problem*

$$\lambda\Phi(w) - L\Phi(w) - \big(b(w), D\Phi(w)\big)_E = F(w), \quad w \in E, \tag{5.9}$$

where $\lambda > \frac{K_1}{2} + r^2$. Then (5.9) has a unique solution $\Phi \in \mathcal{D}(L, C_{b,2}^1(E)) \cap C_b^2(E)$, which is given by

$$\Phi(w) = \int_0^\infty e^{-\lambda t} P_t F(w) dt, \quad w \in E. \tag{5.10}$$

Proof. We show Φ which is given in (5.10) belongs to $\mathcal{D}(L, C^1_{b,2}(E))$. By (5.4) and (5.5), we have the following estimate:

$$
\begin{aligned}
\big|(D\Phi(w), k)_E\big| &\le \int_0^\infty e^{-\lambda t}\mathbb{E}\Big[\|DF(X_t^w)\|_E \|Z_t(w;k)\|_E\Big]dt \\
&\le \int_0^\infty e^{-\lambda t}\mathbb{E}\Big[\|DF(X_t^w)\|_E \cdot \big\{e^{(K_1+2r^2)t/2}\|k\|_E\big\}\Big]dt \\
&\le \Big(\int_0^\infty e^{-(2\lambda - K_1 - 2r^2)t/2}dt\Big)\cdot \|DF\|_\infty \|k\|_E \\
&= \frac{2}{2\lambda - K_1 - 2r^2}\cdot \|DF\|_\infty \|k\|_E, \quad k, w \in E.
\end{aligned}
$$

This implies the estimate

$$
\big\|D\Phi(w)\big\|_E \le \frac{2}{2\lambda - K_1 - 2r^2}\cdot \|DF\|_\infty, \quad w \in E. \tag{5.11}
$$

Next, we aim to check conditions (4.6) and (4.7) in Remark 4.4 as in the proof of Proposition 4.5. We set

$$
S(b)(w)_t := \int_0^t S_{t-s}b(X_s^w)ds, \quad w \in E,\ t \ge 0.
$$

By the mean value theorem, we have

$$
\begin{aligned}
&\frac{1}{t}\big(R_t\Phi(w) - \Phi(w)\big) \\
&= \frac{1}{t}\mathbb{E}\big[\Phi(Y_t^w) - \Phi(w)\big] \\
&= \frac{1}{t}\mathbb{E}\Big[\Phi\big(X_t^w - S(b)(w)_t\big) - \Phi(w)\Big] \\
&= \frac{1}{t}\mathbb{E}\big[\Phi(X_t^w) - \Phi(w)\big] \\
&\qquad - \frac{1}{t}\mathbb{E}\Big[\int_0^1 \Big(D\Phi\big(X_t^w - \theta S(b)(w)_t\big), S(b)(w)_t\Big)_E d\theta\Big] \\
&= \frac{1}{t}\big(P_t\Phi(w) - \Phi(w)\big) \\
&\qquad - \int_0^1 \mathbb{E}\Big[\Big(D\Phi\big(X_t^w - \theta S(b)(w)_t\big), \frac{1}{t}S(b)(w)_t\Big)_E\Big]d\theta. \quad w \in E.
\end{aligned} \tag{5.12}
$$

By letting $t \searrow 0$ on the right-hand side of (5.12), we obtain

$$
\begin{aligned}
\frac{1}{t}\big(P_t\Phi(w) - \Phi(w)\big) &= \frac{1}{t}\Big(\int_0^\infty e^{-\lambda s}P_{s+t}F(w)ds - \int_0^\infty e^{-\lambda s}P_{s+t}F(w)ds\Big) \\
&= \frac{e^{\lambda t} - 1}{t}\int_t^\infty e^{-\lambda s}P_sF(w)ds - \frac{1}{t}\int_0^t e^{-\lambda s}P_sF(w)ds \\
&\longrightarrow \lambda\Phi(w) - F(w) \quad \text{as } t \searrow 0, \quad w \in E.
\end{aligned} \tag{5.13}
$$

and

$$\int_0^1 \mathbb{E}\Big[\Big(D\Phi\big(X_t^w - \theta S(b)(w)_t\big), \frac{1}{t}S(b)(w)_t\Big)_E\Big] d\theta$$
$$\longrightarrow \int_0^1 \mathbb{E}\big[\big(D\Phi(X_0^w), b(X_0^w)\big)_E\big] d\theta = \big(D\Phi(w), b(w)\big)_E \quad \text{as } t \searrow 0, \quad w \in E. \tag{5.14}$$

Hence by combining (5.12), (5.13) and (5.14), we obtain the point-wise convergence

$$\lim_{t\searrow 0} \frac{1}{t}\big(R_t\Phi(w) - \Phi(w)\big) = \lambda\Phi(w) - F(w) - \big(D\Phi(w), b(w)\big)_E, \quad w \in E. \tag{5.15}$$

On the other hand, we can see that $\Phi \in C_b^2(E)$ by (5.10) and recalling Proposition 5.2. Then we obtain that the right-hand side of (5.15) belongs to $C_{b,2}^1(E)$ by recalling (5.11) and the drift b has a linear growth with respect to $\|w\|_E$. So, we can also check the second condition (4.7) in Remark 4.4. Therefore, we conclude that $\Phi \in \mathcal{D}(L, C_{b,2}^1(E)) \cap C_b^2(E)$ and it satisfies (5.9).

Finally, we show uniqueness. We assume that there exists another solution $\Phi' \in \mathcal{D}(L, C_{b,2}^1(E)) \cap C_b^2(E)$ to (5.9). Then by Proposition 4.6, it follows that Φ and Φ' satisfy

$$F = (\lambda - \overline{\mathcal{L}}_0)\Phi = (\lambda - \overline{\mathcal{L}}_0)\Phi', \quad \lambda > \frac{K_1}{2} + r^2. \tag{5.16}$$

Then by multiplying both sides of (5.16) by $\Phi - \Phi'$ and by integrating with respect to μ, we obtain

$$\lambda\|\Phi - \Phi'\|_{L^2(\mu)}^2 - \big(\overline{\mathcal{L}}_0(\Phi - \Phi'), \Phi - \Phi'\big)_{L^2(\mu)} = 0, \quad \lambda > \frac{K_1}{2} + r^2. \tag{5.17}$$

Moreover, by using the dissipativity of $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$, (5.17) leads us to $\|\Phi - \Phi'\|_{L^2(\mu)}^2 \le 0$. This completes the proof of uniqueness. ∎

5.2 Proof of Theorem 5.1

In this subsection, we give a proof of Theorem 5.1. We note that by Remark 4.9 it is sufficient to prove only that $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ generates a C_0-semigroup on $L^2(\mu)$. Since such generators are maximal, it follows that we have the equality in (4.19) and all is proved. For the proof, we use the result of the above subsection. So, we give an approximation scheme of the drift $\widetilde{b}$ as follows:

Firstly, we introduce the Yosida approximation of $\partial_0\widetilde{U}$. By (5.1), it is a maximal dissipative mapping (see e.g. Proposition 1.5 of Chapter IV in Showalter [22]). For any $\alpha > 0$, we set

$$J_\alpha(z) := \big(I_{\mathbb{R}^d} + \alpha\partial_0\widetilde{U}\big)^{-1}(z), \quad z \in \mathbb{R}^d,$$

and define the Yosida approximation $(\partial_0\widetilde{U})_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ by

$$(\partial_0\widetilde{U})_\alpha(z) := (\partial_0\widetilde{U})(J_\alpha(z)), \quad z \in \mathbb{R}^d.$$

Then $(\partial_0\widetilde{U})_\alpha$ is monotone and the following Lipschitz continuity holds:

$$\big|(\partial_0\widetilde{U})_\alpha(z_1)-(\partial_0\widetilde{U})_\alpha(z_2)\big|\le\frac{2}{\alpha}|z_1-z_2|,\quad z_1,z_2\in\mathbb{R}^d,$$

Moreover, it is well-known that

$$\big|(\partial_0\widetilde{U})_\alpha(z)\big|\le\big|\partial_0\widetilde{U}(z)\big|,\quad z\in\mathbb{R}^d, \tag{5.18}$$

$$\lim_{\alpha\searrow 0}(\partial_0\widetilde{U})_\alpha(z)=\partial_0\widetilde{U}(z),\quad z\in\mathbb{R}^d. \tag{5.19}$$

$\widetilde{b}_\alpha:E\to E$ is defined in the same way as $\widetilde{b}$ with $\partial_0\widetilde{U}$ replaced by $(\partial_0\widetilde{U})_\alpha$. Note that $\widetilde{b}_\alpha$ is Lipschitz continuous and dissipative on E.

Secondly, we introduce a further regularization. Let $B:\mathrm{Dom}(B)\subset E\to E$ be a self-adjoint negative definite operator such that B^{-1} is of trace class. For any $\alpha,\beta>0$, we set

$$\widetilde{b}_{\alpha,\beta}(w):=\int_E e^{\beta B}\widetilde{b}_\alpha\big(e^{\beta B}w+y\big)N_{\frac{1}{2}B^{-1}(e^{2\beta B}-1)}(dy),\quad w\in E.$$

Then by Theorem 9.19 in [9], we can see that $\widetilde{b}_{\alpha,\beta}$ satisfies condition **(D)** and

$$\lim_{\beta\searrow 0}\widetilde{b}_{\alpha,\beta}(w)=\widetilde{b}_\alpha(w),\quad w\in E. \tag{5.20}$$

We also see that for any $\alpha>0$, there exists $C_\alpha>0$ such that

$$\|\widetilde{b}_{\alpha,\beta}(w)\|_E\le C_\alpha(1+\|w\|_E),\quad w\in E. \tag{5.21}$$

Finally, we are in a position to give the proof for our main result.

Proof of Theorem 5.1. Let $F\in C_b^2(E)$ and we consider the function

$$\Phi_{\alpha,\beta}(w):=\int_0^\infty e^{-\lambda t}P_t^{\alpha,\beta}F(w)dt,\quad w\in E,$$

where $\lambda>\frac{K_1}{2}+r^2$ and $\{P_t^{\alpha,\beta}\}_{t\ge 0}$ is the transition semigroup defined as $\{P_t\}_{t\ge0}$ with $\widetilde{b}$ replaced by $\widetilde{b}_{\alpha,\beta}$. Then by Propositions 4.6 and 5.3, we know that $\Phi_{\alpha,\beta}\in\mathcal{D}(L,C_{b,2}^1(E))\subset\mathrm{Dom}(\overline{\mathcal{L}}_0)$ and we have

$$(\lambda-\overline{\mathcal{L}}_0)\Phi_{\alpha,\beta}=F+\big(\widetilde{b}_{\alpha,\beta}-\widetilde{b},D\Phi_{\alpha,\beta}\big)_{L^2(\mu)}. \tag{5.22}$$

The right-hand side of (5.22) can be estimated as follows:

$$\begin{aligned}
I_{\alpha,\beta} &:= \int_E\big(\widetilde{b}_{\alpha,\beta}(w)-\widetilde{b}(w),D\Phi_{\alpha,\beta}(w)\big)_E^2\mu(dw)\\
&\le \int_E\big\|\widetilde{b}_{\alpha,\beta}(w)-\widetilde{b}(w)\big\|_E^2\cdot\big\|D\Phi_{\alpha,\beta}(w)\big\|_E^2\mu(dw)\\
&\le \Big(\frac{2}{2\lambda-K_1-2r^2}\|DF\|_\infty\Big)^2\int_E\big\|\widetilde{b}_{\alpha,\beta}(w)-\widetilde{b}(w)\big\|_E^2\mu(dw).
\end{aligned} \tag{5.23}$$

Hence by recalling (5.20) and (5.21), we have

$$\limsup_{\beta\searrow 0} I_{\alpha,\beta} \leq \Big(\frac{2}{2\lambda - K_1 - 2r^2}\|DF\|_\infty\Big)^2 \int_E \big\|\widetilde{b}_\alpha(w) - \widetilde{b}(w)\big\|_E^2 \mu(dw).$$

Moreover, by recalling (5.18), (5.19) and using Lebesgue's dominated convergence theorem, we have

$$\lim_{\alpha\searrow 0}\lim_{\beta\searrow 0} I_{\alpha,\beta} = \lim_{\alpha\searrow 0}\big(\limsup_{\beta\searrow 0} I_{\alpha,\beta}\big) = 0. \tag{5.24}$$

Therefore, by combining (5.22) and (5.24), we see

$$\lim_{\alpha\searrow 0}\lim_{\beta\searrow 0}(\lambda - \overline{\mathcal{L}}_0)\Phi_{\alpha,\beta} = F \quad \text{in } L^2(\mu).$$

This means the closure of Range$(\lambda - \overline{\mathcal{L}}_0)$ contains $C_b^2(E)$. Since $C_b^2(E)$ is dense in $L^2(\mu)$, Range$(\lambda - \overline{\mathcal{L}}_0)$ is also dense in $L^2(\mu)$. Then by the Lumer-Philips theorem (see Theorem 1.1 and Theorem 1.2 in [11] for details), we can see $(\overline{\mathcal{L}}_0, \mathrm{Dom}(\overline{\mathcal{L}}_0))$ generates the C_0-semigroup $\{P_t\}_{t\geq 0}$. This completes the proof. ∎

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