Markov processes associated with L^p-resolvents and applications to stochastic differential equations on Hilbert space

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Abstract. We give general conditions on a generator of a C_0-semigroup (resp. of a C_0-resolvent) on $L^p(E, \mu)$, $p \geq 1$, where E is an arbitrary (Lusin) topological space and μ a σ-finite measure on its Borel σ-algebra, so that it generates a sufficiently regular Markov process on E. We present a general method how these conditions can be checked in many situations. Applications to solve stochastic differential equations on Hilbert space in the sense of a martingale problem are given.

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Dedicated to Giuseppe Da Prato on the occasion of his 70th birthday.

0 Introduction

In this paper we study stochastic differential equations (SDE) in infinite dimensions, e.g. on a Hilbert space H, with possibly very singular coefficients. For such equations, strong or mild solutions (cf. [8]) do not exist in general, but it is only possible to construct weak solutions or even only martingale solutions, i.e. a Markov process that solves the martingale problem for the (partial) differential operator ("Kolmogorov operator") associated with the SDE (cf. Proposition 1.4 below and [18] for the general theory in finite dimensions). The latter notion of solution we shall briefly call "martingale solutions". The notions of weak and martingale solutions are only equivalent in finite dimensions (cf. [18]), but not in infinite dimensions. Under additional hypotheses, however, one can construct weak solutions from martingale solutions also in the infinite dimensional case. We refer to [2] and [14] for a detailed discussion.

Given an SDE on a Hilbert space H (by heuristically applying Itô's formula, see Section 5 below) one can always write down the corresponding Kolmogorov operator L_0 on a space of nice test functions. If one can prove that its closure L generates a C_0-semigroup $P_t = e^{tL}$, $t \geq 0$, on $L^p(H, \mu)$ for some suitably chosen measure μ (see Section 5) and if this semigroup is sufficiently regular, then one can prove that there exists a Markov process with transition probabilities given by P_t, $t \geq 0$. This process then automatically solves the martingale problem for L, and thus is a martingale solution to the SDE.

The main results of this paper give general conditions, which are checkable in applications, on a given Kolmogorov operator L, more precisely, on the generator of a C_0-semigroup (or C_0-resolvent) on $L^p(H, \mu)$, $p \geq 1$, so that an associated sufficiently regular Markov process (namely, a μ-standard right process) exists giving the desired solution to the martingale problem determined by L (cf. Theorems 1.1 and 1.3 below). Those two general theorems are formulated for general C_0-resolvents on $L^p(E,\mu)$ for abstract (Lusin) topological spaces E with Borel σ-algebra $\mathcal{B}$ and σ-finite measures μ on $(E,\mathcal{B})$. In particular, Theorem 1.3 generalizes the corresponding results in [13] and [16], [17] and is the first of its kind on L^p-measure spaces for arbitrary $p \geq 1$ in the theory of Markov processes giving conditions on the generator directly, which can be verified in many models for stochastic dynamics.

The price we pay for including non-regular (in particular, non-continuous) coefficients, is, that we can only solve the martingale problem for a restricted class of initial distributions (see Proposition 1.4 below).

The main general results are given in Section 1. Sections 2, 3 and Appendix A contain preparations for their proofs, which, in turn, are contained in Section 4. In Appendix B, we recall all notions from the theory of Markov processes used in this paper.

Another substantial part of the paper, namely Section 5, is devoted to applications to a class of SDE on Hilbert space, where we implement the entire approach summarized above, including the construction of the reference measure μ and the reasoning why the closure L of the underlying Kolmogorov operator L_0 does, in fact, generate a C_0-semigroup on $L^p(H,\mu)$, $p \geq 1$. Let us briefly describe this here and also compare our results with others in the literature.

Consider the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \big[AX(t) + F_0\big(X(t)\big)\big]\,\mathrm{d}t + \sqrt{C}\,\mathrm{d}W(t) \tag{1}$$

on a Hilbert space H. Here $W(t)$, $t \geq 0$, is a cylindrical Brownian motion on H, C is a positive definite self-adjoint linear operator on H, $A : D(A) \subset H \to H$ the infinitesimal generator of a C_0-semigroup on H, and $F_0 : D(F_0) \subset H \to H$ a measurable map. For sufficiently regular functions F_0 (e.g. if F_0 is Lipschitz) there is an elaborate theory to solve these equations (see e.g. [8], and in case $\operatorname{Tr} C < \infty$, A is self-adjoint, negative definite, $D(F_0) = D\big((-A)^{1/2}\big)$ and F_0 is m-dissipative and hemicontinuous see [12]). If $\dim H = \infty$, for irregular F_0 there are very few results on existence of solutions to (1), at least if F_0 is not the gradient of a function (see [2] for the latter case).

If $\dim H < \infty$, it is well-known that SDE (1) has a weak solution under very general conditions on F_0 (cf. e.g. [9], [13] and [16]) and even a strong solution if C is invertible and $F_0 \in L^p_{\mathrm{loc}}(\mathrm{d}x)$, where $\mathrm{d}x$ denotes Lebesgue measure, and

$$p > \dim H\ , \tag{2}$$

and if an appropriate non-explosion (e.g. coercivity) condition holds (cf. [11]). If F_0 is not the gradient of a function or does not satisfy a certain smallness condition (more precisely, if the Kolmogorov operator associated to (1) fails to satisfy the weak sector condition, cf. [13]), condition (2) becomes crucial, because Sobolev embedding theorems are essential in the proofs for getting regularity of the transition probabilities or for applying Girsanov's theorem.

If $\dim H = \infty$, so (2) is meaningless, there are no Sobolev embedding theorems (not even for Gaussian measures on H) and there is, of course, no analogue of Lebesgue measure. Also, at least if C is trace class, Girsanov's theorem does not hold. So, it is a challenging problem to construct solutions for SDE (1) in case F_0 is merely measurable and unbounded.

Quite recently unique martingale solutions to (1), which are strongly Feller, have been constructed in [7], under the assumption that F_0 is m-dissipative and C has an inverse in $L(H)$ (:= all bounded linear operators from H to H) and assuming that an infinitesimally invariant measure exists for the underlying Kolmogorov operator (see hypothesis (H2) in Section 5 below), serving as a substitute for a Lebesgue measure on H. In Section 5 of this paper applying our general results from Section 1 we partially extend those results, in particular by including also the case where $\operatorname{Tr} C < \infty$. We also give sufficient conditions on A, C, F_0 for (H2) to hold (cf. Proposition 5.2 and also Remark 5.3(iii) below). We get a martingale solution to (1), however, compared with [7] we loose the strong Feller property as one usually does, if $\operatorname{Tr} C < \infty$.

Finally, we would like to mention that the main motivation for this paper (apart from aiming to extend the results in [13] and [16], [17]) came from [7]. The entire Section 5 of this paper is devoted to extend the results therein to the present case drawing a lot of ideas from [7]. The last named author had the pleasure to be a coauthor of Giuseppe Da Prato for this paper. So, let us conclude this introduction with a

"Happy Birthday, Beppe!"

and the wish to him for many more years to come as productive and as filled with inspiring work as in the past.

1 Main results

Let E be a Lusin topological space (i.e. the continuous one-to-one image of a Polish space). We shall denote by $\mathcal{T}$ the topology, by $\mathcal{B}$ the Borel σ-algebra on E and let μ be a σ-finite measure on $(E, \mathcal{B})$. We use the following notation: $C(E)$ and $bC(E)$ denote the spaces of continuous and bounded continuous real-valued functions respectively; $p\mathcal{B}$ and $b\mathcal{B}$ denote the spaces of positive and bounded $\mathcal{B}$-measurable real-valued functions, respectively. Let $p \in [1, \infty)$, $(V_\alpha)_{\alpha>0}$ be a strongly continuous sub-Markovian (i.e. if $f \in L^p(E, \mu)$, $0 \leq f \leq 1$, then $0 \leq \alpha V_\alpha f \leq 1$ for all $\alpha > 0$) resolvent of contractions on $L^p(E, \mu)$ and L the infinitesimal generator of $(V_\alpha)_{\alpha>0}$, having the domain $D(L) = V_\beta\big(L^p(E, \mu)\big)$, $L(V_\alpha f) = \alpha V_\alpha f - f$ for all $f \in L^p(E, \mu)$.

An element $u \in L^p_+(E, \mu)$ is called *β-excessive* if $\alpha V_{\beta+\alpha} u \leq u$ for all $\alpha > 0$ and we shall denote by $\mathcal{E}_\beta$ the set of all β-excessive elements. Notice that $\mathcal{E}_\beta$ is min-stable and every decreasing family $\mathcal{F}$ from $\mathcal{E}_\beta$ has an infimum denoted by $\bigwedge \mathcal{F}$. If $f \in L^p(E, \mu)$ is such that there exists $u \in \mathcal{E}_\beta$ with $u \geq f$, we shall denote by $R_\beta f$ the reduced of f in $\mathcal{E}_\beta$, $R_\beta f := \bigwedge\{u \in \mathcal{E}_\beta \mid u \geq f\}$.

An increasing sequence $(F_k)_k$ of $\mathcal{T}$-closed sets in E is called *μ-nest* provided that in $L^p(E, \mu)$

$$\lim_k R_1(u 1_{E \setminus F_k}) = 0$$

for all $u \in D(L) \cap \mathcal{E}_1$.

A function u on E is called *μ-quasi continuous* (resp. *μ-quasi lower semicontinuous*) in $\mathcal{T}$ (with respect to $(V_\alpha)_{\alpha>0}$) if there exists a μ-nest $(F_n)_n$ such that $u|_{F_n}$ is a $\mathcal{T}$-continuous (resp. $\mathcal{T}$-lower semicontinuous) real function for all n.

We can state now the main results of this paper, which associate to $(V_\alpha)_{\alpha>0}$ a μ-standard right process. The proofs will be given in Section 4 below. For the reader not familiar with Markov process theory, we present in Appendix B below a brief description of the related notions.

Theorem 1.1. *Assume that the following two conditions hold.*

(I) There exists a μ-nest of $\mathcal{T}$-compact sets.

(II) There exists a countable $\mathbb{Q}$-linear space $\mathcal{A} \subset bC(E) \cap D(L)$ such that $\mathcal{A}$ is dense in $D(L)$ in the graph norm, $\mathcal{A}$ separates the points of E and $u \wedge \alpha$ belongs to the closure of $\mathcal{A}$ in the uniform norm for all $\alpha > 0$ and $u \in \mathcal{A}$.

Let $\mathcal{T}_0$ be the (metrizable Lusin) topology on E generated by $\mathcal{A}$. Then the following assertions hold.

(a) There exists a μ-standard right process with state space E (endowed with the topology $\mathcal{T}_0$) whose resolvent regarded on $L^p(E,\mu)$ coincides with $(V_\alpha)_{\alpha>0}$.

(b) The process is càdlàg in the topology $\mathcal{T}$ P^μ-a.e.

(c) Every element from $D(L)$ has a μ-quasi continuous version (with respect to the topology $\mathcal{T}_0$).

Remark 1.2. 1. The following condition implies that the above assumption (I) holds: *There exists a function $u \in \mathcal{C}(E) \cap D(L)$ such that $(L-\beta)u \leq 0$ for some $\beta > 0$ and the level sets $[u \leq n]$ are $\mathcal{T}$-compact.* Indeed, we remark that since the function u given by (I) is such that $u = V_\beta g$ with $g = (\beta - L)u$, it follows that $u \in \mathcal{E}_\beta$ and then we apply Remark 3.3 below.

2. Assumption (II) is implied by the following hypothesis: *There exists a countable $\mathbb{Q}$-algebra $\mathcal{A} \subset b\mathcal{C}(E) \cap D(L)$ such that $\mathcal{A}$ is dense in $D(L)$ in the graph norm and $\mathcal{A}$ separates the points of E.*

A set M is called *μ-exceptional* if $M \subset \bigcap_n (E \setminus F_n)$ for some μ-nest $(F_n)_n$. We say that a property of points in E holds *μ-quasi everywhere* (abbreviated *μ-q.e.*) if it holds outside some μ-exceptional set. Every μ-exceptional set is clearly μ-negligible.

Theorem 1.3. *Assume that condition (I) from Theorem 1.1 is satisfied and that the following condition holds.*

(II') There exist a countable $\mathbb{Q}$-linear space $\mathcal{A} \subset D(L) \cap L^\infty(E,\mu)$ and a μ-exceptional set M such that:

– $\mathcal{A}$ is dense in $D(L)$ in the graph norm;

– every element u of $\mathcal{A}$ possesses a μ-quasi continuous version $\widetilde{u}$ and the set $\{\widetilde{u} |\ u \in \mathcal{A}\}$ separates the points of $E \setminus M$;

– for all $u \in \mathcal{A}$ and $\alpha \in \mathbb{R}_+$ the element $u \wedge \alpha$ belongs to the closure of $\mathcal{A}$ in $L^\infty(E,\mu)$.

Then there exists a Lusin topology $\mathcal{T}_0$ on E such that the conclusions (a), (b) and (c) from Theorem 1.1 hold. Moreover there exists a μ-nest $(K_n)_n$ such that on each set K_n the topologies $\mathcal{T}$ and $\mathcal{T}_0$ coincide.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the Markov process associated with $(V_\alpha)_{\alpha>0}$ in assertion (a) of Theorem 1.1, let $g_0 \in L^{p'}_+(E,\mu)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) be such that $\int_E g_0 \,\mathrm{d}\mu = 1$ and put $\nu = g_0 \cdot \mu$. The next result shows that the process X solves the martingale problem for $\big(L, D(L)\big)$ under $P^\nu = \int_E P^x \nu(\mathrm{d}x)$.

Proposition 1.4. *For every $f \in D(L)$,*

$$f(X_t) - \int_0^t Lf(X_s)\,\mathrm{d}s$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale under P^ν.

Remark 1.5. *The main theorem in [10] gives a necessary and sufficient condition on a strongly continuous contraction resolvent to be associated with a reasonably regular (more precisely, μ-tight μ-special standard) Markov process. The condition is a modification of the quasi-regularity property of a Dirichlet form (cf. [13]), formulated in terms of the resolvent only, without using any associated Dirichlet form, but so that the main techniques from [13] still apply. However, the* *conditions on the resolvent in [10] are purely abstract and almost impossible to check directly in most applications, since in most cases one knows almost nothing about the resolvent a-priori. In contrast to that, the above Theorems 1.1 and 1.3 are taylored for application by being based on a condition in terms of the generator directly and a condition that can be checked by a general method (cf. Section 5 below), which is also only based on knowing the generator on a suitably chosen space of test functions. The latter is always the case in applications, of which an instance is presented in Section 5 below.*

2 Preliminaries on the reduction operation and quasi-continuity

Let $p \in [1, \infty)$ and $(V_\alpha)_{\alpha>0}$ be a strongly continuous sub-Markovian resolvent of contractions on $L^p(E, \mu)$, where μ is a σ-finite measure on $(E, \mathcal{B})$.

If $u \in D(L)$ then we consider the *graph norm* $\|u\|_{D(L)}$ of u,

$$\|u\|_{D(L)} = \|u\|_{L^p} + \|Lu\|_{L^p} .$$

Proposition 2.1. *For every $u \in D(L)$ we have $R_1 u \in D(L)$ and*

$$\|R_1 u\|_{D(L)} \leq 3\|u\|_{D(L)} .$$

Proof. Let us prove firstly that if $\beta > 0$ then for all $f \in L^p(E, \mu)$ we have

$$\frac{\beta}{2\beta + 1} \cdot \|V_\beta f\|_{D(L)} \leq \|f\|_{L^p} \leq \sup(\beta, 1) \cdot \|V_\beta f\|_{D(L)} .$$

Indeed, we have $L(V_\beta f) = \beta V_\beta f - f$ and so

$$\|f\|_{L^p} \leq \|f - \beta V_\beta f\|_{L^p} + \|\beta V_\beta f\|_{L^p} \leq \sup(\beta, 1)\big(\|f - \beta V_\beta f\|_{L^p} + \|V_\beta f\|_{L^p}\big) = \sup(\beta, 1)\|V_\beta f\|_{D(L)} ,$$

$$\|V_\beta f\|_{L^p} + \|f - \beta V_\beta f\|_{L^p} \leq \frac{1}{\beta}\|f\|_{L^p} + \|f\|_{L^p} + \|\beta V_\beta f\|_{L^p} \leq \left(2 + \frac{1}{\beta}\right)\|f\|_{L^p} .$$

Let now $u \in D(L)$, $u = V_1 f$, with $f \in L^p(E, \mu)$. From $V_1 f = V_1(f^+) - V_1(f^-)$ we get $R_1(V_1 f) \preccurlyeq V_1(f^+)$, where $\preccurlyeq$ denotes the specific order in $\mathcal{E}_1$. Therefore (cf. [6]) there exists $f_1 \in p\mathcal{B}$, $f_1 \leq f^+$, such that $R_1(V_1 f) = V_1 f_1$. Consequently $R_1(V_1 f) \in D(L)$ because $f_1 \in L^p(E, \mu)$. By the first part of the proof (for $\beta = 1$) we get $\big\|R_1(V_1 f)\big\|_{D(L)} \leq 3\|f_1\|_{L^p} \leq 3\|f\|_{L^p} \leq 3\|V_1 f\|_{D(L)}$. $\square$

Remark 2.2. *(i) If $f_0 \in L^p(E, \mu)$, $f_0 > 0$, then an increaing sequence of $\mathcal{T}$-closed sets $(F_n)_n$ is a μ-nest if and only if $\lim_n R_\beta(1_{E \setminus F_n} V_\beta f_0) = 0$ for one $\beta > 0$.*
(ii) Let $(F_n)_n$ be a μ-nest and $(F'_n)_n$ an increasing sequence of $\mathcal{T}$-closed sets such that $\mu(F_n \triangle F'_n) = 0$ for all n. Then $(F'_n)_n$ is also a μ-nest. Particularly every μ-negligible $\mathcal{T}$-open set is μ-exceptional.

Remark 2.3. *(i) If u is a μ-quasi lower semicontinuous function on E and $u \leq 0$ μ-a.e. then $u \leq 0$ μ-q.e.*
(ii) Let $(u_n)_n$ be a sequence of μ-quasi continuous functions on E. Then there exists a μ-nest $(F_k)_k$ such that $u_n|_{F_k}$ is $\mathcal{T}$-continuous for all n and k.

A sequence $(u_n)_n$ of numerical functions on E converges *μ-quasi uniformly* (abbreviated *μ-q.u.*) to a function u on E if there exists a μ-nest $(F_k)_k$ such that $u|_{F_k}$, $u_n|_{F_k}$ are real valued functions and $\big(u_n|_{F_k}\big)_n$ converges uniformly to $u|_{F_k}$ for all k.

Notice that if $(u_n)_n$ converges μ-quasi uniformly to u and every function u_n is μ-quasi continuous then the function u is also μ-quasi continuous.

Proposition 2.4. (W. Stannat [16]) *Let $(u_n)_n \subset D(L)$ be such that every u_n possesses a μ-quasi continuous version $\widetilde{u}_n$. If $(u_n)_n$ converges to $u \in D(L)$ in the graph norm, then u possesses a μ-quasi continuous version $\widetilde{u}$ and a subsequence of $(\widetilde{u}_n)_n$ converges μ-quasi uniformly to $\widetilde{u}$.*

Proof. Passing to a subsequence, we may assume that $\|u - u_n\|_{D(L)} \leq \frac{1}{4^n}$ for all n and let $(F_n)_n$ be a μ-nest such that $\widetilde{u}_i|_{F_n}$ is continuous for all i and n. We set

$$\Gamma_n = \bigcup_{i\geq n}\left[|\widetilde{u}_{i+1} - \widetilde{u}_i| > \frac{1}{2^i}\right]$$

and since $\Gamma_n \cup (E \setminus F_n)$ is open for all n, it follows that the sequence $(F'_n)_n$ defined by $F'_n = F_n \cap (E \setminus \Gamma_n)$ is increasing and if $f_0 \in L^p(E,\mu)$, $0 < f_0 \leq 1$, then we have

$$R_1(1_{\Gamma_n} V_1 f_0) \leq \sum_{i\geq n} 2^i \big(R_1(u_{i+1} - u_i) + R_1(u_i - u_{i+1})\big)\ .$$

From

$$||\sum_{i\geq n} 2^i (R_1(u_{i+1} - u_i) + R_1(u_i - u_{i+1}))||_{D(L)} \leq 3\sum_{i\geq n} 2^{i+1}||u_{i+1} - u_i||_{D(L)} \leq \sum_{i\geq n} 2^{i+1}\frac{6}{4^i} = \frac{3}{2^{n-3}}$$

it follows that $\lim_n R_1(1_{E\setminus F'_n} V_1 f_0) = 0$. Consequently $(F'_n)_n$ is a μ-nest and on F'_n we have $|\widetilde{u}_{i+1} - \widetilde{u}_i| \leq \frac{1}{2^i}$ if $i > n$. We conclude that the sequence $(\widetilde{u}_i|_{F'_n})_i$ is uniformly convergent on F'_n to $\widetilde{u}|_{F'_n}$. □

Proposition 2.5. *Let $(u_n)_n$ be a sequence in $L^\infty(E,\mu)$, converging in $L^\infty(E,\mu)$ to u, such that every u_n possesses a μ-quasi continuous version $\widetilde{u}_n$. Then u possesses a μ-quasi continuous version $\widetilde{u}$ and there exists a μ-nest $(F_k)_k$ such that $\widetilde{u}_n|_{F_k}$, $\widetilde{u}|_{F_k}$ are finite continuous for all n and k, and $(\widetilde{u}_n)_n$ converges uniformly to $\widetilde{u}$ on $\bigcup_k F_k$.*

Proof. Let $(F_k)_k$ be a μ-nest such that $\widetilde{u}_n|_{F_k}$ is a finite continuous function for all n and k. Since $(u_n)_n$ is converging in $L^\infty(E,\mu)$ to u,it follows that there exists a sequence $(\varepsilon_n)_n$ in $\mathbb{R}_+$ converging to zero and such that for all $k \geq n$ we have μ-a.e.: $|\widetilde{u}_k - u| \leq \frac{\varepsilon_n}{2}$ and thus $|\widetilde{u}_{n+p} - \widetilde{u}_n| \leq \varepsilon_n$ for all p and n. Consequently the set $G = \bigcup_{n,p}[|\widetilde{u}_{n+p} - \widetilde{u}_n| > \varepsilon_n]$ is μ-negligible, $F - K \setminus G$ is $\mathcal{T}$-closed for all k and therefore the sequence $(F_k \setminus G)_k$ is also a μ-nest and $|\widetilde{u}_{n+p} - \widetilde{u}_n| \leq \varepsilon_n$ on $\bigcup_k (F_k \setminus G)$. We conclude that on this set the sequence $(\widetilde{u}_n)_n$ converges uniformly and thus there exists a μ-quasi continuous version $\widetilde{u}$ of u satisfying the required conditions. □

Proposition 2.6. *Let $(s_n)_n$ be a decreasing sequence in $\mathcal{E}_\beta$ such that $\bigwedge_n s_n = 0$ and each s_n possesses a μ-quasi lower semicontinuous version $\widetilde{s}_n$. Then a subsequence of $(\widetilde{s}_n)_n$ converges to zero μ-quasi uniformly.*

Proof. Let $f_0 \in L^p(E,\mu)$, $f_0 > 0$ with $V_1 f_0 \le 1$, $\varphi \in L^1(E,\mu)$ and $(F_n)_n$ be a μ-nest such that $\widetilde{s}_i|_{F_n}$ is $\mathcal{T}$-lower semicontinuous for all i and n, and $\mu\big(\varphi R_1(1_{E\setminus F_n} V_1 f_0)\big) < \frac{1}{2^n}$. There exists a subsequence $(s_{i_n})_n$ of $(s_n)_n$ such that $\mu(\varphi s_{i_n}) \le \frac{1}{4^n}$ for all n. We consider the set

$$G_n = \left[\widetilde{s}_{i_n} > \frac{1}{2^n}\right] \cup [E \setminus F_n]$$

and notice that G_n is open. Since $V_1 f_0 \le 2^n s_{i_n} + R_1(1_{E\setminus F_n} V_1 f_0)$ on G_n, we deduce that $R_1(1_{G_n} V_1 f_0) \le 2^n s_{i_n} + R_1(1_{E\setminus F_n} V_1 f_0)$ and so $\mu(\varphi R_1(1_{G_n} V_1 f_0)) \le \frac{1}{2^{n-1}}$ for all n. The set $F'_n = \bigcap_{k\ge n+1}(E \setminus G_k)$ is $\mathcal{T}$-closed and

$$\mu(\varphi R_1(1_{E\setminus F'_n} V_1 f_0)) \le \sum_{k\ge n+1} \mu(\varphi R_1(1_{G_k} V_1 f_0)) \le \frac{1}{2^{n-1}} \ .$$

Consequently $(F'_n)_n$ is a μ-nest and $\widetilde{s}_{i_{n+1}}|_{F_n} \le \frac{1}{2^{n+1}}$ for all n and we conclude that the sequence $(s_{i_n})_n$ converges to zero μ-quasi uniformly. □

In the sequel we assume that each $u \in D(L)$ possesses a μ-quasi continuous version $\widetilde{u}$.

Proposition 2.7. *Every $s \in \mathcal{E}_\beta$ possesses a μ-quasi lower semicontinuous version $\widetilde{s}$ and if $(s_n)_n \subset \mathcal{E}_\beta$ is a sequence decreasing to zero in $L^p(E,\mu)$, then a subsequence of $(\widetilde{s}_n)_n$ converges μ-quasi uniformly to zero. Particularly, if $(F_n)_n$ is a μ-nest, then for every $u \in D(L)$ a subsequence of $\big(\widetilde{R_\beta(1_{E\setminus F_n}} V_\beta f_0)\big)_n$ converges to zero μ-quasi uniformly.*

Proof. The assertion follows by Proposition 2.6 since for every $s \in \mathcal{E}_\beta$ there exists an increasing sequence $(u_n)_n \subset D(L)$ such that $(u_n)_n$ increases to s. We get $\widetilde{s} = \sup_n \widetilde{u}_n$. □

Proposition 2.8. *Let $(u_n)_n \subset D(L)$ converging to zero in the graph norm. Then $\big(\widetilde{R}_\beta(|u_n|)\big)_n$ has a subsequence converging to zero μ-q.e., where $\widetilde{R_\beta\big(|u_n|\big)}$ denotes a μ-quasi lower semicontinuous version of the element $R_\beta\big(|u_n|\big) \in \mathcal{E}_\beta$.*

Proof. First we note that if $u \in D(L)$ by Proposition 2.1 we get $R_\beta(u) \in D(L)$ and therefore there exists a μ-quasi continuous version $\widetilde{R}_\beta(u)$ of $R_\beta(u)$. It follows that

$$R_\beta\big(|u_n|\big) \le R_\beta(u_n^+) + R_\beta(u_n^-) = R_\beta(u_n) + R_\beta(-u_n) \ ,$$

and because the function $\widetilde{R}_\beta\big(|u_n|\big) - \widetilde{R}_\beta u_n - \widetilde{R}_\beta(-u_n)$ is μ-quasi lower semicontinuous and negative μ-a.e., we deduce by Remark 2.3 that $\widetilde{R}_\beta\big(|u_n|\big) \le \widetilde{R}_\beta u_n + \widetilde{R}_\beta(-u_n)$ μ-q.e. Again by Proposition 2.1 there exists a constant K such that $\|R_\beta(u)\|_{D(L)} \le K\|u\|_{D(L)}$ for all $u \in D(L)$ and therefore the sequences $(R_\beta(u_n))_n$ and $\big(R_\beta(-u_n)\big)_n$ are converging to zero in graph norm. From Proposition 2.4 we deduce that a subsequence of $(\widetilde{R}_\beta(u_n))_n$ (resp. $\big(\widetilde{R}_\beta(-u_n)\big)_n$) is converging to zero μ-quasi uniformly. □

3 The associated resolvent of kernels

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on the Lusin measurable space $(E,\mathcal{B})$ (i.e. it is measurably isomorphic with a Borel subset of a metrizable compact space endowed with the Borel σ-algebra). Recall that a function $s \in p\mathcal{B}$ is termed *$\mathcal{U}$-excessive* if $\alpha U_\alpha s \le s$ for all

$\alpha > 0$ and $\sup_{\alpha>0} \alpha U_\alpha s = s$. If $\beta > 0$ then the family $\mathcal{U}_\beta = (U_{\beta+\alpha})_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$, having U_β as (bounded) initial kernel. We denote by $\mathcal{E}(\mathcal{U}_\beta)$ the set of all $\mathcal{B}$-measurable $\mathcal{U}_\beta$-excessive functions on E.

By Remark 2.3 in [4] there exists a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that $U_\alpha = V_\alpha$ as operators on $L^p(E, \mu)$ for all $\alpha > 0$, and the following condition is satisfied for one (and therefore for all) $\beta > 0$:

(A) $\qquad \mathcal{E}(\mathcal{U}_\beta)$ is min-stable, $1 \in \mathcal{E}(\mathcal{U}_\beta)$ and $\sigma\big(\mathcal{E}(\mathcal{U}_\beta)\big) = \mathcal{B}$.

Such a resolvent of kernels $\mathcal{U}$ will be named *associated* with $(V_\alpha)_{\alpha>0}$. In the sequel we assume that $\mathcal{U}$ is associated with $(V_\alpha)_{\alpha>0}$. Note that $\mu \circ U_\beta \ll \mu$ for all $\beta > 0$.

Let $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be a second sub-Markovian resolvent of kernels on $(E, \mathcal{B})$. We say that $\mathcal{U}$ and $\mathcal{U}'$ are *μ-equivalent* provided that $U_\alpha f = U'_\alpha f$ μ-a.e. for all $f \in p\mathcal{B}$ and $\alpha > 0$.

Trivial modification of $\mathcal{U}$.
Let $M \in \mathcal{B}$ be such that $U_\alpha(1_M) = 0$ on $E \setminus M$ for one (and therefore for all) $\alpha > 0$.

For all $\alpha > 0$ we define the kernel

$$U'_\alpha f = 1_{E\setminus M} U_\alpha f + \frac{1}{1+\alpha} 1_M f \ , \quad f \in p\mathcal{B}.$$

Then the family $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ satisfying condition (A), called the *trivial modification* of $\mathcal{U}$ on M. If $\mu(M) = 0$ then $\mathcal{U}'$ and $\mathcal{U}$ are μ-equivalent, particularly $\mathcal{U}'$ is also associated with $(V_\alpha)_{\alpha>0}$.

The family $\mathcal{U}|_{E\setminus M} = \big(U_\alpha|_{E\setminus M}\big)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $\big(E \setminus M, \mathcal{B}|_{E\setminus M}\big)$ which satisfies (A), called the *restriction* of $\mathcal{U}$ to $E \setminus M$. A function $s \in p\mathcal{B}$ will be $\mathcal{U}'_\beta$-excessive if and only if $s|_{E\setminus M}$ is $\mathcal{U}_\beta|_{E\setminus M}$-excessive.

If $s \in \mathcal{E}(\mathcal{U}_\beta)$ and $A \in \mathcal{B}$, we denote by $R^A_\beta s$ the *reduced function* of s on A, $R^A_\beta s = \inf\{t \in \mathcal{E}(\mathcal{U}_\beta) \mid t \geq s \text{ on } A\}$ and note that $R^A_\beta s$ is universally $\mathcal{B}$-measurable.

The topology $\mathcal{T}$ is named *natural with respect to* $\mathcal{U}$ provided that every $\mathcal{T}$-open set is finely open with respect to $\mathcal{U}_\beta$ for some $\beta > 0$. The *fine topology* is the topology on E generated by $\mathcal{E}(\mathcal{U}_\beta)$.

Remark 3.1. *Let $(s_n)_n \subset \mathcal{E}(\mathcal{U}_\beta)$ and $(f_n)_n \subset p\mathcal{B}$ be such that $s_n = f_n$ μ-a.e. Then there exists a trivial modification $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{U}$ and $\mathcal{U}'$ are μ-equivalent and $(f_n)_n \subset \mathcal{E}(\mathcal{U}'_\beta)$.* Indeed, let $M = \bigcup_n [f_n \neq s_n]$. Then $\mu(M) = 0$ and since $\mu \circ U_\beta \ll \mu$, by Lemma 2.1 in [4] there exists a set $M_0 \in \mathcal{B}$, $M_0 \supset M$, such that $U_\alpha(1_{M_0}) = 0$ on $E \setminus M_0$ for all $\alpha > 0$. The trivial modification $\mathcal{U}'$ of $\mathcal{U}$ on M_0 satisfies the required condition.

We note that since $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a resolvent of kernels on $(E, \mathcal{B})$ associated with $(V_\alpha)_{\alpha>0}$, then for every $u \in \mathcal{E}_\beta$ there exists a μ-version of u which is a $\mathcal{U}_\beta$-excessive function. Also, if $f \in \mathcal{E}(\mathcal{U}_\beta) \cap L^p(E, \mu) - \mathcal{E}(\mathcal{U}_\beta) \cap L^p(E, \mu)$, then the reduced function of f in $\mathcal{E}(\mathcal{U}_\beta)$ is a μ-version of $R_\beta[f]$, where $[f]$ denotes the element of $L^p(E, \mu)$ having f as μ-version. Moreover, if $f \in p\mathcal{B} \cap L^p(E, \mu)$ and G is finely open, then $R^G_\beta U_\beta f$ is a μ-version of the element $R_\beta(1_G U_\beta f)$ of $\mathcal{E}_\beta$. Consequently, if G_1, G_2 are two finely open sets such that $\mu(G_1 \triangle G_2) = 0$ then $R^{G_1}_\beta U_\beta f = R^{G_2}_\beta U_\beta f$ μ-a.e.

Remark 3.2. *Let $(F_n)_n$ be an increasing sequence of $\mathcal{T}$-closed sets in E and assume that the topology $\mathcal{T}$ is natural with respect to $\mathcal{U}$. Then the following assertions are equivalent:*

(i) $(F_n)_n$ *is a* μ*-nest.*

(ii) $\inf_n R_\beta^{E\setminus F_n} U_\beta g_0 = 0$ μ*-a.e. for one* $\beta > 0$ *and a strictly positive function* g_0 *such that* $U_\beta g_0 < \infty$ μ*-a.e.*

Remark 3.3. *Let* $s \in \mathcal{E}(\mathcal{U}_\beta)$ *be such that* $s < \infty$ μ*-a.e. and the level set* $K_n = [s \leq n]$ *is compact for all* n*. Then* $(K_n)_n$ *is a* μ*-nest (of compact sets).* Indeed, since $1 \leq \frac{s}{n}$ on $E \setminus K_n$, we deduce that $\inf_n R_\beta^{E\setminus K_n} s_0 = 0$ on the set $[s < \infty]$, hence μ-a.e. (here $s_0 = U_\beta f_0 \leq 1$, with $f_0 > 0$, $f_0 \in L^p(E, \mu)$).

Recall that a *Ray cone associated with* $\mathcal{U}_\beta$ is a convex cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_{\beta+\alpha}(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, $\mathcal{R}$ is min-stable, separable in the uniform norm and contains the positive constant functions; see e.g. [3]. The topology on E generated by a Ray cone is called *Ray topology.* Clearly, every Ray topology is natural with respect to $\mathcal{U}$.

Remark 3.4. *If* $M \in \mathcal{B}$ *is a set such that* $U_\alpha(1_M) = 0$ *on* $E \setminus M$ *then the following assertions hold.*
(i) Let $\mathcal{R}$ *be a Ray cone associated with* $\mathcal{U}_\beta$ *and* $\mathcal{U}'$ *be the trivial modification of* $\mathcal{U}$ *on* M*. Then there exists a Ray cone* $\mathcal{R}'$ *with respect to* $\mathcal{U}'_\beta$ *such that* $\mathcal{R} \subset \mathcal{R}'$.
(ii) If $\mathcal{R}^\circ$ *is a Ray cone associated with* $\mathcal{U}_\beta|_{E\setminus M}$ *then there exists a Ray cone* $\mathcal{R}$ *with respect to* $\mathcal{U}_\beta$ *such that* $\mathcal{R}|_{E\setminus M} = \mathcal{R}^\circ$.

We now collect some results on the saturated set E_1 of E with respect to $\mathcal{U}_\beta$; cf. [3] and [4].

$(*1)$ E is a finely dense subset of E_1 and therefore if $G \subset E_1$, $G \in \mathcal{B}_1$, is finely open, then $R_\beta^{G\cap E} s = R_\beta^G s$ for all $s \in \mathcal{E}(\mathcal{U}_\beta^1)$, where $\mathcal{U}_\beta^1$ is the extension of the resolvent $\mathcal{U}_\beta$ to E_1.

$(*2)$ If ξ is a $\mathcal{U}_\beta$-excessive measure (i.e. ξ is a σ-finite measure such that $\alpha U_{\beta+\alpha}\xi \leq \xi$ for all $\alpha > 0$), then E is ξ-semisaturated with respect to $\mathcal{U}_\beta$ (i.e. every $\mathcal{U}_\beta$-excessive measure dominated by a potential is also a potential; recall that a *potential* is a $\mathcal{U}_\beta$-excessive measure of the form $\nu \circ U_\beta$, where ν is a positive measure on $(E, \mathcal{B})$) if and only if the set $E_1 \setminus E$ is ξ-polar (i.e. $R_\beta^{E_1\setminus E} 1 = 0$ ξ-a.e.).

$(*3)$ If ξ is a $\mathcal{U}_\beta$-excessive measure and E is ξ-semisaturated, then there exists a second sub-Markovian resolvent $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ which is ξ-equivalent with $\mathcal{U}$, $\mathcal{U}'$ is a trivial modification of $\mathcal{U}$ and E is semisaturated with respect to $\mathcal{U}'$. Particularly, $\mathcal{U}'$ is the resolvent of a right process with state space E, endowed with any Ray topology with respect to $\mathcal{U}$ (see Remark 3.4 *(i)*).

Let $\mu' = \varphi \cdot \mu$, where $\varphi > 0$, $\varphi \in L^1(E, \mu)$, and consider the $\mathcal{U}_\beta$-excessive measure $\xi = \mu' \circ U_\beta$.

Lemma 3.5. *Assume that* E *is endowed with a Ray topology and suppose that there exists an increasing sequence* $(K_n)_n$ *of Ray compact sets in* E *which is a* ξ*-nest. Then* E *is* ξ*-semisaturated with respect to* $\mathcal{U}_\beta$.

Proof. The set $E_1 \setminus K_n$ is Ray open in E_1, and by the above assertion $(*1)$ we have $R_\beta^{E\setminus K_n} U_\beta^1 f_0 = R_\beta^{E_1\setminus K_n} U_\beta^1 f_0$. It follows that $R_\beta^{E_1\setminus E} U_\beta^1 f_0 = \inf_n R_\beta^{E\setminus K_n} U_\beta^1 f_0 = 0$ ξ-a.e. Therefore the set $E_1 \setminus E$ is ξ-polar and by $(*2)$ we conclude that E is ξ-semisaturated. $\square$

4 Proof of the main results

Proof of Theorem 1.1.
Notice firstly that $\sigma(\mathcal{A}) = \mathcal{B}$ and therefore $(E, \mathcal{B})$ is a Lusin measurable space. We show that assertion (c) holds. Clearly, we have $D(L) \subset \mathcal{E}_\beta - \mathcal{E}_\beta$. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ associated with $(V_\alpha)_{\alpha>0}$. By Remark 3.1 there exists a trivial modification $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{U}'$ is μ-equivalent with $\mathcal{U}$, $\{u \wedge n \mid n \in \mathbb{N}\} \subset b\mathcal{E}(\mathcal{U}'_\beta)$ for all $u \in \mathcal{A}$, and $\mathcal{A} \subset b\mathcal{E}(\mathcal{U}'_\beta) - b\mathcal{E}(\mathcal{U}'_\beta)$. Consequently, $\mathcal{T}_0 = \mathcal{T}(\mathcal{A})$ is a natural topology with respect to $\mathcal{U}'$, and therefore by Proposition 2.4 and assumption (II) it follows that every element $v \in D(L)$ possesses a μ-quasi continuous version $\widetilde{v}$ (with respect to $\mathcal{T}_0$). Notice also that by hypothesis (I) there exists a μ-nest of $\mathcal{T}$-compact (and therefore $\mathcal{T}_0$-compact) sets.

We shall prove now assertion (a) in four steps.

Step I. We prove that if $f \in bp\mathcal{B}$ and $\alpha > 0$, then $U'_\alpha f$ has a μ-quasi continuous version. Indeed, we may suppose that $\alpha = \beta$ and let $f_n \in bL^p_+(E, m)$ be such that $U'_\beta f = \sum_n U'_\beta f_n$. Since $U'_\beta f_n \in D(L)$, there exists a μ-quasi continuous version $\widetilde{U'_\beta} f_n$ of $U'_\beta f_n$. Let $s_n = \sum_{k\geq n} \widetilde{U'_\beta} f_k$, then $(s_n)_n \subset \mathcal{E}(\mathcal{U}'_\beta)$, it is decreasing to zero μ-a.e. and each s_n is μ-quasi lower semicontinuous. By Proposition 2.6 it results that a subsequence of $(s_n)_n$ converges μ-quasi uniformly to zero and consequently $\sum_n \widetilde{U'_\beta} f_n$ is a μ-quasi continuous version of $U'_\beta f$.

Step II. We show now that there exists a resolvent of kernels $\widetilde{\mathcal{U}} = (\widetilde{U}_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that $\widetilde{\mathcal{U}}$ is μ-equivalent with $\mathcal{U}$ and $\widetilde{U}_\alpha f$ is μ-quasi continuous for all $\alpha > 0$ and $f \in bp\mathcal{B}$.

We denote by $qC(E)$ the linear space of all μ-quasi continuous real valued functions. For every $\alpha > 0$ and $f \in b\mathcal{B}$, we denote by $T_\alpha f$ an element of $bqC(E)$ which is a μ-version of $U'_\alpha f$. The map $T_\alpha : b\mathcal{B} \to bqC(E)$ is quasi linear positive in the sense of [1], with respect to the outer capacity c^α_μ defined by

$$c^\alpha_\mu(G) = \int \varphi R^G_\alpha(U'_\alpha f_0)\, \mathrm{d}\mu\ , \quad G \in \mathcal{T},$$

where $\varphi \in L^1(E, \mu) \cap L^{p'}(E, \mu)$, $\varphi > 0$, $f_0 \in bp\mathcal{B} \cap L^p(E, \mu)$, $f_0 > 0$, with $U'_\alpha f_0 \leq 1$. By Remark 3.2 we have: an increasing sequence $(F_k)_k$ of $\mathcal{T}$-closed sets in E is a μ-nest if and only if $\inf_k c^\alpha_\mu(E \setminus F_k) = 0$. We check that if $f_n \searrow 0$ then $(T_\alpha f_n)_n$ converges to zero in capacity, i.e. $\lim_n c^\alpha_\mu\big([|T_\alpha f_n| > \varepsilon]\big) = 0$ for all $\varepsilon > 0$. Indeed, let $(F_k)_k$ be a μ-nest such that $T_\alpha f_n|_{F_k}$ is continuous and $c^\alpha_\mu(E \setminus F_k) \leq \frac{1}{2^k}$ for all n and k. We get

$$c^\alpha_\mu\big([|T_\alpha f_n| > \varepsilon]\big) \leq \frac{1}{\varepsilon} \int \varphi U'_\alpha f_n\, \mathrm{d}\mu + \sum_{k\geq n} c^\alpha_\mu(E \setminus F_k) \leq \frac{1}{\varepsilon} \|\varphi\|_{L^{p'}} \cdot \|U'_\alpha f_n\|_{L^p} + \frac{1}{2^{n-1}}\,.$$

By Theorem 4.4 in [1] there exists a kernel $\widetilde{T}_\alpha$ on $(E, \mathcal{B})$ such that $\widetilde{T}_\alpha f = T_\alpha f$ μ-q.e. for all $f \in bp\mathcal{B}$. By Lemma A.1 (in Appendix A) we conclude that there exists a resolvent of kernels $\widetilde{\mathcal{U}} = (\widetilde{U}_\alpha)_{\alpha>0}$ as claimed.

Step III. We show that there exists a resolvent $\mathcal{U}'' = (U''_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ which is associated with $(V_\alpha)_{\alpha>0}$, $U''_\alpha f \in bqC(E)$ for all $f \in bp\mathcal{B}$ and $\alpha > 0$, and $\mathcal{A} \subset b\mathcal{E}(\mathcal{U}''_\beta) - b\mathcal{E}(\mathcal{U}''_\beta)$.

Lemma 4.1. *If M is a μ-exceptional set, then there exists a μ-exceptional set $F \supset M$ such that $\widetilde{U}_\alpha(1_F) = 0$ on $E \setminus F$. Particularly, the trivial modification $\widetilde{\mathcal{U}}' = (\widetilde{U}'_\alpha)_{\alpha>0}$ of $\widetilde{\mathcal{U}}$ on F is a sub-Markovian resolvent on $(E, \mathcal{B})$ which is μ-equivalent with $\mathcal{U}$ and $\widetilde{U}'_\alpha f \in bqC(E)$ for all $f \in bp\mathcal{B}$.*

Proof. We have $\widetilde{U}_\alpha(1_M) = 0$ μ-a.e., because $\mu(M) = 0$. By Lemma 2.1 in [4] there exists a set $F \in \mathcal{B}$ such that $\widetilde{U}_\alpha(1_F) = 0$ on $E \setminus F$ for all $\alpha > 0$, more precisely we have $F = \bigcup F_n$ where $F_{n+1} = F_n \cup \big[\widetilde{U}_\alpha(1_{F_n}) > 0\big]$ for all $n \geq 0$, with $F_0 = M$. Since the function $\widetilde{U}_\alpha(1_{F_n})$ is μ-quasi continuous and $\widetilde{U}_\alpha(1_{F_n}) = 0$ μ-a.e., it follows that the set $\big[\widetilde{U}_\alpha(1_{F_n}) > 0\big]$ is μ-exceptional for all n, hence F is also μ-exceptional. We conclude that the trivial modification $\widetilde{\mathcal{U}}'$ of $\widetilde{\mathcal{U}}$ on F has the property: $\widetilde{U}_\alpha f = \widetilde{U}'_\alpha f$ μ-q.e. and therefore $\widetilde{U}'_\alpha f$ is also a μ-quasi continuous function, for all $f \in bp\mathcal{B}$ and $\alpha > 0$. □

Let us denote by $\bar{\mathcal{A}}$ the closure of $\mathcal{A}$ in the uniform norm. By hypothesis (II) it follows that if $u \in \mathcal{A}$ and $\alpha \geq 0$ then $u \wedge \alpha \in \bar{\mathcal{A}}$ and therefore $|u| \in \bar{\mathcal{A}}$. Consequently, $\bar{\mathcal{A}}$ is a vector lattice with respect to the pointwise infimum and $u \wedge \alpha \in \bar{\mathcal{A}}$ for all $u \in \bar{\mathcal{A}}$ and $\alpha > 0$.

For a function $w \in p\mathcal{B} \cap \mathcal{E}_\beta$, we shall denote by $\widehat{w}$ its *μ-quasi lower semicontinuous regularization*,

$$\widehat{w} := \sup_n n\widetilde{U}_{\beta+n} w \ .$$

Lemma 4.2. *The following assertions hold.*

(i) If $t \in \bar{\mathcal{A}}$ then a subsequence of $(\alpha_n \widetilde{U}_{\beta+\alpha_n} t)_n$ converges μ-quasi uniformly to t, where $\alpha_n \nearrow +\infty$.

(ii) Let $v \in D(L)$ and $u \in \bar{\mathcal{A}}_+$. Then there exists a sequence $(\alpha_k)_k \subset \mathbb{R}_+$, $\alpha_k \nearrow \infty$, such that $\big(\alpha_k \widetilde{U}_{\beta+\alpha_k}(\widetilde{v} \wedge u)\big)_k$ converges μ-q.e. to $\widetilde{v} \wedge u$.

(iii) Let $u \in q\mathcal{C}(E) \cap \mathcal{E}_\beta$ such that there exists a sequence $(u_n)_n \subset \mathcal{A}$ converging to u μ-q.u. Then $u = \widehat{u}$ μ-q.e.

Proof. (i) Assume firstly that $t \in \mathcal{A}$. Then $\alpha_n \widetilde{U}_{\beta+\alpha_n} t$ converges to t in $L^p(E, \mu)$ and since $t \in D(L)$, it follows that the above convergence holds in the graph norm. By Proposition 2.4 we deduce that a subsequence converges μ-quasi uniformly to $\widetilde{t} = t$. If $t \in \bar{\mathcal{A}}$ then we consider a sequence $(t_k)_k \subset \mathcal{A}$ converging to t uniformly. By the first part of the proof, we may assume (passing to a subsequence) that $\alpha_n \widetilde{U}_{\beta+\alpha_n} t_k \xrightarrow{n\to\infty} t_k$ μ-quasi uniformly for all k and let $(F_i)_i$ be a μ-nest such that $\alpha_n U_{\beta+\alpha_n} t_k|_{F_i} \xrightarrow{n\to\infty} t_k|_{F_i}$ uniformly for all k and i. From

$$|t - \alpha_n \widetilde{U}_{\beta+\alpha_n} t| \leq |t - t_k| + |t_k - \alpha_n \widetilde{U}_{\beta+\alpha_n} t_k| + \big|\alpha_n \widetilde{U}_{\beta+\alpha_n}(t_k - t)\big| \leq 2\|t - t_k\|_\infty + |t_k - \alpha_n \widetilde{U}_{\beta+\alpha_n} t_k|$$

we deduce that $(\alpha_n \widetilde{U}_{\beta+\alpha_n} t)_n$ converges uniformly to t on each F_i.

(ii) Let $(v_n)_n \subset \mathcal{A}$ be such that $(v_n)_n$ converges to v in the graph norm and put $w_n = R_\beta\big(|v - v_n|\big)$. By Proposition 2.8, passing to a subsequence, there exists a μ-exceptional set M_1 such that $\lim_n \widehat{w}_n = 0$ and $\lim_n v_n = \widetilde{v}$ on $E \setminus M_1$. Since $(v_n \wedge u)_n \subset \bar{\mathcal{A}}$, from (i) it follows that there exist a sequence $(\alpha_k)_k \subset \mathbb{R}_+$, $\alpha_k \nearrow \infty$, and a μ-exceptional set $M_2 \supset M_1$, such that $\lim_k \alpha_k \widetilde{U}_{\beta+\alpha_k}(v_n \wedge u^+) = v_n \wedge u^+$ on $E \setminus M_2$, for all n. On the other hand, since we have μ-q.e.

$$\big|\alpha \widetilde{U}_{\beta+\alpha}(\widetilde{v} \wedge u - v_n \wedge u)\big| \leq \alpha \widetilde{U}_{\beta+\alpha}\big(|\widetilde{v} - v_n|\big) \leq \widehat{w}_n \ ,$$

there exists a μ-exceptional set $M \supset M_2$ such that $\big|\alpha_k \widetilde{U}_{\beta+\alpha_k}(\widetilde{v} \wedge u - v_n \wedge u)\big| \leq \widehat{w}_n$ on $E \setminus M$ for all k and n. Consequently, the claimed convergence holds on $E \setminus M$.

(iii) We may assume that there exists $v \in qC(E) \cap \mathcal{E}_\beta \cap D(L)$ with $u \leq v$. Since the map $\alpha \mapsto \alpha \widetilde{U}_{\beta+\alpha} u$ is increasing (the inequalities being μ-q.e.), it suffices to show that for a sequence $(\alpha_n)_n \nearrow \infty$ we have that $(\alpha_n \widetilde{U}_{\beta+\alpha_n} u)_n$ converges to u μ-q.e. Let $s_0 = \widetilde{U}_\beta f_0$, $f_0 \in L^p_+(E, \mu)$,

$f_0 > 0$, and $(F_k)_k$ be a μ-nest such that $s_0|_{F_k}$ is continuous, $s_0 \geq \frac{1}{k}$ on F_k, and $\left(u_n|_{F_k}\right)_n$ converges uniformly on F_k to $u|_{F_k}$ for all k. Clearly $\left(v \wedge u_n^+|_{F_k}\right)_n$ converges uniformly on F_k to $u|_{F_k}$. Passing to a subsequence, we may assume that for all n we have $|v \wedge u_n^+ - u| \leq \frac{1}{n}s_0$ on F_n. We put $v_n = \frac{1}{n}s_0 + R_\beta(v1_{E\setminus F_n})$. By Proposition 2.7 there exists a μ-exceptional set $M' \supset \bigcap_k E \setminus F_k$ such that the sequence $\left(\widehat{R_\beta(v1_{E\setminus F_n})}\right)_n$ converges pointwise to zero on $E \setminus M'$. We have μ-q.e.

$$\left|\alpha \widetilde{U}_{\beta+\alpha}(v \wedge u_n^+ - u)\right| \leq \alpha \widetilde{U}_{\beta+\alpha}\left(|v \wedge u_n^+ - u|\right) \leq \widehat{v}_n$$

and therefore there exists a μ-exceptional set $M'' \supset M'$ such that $\left|\alpha \widetilde{U}_{\beta+\alpha}(v \wedge u_n^+ - u)\right| \leq \widehat{v}_n$ on $E \setminus M''$ for all n and $\alpha > 0$. By assertion (ii) there exist a μ-exceptional set $M \supset M''$ and a sequence $(\alpha_n)_n$, $\alpha_n \nearrow \infty$, such that $\lim_n \alpha_n \widetilde{U}_{\beta+\alpha_n}(v \wedge u_n^+) = v \wedge u_n^+$ on $E \setminus M$. We conclude that $(\alpha_n \widetilde{U}_{\beta+\alpha_n} u)_n$ converges to u on $E \setminus M$. □

Proposition 4.3. *After a trivial modification of $\widetilde{\mathcal{U}}$ on a μ-exceptional set, we may assume that $\mathcal{A} \subset b\mathcal{E}(\widetilde{\mathcal{U}}_\beta) - b\mathcal{E}(\widetilde{\mathcal{U}}_\beta)$. Particularly, the set $\mathcal{E}(\widetilde{\mathcal{U}}_\beta)$ separates the points of E and $\widetilde{U}_\beta 1 > 0$ on E.*

Proof. Since $\mathcal{A} \subset D(L)$, for every $u \in \mathcal{A}$ there exists a $u_1, u_2 \in b\mathcal{E}(\mathcal{U}_\beta) \cap L^p(U, \mu)$, $u_1 = \widetilde{U}_\beta f_1$, $u_2 = \widetilde{U}_\beta f_2$, with $f_1, f_2 \in bp\mathcal{B}$, such that the set $M_u = [u \neq u_1 - u_2]$ is μ-negligible. Because $\widetilde{U}_\beta f_1, \widetilde{U}_\beta f_2 \in bqC(E)$, we deduce that the set $M = \bigcup_{u \in \mathcal{A}} M_u$ is μ-exceptional and therefore the trivial modification of $\widetilde{\mathcal{U}}$ on M satisfies the required properties; notice that by Lemma 4.1 it follows that the property of $\widetilde{\mathcal{U}}$ that $\widetilde{U}_\alpha(bp\mathcal{B}) \subset qC(E)$ is preserved. □

Let $D_{\widetilde{\mathcal{U}}_\beta}$ denote the set of all *non-branch* points of $\widetilde{\mathcal{U}}_\beta$:

$$D_{\widetilde{\mathcal{U}}_\beta} = \bigcap_{u,v \in \mathcal{E}(\mathcal{U}_\beta)} \left[\widehat{u \wedge v} = u \wedge v\right] \cap [\widehat{1} = 1]\,.$$

It is known that there exists a countable set $\mathcal{F}_0 \subset bp\mathcal{B} \cap L^p(E, \mu)$ such that

$$D_{\widetilde{\mathcal{U}}_\beta} = \bigcap_{f,g \in \mathcal{F}_0} \left[\widehat{\widetilde{U}_\beta f \wedge \widetilde{U}_\beta g} = \widetilde{U}_\beta f \wedge \widetilde{U}_\beta g\right] \cap [\widehat{1} = 1]$$

(see e.g. [3]).

Corollary 4.4. *The set $E \setminus D_{\widetilde{\mathcal{U}}_\beta}$ is μ-exceptional.*

Proof. If $u = \widetilde{U}_\beta f$, $v = \widetilde{U}_\beta g$ with $f, g \in \mathcal{F}_0$, then the function $u \wedge v$ satisfies the hypothesis of Lemma 4.2 (iii) and therefore $\widehat{u \wedge v} = u \wedge v$ μ-q.e. The set $[\widehat{1} \neq 1]$ is also μ-exceptional. Indeed, we may apply again Lemma 4.2 (iii) to the function $1 \wedge (n\widetilde{U}_\beta f)$, where $f > 0$, since by Proposition 4.3 we get $\widetilde{U}_\beta f > 0$ and we have μ-q.e.

$$\widehat{1} = \sup_n \widehat{1 \wedge (n\widetilde{U}_\beta f)} = \sup_n f \wedge (n\widetilde{U}_\beta f) = 1\,. \qquad \square$$

We can now complete the proof of Step III. It suffices to apply Lemma 4.1 for the set $M = E \setminus D_{\widetilde{\mathcal{U}}_\beta}$, which is μ-exceptional by Corollary 4.4.

Step IV. Let $\mathcal{R}$ be a Ray cone associated with $\mathcal{U}''_\beta$, such that $\mathcal{A} \subset \mathcal{R} - \mathcal{R}$. We claim that every $r \in \mathcal{R}$ is μ-quasi continuous. It suffices to prove this property for every $r \in \mathcal{R}_\infty$, since $\mathcal{R}$ is the closure in the uniform norm of $\mathcal{R}_\infty$. Recall that (see e.g. [3]) $\mathcal{R}_\infty = \bigcup_{n \geq 0} \mathcal{R}_n$, where $\mathcal{R}_n \subset b\mathcal{E}(\mathcal{U}'_\beta)$ is defined inductively as follows:

$$\mathcal{R}_{n+1} = \mathbb{Q}_+ \cup \mathbb{Q}_+ \cdot \mathcal{R}_n \cup \Big(\sum_f \mathcal{R}_n\Big) \cup \Big(\bigwedge_f \mathcal{R}_n\Big) \cup \Big(\bigcup_{\alpha \in \mathbb{R}_+} \widetilde{U}'_{\beta+\alpha}(\mathcal{R}_n)\Big) \cup \widetilde{U}'_\beta\big(p(\mathcal{R}_n - \mathcal{R}_n)\big)$$

with $\mathcal{R}_0 - \mathcal{R}_0 \supset \mathcal{A}$. Since $U''_\alpha(bp\mathcal{B}) \subset bqC(E)$ for all $\alpha > 0$, we get $\mathcal{R}_n \subset qC(E)$ for all n.

Furthermore, let $\mathcal{T}_\mathcal{R}$ be the Ray topology generated by $\mathcal{R}$. We consider a μ-nest $(K_n)_n$ of $\mathcal{T}_0$-compact sets such that $r|_{K_n}$ is $\mathcal{T}_0$-continuous for all $r \in \mathcal{R}_\infty$. Because $\mathcal{T}_\mathcal{R} = \mathcal{T}(\mathcal{R}_\infty)$ we deduce that on each K_n the traces of the Ray topology, $\mathcal{T}_0$ and $\mathcal{T}$ coincide, hence $(K_n)_n$ is a μ-nest of Ray compact sets. By Lemma 3.5 we conclude that E is ξ-semisaturated with respect to $\mathcal{U}''_\beta$. Assertion (a) follows now by $(*3)$ and since there exists a μ-nest of $\mathcal{T}_0$-compact sets (see e.g. [3]).

Because $(K_n)_n$ is a μ-nest, it follows that $\sup_n T_{E\setminus K_n}(\omega) \geq \zeta(\omega)$ for all $\omega \in \Omega_0$, with $P^\mu(\Omega \setminus \Omega_0) = 0$. Hence if $\omega \in \Omega_0$ and $t_0 < \zeta(\omega)$ then there exists $n \geq 1$ such that $t_0 < T_{E\setminus K_n}(\omega)$, i.e. $X_t(\omega) \in K_n$ for all $t \in [0, t_0)$. But $\mathcal{T}_0|_{K_n} = \mathcal{T}|_{K_n}$ and thus the trajectory $t \mapsto X_t(\omega)$ is càdlàg also in the topology $\mathcal{T}$, for all $\omega \in \Omega_0$, hence the process is càdlàg in the topology $\mathcal{T}$, P^μ-a.e. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3.
Let $(F_n)_n$ be a μ-nest of $\mathcal{T}$-compact sets such that for every $u \in \mathcal{A}$ the function $\widetilde{u}|_{F_n}$ is real valued and continuous, the set of functions $\widetilde{\mathcal{A}} = \{\widetilde{u} \mid u \in \mathcal{A}\}$ separates the points of $\bigcup_n F_n$ and $\widetilde{\mathcal{A}}|_{\bigcup_n F_n}$ is a $\mathbb{Q}$-linear space. We may assume that all the functions from $\widetilde{\mathcal{A}}$ are bounded. Let $\mathcal{A}_\infty$ be the closure of $\mathcal{A}$ in $L^\infty(E, \mu)$. By Proposition 2.5 it follows that every $u \in \mathcal{A}_\infty$ possesses a bounded μ-quasi continuous version $\widetilde{u}$ and there exists a μ-nest $(K_n)_n$, $K_n \subset F_n$ for all n, such that if $E_\circ = \bigcup_n K_n$ then $\widetilde{u} \wedge \alpha|_{E_\circ}$ belongs to the closure of of $\widetilde{\mathcal{A}}|_{E_\circ}$ in the uniform norm for all $u \in \mathcal{A}$. Let $\mathcal{T}_\circ$ be the topology on $E_\circ$ generated by $\widetilde{\mathcal{A}}|_{E_\circ}$. We remark that $(E_\circ, \mathcal{T}_\circ)$ is a Lusin topological space, $\mathcal{T}_\circ|_{K_n} = \mathcal{T}|_{K_n}$ for all n and the topologies $\mathcal{T}_\circ$ and $\mathcal{T}$ on $E_\circ$ generate the same Borel σ-algebra. Since the set $E \setminus E_\circ$ is μ-negligible, we may consider $(V_\alpha)_{\alpha>0}$ as a strongly continuous sub-Markovian resolvent of contractions on $L^p(E_\circ, \mu)$ and we can apply Theorem 1.1 with $\widetilde{\mathcal{A}}|_{E_\circ}$ as the given countable $\mathbb{Q}$-linear space satisfying condition (II) on $E_\circ$.

Let $\mathcal{U}^\circ = (U^\circ_\alpha)_{\alpha>0}$ be the resolvent of kernels on $(E_\circ, \mathcal{B}|_{E_\circ})$ such that $E_\circ$ is semisaturated with respect to $\mathcal{U}^\circ$ and let $\mathcal{R}^\circ \subset \mathcal{E}(\mathcal{U}^\circ_\beta)$ be a Ray cone associated with $\mathcal{U}^\circ_\beta$ such that the traces of $\mathcal{T}(\mathcal{R}^\circ)$ and $\mathcal{T}_\circ$ coincide on each F_n for some μ-nest $(F_n)_n$ of $\mathcal{T}_\circ$-compact subsets of $E_\circ$. Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the "trivial" extension of $\mathcal{U}^\circ = (U^\circ_\alpha)_{\alpha>0}$ from $E_\circ$ to E,

$$U_\alpha f = 1_{E_\circ} U^\circ_\alpha(f|_{E_\circ}) + \frac{1}{\alpha+1} 1_{E\setminus E_\circ} f, \quad f \in p\mathcal{B}.$$

Then $\mathcal{U}$ is resolvent of kernels on E associated with $(V_\alpha)_{\alpha>0}$ and $\mathcal{U}^\circ = \mathcal{U}|_{E_\circ}$. By Remark 3.4 (ii) there exists a Ray cone $\mathcal{R}$ associated with $\mathcal{U}_\beta$, such that $\mathcal{R}|_{E_\circ} = \mathcal{R}^\circ$. Then $(F_n)_n$ is a μ-nest of Ray compact sets in E and $\mathcal{U}$ is the resolvent of a right process with state space E, satisfying the claimed properties (a), (b) and (c). $\square$

Now we are going to give a proof of Proposition 1.4 which apart from integrability issues is standard, we include it for the reader's convenience.

Proof of Proposition 1.4.
Let us first observe that the function

$$M_t := f(X_t) - \int_0^t Lf(X_s)\ \mathrm{d}s$$

is well-defined P^ν-a.e. More precisely, we prove that if $g \in p\mathcal{B}$ equals zero μ-a.e. then $\int_0^t g(X_s)\ \mathrm{d}s$ vanishes P^μ-a.e. Indeed, we have

$$E^x \int_0^t g(X_s)\ \mathrm{d}s = \int_0^t p_s g(x)\ \mathrm{d}s$$

and since $U_\beta g = 0$ μ-a.e. for all $\beta > 0$, we get

$$\begin{aligned} E^\nu\left(\int_0^t g(X_s)\ \mathrm{d}s\right) &= \int_E \nu(\mathrm{d}x) \int_0^t p_s g(x)\ \mathrm{d}s = \sup_\beta \int_E \nu(\mathrm{d}x) \int_0^t e^{-\beta s}\, p_s g(x)\ \mathrm{d}s \\ &\leq \sup_\beta \int_E U_\beta g(x)\ \nu(\mathrm{d}x) = 0\ . \end{aligned}$$

We shall denote by $(P_t)_{t\geq 0}$ the family of linear operators on $L^p(E,\mu)$ induced by the transition function $(p_t)_{t\geq 0}$ of the process X. Consequently, $(P_t)_{t\geq 0}$ is a strongly continuous sub-Markovian semigroup of contractions on $L^p(E,\mu)$, having $\big(L, D(L)\big)$ as infinitesimal generator. Therefore, for every $f \in D(L)$ and $t > 0$ we have $P_t f \in D(L)$, $LP_t f = P_t Lf$ and $\int_0^t P_s Lf\ \mathrm{d}s = P_t f - f$.

We now show that for all t, M_t is P^ν-integrable. We have

$$\begin{aligned} E^\nu\big(|M_t|\big) &\leq E^\nu\big(|f(X_t)|\big) + E^\nu\left(\left|\int_0^t Lf(X_s)\ \mathrm{d}s\right|\right) \\ &\leq \int_E g_0\, P_t\big(|f|\big)\ \mathrm{d}\mu + \int_0^t \left(\int_E g_0\, P_s\big(|Lf|\big)\ \mathrm{d}\mu\right) \mathrm{d}s \\ &\leq \|g_0\|_{L^{p'}} \big(\|f\|_{L^p} + t\,\|Lf\|_{L^p}\big) < \infty\ . \end{aligned}$$

We check finally the martingale property of M_t under P^ν. If $s,t \geq 0$, $s < t$, and $G \in bp\mathcal{F}_s$, then using the Markov property of X we obtain:

$$E^\nu\big(G \cdot f(X_t)\big) = E^\nu\big(G \cdot p_{t-s} f(X_s)\big)$$

and

$$\begin{aligned} E^\nu\left(G \cdot \int_s^t Lf(X_u)\ \mathrm{d}u\right) &= E^\nu\left(G \cdot \int_0^{t-s} Lf(X_v)\circ\theta_s\ \mathrm{d}v\right) = E^\nu\left(G \cdot \int_0^{t-s} E^{X_s}\big(Lf(X_v)\big)\ \mathrm{d}v\right) \\ &= E^\nu\left(G \cdot \int_0^{t-s} P_v Lf(X_s)\ \mathrm{d}v\right) = E^\nu\Big(G \cdot \big(P_{t-s} f(X_s) - f(X_s)\big)\Big)\ , \end{aligned}$$

and so

$$E^\nu(G\cdot M_t) = E^\nu\left(G \cdot \left(f(X_t) - \int_s^t Lf(X_u)\ \mathrm{d}u\right)\right) - E^\nu\left(G \cdot \int_0^s Lf(X_u)\ \mathrm{d}u\right) = E^\nu(G \cdot M_s)\ .$$

$\square$

5 Applications

In this section we want to apply the above results to construct martingale solutions to stochastic differential equations on a Hilbert space H (with inner product $\langle\ ,\ \rangle$ and norm $|\cdot|$) of type

$$\mathrm{d}X(t) = \big[AX(t) + F_0\big(X(t)\big)\big]\ \mathrm{d}t + \sqrt{C}\ \mathrm{d}W(t)\ . \tag{3}$$

Here $W(t)$, $t \geq 0$, is a cylindrical Brownian motion on H, C is a positive definite self-adjoint linear operator on H and $A : D(A) \subset H \to H$ the infinitesimal generator of a C_0-semigroup on H. Furthermore,

$$F_0(x) := y_0\ , \quad x \in D(F), \tag{4}$$

where $y_0 \in F(x)$ such that $|y_0| = \min_{y \in F(x)} |y|$, and $F : D(F) \subset H \to 2^H$ is an m-dissipative map. This means that $D(F)$ is a Borel set in H and

$$\langle u - v, x - y\rangle \leq 0 \quad \forall\, x, y \in D(F),\ u \in F(x),\ v \in F(y), \tag{5}$$

and

$$\mathrm{Range}(I - F) := \bigcup_{x \in D(F)} \big(x - F(x)\big) = H\ .$$

Since for any $x \in D(F)$ the set $F(x)$ is closed, non-empty and convex, F_0 is well-defined by (4). Such equations have been studied in [7], the main novelty being that F_0 has no continuity properties. In [7], however, a martingale solution to (3) was only constructed under the assumption that the inverse C^{-1} of C exists and is bounded and that $A = A^*$ where $\big(A^*, D(A^*)\big)$ denotes the adjoint of $\big(A, D(A)\big)$. Hence, in particular, the case, where C is trace class, was not covered in the final result. Those results, however, which concern the underlying operator ("Kolmogorov operator") from [7], were proved without the assumption that $C^{-1} \in L(H)$ and we will use them below (among other things) in an essential way to verify the assumptions of our general results formulated in Section 1.

Let us first write the underlying Kolmogorov operator L_0 and then formulate precise conditions on the coefficients on A, F_0 and C in (3) which will ensure the existence of a measure μ as in Theorem 1.1 and imply that all its conditions are satisfied for the resolvent generated by the closure of L of L_0. We recall that by Proposition 1.4 the process in Theorem 1.1 satisfies the martingale problem determined by L, hence is a martingale solution for (3).

A heuristic application of Itô's formula to a solution of (3) implies that the Kolmogorov operator on test functions

$$\varphi \in \mathcal{E}_A(H) := \mathrm{lin.\,span}\big\{\sin\langle h, x\rangle,\ \cos\langle h, x\rangle\ \big|\ h \in D(A^*)\big\} \tag{6}$$

has the following form:

$$L_0\varphi(x) = \frac{1}{2} \cdot \mathrm{Tr}\big[CD^2\varphi(x)\big] + \big\langle x, A^* D\varphi(x)\big\rangle + \big\langle F_0(x), D\varphi(x)\big\rangle\ , \quad x \in H, \tag{7}$$

where $D\varphi(x)$, $D^2\varphi(x)$ denote the first and second Fréchet derivatives of φ at $x \in H$ considered as an element in H and as an operator on H, respectively. We note that by the chain rule $D\varphi(x) \in D(A^*)$ for all $\varphi \in \mathcal{E}_A(H)$, $x \in H$. Clearly, L_0 is well-defined for all φ of the form

$$\varphi(x) = f\big(\langle h_1, x\rangle, \ldots, \langle h_M, x\rangle\big)\ , \quad x \in H,$$

with $f \in C^2(\mathbb{R}^M)$, $M \in \mathbb{N}$, $h_1, \ldots, h_M \in D(A^*)$. We shall use this below frequently. As in [7], from now on we make the following assumptions:

(H1) (i) A is the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, on H, and there exists a constant $\omega > 0$ such that

$$\langle Ax, x\rangle \leq -\omega\,|x|^2 \quad \forall\, x \in H \ .$$

(ii) C is self-adjoint, nonnegative definite and such that $\operatorname{Tr} Q < \infty$, where

$$Qx := \int_0^\infty e^{tA}\, C\, e^{tA^*}\, x\, \mathrm{d}t\ , \qquad x \in H.$$

(H2) There exists a probability measure μ on the Borel σ-algebra $\mathcal{B}(H)$ of H such that

(i) $\displaystyle\int_{D(F)} \big(|x|^4 + |F_0(x)|^2 + |x|^4 \cdot |F_0(x)|^2\big)\ \mu(\mathrm{d}x) < \infty\ .$

(ii) For all $\varphi \in \mathcal{E}_A(H)$ we have $L_0\varphi \in L^2(H,\mu)$ and

$$\int L_0\varphi\ \mathrm{d}\mu = 0 \quad (\text{“infinitesimal invariance”}).$$

(iii) $\mu\big(D(F)\big) = 1$.

Remark 5.1. *(i) In this section, for simplicity, we shall apply Theorem 1.1 only in the case $p = 2$. To handle $p \geq 1$, we would need in (H2)(i) that*

$$\int_{D(F)} \big(|x|^{2p} + |F_0(x)|^p + |x|^{2p} \cdot |F_0(x)|^p\big)\ \mu(\mathrm{d}x) < \infty\ ,$$

and in (H2)(ii) that $L_0\varphi \in L^p(H,\mu)$ for all $\varphi \in \mathcal{E}_A(H)$.

(ii) If $0 \in D(F)$, by an elementary calculation one derives from (H1)(i) and (5) that for all $\eta \in (0,\omega)$ there exists a constant C_η such that

$$\big\langle Ax + F_0(x), x\big\rangle \leq -\eta\,|x|^2 + C_\eta \quad \forall\, x \in D(A) \cap D(F)\ .$$

Now we want to give a sufficient condition on the coefficients A and F_0, so that (H2) holds.

Proposition 5.2. *Assume that*

(i) $D(F) = H$ and F_0 is hemicontinuous, i.e. for all $x, y, z \in H$

$$\mathbb{R} \ni \lambda \mapsto \big\langle F_0(x + \lambda y), z\big\rangle$$

is continuous.

(ii) $H \ni x \mapsto -\langle x, A^*x\rangle$ is nonnegative and has compact level sets in H, where we set $-\langle x, A^*x\rangle := +\infty$ if $x \in H \setminus D(A^*)$.

(iii) There exist constants $\gamma, \kappa^* > 0$, such that

$$\big|F_0(x)\big| \leq \gamma\, e^{\frac{1}{2}\kappa^*|x|^2}\, \big(1 - \langle x, A^*x\rangle\big)^{1/2}\ .$$

(iv) $\operatorname{Tr} C < \infty$.

Then there exists a probability measure μ on $\big(H, \mathcal{B}(H)\big)$ satisfying (H2) and, in addition, we have that the function Θ_κ defined in (11) *below, is in $L^1(H,\mu)$ for all $\kappa \in (0,\infty)$.*

Proof. Fix $\kappa \in (0,\infty)$ and define

$$\tilde{V}_\kappa(x) := e^{\kappa |x|^2} \ , \quad x \in H. \tag{8}$$

Let $e_j \in D(A^*)$, $j \in \mathbb{N}$, such that $\operatorname{lin.span}\{e_j \mid j \in \mathbb{N}\}$ is dense in $D(A^*)$ with respect to the norm. $|\cdot|_{A^*} := |A^* \cdot| + |\cdot|$, i.e. the graph norm of $\big(A^*, D(A^*)\big)$, and $\{e_j \mid j \in \mathbb{N}\}$ is an orthonormal basis of H. For $N \in \mathbb{N}$ define $E_N := \operatorname{lin.span}\{e_j \mid j \le N\}$ and

$$B_j(x) := \langle x, A^* e_j\rangle + \big\langle F_0(x), e_j\big\rangle \ , \quad x \in H,\ j \in \mathbb{N}. \tag{9}$$

Then for $P_N :=$ orthogonal projection onto E_N in H we have

$$L_0 \tilde{V}_\kappa(x) = \kappa\, e^{\kappa|x|^2} \big[2\kappa \langle Cx, x\rangle + \operatorname{Tr}(P_N C) + \langle x, A^* x\rangle + \langle F_0(x), x\rangle\big] \ . \tag{10}$$

Define $\Theta_\kappa : H \to [0,\infty]$ by

$$\Theta_\kappa(x) := \begin{cases} e^{\kappa|x|^2}\big(1 - \langle x, A^* x\rangle\big) & \text{if } x \in D(A^*), \\ +\infty & \text{if } x \in H \setminus D(A^*). \end{cases} \tag{11}$$

Then by (5)

$$L_0\, \tilde{V}_\kappa(x) \le \kappa\, e^{\kappa|x|^2} \big[2\kappa\, \|C\| \cdot |x|^2 + \operatorname{Tr} C + 1 + |F_0(0)| \cdot |x|\big] - \kappa\, \Theta_\kappa \ .$$

Hence, by assumption (ii) we can find constants $c, m > 0$ such that

$$L_0\, \tilde{V}_\kappa(x) \le c - m\, \Theta_\kappa(x) \quad \text{for all } x \in E_N,\ N \in \mathbb{N}. \tag{12}$$

Furthermore, since F_0 is dissipative and hemicontinuous, it follows by [19, Proposition 26.4] that B_j is continuous on H for all $j \in \mathbb{N}$. Finally, the rextriction of $\tilde{V}_\kappa$ to E_N has compact level sets for all $N \in \mathbb{N}$ and for all $j \in E_N$ and $\kappa \in [\kappa^*, \infty)$

$$\big|B_j(x)\big| \le |A^* e_j| \cdot |x| + \gamma\, \Theta_{\kappa^*}^{1/2}(x) \, , \quad x \in H. \tag{13}$$

Now in the same way as to prove Theorem 5.1 in [5] or Theorem 2.4 in [14], it follows that there exists a probability measure μ on $\big(H, \mathcal{B}(H)\big)$, such that for all $\varphi : H \to \mathbb{R}$ of the form

$$\varphi(x) = f\big(\langle e_1, x\rangle, \ldots, \langle e_M, x\rangle\big) \ , \quad x \in H,$$

$f \in C_b^\infty(\mathbb{R}^M)$, $M \in \mathbb{N}$, we have $L_0 \varphi \in L^2(H,\mu)$ and

$$\int L_0 \varphi \, \mathrm{d}\mu = 0 \tag{14}$$

as well as

$$\int \Theta_\kappa \, \mathrm{d}\mu < \infty \ . \tag{15}$$

Since our orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ is dense in $D(A^*)$ with respect to $|\cdot|_{A^*}$, it follows by (15) and a simple approximation argument that $L_0\varphi \in L^2(H,\mu)$ and (14) holds for all $\varphi \in \mathcal{E}_A(H)$. Clearly, assumption (iii) and (15) also imply (H2)(i) and thus all assertions are proved. $\square$

Remark 5.3. *(i) A typical example for the above situation is the following: A = Dirichlet Laplacian on an open, bounded set in $\mathbb{R}^d$ and $H = L^2(\mathbb{R}^d, \mathrm{d}x)$, where $\mathrm{d}x$ denotes Lebesgue measure.*

(ii) Instead of (13) *it would have been enough for the above proof that for all $j \in \mathbb{N}$*

$$\big|B_j(x)\big| \le |A^* e_j| \cdot |x| + \delta_j\big(\Theta_\kappa(x)\big)\,\Theta_\kappa(x)\ , \quad x \in H,$$

for some bounded Borel function $\delta_j : [0,\infty) \to [0,\infty)$ such that $\lim_{r\to\infty} \delta_j(r) = 0$. We refer to [5, Theorem 5] for details.

(iii) The assumption $\operatorname{Tr} C < \infty$ in Proposition 5.2 is not essential. Taking $\tilde{V}_\kappa$ in its proof to be a properly "weighted norm", we can compensate for $\operatorname{Tr} C = \infty$. We refer to [5, Proposition 7.2] for an example.

By assumption (H2)(ii) it is easy to prove that $\big(L_0, \mathcal{E}_A(H)\big)$ is dissipative on $L^2(H,\mu)$ (cf. [7, Proposition 2.1]), hence closable. Let $\big(L, D(L)\big)$ denote its closure. The first main result in [7], however, is that (H1) and (H2) imply that $\big(L, D(L)\big)$ is m-dissipative (cf. [7, Theorem 2.3]), hence generates a C_0-semigroup $P_t := e^{tL}$, $t \ge 0$, on $L^2(H,\mu)$. By [7, Corollary 2.5], $(P_t)_{t\ge0}$ is Markovian, i.e. positivity preserving and $P_t 1 = 1$ for all $t \ge 0$. Clearly, μ is invariant for $(P_t)_{t\ge0}$, i.e.

$$\int P_t f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu \quad \forall\, t \ge 0,\ f \in L^2(H,\mu).$$

Define for $f \in L^2(H,\mu)$

$$V_\alpha f := \int_0^\infty e^{-\alpha t}\, P_t f \,\mathrm{d}t\ , \quad \alpha > 0. \tag{16}$$

Then $(V_\alpha)_{\alpha>0}$ is a strongly continuous Markovian contraction resolvent as in Sections 1–4 above. By Proposition 1.4, a right process associated with $(V_\alpha)_{\alpha>0}$ is a martingale solution of (3). So, it remains to prove that such a right process exists, i.e. we have to check that conditions (I) and (II) in Theorem 1.1 hold. We shall do that in the next proposition, which is the main result of this section. For the underlying topology $\mathcal{T}$ on H we take

$$\mathcal{T} := \text{weak topology on } H. \tag{17}$$

We need, however, the following additional condition:

(H3) (i) There exists an orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ of H so that $\bigcup_{N\in\mathbb{N}} E_N$ with $E_N := \operatorname{lin.span}\{e_j \mid 1 \le j \le N\}$ is dense in $D(A^*)$ with respect to $|\cdot|_{A^*}$ and such that for the orthogonal projection P_N onto E_N in H we have that the function

$$H \ni x \mapsto \langle P_N x, A^*\, P_N x\rangle$$

converges in $L^1(H,\mu)$ to

$$H \ni x \mapsto \langle x, A^* x\rangle$$

(defined to be $+\infty$ if $x \in H \setminus D(A^*)$).

(ii) There exist two increasing Borel functions $\varrho_1, \varrho_2 : [1,\infty) \to (0,\infty)$ such that

$$\big|F_0(x)\big|^2 \le \varrho_1\big(|x|\big) + \varrho_2\big(|x|\big)\,\big|\langle x, A^* x\rangle\big| \quad \forall\, x \in H\ ,$$

and the function on the right hand side is in $L^1(H,\mu)$.

Remark 5.4. *Consider the situation of Proposition 5.2 and, in addition, assume that there exists an orthonormal basis $\{e_j \mid j \in \mathbb{N}\}$ consisting of eigenvectors of $\big(A^*, D(A^*)\big)$ with eigenvalues λ_j, $j \in \mathbb{N}$, such that*

$$\sum_{j=1}^{N} \lambda_j \langle x, e_j\rangle e_j \to A^* x \quad \text{in } H \text{ as } N \to \infty.$$

Then (H3) is obviously fulfilled.

Proposition 5.5. *Assume (H1)–(H3) and let $(V_\alpha)_{\alpha>0}$ be as defined above. Then conditions (I) and (II) in Theorem 1.1 hold.*

Proof. Since $D(L)$ is separable with respect to the graph norm $\|\cdot\|_L := \big(\|L\cdot\|^2_{L^2(H,\mu)} + \|\cdot\|^2_{L^2(H,\mu)}\big)^{1/2}$ we can construct a countable $\mathbb{Q}$-algebra in $\mathcal{E}_A(H)$ dense with respect to $\|\cdot\|_L$ in $\mathcal{E}_A(H)$, hence dense with respect to $\|\cdot\|_L$ in $D(L)$. So, condition (II) holds by Remark 1.2.2.

To show condition (I) define $V(x) := |x|^2$, $x \in H$, and let $\{e_j \mid j \in \mathbb{N}\}$, E_N, P_N, $N \in \mathbb{N}$, be as in (H3). Then for all $x \in E_N$

$$\begin{aligned}(1 - L_0)\, V(x) &= V(x) - \operatorname{Tr} P_N\, C - 2\langle x, A^* x\rangle - 2\big\langle F_0(x), x\big\rangle \\ &\le g_0(x)\ ,\end{aligned} \tag{18}$$

where for $x \in H$

$$g_0(x) := 2\,|x|^2 + \big(2 + \varrho_2(|x|)\big)\, \big|\langle x, A^* x\rangle\big| + \varrho_1\big(|x|\big)\ .$$

By (H2)(i) and (H3)(ii) we have $g_0 \in L^1(H,\mu)$.

We recall that by [7, Theorem 2.3] the set $(1 - L_0)\,\mathcal{E}_A(H)$ is dense in $L^2(H,\mu)$, hence also in $L^1(H,\mu)$. Hence the closure $\big(L_1, D(L_1)\big)$ of $\big(L_0, \mathcal{E}_A(H)\big)$ (which exists since $\big(L_0, \mathcal{E}_A(H)\big)$ is also dissipative on $L^1(H,\mu)$) also generates a C_0-semigroup $P^{(1)}_t := e^{tL_1}$, $t \ge 0$, on $L^1(H,\mu)$. Let $V^{(1)}_\alpha$, $\alpha > 0$, denote the corresponding resolvent. For $\alpha > 0$ by definition $V_\alpha = V^{(1)}_\alpha$ on $(\lambda - L_0)\,\big(\mathcal{E}_A(H)\big)$, hence

$$V_\alpha f = V^{(1)}_\alpha f \quad \text{for all } f \in L^2(H,\mu) \tag{19}$$

by continuity. Since $g_0 \in L^1(H,\mu)$, an easy approximation argument shows that $V \circ P_N \in D(L_1)$ for all $N \in \mathbb{N}$, so (18) can be written as

$$(1 - L_1)\,(V \circ P_N) \le g_0 \circ P_N\ .$$

Applying $V^{(1)}_1$ we obtain that

$$V \circ P_N \le V^{(1)}_1 (g_0 \circ P_N) \quad \text{for all } N \in \mathbb{N}.$$

By (H3) we can take $N \to \infty$ to obtain

$$V \le V^{(1)}_1 g_0 =: g \in L^1(H,\mu)\ . \tag{20}$$

But then for all $\alpha > 0$

$$\alpha\, V^{(1)}_{\alpha+1} g^{1/2} \le \frac{\alpha}{\alpha+1}\,\big((\alpha+1)\, V^{(1)}_{\alpha+1} g\big)^{1/2} = \frac{\alpha^{1/2}}{(\alpha+1)^{1/2}}\,(\alpha\, V^{(1)}_{\alpha+1} g)^{1/2} \le g^{1/2}\ .$$

Hence by (19), $g^{1/2}$ is a 1-excessive function in $L^2(H, \mu)$, which by (20) dominates the function $|\cdot|$ which has $\mathcal{T}$-compace level sets. Therefore, for $k \in \mathbb{N}$,

$$R_1\big(1_{\{|\cdot|>k\}}\big) \leq \frac{1}{k}\, g^{1/2}\ ,$$

hence by Remark 2.2 (applied to $f_0 \equiv 1$, so $V_1 1 = 1$) we obtain that $F_k := \big\{|\cdot| \leq k\big\}$, $k \in \mathbb{N}$, is a μ-nest of $\mathcal{T}$-compact sets, and condition (I) is proved. $\square$

By Theorem 1.1(b) we know that our process $(X_t)_{t\geq 0}$ is càdlàg in the weak topology P_μ-a.e. Since our Kolmogorov operator is a differential operator, the paths are moreover weakly continuous P_μ-a.e.

Proposition 5.6. *Consider the situation of Proposition 5.5. Then $(X_t)_{t\geq 0}$ is a weakly continuous process P_μ-a.e.*

Proof. By (the proof of) [7, Theorem 6.3] for all $\varphi \in \mathcal{E}_A(H)$ we have that there exists a constant $c(\varphi) > 0$ such that for all $t, s > 0$

$$\int \big|\varphi(X_t) - \varphi(X_s)\big|^4 \,\mathrm{d}P_\mu \leq c(\varphi)\,|t-s|^{3/2}.$$

(We emphasize that the condition that $L^{-1} \in L(H)$ is not necessary for this part of the proof of [7, Theorem 6.3]). By the Kolmogorov-Chentsov theorem it follows that there exists $\Omega_0 \in \mathcal{F}$ with $P_\mu(\Omega_0) = 1$ and $t \mapsto \varphi\big(X_t(\omega)\big)$ is Hölder continuous on the dyadics for all $\varphi \in \mathcal{M}$, where

$$\mathcal{M} := \Big\{\cos\big(k\langle h_n,\,\cdot\,\rangle\big),\ \sin\big(k\langle h_n,\,\cdot\,\rangle\big)\ \Big|\ k, n \in \mathbb{N}\Big\}$$

with $h_n \in D(A^*)$, $n \in \mathbb{N}$, forming a dense subset of H. Since we already know that $(X_t)_{t\geq 0}$ is cadlag P_μ-a.s., it follows that $X_t = X_{t-}$ for all $t > 0$ P_μ-a.e. $\square$

To keep the size of this paper within reasonable limits, we do not include explicit examples for the coefficients A, F, C here so that all the above applies, but rather refer to [7, Section 9].

A Appendix

The following result is a version of Proposition 1.4.13 from [3].

Lemma A.1. *Let $(W_\alpha)_{\alpha>0}$ be a family of kernels on $(E, \mathcal{B})$ such that $W_\alpha(bp\mathcal{B}) \subset qC(E)$ for all $\alpha > 0$. Assume that for $\beta > 0$ and $f, g \in bp\mathcal{B}$ the following properties hold μ-q.e.*

- *$W_\alpha f = W_\alpha g$ provided that $f = g$ μ-a.e.*
- *$W_\alpha f = W_\beta f + (\beta - \alpha) W_\alpha W_\beta f$, if $\alpha < \beta$, and $W_\alpha W_\beta f = W_\beta W_\alpha f$.*
- *$\alpha W_\alpha 1 \leq 1$.*

Then there exists a sub-Markovian resolvent of kernels $(\bar{W}_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that $W_\alpha f = \bar{W}_\alpha f$ μ-q.e. for all $f \in p\mathcal{B}$ and $\alpha > 0$.

Proof. We consider a countable subset $\mathcal{H}$ of $bp\mathcal{B}$ such that $\mathcal{H}-\mathcal{H}$ is a $\mathbb{Q}$ vector space and a lattice with respect to the pointwise order, $1 \in \mathcal{H}$, $\sigma(\mathcal{H}) = \mathcal{B}$ and $W_\alpha(\mathcal{H}) \subset \mathcal{H}$ for all $\alpha \in \mathbb{Q}_+^*$. By hypothesis there exist a μ-exceptional set $M_0 \in \mathcal{B}$ such that for all $\alpha, \beta \in \mathbb{Q}_+^*$ and $h \in \mathcal{H}$ we have on $E \setminus M_0$:

$$W_\alpha h = W_\beta h + (\beta - \alpha) W_\alpha W_\beta h \ ,$$

if $\alpha < \beta$, $W_\alpha W_\beta h = W_\beta W_\alpha h$, $\alpha W_\alpha 1 \le 1$.

Let now $M = \bigcup_{n \ge 0} M_n$, where M_n is defined inductively by

$$M_{n+1} = M_n \cup \bigcup_{\alpha \in \mathbb{Q}_+^*} \big[W_\alpha(1_{A_n}) > 0\big] \ .$$

We deduce that for all n the set M_n (and therfore M) is ξ-exceptional and $M_0 \cup \big[W_\alpha(1_M) > 0\big] \subset M$ for all $\alpha > 0$.

By construction it follows that $W_\alpha f = W_\alpha(f 1_{E\setminus M})$ on $E \setminus M$ for each $f \in p\mathcal{B}$ and $\alpha \in \mathbb{Q}_+^*$ and consequently for all $\alpha, \beta \in \mathbb{Q}_+^*$, $\alpha < \beta$ and $h \in \mathcal{H}$ we have on $E \setminus M$:

$$W_\alpha W_\beta h = W_\alpha\big(1_{E\setminus M} W_\beta(h 1_{E\setminus M})\big) = W_\beta\big(1_{E\setminus M} W_\alpha(h 1_{E\setminus M})\big) \ ,$$

$$W_\alpha(h 1_{E\setminus M}) = W_\beta(h 1_{E\setminus M}) + (\beta - \alpha) W_\alpha\big(1_{E\setminus M} W_\beta(h 1_{E\setminus M})\big) \ .$$

If $\alpha \in \mathbb{Q}_+^*$, then we define the kernel $\bar{W}_\alpha$ on $(E, \mathcal{B})$ by

$$\bar{W}_\alpha f = 1_{E\setminus M} W_\alpha(f 1_{E\setminus M}) \ , \quad f \in bp\mathcal{B}.$$

We have $\alpha \bar{W}_\alpha 1 \le 1$ and $\bar{W}_\alpha \bar{W}_\beta = \bar{W}_\beta \bar{W}_\alpha$, $\bar{W}_\alpha = \bar{W}_\beta + (\beta - \alpha)\bar{W}_\alpha \bar{W}_\beta$. If $\alpha \in \mathbb{R}_+^*$ then we set $\bar{W}_\alpha = \sup_{\mathbb{Q} \ni \beta > \alpha} \bar{W}_\beta$. In this way the family $(\bar{W}_\alpha)_{\alpha > 0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that $\bar{W}_\alpha f = W_\alpha f$ on $E \setminus M$ (hence μ-q.e.) for all $f \in p\mathcal{B}$ and $\alpha > 0$. □

B Appendix

Let E be a metrizable Lusin topological space and $\mathcal{B}$ the Borel σ-algebra on E.

A *transition function* on E is a family $(p_t)_{t \ge 0}$ of kernels on $(E, \mathcal{B})$ which are sub-Markovian (i.e. $p_t 1 \le 1$ for all $t \ge 0$), such that $p_0 f = f$ and $p_s(p_t f) = p_{s+t} f$ for all $s, t \ge 0$ and $f \in p\mathcal{B}$. We assume that for all $f \in p\mathcal{B}$ the function $(t, x) \mapsto p_t f(x)$ is $\mathcal{B}\big([0, \infty)\big) \otimes \mathcal{B}$-measurable. We denote by $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ the family of kernels on $(E, \mathcal{B})$ given by

$$U_\alpha f = \int_0^\infty e^{-\alpha t} p_t f \, \mathrm{d}t \ .$$

Consequently, $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ and it is called *associated with* $(p_t)_{t \ge 0}$.

A *right process with state space* E (associated with the transition function $(p_t)_{t \ge 0}$) is a collection $X = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \theta_t, P^x)$ where: $(\Omega, \mathcal{G})$ is a measurable space, $(\mathcal{G}_t)_{t \ge 0}$ is a family of sub σ-algebras of $\mathcal{G}$ such that $\mathcal{G}_s \subseteq \mathcal{G}_t$ if $s < t$; for all $t \ge 0$, $X_t : \Omega \to E_\Delta$ is a $\mathcal{G}_t/\mathcal{B}_\Delta$-measurable map such that $X_t(\omega) = \Delta$ for all $t > t_0$ if $X_{t_0}(\omega) = \Delta$, where Δ is a cemetery state adjoined to E as an isolated point of $E_\Delta := E \cup \{\Delta\}$ and $\mathcal{B}_\Delta$ is the Borel σ-algebra on E_Δ; we set $\zeta(\omega) := \inf\big\{t \bigm| X_t(\omega) = \Delta\big\}$; for each $t \ge 0$, the map $\theta_t : \Omega \to \Omega$ is such that $X_s \circ \theta_t = X_{s+t}$ for all $s > 0$; for all $x \in E_\Delta$, P^x is a probability measure on $(\Omega, \mathcal{G})$ such that $x \mapsto P^x(F)$ is universally $\mathcal{B}$-measurable for all $F \in \mathcal{G}$; $E^x(f \circ X_0) = f(x)$ and the following *Markov property* holds:

$$E^x(f \circ X_{s+t} \cdot G) = E^x(p_t^\Delta f \circ X_s \cdot G)$$

for all $x \in E_\Delta$, $s, t \geq 0$, $f \in p\mathcal{B}_\Delta$ and $G \in p\mathcal{G}_s$, where p_t^Δ is the Markovian kernel on $(E_\Delta, \mathcal{B}_\Delta)$ such that $p_t^\Delta 1 = 1$ and $p_t^\Delta|_E = p_t$; for all $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$; the filtration $(\mathcal{G}_t)_{t\geq 0}$ is *right continuous* (i.e. $\mathcal{G}_t = \mathcal{G}_{t+} := \bigcap\limits_{s>t} \mathcal{G}_s$) and *augmented* (i.e. $\mathcal{G}_t = \tilde{\mathcal{G}}_t := \bigcap_\mu \mathcal{G}_t^\mu$, where for every probability measure μ on $(E, \mathcal{B})$, $\mathcal{G}^\mu$ is the completion of $\mathcal{G}$ with respect to the probability measure $P^\mu := \int P^x \, \mu(\mathrm{d}x)$ on $(\Omega, \mathcal{G})$ and $\mathcal{G}_t^\mu$ is the completion of $\mathcal{G}_t$ in $\mathcal{G}^\mu$ with respect to P^μ); we assume that for all $\alpha > 0$, every function u which is α-excessive with respect to the resolvent associated with $(p_t)_{t\geq 0}$ and each probability measure μ on $(E, \mathcal{B})$, the function $t \mapsto u \circ X_t$ is right continuous on $[0, \infty)$ P^μ-a.s.

We consider the *natural filtration* associated with X: $\mathcal{F} := \tilde{\mathcal{F}}^0$, $\mathcal{F}_t := \tilde{\mathcal{F}}_t^0$, where $\mathcal{F}^0 := \sigma(X_s \mid s < \infty)$, $\mathcal{F}_t^0 := \sigma(X_s \mid s \leq t)$. It is known that always a right process may be considered with respect to its natural filtration:

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x) \, .$$

The sub-Markovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ associated with $(p_t)_{t\geq 0}$ is called the *resolvent of the process* X and for all $f \in p\mathcal{B}$, $\alpha > 0$ and $x \in E$ we have

$$U_\alpha f(x) = E^x \int_0^\zeta e^{-\alpha t} f \circ X_t \, \mathrm{d}t \, ,$$

with the convention $f(\Delta) = 0$.

A *stopping time* is a map $T : \Omega \to \bar{\mathbb{R}}_+$ such that the set $[T \leq t]$ belongs to $\mathcal{F}_t$ for all $t \geq 0$.

Let μ be a σ-finite measure on $(E, \mathcal{B})$. The right process X is called *μ-standard*, if it possesses left limits in E P^μ-a.e. on $(0, \zeta)$ and for every increasing sequence $(T_n)_n$ of stopping times, $T_n \nearrow T$, the sequence $(X_{T_n})_n$ converges to X_T P^μ-a.e. on $[T < \zeta]$.

Let $\mathcal{T}$ be a topology on E. The right process X is named *càdlàg in the topology $\mathcal{T}$ P^μ-a.e.* provided that P^μ-a.e. $t \mapsto X_t$ is right continuous and has left limits in E on $(0, \zeta)$.

For more details on right processes see e.g. [15] and [13].

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