SOME PARABOLIC PDEs WHOSE DRIFT IS AN IRREGULAR RANDOM NOISE IN SPACE

Francesco RUSSO [(1)] and Gerald TRUTNAU [(2)]

(1) Université Paris 13
Institut Galilée, Mathématiques
99, avenue J.B. Clément
F-93430 Villetaneuse

(2) Universität Bielefeld
Fakultät für Mathematik
Postfach 10 01 31
D-33501 Bielefeld

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Abstract: We consider a new class of random partial differential equation of parabolic type where the stochastic term is constituted by an irregular noisy drift, not necessarily Gaussian. We provide a suitable interpretation and we study existence. After freezing a realization of the drift (stochastic process), we study existence and uniqueness (in some suitable sense) of the associated parabolic equation and we investigate probabilistic interpretation.

Key words: Singular drifted PDEs, Dirichlet processes, martingale problem, stochastic partial differential equations, distributional drift.

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1 Introduction

This paper will focus on a random partial differential equations consisting on a parabolic PDE with irregular noise in the drift. That equation is motivated by *random irregular media models*. It is devoted to the formulation, the study of the existence (with somehow uniqueness) and its double probabilistic representation.

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\eta(x)$, a noise type generalized random field and u^0 be a real function. The equation is

$$\begin{cases} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\partial^2_{xx} u(t,x) + \dot{\eta}(x)\partial_x u(t,x) & = & 0, \\ u(T,x) & = & u^0(x) \end{cases} \tag{1.1}$$

The product has to be suitably interpreted since $\dot{\eta}$ will be the derivative in the sense of distributions of a continuous process. Among the examples of η one could imagine different possibilities of continuous processes as classical Wiener process, (multi)fractional Brownian motion but also non-Gaussian processes. The derivative in the sense of distributions $\dot{\eta}(x)$ will be the associated noise. (1.1) is a new type of SPDE, not yet studied even when η is a classical Brownian motion.

In the case when $\dot{\eta}(x)$ is replaced by a space-time noise $\dot{\eta}(t,x)$ is a white noise in time, or a space-time white noise, some relevant work was done by Nualart and Viens, see e.g. [18]. In that case, the dependence in time helps for the corresponding stochastic integration.

The idea of the paper is to freeze first the realization ω and to set $b(x) = \eta(x)(\omega)$ and we consider the deterministic Cauchy problem

$$\begin{cases} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\partial_{xx} u(t,x) + b'(x)\partial_x u(t,x) & = & 0, \\ u(T,x) & = & u^0(x) \end{cases} \tag{1.2}$$

The difficulty is that a product of a distribution and a continuous function is not defined in the theory of Schwarz distributions so one has to invent some substitution tools. Ideally, the parabolic PDE can be probabilistically represented through a diffusion process through a generalized Feynmann-Kac formula. That diffusion is morally the solution of a stochastic differential equation with generalized drift of the type

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt. \tag{1.3}$$

We will give a meaning to (1.3) at three different levels.

- The level of a suitable martingale problem.
- The level of a stochastic differential equation in the sense of distribution laws.
- The level of stochastic differential equation in the strong sense.

At each one of these levels we will provide conditions for equation (1.3), with suitable initial condition, to be well-posed. Later we define the notion of C_b^0-solution to the generalized parabolic PDE (1.2): for that we show existence, uniqueness and probabilistic representation.

In the last part of the paper, we show that, when η is a strong finite cubic variation process and $\sigma = 1$, then the solutions to (1.2) obtained for $b = \eta(\omega)$ provide solutions to the SPDE (1.1). A typical example of strong zero cubic variation process is fractional Brownian motion with Hurst index $H \geq \frac{1}{3}$. The equation will be understood in the following *weak distributional* sense that we can formally reconstruct as follows.

We integrate equation (1.1) from t to T in time and against a test function α smooth with compact support in space. We formally get the following:

$$\begin{aligned} -\int_{\mathbb{R}} dx\alpha(x)u(t,x) &= -\int_{\mathbb{R}} dx\alpha(x)u^0(x) - \int_t^T ds\frac{1}{2}\int_{\mathbb{R}} dx\alpha'(x)\partial_x u(s,x) \\ &+ \int_t^T ds \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\partial_x u(s,x) \end{aligned} \tag{1.4}$$

where previous integral with respect to $d^\circ\eta$ has to be interpreted since b is not of bounded variation.

We will show that the *probabilistic* solutions that we construct will in fact solve equation (1.4) where the stochastic integral element $d\eta$ will be a regularization symmetric (Stratonovich) type integral $d^0\eta$ in the sense of [22].

Diffusions in the generalized sense were studied by several authors starting, at least at our knowledge by [20]; later on many authors considered special cases of stochastic differential equations with generalized coefficients, it is difficult to quote all of them: in particular, we think to the case when b is a measure: [7], [17], [19]. In all those cases solutions were semimartingales. More recently [8] considered special cases of non-semimartingales solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes.

[10] and [11] treated well-posedness of the martingale problem, weak Itô formula, semimartingale characterization and Lyons-Zheng decomposition. The only assumption was the strict positivity of σ and the existence of the function $\Sigma(x) = 2\int_0^x \frac{b'}{\sigma^2} dy$ with suitable regularizations. Bass and Chen [2] were also interested in (1.3) and they provided a well stated framework when σ is $\frac{1}{2}$-Hölder continuous and b is γ-Hölder continuous, $\gamma > \frac{1}{2}$.

In the present paper we put truely emphasis on the formulation of a stochastic differential equations which can be solved adding some more assumptions on the coefficients. Several examples are provided for the case of weak and strong solutions.

The paper is organized as follows. Section 2 is devoted to basic preliminaries, including recalls about Young integrals, Section 3 is devoted to useful recalls in stochastic calculus via regularization. In Section 4, we introduce the formal *elliptic* operator L, we recall the notion of C^1-generalized solution of $Lf = \ell$ for continuous real functions ℓ, we introduce a basic technical assumption $\mathcal{A}(\nu_0)$ for L and we illustrate several examples where it is verified. In Section 5 we discuss different notions of martingale problems and Section 6 provides suitable notions of solution of *stochastic differential equation* with distributional drift and its connections with different types of martingale problem. Chapter 7 presents the notion of C_b^0-solution for a parabolic equation $\mathcal{L}u = \lambda$ where λ is bounded and continuous and $\mathcal{L} = \partial_t + L$. We also provide existence, uniqueness and probabilistic representation; chapter 8 discusses *mild* solutions for previous parabolic equations and useful integrability properties for the solutions. In Section 9, when $\sigma = 1$, finally one shows that the C_b^0 solutions are also in fact true *weak* solutions of SPDE

(1.1).

2 Preliminaries

In this paper T will be a fixed horizon time, unless something else is specified. A function f defined on $[0,T]$ (resp. $\mathbb{R}_+$) is extended without mention setting $f(0)$ for $t \leq 0$ and $f(T)$ for $t \geq T$ (resp. $f(0)$ for $t \leq 0$).

C^0 will indicate the set of continuous functions defined on $\mathbb{R}$. We denote by C_0^0 (resp. C_0^1) the space of continuous (continuous differentiable) functions vanishing at zero. We denote by $C_b^0([0,T] \times \mathbb{R})$ the space of real continuous bounded functions defined on $[0,T] \times \mathbb{R}$. $C_b^0(\mathbb{R})$ or simply C_b^0 indicates the continuous bounded functions defined on $\mathbb{R}$.

We will consider functions $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, which are continuous. A sequence (u_n) in C_b^0 will be said to converge in **a bounded way** to u if

- $\lim_{n\to\infty} u_n(t,x) = u(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R}$
- There is $c > 0$, independent of the sequence such that

$$\sup_{t \leq T, x \in \mathbb{R}} |u_n(t,x)| \leq c, \forall n \in \mathbb{N}. \tag{2.1}$$

If the sequence (u_n) does not depend on t we define similarly the convergence of $(u_n) \in C_b^0(\mathbb{R})$ to $u \in C_b^0(\mathbb{R})$ in a bounded way.

In the whole article if u', u'' will denote spaces derivatives $\partial_x u, \partial_{xx}^2 u$. The composition notation $u_1 \circ u_2$, means $(u_1 \circ u_2)(t,x) = u_1(t, u_2(t,x))$.

For positive integers m, k, $C^{m,k}$ will indicate functions in the corresponding class of differentiability. For instance, $C^{1,2}([0,T[\times\mathbb{R})$ will be the space of $(t,x) \mapsto u(t,x)$ functions which are C^1 on $[0,T[\times\mathbb{R}$ (i.e. one times continuously differentiable) and such that u'' exists and is continuous.

$C_b^{m,k}$ will indicate the set of functions $C^{m,k}$ such that the partial derivatives of all order are bounded.

The vector spaces $C^0(\mathbb{R})$ and $C^p(\mathbb{R})$ are topological Fréchet spaces or F-type space according to the terminology of [4] chapter 1.2. They are equipped with the following natural topology. A sequence f_n belonging to $C^0(\mathbb{R})$ (resp. $C^p(\mathbb{R})$) is said to converge to f in the $C^0(\mathbb{R})$ (resp. $C^p(\mathbb{R})$) sense if f_n (resp. f_n and all the derivatives up to order p) converges (resp. converge) to f (resp. to f and all its derivative) uniformly on each compact of $\mathbb{R}$. If I is a compact real interval and $\gamma \in]0,1[$, we denote by $C^\gamma(I)$ the vector space of real functions defined on I being Hölder with parameter γ. We denote by $C^\gamma(\mathbb{R})$, or simply C^γ, the space of locally Hölder functions, i.e. Hölder on each compact real interval.

Suppose $I = [\tau, T]$, τ, T being two real numbers such that $\tau < T$. Here T does not need to be necessarily positive. Recall that $f : I \mapsto \mathbb{R}$ belongs to $C^\gamma(I)$ if

$$N_\gamma(f) := \sup_{\tau \le s,t \le T} \frac{|f(t) - f(s)|}{|t-s|^\gamma} < \infty.$$

Clearly $f \mapsto |f(\tau)| + |N_\gamma(f)|$ defines a norm on $C^\gamma(I)$ which makes it a Banach space.

We make also some recalls about the so called **Young integrals**, see [29], remaining however in a simplified framework as in [9] or [25]. We recall the basic inequality, stated for instance in [9]. If $f \in C^1(I), g \in C^1(I)$, then

$$\Big| \int_a^b (f(x) - f(a)) dg(x) \Big| \le C_\rho (b-a)^{1+\rho} N_\gamma(f) N_\beta(g), \tag{2.2}$$

for any $[a,b] \subset I$ and $\rho \in]0, \gamma + \beta - 1[$, where C_ρ is a universal constant. The bilinear map sending (f,g) to $\int_0^\cdot f dg$ can continuously extended to $C^\gamma \times C^\beta$ with values in $C^0(I)$. By definition that object will be called the **Young integral** of f with respect to g. We also denote it $\int_0^\cdot f d^{(y)} g$. By additivity we set for $a, b \in [0,T]$,

$$\int_a^b f d^{(y)} g = \int_0^b f d^{(y)} g - \int_0^a f d^{(y)} g.$$

Remark 2.1 *Inequality (2.2) remains true for* $f \in C^\gamma(I), g \in C^\beta(I)$. *In particular* $t \mapsto \int_\tau^t f d^{(y)} g$ *belongs to* $C^\beta(I)$. *In fact*

$$\left| \int_a^b f dg \right| \le \left| \int_a^b (f - f(a)) dg \right| + |f(a)(g(b) - g(a))| .$$

Through the extension of the bilinear operator sending (f,g) to $\int_0^\cdot f dg$ it is possible to get the following chain rule for Young integrals.

Proposition 2.2 *Let $f, g, F : I \to \mathbb{R}, I = [\tau, T]$. We suppose $g \in C^\beta(I), f \in C^\gamma(I), F \in C^\delta(I)$ with $\gamma + \beta > 1, \quad \delta + \beta > 1$. We define $G(t) = \int_\tau^t f d^{(y)} g$. Then*

$$\int_\tau^t F d^{(y)} G = \int_\tau^t F f d^{(y)} g.$$

Proof. If $g \in C^1(I)$ then the result is obvious. We remark that $G \in C^\gamma(I)$. Using successively inequality (2.2) one can show that the two linear maps $g \mapsto \int_\tau^\cdot F d^{(y)} G$ and $g \mapsto \int_\tau^\cdot F f d^{(y)} g$ are continuous from $C^\delta(I)$ to $C^0(I)$. This concludes the proof of the Proposition. ■

By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ (i.e. a C^∞-function which tends faster to zero than any power of $|x|^{-1}$ as $|x| \to \infty$) with $\int \Phi(x)\, dx = 1$. We set $\Phi_n(x) := n\Phi(nx)$.

The result below shows that mollifications of a Hölder function f converge to f with respect to the Hölder topology.

Proposition 2.3 *Let $f \in C^{\gamma'}(I)$. We denote $f_n = \Phi_n * f$. Then $f_n \to f$ in the $C^\gamma(I)$ topology for any $\gamma < \gamma'$.*

Proof. We need to show that $N_\gamma(f - f_n)$ converges to zero. We set $\Delta_n(t) = (f - f_n)(t)$. Let $a, b \in I$. We will establish that

$$|\Delta_n(b) - \Delta_n(a)| \le \text{const}|b - a|^\gamma \left(\frac{1}{n}\right)^{\gamma' - \gamma}. \tag{2.3}$$

Without restriction of generality we can suppose $a < b$. We distinguish two cases.

Case $a < a + \frac{1}{n} < b$.

We have

$$\begin{aligned} |\Delta_n(b) - \Delta_n(a)| &\le \left| \int (f(b - \frac{y}{n}) - f(b))\Phi(y) dy \right| \\ &+ \left| \int (f(a - \frac{y}{n}) - f(a))\Phi(y) dy \right| \\ &\le 2 \int \left| \frac{y}{n} \right|^{\gamma'} |\Phi(y)| dy \\ &\le 2 \int |\Phi(y)||y|^{\gamma'} dy\, (b - a)^\gamma \left(\frac{1}{n}\right)^{\gamma' - \gamma} \end{aligned}$$

Case $a < b \le a + \frac{1}{n}$.

In this case we have

$$\begin{aligned} |\Delta_n(b) - \Delta_n(a)| &\le \int |f(b) - f(a)||\Phi(y)|dy + \int |f(b + \frac{y}{n}) - f(a + \frac{y}{n})||\Phi(y)|dy \\ &\le 2(b-a)^{\gamma'} \int |\Phi(y)|dy \le 2 \int |\Phi(y)|dy\, (b-a)^{\gamma} \left(\frac{1}{n}\right)^{\gamma' - \gamma} \end{aligned}$$

Therefore (2.3) is verified with $const = 2 \int |\Phi(y)|(1 + |y|^{\gamma'})dy$. This implies

$$N_\gamma(f - f_n) \le \text{const} \left(\frac{1}{n}\right)^{\gamma' - \gamma},$$

which allows to conclude. ■

For convenience we introduce the topological vector space defined by

$$D^\gamma = \bigcup_{\gamma' > \gamma} C^{\gamma'}(\mathbb{R}).$$

It is also a vector algebra. It is not a metric space but an inductive limit of the F-spaces C^γ. Consequently the weak version of Banach-Steinhaus theorem for F-spaces can be adapted.

A direct consequence of Banach-Steinhaus theorem of [4] section 2.1 is the following.

Theorem 2.4 *Let $E = \bigcup_n E_n$ be an inductive limit of F-spaces E_n and F another F-space. Let (T_n) be a sequence of linear continuous operator $T_n : E \to F$. Suppose that $Tf := \lim_{n\to\infty} T_n f$ exists for any $f \in E$. Then $T : E \to F$ is again a continuos (linear) operator.*

Next Corollary is a consequence of the definition of Young integral and Remark 2.1.

Corollary 2.5 *Let $f \in D^\gamma, g \in D^\beta$ with $\gamma + \beta \ge 1$. Then $t \mapsto \int_0^t f d^{(y)} g$ is well defined and it belongs to D^β.*

3 Recalls on stochastic calculus via regularization

We recall here a few notions related to stochastic calculus via regularization. The theory was started in [22] but we refer to some survey paper [25].

The considered stochastic processes may be defined on $[0,T], \mathbb{R}_+$ or $\mathbb{R}$. Let $X = (X_t, t \in \mathbb{R})$ be a continuous process and $Y = (Y_t, t \in \mathbb{R})$ be a process with paths in L^1_{loc}. For the paths process Y with parameters on $[0,T]$ (resp. $\mathbb{R}_+$) we apply the same convention of the beginning of previous section for functions. So we extend without other mention setting Y_0 for $t \le 0$ and Y_T for $t \ge T$ (resp. Y_0 for $t \le 0$). $\mathcal{C}$ will denote the vector algebra of continuous processes. It is an F-space if equipped with the topology of the ucp (uniform convergence in probability) convergence.

We recall in the sequel the most useful rules of calculus, see for instance [25] or [24].

The forward integral and the covariation process are defined by the following limits in the ucp sense whenever they exist

$$\int_0^t Y \, d^- X := \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \, ds \tag{3.4}$$

$$\int_0^t Y_s \, d^\circ X_s := \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{s-\varepsilon}}{2\varepsilon} \, ds \tag{3.5}$$

$$[X, Y]_t := \lim_{\varepsilon \to 0+} C^\varepsilon(X, Y)_t, \tag{3.6}$$

where

$$C^\varepsilon(X, Y)_t := \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) \, ds.$$

All stochastic integrals and covariation processes will be of course elements of $\mathcal{C}$. If $[X, Y]$, $[X, X]$, $[Y, Y]$ exist we say that (X, Y) **has all its mutual covariations**.

Remark 3.1 *If X is (locally) of bounded variation, we have*

- $\int_0^t X \, d^- Y = \int_0^t X_s \, d^\circ Y_s = \int_0^t X_s \, dY_s$ *where the third integral is meant in the Lebesgue-Stieltjes sense.*

- $[X, Y] \equiv 0$.

Remark 3.2 *a)* $\int_0^t Y_s\, d^\circ X_s = \int_0^t Y_s\, d^\circ X_s + \frac{1}{2}[X,Y]$ *provided that two among the three integrals or covariations exist.*

b) $X_tY_t = X_0Y_0 + \int_0^t Y_s\, d^- X_s + \int_0^t X_s\, d^- Y_s + [X,Y]_t$ *provided that two of the three integrals or covariations exist.*

c) $X_tY_t = X_0Y_0 + \int_0^t Y\, d^\circ X + \int_0^t X_s\, d^\circ Y_s$ *provided that one of the two integrals exists.*

Remark 3.3 *a) If* $[X,X]$ *exists then it is always an increasing process and* X *is called a* **finite quadratic variation process***. If* $[X,X]=0$ *then* X *is said to be a* **zero quadratic variation process***.*

b) Let X*,* Y *be continuous processes such that* (X,Y) *has all its mutual covariations. Then* $[X,Y]$ *has bounded (total) variation. If* $f,g \in C^1$ *then*

$$[f(X), g(Y)]_t = \int_0^t f'(X)g'(Y)\, d[X,Y].$$

c) If A *is a zero quadratic variation process and* X *is a finite quadratic variation process then* $[X,A] \equiv 0$*.*

d) A bounded variation process is a zero quadratic variation process.

*e) (***Classical Itô formula***) Let* Y *be a cadlag process. If* $f \in C^2$ *then* $\int_0^\cdot f'(X)\, d^- X$ *exists and is equal to*

$$f(X) - f(X_0) - \frac{1}{2}\int_0^\cdot f''(X)\, d[X,X].$$

f) If $g \in C^1$ *and* $f \in C^2$ *then the forward integral* $\int_0^\cdot g(X) d^- f(X)$ *is well defined.*

In this paper all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale, Y is $\mathbb{F}$-adapted cadlag process, then $\int_0^\cdot Y\, d^- X$ is the usual Itô integral. If Y is $\mathbb{F}$-semimartingale then $\int_0^\cdot Y\, d^0 X$ is the classical Fisk-Stratonovich integral and $[X,Y]$ the usual covariation process $\langle X, Y\rangle$.

We introduce now the notion of Dirichlet process which were introduced by H. Föllmer [13] and considered by many authors, see for instance [3, 26] for classical properties.

In the present section, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion.

Definition 3.4 *Let X be an $(\mathcal{F}_t)$-adapted (continuous) process X . An $(\mathcal{F}_t)$-**Dirichlet process** is the sum of an $(\mathcal{F}_t)$- local martingale M plus a zero quadratic variation process A. For simplicity we will suppose $A_0 = 0$ p.s.*

Remark 3.5 *(i) Process (A_t) in previous decomposition is an $(\mathcal{F}_t)$-adapted process.*

(ii) An $(\mathcal{F}_t)$-semimartingale is an $(\mathcal{F}_t)$-Dirichlet process.

(iii) The decomposition $M + A$ is unique.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ of class C^1, X an $(\mathcal{F}_t)$-Dirichlet process. Then $f(X)$ is again a $(\mathcal{F}_t)$-Dirichlet process with local martingale part $M_t^f = f(X_0) + \int_0^t f'(X)dM$.

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations. Remark 3.3 b) says that finite quadratic variation processes are stable through C^1 transformations. The last point of previous remark states that C^1 stability also holds for Dirichlet processes.

Young integrals introduced in the Preliminaries can be connected with the integral via regularization. The next proposition has been proved in [25].

Proposition 3.6 *Let I be a compact real interval. Let X, Y be processes whose paths are respectively in $C^\gamma(I)$ and in $C^\beta(I)$, with $\gamma > 0, \beta > 0$ and $\alpha + \beta > 1$.*

For any symbol $\star \in \{+, -, \circ\}$ the integral $\int_0^\cdot X d^\star Y$ coincides with the Young integral $\int_0^\cdot X d^{(y)} Y$.

Remark 3.7 *Suppose that X and Y verifies the conditions of Proposition 3.6, then Remark 3.1 implies that $[X,Y]=0$. Note that it is not supposed that either $[X,X]$ or $[Y,Y]$ exists.*

We need an extension of stochastic calculus via regularization in the direction of higher $n-$variation. The properties concerning higher variation than 2 can be found for instance in [5].

We set

$$[X,X,X]_t^{\epsilon} = \frac{1}{\varepsilon}\int_0^t (X_{s+\epsilon}-X_s)^3\, ds.$$

We define also

$$\left\|[X,X,X]^{\epsilon}\right\|_t = \frac{1}{\varepsilon}\int_0^t |X_{s+\epsilon}-X_s|^3\, ds.$$

If the limit when $\varepsilon \to 0$ in probability of $[X,X,X]_t^{\epsilon}$ exists for any t, we denote it by $[X,X,X]_t$. If the limiting process $[X,X,X]$ has a continuous version, we say that X is a **finite cubic variation process**.

If moreover, there is a positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ coverging to zero such that

$$\sup_{\varepsilon_n} \left\|[X,X,X]^{\varepsilon_n}\right\|_T < +\infty, \tag{3.7}$$

we will say that it X is a **strong finite cubic variation process**. If X is a (strong) finite cubic variation process such that $[X,X,X]=0$, X will be said (strong) zero **finite cubic variation process**.

For instance, if $X = B^H$, a fractional Brownian motion with Hurst index then X is a finite quadratic variation process if and only if $H \geq \frac{1}{2}$, see [24]. It is a strong zero cubic variation process if and only if $H \geq \frac{1}{3}$, see [5]. On the other hand B^H is a zero cubic variation process if and only if $H > \frac{1}{6}$, see [14].

It is clear that a finite quadratic variation process is a strong zero cubic variation process. On the other processes whose paths are Hölder continuous with parameter greater than $\frac{1}{3}$ are strong zero cubic variation processes.

As for finite quadratic variation and Dirichlet processes, the C^1-stability also holds for finite cubic variation processes. The next Proposition is a particular case of a result contained in [5].

Proposition 3.8 *Let X be a strong finite cubic variation process, V a bounded variation process and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1. Then $Z = f(V, X)$ is again a strong finite cubic variation process and*

$$[Z, Z, Z]_t = \int_0^t \partial_x f(V_s, X_s) d[X, X, X]_s.$$

Moreover a Itô chain rule property holds as follows.

Proposition 3.9 *Let X be a strong finite cubic variation process, V a bounded variation process and a cadlag process Y. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class $C^{1,3}$.*

$$\begin{aligned} \int_0^t Y d^0 f(V, X) &= \int_0^t Y \partial_v f(V_s, X_s) dV_s \\ &+ \int_0^t Y \partial_x f(V_s, X_s) d^0 X_s \\ &- \frac{1}{12} \int_0^t Y \partial^3_{xxx} f(V_s, X_s) d[X, X, X]_s \end{aligned}$$

We deduce in particular that a C^1 transformation of a strong zero cubic variation process is again a strong zero cubic variation process.

4 About the PDE operator L

Let σ, $b \in C^0(\mathbb{R})$ such that $\sigma > 0$. Without restriction of generality we will suppose $b(0) = 0$.

We consider formally a PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2} g'' + b' g'. \tag{4.1}$$

If b is of class C^1, so that b' is continuous, we will say that L is a **classical** PDE operator.

For a given mollifier Φ we denote

$$\sigma_n^2 := (\sigma^2 \wedge n) * \Phi_n \quad b_n := (-n \wedge (b \vee n)) * \Phi_n.$$

We then consider

$$\begin{aligned} L_n g &= \frac{\sigma_n^2}{2} g'' + b_n' g', \text{for } g \in C^2(\mathbb{R}) \\ \mathcal{L}_n u &= \partial_t u + L_n u, \text{for } u \in C^{1,2}([0,T[\times\mathbb{R}) \end{aligned} \tag{4.2}$$

where L_n acts on x. A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

Previous definitions are slightly different than in the papers [10, 11] but most of the result about the analysis of L and the study of the martingale problem can be adapted. In those papers there was only regularization but no troncation; here troncation is used to study associated parabolic equations.

Definition 4.1 *A function $f \in C^1(\mathbb{R})$ is said to be a C^1-generalized solution to*

$$Lf = \dot{\ell}, \tag{4.3}$$

where $\dot{\ell} \in C^0$, if, for any mollifier Φ, there are sequences (f_n) in C^2, $(\dot{\ell}_n)$ in C^0 such that

$$L_n f_n = \dot{\ell}_n, \quad f_n \to f \text{ in } C^1, \quad \dot{\ell}_n \to \dot{\ell} \text{ in } C^0. \tag{4.4}$$

Proposition 4.2 *There is a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy$$

exists in C^0, independently from the mollifier. Moreover, in this case, any solution f to $Lf = 0$ fulfills

$$f'(x) = e^{-\Sigma(x)} f'(0). \tag{4.5}$$

Proof. This result follows very similarly to the proof of Proposition 2.3 of [10]: first at the level of regularization and then passing to the limit.

■

From now on, throughout the whole paper, we will suppose the existence of this function Σ. We will set

$$h'(x) := \exp(-\Sigma(x)), h(0) = 0. \tag{4.6}$$

Thus, $h'(0) = 1$ holds. Even though in [10] we discuss the general case with related non-explosion conditions, here in order to ensure conservativeness we suppose that

$$\begin{aligned} \int_{-\infty}^{0} e^{-\Sigma(x)} dx &= \int_{0}^{\infty} e^{-\Sigma(x)} dx = +\infty \\ \int_{-\infty}^{0} \frac{e^{\Sigma(x)}}{\sigma^2} dx &= \int_{0}^{\infty} \frac{e^{\Sigma(x)}}{\sigma^2} dx = +\infty. \end{aligned} \tag{4.7}$$

Previous assumptions are of course verified if σ is lower bounded by a positive constant and b is constant outside a compact interval.

The condition (4.7) implies that the image set of h is $\mathbb{R}$.

Remark 4.3 *Proposition 4.2 implies the uniqueness of the problem*

$$Lf = \dot{\ell}, \quad f \in C^1, \quad f(0) = x_0, \quad f'(0) = x_1 \tag{4.8}$$

for every $\dot{\ell} \in C^0$, $x_0, x_1 \in \mathbb{R}$.

Remark 4.4 *We present four basic examples where* Σ *exists.*

a) If $b(x) = \alpha \left(\frac{\sigma^2(x)}{2} - \frac{\sigma^2(0)}{2} \right)$ *for some* $\alpha \in]0,1]$ *then*

$$\Sigma(x) = \alpha \log \left(\frac{\sigma^2(x)}{\sigma^2(0)} \right)$$

and

$$h'(x) = \frac{\sigma^{2\alpha}(0)}{\sigma^{2\alpha}(x)}.$$

If $\alpha = 1$ *the operator* L *can be formally be expressed in divergence form as* $Lf = (\frac{\sigma^2}{2} f')'$.

b) Suppose that b *is of bounded variation. Then we get*

$$\int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2},$$

since $db_n \to db$ *weakly-*$*$ *and* $\frac{1}{\sigma^2}$ *is continuous.*

c) If σ has bounded variation then we have

$$\Sigma(x) = -2\int_0^x b\, d(\frac{1}{\sigma^2}) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the case where $\sigma = 1$ for any b.

d) Suppose that σ is locally Hölder continuous with parameter γ and b is locally Hölder continuous with parameter β so that $\beta + \gamma > 1$. Since σ is locally bounded, than σ^2 is also locally Hölder continuous with parameter γ. Proposition 2.3 implies that $\sigma_n^2 \to \sigma^2$ locally in $C^{\gamma'}$ and $b_n \to b$ locally in $C^{\beta'}$ for every $\gamma' < \gamma$ and $\beta' < \beta$. Since σ is strictly positive on each compact, $\frac{1}{\sigma_n^2} \to \frac{1}{\sigma^2}$ locally in $C^{\gamma'}$. By Remark 2.1, Σ is well defined and it is locally Hölder continuous with parameter β'.

Again the following lemma can be proved at the level of regularizations, see also Lemma 2.6 in [10].

Lemma 4.5 *The unique solution to problem (4.8) is given by*

$$\begin{aligned} f(0) &= x_0, \\ f'(x) &= h'(x)\left(2\int_0^x \frac{\dot{\ell}(y)}{(\sigma^2 h')(y)}\, dy + x_1\right). \end{aligned}$$

Remark 4.6 *If $b' \in C^0(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ is a classical solution to $Lf = \dot{\ell}$ then f is immediately seen also to be a C^1-generalized solution.*

Remark 4.7 *Given $\ell \in C^1$, we denote by $T\ell$ the function $f \in C^1$ verifying previous definition with $x_0 = 0, x_1 = 0$. The unique solution to problem (4.8) is given by*

$$f = x_0 + x_1 h + T\ell.$$

We denote $T^{x_1}\ell = T\ell + x_1 h$, i.e. with $x_0 = 0$.

Remark 4.8 *Let $f \in C^1$. There is at most one $\dot{\ell} \in C^0$ such that $Lf = \dot{\ell}$. In fact, to see this, it is enough to suppose that $f = 0$. Lemma 4.5 implies that*

$$2\int_0^x \frac{\dot{\ell}}{\sigma^2 h'}(y) dy \equiv 0$$

consequently $\dot{\ell}$ is forced to be zero.

This consideration allows to define without ambiguity $L : \mathcal{D}_L \to C^0$. We will denote by $\mathcal{D}_L$ the set of all $f \in C^1(\mathbb{R})$ such that there exists some $\dot{\ell} \in C^0$ with $Lf = \dot{\ell}$ in the C^1-generalized sense. In particular $T\ell \in \mathcal{D}_L$.

A direct consequence of Lemma 4.5 is the following useful result.

Lemma 4.9 *$\mathcal{D}_L$ is the set of $f \in C^1$ such there is $\psi \in C^1$ with $f' = e^{-\Sigma}\psi$.*

In particular it gives us the following density proposition.

Proposition 4.10 *$\mathcal{D}_L$ is dense in C^1.*

Proof. It is enough to show that every C^2-function is the C^1-limit of a sequence of functions in $\mathcal{D}_L$. Let (ψ_n) be a sequence in C^1 converging to $f'e^{\Sigma}$ in C^0. It follows that

$$f_n(x) := f(0) + \int_0^x e^{-\Sigma}(y)\psi_n(y)dy, \quad x \in \mathbb{R}$$

converges to $f \in C^1$ and $f_n \in \mathcal{D}_L$. □

We need now to discuss technical aspects of the way L and its domain $\mathcal{D}_L$ are transformed by h. We recall that $Lh = 0$ and h' is strictly positive. The condition (4.7) implies that the image set of h is $\mathbb{R}$.

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\tilde{\sigma}_h^2}{2}\phi'', \quad \phi \in C^2, \tag{4.9}$$

where

$$\tilde{\sigma}_h(y) = (\tilde{\sigma}h')(h^{-1}(y)), y \in \mathbb{R}.$$

L^0 is a classical PDE map; however we can also consider it at the formal level and introduce $\mathcal{D}_{L^0}$.

Proposition 4.11 *a) $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2\sigma^2$,*

b) $\mathcal{D}_{L^0} = C^2$,

c) $\phi \in \mathcal{D}_{L^0}$ holds if and only if $\phi \circ h \in \mathcal{D}_L$. Moreover, we have

$$L(\phi \circ h) = (L^0\phi) \circ h \tag{4.10}$$

for every $\phi \in C^2$.

We try now to proceed in the interpretation of operator L in an alternative way. Formally we will proceed defining $\hat{L}f(x) = \int_0^x Lf(y)dy$. In our context, it could be defined on $\mathcal{D}_L$, just as taking the primitive of Lf vanishing at zero. But we will proceed first differently and defining an operator on C^2. Suppose first that b' is continuous. Then, integrating by parts we would get

$$\int_0^x Lf(y)dy = \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + (bf')(x) - (bf')(0). \tag{4.11}$$

We remark that the right member of previous expression makes sense for any $f \in C^2$ and continuous b. We will define $\hat{L} : C^2 \to C_0^0$ as

$$\hat{L}f := \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + (bf')(x). \tag{4.12}$$

One may ask if in the general case, the two definitions above are somehow compatible. Now, in general this will not be the case since $\mathcal{D}_L \bigcap C^2$ may be empty. We will see later that, under Assumption $\mathcal{A}(\nu_0)$ this will be however possible.

So far, we have learnt how to eliminate the first order term in a PDE operator through a transformation which is called of Zvonkin type (see [30]). Now we would like to introduce a transformation which puts the PDE operator in a divergence form.

Let L be a PDE operator which is formally of type (4.1)

$$Lg = \frac{\sigma^2}{2} g'' + b'g'.$$

We consider a function of class C^1, namely $k : \mathbb{R} \to \mathbb{R}$ such that

$$k(0) = 0 \quad \text{and} \quad k'(x) = \sigma^{-2}(x) \exp(\Sigma(x)). \tag{4.13}$$

According to assumptions (4.7) k is bijective on $\mathbb{R}$.

Remark 4.12 *If there is no drift term, i.e.* $b = 0$*, then we have* $k'(x) = \sigma^{-2}(x)$*.*

Lemma 4.13 *We consider the formal PDE operator given by*

$$L^1 g = \frac{\bar{\sigma}_k^2}{2} g'' + \left(\frac{\bar{\sigma}_k^2}{2}\right)' g' = (\frac{\bar{\sigma}_k^2}{2} g')' \tag{4.14}$$

where

$$\bar{\sigma}_k(z) = (\sigma k') \circ k^{-1}(z), \quad z \in \mathbb{R}$$

Then

(i) $g \in \mathcal{D}_{L^1}$ *if and only if* $g \circ k \in \mathcal{D}_L$,

(ii) for every $g \in \mathcal{D}_{L^1}$ *we have* $L^1 g = L(g \circ k) \circ k^{-1}$.

Proof. It is practically the same as in Lemma 2.16 of [10]. ■

We give now a lemma whose proof can be easily established by investigation. Suppose that L is a classical PDE operator. Then $\mathcal{L} = \partial_t + L$ is well defined for $C^{1,2}(]0,T[\times\mathbb{R})$ functions where L acts on the second variable. Given a function $\varphi \in C([0,T] \times \mathbb{R})$ from now on we will set $\tilde{\varphi} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\tilde{\varphi}(t,y) = \varphi(t, h^{-1}(y))$.

Lemma 4.14 *Let us suppose* $h \in C^2(\mathbb{R})$. *We set* $\sigma_h = \sigma h'$.
We define the PDE operator $\mathcal{L}^0$ *by* $\mathcal{L}^0 \varphi = \partial_t \varphi + L^0 \varphi$ *where*

$$L^0 f = \frac{\tilde{\sigma}_h^2}{2} f''$$

is again a classical operator.
If $f \in C^{1,2}([0,T[\times\mathbb{R})$ *and* $\mathcal{L} f = \gamma$ *in the classical sense then* $\mathcal{L}^0 \tilde{f} = \tilde{\gamma}$.

We will now formulate a supplementary assumption which will be useful when we will study singular stochastic differential equations in the proper sense and not only under the form of martingale problem.

Technical Assumption $\mathcal{A}(\nu_0)$.

Let ν_0 be a topological F-space which is a linear topological subspace of $C^0(\mathbb{R})$ (or eventually an inductive limit of sub F-spaces). The ν_0-convergence implies convergence in C^0 and therefore pointwise convergence.

We say that L fulfills that Assumption $\mathcal{A}(\nu_0)$ if

(i) $C^1 \subset \nu_0$ which is dense.

(ii) For every $g \in C^1(\mathbb{R})$, the following multiplicative operator $\phi \to g\phi$ maps ν_0 into itself.

(iii) Let $T : C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ as defined in lemma 4.5, i.e. $f = T\ell$ is such that

$$\begin{cases} f(0) &= 0, \\ f'(x) &= e^{-\Sigma}(x) \left(2 \int_0^x \frac{e^{\Sigma}(y)\ell'(y)}{\sigma^2(y)} \, dy \right). \end{cases}$$

We recall that $f = T\ell$ solves the problem $Lf = \ell'$ with $f(0) = f'(0) = 0$. We suppose that T admits a continuous extension to ν_0.

(iv) Let $x_1 \in \mathbb{R}$. For every $f \in C^2, f(0) = 0, f'(0) = x_1$, so that $\hat{L}f = \ell$ we have $\ell \in \nu_0$ and $T^{x_1}\ell = f$, where T^{x_1} denotes the continuous extension of T^{x_1} (see Remark 4.7) to ν_0 which exists by (iii).

(v) The set $\hat{L}C^2$ is dense into $\{\ell \in \nu_0 | \ell(0) = 0\}$.

Remark 4.15 *Let $x_1 \in \mathbb{R}$.*

(i) Remark 4.7 and point (iii) imply that $T^{x_1} : C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ prolongates continuously to ν_0. Moreover

$$\{f \in C^2 | f(0) = 0, f'(0) = x_1\} \subset Im T^{x_1}.$$

(ii) Point (iv) of previous Technical Assumption shows that $b \in \nu_0$ and $T^1 b = id$ where $id(x) = x$: in fact $id(0) = 0, id'(1) = 1$ and (4.12) implies that $\hat{L}id = b$.

(iii) Point (i) is verified for instance if the map T is closable, as a map from C^0 to C^1. In that case ν_0 may be defined as the domain of the closure of C^1 equipped with the graph topology related to $C^0 \times C^1$.

Below we give some sufficient conditions for the verification of points (iv) and (v) of the Technical Assumption.

We define by $C^1_{\nu_0}$ the vector space of functions $f \in C^1$ such that $f' \in \nu_0$; it will be an F-space if equipped with the following topology. A sequence (f_n)

will be said to converge to f in $C^1_{\nu_0}$ if $f_n(0) \to f(0)$ and (f_n') converges to f' in ν_0. In particular a sequence converging according to $C^1_{\nu_0}$, also converges with respect to C^1. On the other hand $C^2 \subset C^1_{\nu_0}$ and a a sequence converging in C^2, also converges with respect to $C^1_{\nu_0}$. Moreover C^2 is dense $C^1_{\nu_0}$ because C^1 is dense in ν_0.

Lemma 4.16 *Suppose that points (i) to (iii) of previous Technical Assumption are fulfilled. We suppose moreover:*

a) $h \in C^1_{\nu_0}$.

b) For every $f \in C^2, f(0) = 0, f'(0) = 0, \quad \hat{L}f = \ell$ *we have* $\ell \in \nu_0$ *and* $T\ell = f$.

c) $\hat{L} : C^2 \to \nu_0$ *is well-defined and it has a continuous extension to* $C^1_{\nu_0}$, *still denoted by* $\hat{L}$. *Moreover* $\hat{L}h = 0$.

d) $ImT \subset C^1_{\nu_0}$.

e) $\hat{L}T$ *is the identity map on* $\{\ell \in \nu_0 | \ell(0) = 0\}$.

Then T, T^{x_1} *for every* $x_1 \in \mathbb{R}$ *are injective and points (iv), (v), of the technical assumption are verified.*

Proof. The injectivity of T follows from point e). The injectivity of T^{x_1} is a consequence of Remark 4.7.

We prove point (iv). Point c) says that $\hat{L}h = 0$. We set $\hat{f} = f - x_1 h$, $f \in C^2$, where $f(0) = 0$, $f'(0) = x_1$. Clearly $\hat{L}\hat{f} = \hat{L}f = \ell$ and $\hat{f}(0) = 0, \hat{f}'(0) = 0$. Point b) implies that $T\ell = \hat{f}$. Hence $T^{x_1}\ell = T\ell + x_1 h = f$, and (iv) is satisfied.

Concerning point (v), let $\ell \in \nu_0$ with $\ell(0) = 0$ and set $f = T\ell$. Since f belongs to $C^1_{\nu_0}$ by c), f' belongs to ν_0. Point (i) of the Technical Assumption implies that there is a sequence (f_n') of C^1 functions converging to f' in the ν_0 sense, and thus also in C^0. Let (f_n) be the sequence of primitives of (f_n') (which are of class C^2) such that $f_n(0) = 0$. In particular we have that (f_n) converges to f in the $C^1_{\nu_0}$-sense. By c) there exists λ in ν_0 being limit of $\hat{L}f_n$

in the ν_0-sense. Observe that, because of b), $T(\hat{L} f_n) = f_n$. On the other hand $\lim_{n\to+\infty} f_n = f$ in C^1. Applying T and using iii) of the Technical Assumption, we obtain

$$T\lambda = \lim_{n\to+\infty} T(\hat{L} f_n) = \lim_{n\to+\infty} f_n = f = T\ell.$$

The injectivity of T allows to conclude that $\ell = \lambda$.

■

Remark 4.17 *Under the assumptions of Lemma 4.16, we have*

- $\mathcal{D}_L \subset C^1_{\nu_0}$*;*
- $\hat{L} f = \int_0^x L f(y) dy$*.*

In fact, let $f \in \mathcal{D}_L$. Without restriction of generality we can suppose $f(0) = 0$. Let $x_1 = f'(0)$ and we set $\hat{f} = f + x_1 h$ so that $\hat{f}(0) = \hat{f}'(0) = 0$. Setting $\dot{\ell} = L\hat{f}$, Lemma 4.5 implies that $\hat{f} = T\ell$ where $\ell = \int_0^x \dot{\ell}(y) dy$. So $\hat{f} \in ImT \subset C^1_{\nu_0}$. Since $h \in C^1_{\nu_0}$, $f \in C^1_{\nu_0}$, by additivity.

On the other hand,

$$\begin{aligned} Lf &= L\hat{f} + x_1 Lh = \hat{L} f = \dot{\ell}, \\ \hat{L} f &= \hat{L}\hat{f} + x_1 \hat{L} h = \hat{L} T\ell = \ell, \end{aligned}$$

because of point e) of Lemma 4.16.

Example 4.18 *We provide here a series of four significant examples when the Technical Assumption $\mathcal{A}(\nu_0)$ is verified. We only comment the points which are not easy to verify.*

(i) First example is simple. It concerns the case when the drift b' is continous. Our future studied object will correspond an ordinary SDE. In this case one has

$$\nu_0 = C^1, C^1_{\nu_0} = C^2, \hat{L} f = \int_0^{\cdot} L f(y) dy.$$

(ii) L is **close to divergence type**, *i.e.* $b = \frac{\sigma^2 - \sigma^2(0)}{2} + \beta$ *and* β *is bounded variation vanishing at zero. The operator is of divergence type plus a Radon measure term: we have* $\Sigma = \ln \sigma^2 + 2\int_0^x \frac{d\beta}{\sigma^2}$. *In this case we have* $\nu_0 = C^0$. *Points (i) and (ii) of the Technical Assumption are trivial.*

We have in fact

$$h'(x) = e^{-\Sigma} = \frac{1}{\sigma^2(x)} \exp\Big(- 2 \int_0^x \frac{d\beta}{\sigma^2} \Big).$$

T defined at point (iii) of the Technical Assumption writes $T\ell = f$ *where* $f(0) = 0$ *and*

$$f'(x) = \frac{2\sigma^2(0)}{\sigma^2(x)} \exp\Big(- 2 \int_0^x \frac{d\beta}{\sigma^2} \Big) \int_0^x \ell'(y) \exp\Big(2 \int_0^y \frac{d\beta}{\sigma^2} \Big)\, dy. \quad (4.15)$$

Consequently the prolongation of T to $\nu_0 = C^0$, *always denoted by this letter, is given by* $f = T\ell$ *with* $f(0) = 0$ *and*

$$\begin{aligned} f'(x) &= \frac{2}{\sigma^2(x)} \Big\{ \ell(x) - 2 \exp\Big(- 2 \int_0^x \frac{d\beta}{\sigma^2} \Big) \cdot \Big(\ell(0) \\ &+ \int_0^x \ell(y) \exp\Big(2 \int_0^y \frac{d\beta}{\sigma^2} \Big) \frac{1}{\sigma^2(y)}\, d\beta(y) \Big) \Big\}. \end{aligned} \quad (4.16)$$

We verify points (iv) and (v) through lemma 4.16. We have $C^1_{\nu_0} = C^1$. *Pont a) is obvious since* $h' \in C^0$ *and so* $h \in C^1_{\nu_0}$. *Let* $f \in C^2$. *Using Lebesgue-Stieltjes calculus, we can easily show that*

$$\ell(x) = \hat{L}f(x) = \frac{\sigma^2(x)}{2} f'(x) - \frac{\sigma^2(0)}{2} f'(0) + \int_0^x f' d\beta. \quad (4.17)$$

This shows that $\ell \in C^0 = \nu_0$ *and therefore the first part of b). We remark that we can in fact consider* $\hat{L} : C^2 \to \nu_0$ *because*

$$\hat{L}f = \hat{L}(f - x_1 h) + x_1 \hat{L}h = \hat{L}(f - x_1 h) \in \nu_0.$$

The expression of $\hat{L}f$ *prolongates continuously to* $f \in C^1$, *which yields the first part of point c). Moreover inserting the expression of* h' *into* f' *in (4.17), one shows* $\hat{L}h = 0$.

Suppose now in expression (4.17) $f \in C^2, f(0) = 0, f'(0) = 0$. *A simple investigation allows to show that* $T\ell = f$, *so the second part of* *point b) is fulfilled; point d) is also clear because of (4.16). Finally point d) holds because one can prove by inspection that* $\hat{L}T$ *is the identity on* C_0^0.

(iii) *We recall the notation* $D^\gamma(\mathbb{R})$ *which indicates the topological vector space of locally Hölder continuous functions defined on* $\mathbb{R}$ *with parameter* $\alpha > \gamma$. *We recall that* $D^\gamma(\mathbb{R})$ *is a vector algebra.*

Suppose $\sigma \in D^{\frac{1}{2}}$ *and* $b \in C^{\frac{1}{2}}$ *(or* $\sigma \in C^{\frac{1}{2}}$ *and* $b \in D^{\frac{1}{2}}$*). Remark 4.4 d) implies that* Σ *also belongs to* $D^{\frac{1}{2}}$. *We set* $\nu_0 = D^{\frac{1}{2}}$.

Technical Assumption $\mathcal{A}(\nu_0)$ *is verified for the following reasons.*

Since $\Sigma \in D^{\frac{1}{2}}$, $\quad h' = e^{-\Sigma}$ *belongs to the same space.*

Point (i) follows because of Proposition 2.3, point (ii) because $D^{\frac{1}{2}}$ *is an algebra. Corollary 2.5 yields that, for every* $\ell \in D^{\frac{1}{2}}$, *the function*

$$f'(x) = e^{-\Sigma(x)} \int_0^x 2\frac{e^\Sigma}{\sigma^2}(y) d^{(y)} \ell(y), \tag{4.18}$$

is well-defined and it belongs to $D^{\frac{1}{2}}$. *This shows that* T *can be continuously extended to* ν_0 *and point (iii) is established.*

Concerning points (iv) and (v), we use again Lemma 4.16. We observe that

$$C^1_{\nu_0} = \{f \in C^1 | f' \in D^{\frac{1}{2}}\}.$$

Point a) is obvious since $h' = e^{-\Sigma} \in D^{\frac{1}{2}}$. *Let* $f \in C^2$. *Observing* b *as a deterministic process, the definition of* $\hat{L}$ *as in (4.12), integration by parts in Remark 3.2 c) and Proposition 3.6 imply*

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^0 f' + \int_0^x f' d^0 b \tag{4.19}$$

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^{(y)} f' + \int_0^x f' d^{(y)} b. \tag{4.20}$$

First part of point b) follows because of Proposition 2.2. Of course, previous expression can be extended to $f \in C^1_{\nu_0}$ *and this shows the first part of point c).*

The second part of point c) of Lemma 4.16, consists in verifying $\hat{L}h = 0$. *Plugging* $h' = e^{-\Sigma}$ *in previous expression, through Proposition 2.2,* *we obtain*

$$\ell(x) = -\int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)}\Sigma + \int_0^x e^{-\Sigma} d^{(y)} b = 0.$$

Concerning the second part of point b), let $f \in C^2$ *so that* $f(0) = f'(0) = 0$. *We want to show that* $\varphi = T\ell$ *coincides with* f.

Since $\varphi(0) = 0$, *it remains to check* $\varphi' = f'$. *We recall that*

$$\varphi'(x) = e^{-\Sigma}(x) \left(2 \int_0^x \frac{e^{\Sigma}}{\sigma^2}(y) d^{(y)} \ell(y) \right).$$

Applying twice the chain rule Proposition 2.2, (4.19), the fact that

$$e^{\Sigma}(x) = \int_0^x e^{\Sigma} \frac{2d^{(y)} b}{\sigma^2} + 1,$$

and integration by parts we obtain

$$\begin{aligned} \varphi'(x) &= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x 2 \frac{e^{\Sigma}}{\sigma^2} f' d^{(y)} b \right\} \\ &= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x f' d^{(y)} e^{\Sigma} \right\} \\ &= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x f' d^0 e^{\Sigma} \right\} \\ &= e^{-\Sigma}(x) \left\{ (f' e^{\Sigma})(x) - (f' e^{\Sigma})(0) \right\} \\ &= f'(x). \end{aligned}$$

Point b) is therefore completely established.

Point d) follows because in (4.18), when $\ell \in \nu_0$, $f' \in \nu_0$ *also.*

Clearly, as for previous example, $ImT \subset C^1_{\nu_0}$. *It remains to show that* $\hat{L}T$ *is the identity map* $\{f \in D^{\frac{1}{2}} | f(0) = 0\}$.

For this we first remark that

$$\hat{L}f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)} \left(f' e^{\Sigma} \right) \tag{4.21}$$

In fact, by Proposition 3.6 and integration by parts Remark 3.2 c), we obtain

$$f'(x) e^{\Sigma(x)} = f'(0) + \int_0^x e^{\Sigma} d^{(y)} f' + \int_0^x f' d^{(y)} e^{\Sigma}.$$

By the chain rule Proposition 2.2, we obtain the right member of (4.21).

At this point, by definition, if $f = T\ell$, we have

$$f'(x)e^{\Sigma(x)} = \int_0^x 2\frac{e^{\Sigma}}{\sigma^2} d^{(y)}\ell.$$

Therefore (4.21) and Proposition 2.2 allow to conclude

$$\hat{L}f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} 2\frac{e^{\Sigma}}{\sigma^2} d^{(y)}\ell = \ell(x) - \ell(0).$$

(iv) *Suppose b with bounded variation. Then the Technical Assumption is verified for $\nu_0 = BV$ where BV is the space of continuous real functions v equipped with the following topology. A sequence (v_n) in BV converges to v if*

$$\begin{cases} v_n(0) \to v(0) \\ dv_n \to dv \text{ weakly} - *. \end{cases}$$

The arguments for proving that the Technical Assumption is verified are similar but easier than at previous point. Young type calculus is replaced by classical Lebesgue-Stieltjes calculus.

5 Martingale problem

In this section, we consider a PDE operator satisfying the same properties as in the previous section, i.e.

$$Lg = \frac{\sigma^2}{2} g'' + b'g', \tag{5.1}$$

where $\sigma > 0$ and b are continuous. In particular, we assume that

$$\Sigma(x) = \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{5.2}$$

exists in C^0, independently from the chosen mollifier. Then h defined by $h'(x) := \exp(-\Sigma(x))$ and $h(0) = 0$, is a solution to $Lh = 0$ with $h' \neq 0$.

We aim here at introducting different notions of martingale problem trying, when possible, to clarify also the classical notion. For the next two defintions, we consider the following convention. Let $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfilling the *usual conditions*.

Definition 5.1 *A process X is said to solve* **the martingale problem** *related to L (with respect to previous filtered probability space), with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if*

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\, ds$$

is an $(\mathcal{F}_t)_{t\geq 0}$-local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr, \quad t \geq s$$

is an $(\mathcal{F}_t)_{t\geq 0}$-local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X_{t-s}^{s,x}$ solves the martingale problem at time 0.

Definition 5.2 *Let (W_t) be an $(\mathcal{F}_t)$-classical Wiener process. An $(\mathcal{F}_t)$-progressively measurable process $X = (X_t)$ is said to solve* **the sharp martingale problem** *related to L (on the given filtered probability space), with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if*

$$f(X_t) - f(x_0) - \int_0^t Lf(X_r)\, dr = \int_0^t f'(X_r)\sigma(X_r) dW_r$$

for every $f \in \mathcal{D}_L$

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the sharp martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr = \int_s^t f'(X_r^{s,x})\sigma(X_r^{s,x}) dW_r, \quad t \geq s$$

Remark 5.3 *Let (W_t) be an $(\mathcal{F}_t)$-Wiener process. If b' is continuous then a process X solves the (corresponding) sharp martingale problem with respect to L if and only if it is a classical solution of the SDE*

$$X_t = x_0 + \int_0^t b'(X_r)\, dr + \int_0^t \sigma(X_r) dW_r.$$

For this a simple application of the classical Itô formula gives the result.

Remark 5.4 *(i) In general, $f(x) = x$ does not belong to $\mathcal{D}_L$, otherwise a solution to the martingale problem with respect to L would be a semimartingale. According to Remark 5.18, this is generally not the case. In [11] we gave necessary and sufficient conditions on b so that X is a semimartingale.*

(ii) We are interested in the operators

$$\mathcal{A} : \mathcal{D}_L \to \mathcal{C}, \text{ given by } \quad \mathcal{A}(f) = \int_0^\cdot Lf(X_s)\, ds$$

and

$$A : C^1 \to \mathcal{C}, \text{ given by } \quad A(\ell) = \int_0^\cdot \ell'(X_s)\, ds.$$

where $\mathcal{C}$ is the vector algebra of continuous processes.

We may ask whether $\mathcal{A}$ and A are closable in C^1 and in C^0, respectively. We will see that $\mathcal{A}$ admits a continuous extension to C^1. However, A can be extended continuously to some topological vector subspace ν_0 of C^0, where ν_0 includes the drift, only when Assumption $\mathcal{A}(\nu_0)$ is verified.

Similarly to the case of classical stochastic differential equations it is possible to distinguish two types of existence and uniqueness of the martingale problem. Even, if we could distinguish initial conditions which are random $\mathcal{F}_0$-measurable solutions, here we will only discuss deterministic ones. We will symbolize by $MP(L, x_0)$ the martingale problem related to L with initial condition x_0. The notions will only be formulated with respect to the initial condition at time 0.

Definition 5.5 *(***Strong existence***)*

We will say that $MP(L, x_0)$ admits **strong existence** *if the following holds. Given any probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, an $(\mathcal{F}_t)_{t\geq 0}$- Brownian motion $(W_t)_{t\geq 0}, x_0 \in \mathbb{R}$, there is a process $(X_t)_{t\geq 0}$ which is solution to the martingale problem with respect to L and initial condition x_0.*

Definition 5.6 *(***Pathwise uniqueness***) We will say that $MP(L, x_0)$ admits pathwise uniqueness if the following property is fulfilled.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a filtration $(\mathcal{F}_t)_{t\geq 0}$, an $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion $(W_t)_{t\geq 0}$. If two processes $X, \tilde{X}$ are two solutions such that $X_0 = \tilde{X}_0$ a.s., then X and $\tilde{X}$ coincide.

Definition 5.7 *(**Existence in law** or **weak existence**) We will say that $MP(L; x_0)$ admits weak existence if there is a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, and a process $(X_t)_{t\geq 0}$ which is a solution.*

We say that $MP(L)$ admits weak existence if $MP(L; x_0)$ admits weak existence for every x_0.

Definition 5.8 *(**Uniqueness in law**) We say that $MP(L; x_0)$ has a **unique solution in law** if the following holds. We consider an arbitrary probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ on it; we consider also another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ equipped with another filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. We suppose that $X_0 = x_0$, P- a.s. and $\tilde{X}_0 = x_0$, $\tilde{P}$- a.s. Then X and $\tilde{X}$ must have the same law as r.v. with values in $E = C(\mathbb{R}_+)$ (or $C[0, T]$).*

We say that $MP(L)$ has a unique solution in law if $MP(L, x_0)$ has a unique solution in law for every x_0.

In fact similar notions could be formulated combining martingale problem with sharp martingale problem.

Remark 5.9 *Let us suppose b' to be a continuous function, and we do not suppose σ to be strictly positive (only continuous).*

(i) Then the $MP(L, x_0)$ admits strong existence and pathwise uniqueness if the corresponding classical SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b'(X_s)ds$$

admits strong existence and pathwise uniqueness. In this case $\mathcal{D}_L = C^2$ and to establish this, it is enough to use classical Itô formula.

(ii) It is well-known, see [28, 15], that weak existence (resp. uniqueness in law) of the martingale problem is equivalent to that weak existence (resp. uniqueness in law) of the corresponding SDE.

For the rest of the section, let $s \in [0,T]$, $x_0 \in \mathbb{R}$. Moreover let $(\Omega, (\mathcal{F}_t), P)$ be a fixed filtered probability space fulfilling the usual conditions.

The first result on solutions to the martingale problem related to L is the following.

Proposition 5.10 *Let $y_0 = h(x_0)$.*

(i) A process X solves the martingale problem related to L with initial condition x at time s if and only if $Y = h(X)$ is a local martingale which solves on the same probability space

$$Y_t = y_0 + \int_s^t \tilde{\sigma}_h(Y_s)\, dW_s, \tag{5.3}$$

where $\sigma_h(y) = (\sigma h')(h^{-1}(y))$ and (W_t) is an $(\mathcal{F}_t)$-classical Brownian motion.

(ii) Let (W_t) be an $(\mathcal{F}_t)$-classical Brownian motion. If Y is a solution to equation (5.3), then $X = h^{-1}(Y)$ is a solution to the sharp martingale problem with respect to L with initial condition x at time s.

Remark 5.11 *Let X be a solution to the martingale problem with respect to L and set $Y = h(X)$ as at point (i). Since Y is a local martingale, we know from Remark 3.5 (iv) that $X = h^{-1}(Y)$ is a $(\mathcal{F}_t)$-Dirichlet process with martingale part*

$$M_t^X = \int_0^t (h^{-1})'(Y_s)\, dY_s.$$

In particular, X is a finite quadratic variation process with

$$[X,X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Proof (of Propositon 5.10).

For simplicity we will set $s = 0$.

First, let X be a solution to the martingale problem related to L. Since $h \in \mathcal{D}_L$ and $Lh = 0$, we know that $Y = h(X)$ is an $(\mathcal{F}_t)$-local martingale.

In order to calculate its bracket we recall that $h^2 \in \mathcal{D}_L$ and $Lh^2 = \sigma^2 (h')^2$ hold by Proposition 4.11 a). Thus,

$$h^2(X_t) - \int_0^t (\sigma h')^2 (X_s)\, ds$$

is an $(\mathcal{F}_t)$-local martingale. This implies

$$[Y]_t = \int_0^t (\sigma h')^2 (h^{-1}(Y_s))\, ds = \int_0^t \tilde{\sigma}_h^2 (Y_s)\, ds.$$

Finally, Y is a solution to the SDE (5.3) with respect to the standard $\mathcal{F}_Y$-Brownian motion W given by

$$W_t = \int_0^t \frac{1}{\tilde{\sigma}_h(Y_s)}\, dY_s.$$

Now, let $Y = h(X)$ be a solution to (5.3) and $f \in \mathcal{D}_L$. Proposition 4.11 c) says that $\phi := f \circ h^{-1} \in \mathcal{D}_{L^0} \equiv C^2$, where

$$L^0 \phi = \frac{\tilde{\sigma}_h^2}{2} \phi'' = (Lf) \circ h^{-1}. \tag{5.4}$$

So we can apply Itô formula to evaluate $\phi(Y)$ which coincides with $f(X)$. This gives

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s)\, dY_s + \frac{1}{2} \int_0^t \phi''(Y_s)\, d[Y_s].$$

Using $d[Y]_s = \tilde{\sigma}_h^2(Y_s)\, ds$ and taking into account (5.4), we conclude

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \int_0^t Lf(X_s)\, ds. \tag{5.5}$$

This establishes the proposition. ■

Remark 5.12 *From Proposition 5.10 in particular we have the following.*

Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space fulfilling the usual conditions. Let $x_0 \in \mathbb{R}$ and X be a solution to the martingale problem related to L with initial condition x_0. Then, there is a classical Brownian motion (W_t) so that X is a solution to the sharp martingale problem related to L with initial condition x_0.

Corollary 5.13 *Let X be a solution to the martingale problem related to L with initial condition x_0. Then, the map $\mathcal{A}$ admits a continuous extension from $\mathcal{D}_L$ to C^1 with values in $\mathcal{C}$ which we will denote again by $\mathcal{A}$. Moreover, $\mathcal{A}(f)$ is a zero quadratic variation process for every $f \in C^1$.*

Proof. $\mathcal{A}$ has a continuous extension because of (5.5). $\mathcal{A}(f)$ is a zero quadratic variation process because X is a Dirichlet process with martingale part $\int_0^\cdot \sigma(X_s)\, dW_s$ and because of Remark 3.5. ■

Remark 5.14 *The extension of (5.5) to C^1 gives*

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \mathcal{A}(f). \tag{5.6}$$

Choosing $f = id$ in (5.6), we get

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \mathcal{A}(id).$$

We will see that, if there is suitable subspace ν_0 of C^0 such that Technical Assumption $\mathcal{A}(\nu_0)$ is verified then operator A will be extent to ν_0. If b is an element of that space, then it will be possible to write $\hat{L}id = b$ and $\mathcal{A}(id) = A(b)$. In that case we will be able to indicate that X is a solution of the generalized SDE with diffusion coefficient σ and distributional drift b'.

A similar result to Proposition 5.10 can be deduced for the case of a transformation through function k and the divergence type operator introduced at (4.13).

Proposition 5.15 *We consider the application k and the PDE operator L^1 introduced before and at Lemma 4.13.*

A process X solves the martingale problem related to L with initial condition x_0 at time s if and only if $Z = k(X)$ solves the martingale problem related to L^1 with initial condition $k(x_0)$ at time s.

Proof. It is an easy consequence of Lemma 4.13. ■

Let $x_0 \in \mathbb{R}, y_0 = h(x_0)$. Let σ, b, Σ, h as in section 4. We set $\tilde{\sigma}_h = \sigma e^{\Sigma} \circ h^{-1}$. From Proposition 5.10 the following follows.

Corollary 5.16 *(i) Strong existence (resp. pathwise uniqueness) holds for $MP(L, x_0)$ if and only is strong existence (resp. pathwise uniqueness) holds for the SDE*

$$dY_t = \tilde{\sigma}_h(Y_r)\, dW_r,$$

with initial condition $Y_s = h(x_0)$.

(ii) An analogous equivalence holds for weak existence (resp. uniqueness in law).

From Proposition 5.10 we can deduce two other corollaries concerning well-posedness of our martigale problem.

Corollary 5.17 *Under the same assumptions as previous corollary, $MP(L, x_0)$ admits weak existence and uniqueness in law.*

Proof. The statement follows from point (i) of Corollary 5.16 and from the fact that the SDE (5.3) admits weak existence and uniqueness in law because $\tilde{\sigma}_h > 0$, see Th. 5.7, ch. 5 of [15] or [7]. ■

Remark 5.18

- *Corollary 5.11 of [11], allows immediately to see that the solution is a semimartingale for each inital condition if and only if Σ is of bounded variation.*
- *If L is in divergence form, see Remark 4.4 a) with $\alpha = 1$, then the solution corresponds to the process constructed and studied for instance by [27].*

Corollary 5.19 *Suppose either that $\sigma \in D^{\frac{1}{2}}, b \in C^{\frac{1}{2}}$ or $b \in D^{\frac{1}{2}}, \sigma \in C^{\frac{1}{2}}$ and moreover (4.7). Then $MP(L, x_0)$ admits strong existence and pathwise uniqueness.*

Proof. In this case Σ is well defined, see Remark 4.4 d) and σ belongs to $D^{\frac{1}{2}}$. Since h^{-1} is of class C^1, $\tilde{\sigma}_h$ is Hölder continuous with parameter $\frac{1}{2}$. The SDE (5.3) admits pathwise uniqueness because of Theorem 3.5 ii) of [21] and weak existence again through Th. 5.7 of [15]. Yamada-Watanabe theorem, see [15] Corollary 3.23, ch. 5., implies also strong existence for (5.3). The result follows from point (i) of Corollary 5.16. ■

6 A significant stochastic differential equation with distributional drift.

In this section we will discuss the case where the martingale problem is equivalent to a stochastic differential equation to be specified. First of all one would need to give a precise sense to the appearing generalized drift $\int_0^\cdot b'(X_s)ds$, b being a continuous function.

We will introduce a property related to a general process X.

Let ν_1 be a topological F-space (or eventually an inductive limit of F-spaces) which is a topological linear subspace of $C^0(\mathbb{R})$, and $\nu_1 \supset C^1(\mathbb{R})$. We will say that X has the **extended local time property with respect to** ν_1 if there is a mapping $A^X : \nu_1 \to \mathcal{C}$ which extends continuously $\ell \to \int_0^\cdot \ell'(X_s)ds$ from C^1 and whenever $\int_0^\cdot g(X)d^- A(\ell)$ exists for every $g \in C^2$ and every $\ell \in \nu_0$.

Definition 6.1 *Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion, Z be an $\mathcal{F}_0$-measurable random variable. A process X will be called ν_1-solution of the SDE*

$$\begin{cases} dX_t &= b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= Z \end{cases}$$

if

- *X has the extended local time property with respect to ν_1.*
- *$X_t = Z + \int_0^t \sigma(X_s)dW_s + A^X(b)_t$.*
- *X is a finite quadratic variation process.*

Remark 6.2 • *Suppose that $b \in \nu_1$. If $\nu_1 \subset \nu_1'$, a ν_1'-solution is also a ν_1-solution.*

- *Previous definition is also new in the classical case, i.e. when b' is a continuous function. A C^1-solution corresponds to a solution to the SDE in the classical sense. On the other hand a ν_1 solution with ν_1 including C^1, is a solution whose local time has a certain additional regularity.*

Even in this generalized framework it is possible to introduce the notion of *strong ν_1-existence*, *weak ν_1-existence*, *pathwise ν_1-uniqueness* and *ν_1-uniqueness in law.* This can be done similarly as in Definition 5.8 according to the fact that the filtered probability space with the classical Brownian motion is fixed a priori or not.

Lemma 6.3 *We suppose that Technical Assumption $\mathcal{A}(\nu_0)$ is verified. If X is a solution to a martingale problem related to a PDE operator L then it fulfills the extended local time property with respect to $\nu_1 = \nu_0$.*

Proof. Let $\ell \in C^1$. Since X solves the martingale problem with respect to L, setting $f = T\ell$, it follows

$$\begin{aligned} A^X(\ell)_t &= \int_0^t \ell'(X_s)ds = \int_0^t Lf(X_s)ds \\ &= f(X_t) - f(X_0) - \int_0^t f'(X_s)\sigma(X_s)dW_s \end{aligned}$$

The continuity of T on ν_0 imply that A^X can be extended to ν_0.

Let now $\ell \in \nu_0$ and $f = T\ell \in C^1$. Since $f(X)$ equals a local martingale plus $A^X(\ell)$, it remains to show that

$$\int_0^\cdot g(X)d^- f(X) \tag{6.1}$$

exists for any $g \in C^2$. Integrating by parts previous integral, (6.1) equals

$$(gf)(X_\cdot) - (gf)(X_0) - \int_0^\cdot f(X)d^- g(X) - [f(X), g(X)].$$

Remark 3.3 b),f) tells that the right member is well-defined. □.

Lemma 6.4 *Let X be a process having the extended local time property with respect to some space F- space (or inductive limit) ν_1. Suppose that for fixed $g \in C^1$ the application $\ell \to g\ell$ is continuous from ν_1 to ν_1. Then, for every $g \in C^2$ and every $\ell \in \nu_1$, we have*

$$\int_0^{\cdot} g(X) d^- A^X(\ell) = A^X(\Phi(g, \ell)) \tag{6.2}$$

where

$$\Phi(g, \ell)(x) = (g\ell)(x) - (g\ell)(0) - \int_0^x (\ell g')(y)\, dy \tag{6.3}$$

Proof. The Banach-Steinhaus type Theorem 2.4 implies that, for every $g \in C^2$

$$\ell \mapsto \int_0^{\cdot} g(X)\, d^- A^X(\ell) \tag{6.4}$$

is continuous from ν_1 to $\mathcal{C}$. In fact, expression (6.4) is ucp limit of

$$\lim_{\varepsilon \to 0+} \int_0^{\cdot} g(X_s) \frac{A^X(\ell)_{s+\varepsilon} - A^X(\ell)_s}{\varepsilon}\, ds.$$

Note that Φ is a continuous bilinear map from $C^1 \times \nu_1$ to ν_1. Since $A^X : \nu_1 \to \mathcal{C}$ is continuous, the mapping $\ell \to A^X(\Phi(g, \ell))$ is also continuous from ν_1 to $\mathcal{C}$. In order to conclude the proof, we need to check identity (6.2) for $\ell \in C^1$. In that case, since

$$\Phi(g, \ell)(x) = \int_0^x (g\ell')(y)\, dy$$

both members of (6.2) equal

$$\int_0^{\cdot} (g\ell')(X_s) ds.$$

□

We are now going to investigate the relation between the martingale problem associated with L and stochastic differential equations with distributional drift.

Proposition 6.5 *Let $x_0 \in \mathbb{R}$. Suppose that L fullfills the Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, fulfilling the usual conditions and (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion.*

If X solves the sharp martingale problem with respect to L with initial condition x_0, then X is a ν_0-solution to the stochastic differential equation

$$\begin{cases} dX_t &= b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0 \end{cases} \tag{6.5}$$

Remark 6.6 *In particular, if L is close to divergence type as in Example 4.18 (ii) then X is a C^0-solution to previous equation with $b = \frac{\sigma^2}{2} + \beta - \frac{\sigma^2(0)}{2}$.*

Proof. Let X be a solution to the martingale problem related to L. We know by Lemma 6.3 that X fulfills the extended local time property with respect to ν_1. On the other hand, by Remark 5.11, X is a finite quadratic variation process. It remains to show

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A^X(b)_t. \tag{6.6}$$

Let $\ell \in C^1$ and set $f = T^1\ell$. By definition of sharp martingale problem we have

$$T^1\ell(X_t) = T^1\ell(X_0) + \int_0^t ((T^1\ell)'\sigma)(X_s)\, dW_s + A^X(\ell)_t \tag{6.7}$$

According to Remark 4.15 (i), which states the continuity of the map $T^1 : \nu_0 \to C^1$, previous expression can be extended to any $\ell \in \nu_0$.

By Remark 4.15 (ii) $\ell = b \in \nu_0$ and $f = T^1\ell = id$. Replacing this in (6.7) we obtain

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A^X(b).$$

Since $X_0 = Z$ the proof is completed.

■

Corollary 6.7 *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills the Technical Assumption $\mathcal{A}(\nu_0)$. If $MP(L, x_0)$ admits weak (resp. strong) existence then the SDE (6.5) also admits weak (resp. strong) existence).*

Proof. The statement about strong solutions is obvious. Concerning weak solutions, let us admit the existence of a filtered probability space, where there is a solution to the martingale problem with respect to L with initial condition x_0. Then, according to Remark 5.12, the mentioned solution is also a solution to a sharp martingale problem and the result follows. ■

If X is some ν_1-solution to (6.6), is it a solution to the (sharp) martingale problem related to some operator L? The answer is delicate. In the following proposition we only provide the converse of Proposition 6.5 as partial answer.

Proposition 6.8 *Suppose the PDE operator L fulfills the Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, fulfilling the usual conditions and (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion. Let X be a progressively measurable process.*

X solves the sharp martingale problem related to L with respect to some initial condition x_0 if and only if it is a ν_0-solution to the stochastic differential equation

$$\begin{cases} dX_t &= b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0 \end{cases} \tag{6.8}$$

Corollary 6.9 *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Then weak existence and uniqueness in law (resp. strong existence and pathwise uniqueness) holds for the equation (6.8) if and only if the same holds for $MP(L, x_0)$.*

Proof (of the Proposition).

Suppose that X is a ν_0-solution to (6.8) therefore it is a finite quadratic variation process. Let $f \in C^3$. Since X solves (6.6) and $\int_0^\cdot f'(X_s)\, d^- X_s$ always exists by the classical Itô formula (see Remark 3.3 e) of Chapter 1) we know that $\int_0^\cdot f'(X)\, d^- A^X(b)$ also exists and is equal to $\int_0^\cdot f'(X)\, d^- X - \int_0^\cdot (f'\sigma)(X)\, dW$. Therefore, this Itô formula says that

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\sigma(X_s)\, dW_s + \int_0^t f'(X)\, d^- A^X(b) \\ &\quad + \frac{1}{2}\int_0^t f''(X_s)\sigma^2(X_s)\, ds \end{aligned}$$

holds.

By Lemma 6.4, the linearity of mapping A^X and (4.12), we get

$$\begin{aligned} & \int_0^t f'(X) d^- A^X(b) + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s) ds \\ = & \quad A^X\left(\Phi(f',b)\right)_t + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s) ds \\ = & \int_0^t (\frac{\sigma^2}{2} - b)(X_s) f''(X_s) ds + A^X(bf') = A^X(\hat{L}f) \end{aligned}$$

This shows

$$f(X_t) - f(X_0) - \int_0^t (f'\sigma)(X_s) dW_s = A^X(\hat{L}f), \tag{6.9}$$

for every $f \in C^3$. In reality it is possible to show previous equality for any $f \in C^2$. In fact the left member extends continuously to C^2, and even to C^1. The right member is also allowed to be prolongated to C^2: for $f \in C^2$, let (f_n) be a sequence of functions in C^3 converging when $n \to \infty$, according to the C^2 topology to f. In particular the convergence also holds in $C^1_{\nu_0}$. Since $\hat{L}$ is continuous with respect to the $C^1_{\nu_0}$ topology with values in ν_0, we have $\hat{L}f_n \to \hat{L}f$ in ν_0. Finally $A^X(\hat{L}f_n) \to A^X(\hat{L}f)$ ucp because of the extended local time property with respect to ν_0.
We will use in fact the validity of (6.9) for $f \in C^2$ with $f(0) = 0$. We set $x_1 = f'(0)$. We set $\ell = \hat{L}f$. According to Technical Assumption $\mathcal{A}(\nu_0)$ (iv) we have $f = T^{x_1}\ell$. Therefore (6.9) gives

$$T^{x_1}\ell(X_t) = T^{x_1}\ell(X_0) + \int_0^t ((T^{x_1}\ell)'\sigma)(X_s) dW_s + A^X(\ell).$$

Using again the extended local time property with respect to ν_0, and the continuity of T^{x_1} we can state the validity of previous expression to each $\ell \in \nu_0$ with $\ell(0) = 0$, in particular for $\ell \in C^1$ with $\ell(0) = 0$. But in this case, for any $f \in \mathcal{D}_L$ with $f(0) = 0$ and $\ell' = Lf$, we obtain

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s) dW_s + \int_0^t Lf(X_s) ds.$$

This shows the validity of the identity in Definition 5.2 for $f \in \mathcal{D}_L$ and $f(0) = x_0$ and $x_0 = 0$. If $x_0 \neq 0$, we replace f by $f - x_0$ in previous identity, we use that $L(f - x_0) = Lf$ for any $f \in \mathcal{D}_L$.

It follows that X fulfills a sharp martingale problem with respect to L.

This shows the reversed sense of the statement. The direct implication was proved in Proposition 6.5. ■

Corollary 6.10 *We suppose* $\sigma \in D^{\frac{1}{2}}$ *and* $b \in C^{\frac{1}{2}}$ *or* $\sigma \in C^{\frac{1}{2}}$ *and* $b \in D^{\frac{1}{2}}$ *with conditions (4.7). We set* $\nu_0 = D^{\frac{1}{2}}$.

Then equation (6.8) admits ν_0*-strong existence and pathwise uniqueness.*

Proof. The result follows from Corollaries 6.9 and 5.19 ■

7 About C_b^0- generalized solutions of parabolic equations

In this section we want to discuss the related parabolic Cauchy problem with final condition, associated with our stochastic differential equations with distributional drift.

We will make the same assumptions and conventions as in section 4. We consider the formal operator $\mathcal{L} = \partial_t + L$ where L will act, from now on, on the second variable.

Definition 7.1 *Let* λ *be an element of* $C_b^0([0,T] \times \mathbb{R})$ *and* $u^0 \in C_b^0(\mathbb{R})$. *A function* $u \in C_b^0([0,T] \times \mathbb{R})$ *will be said to be a* C_b^0**-generalized solution** *to*

$$\left\{ \begin{array}{rcl} \mathcal{L}u & = & \lambda \\ u(T,\cdot) & = & u^0 \end{array} \right. \tag{7.1}$$

if the following is verified.

(i) For any sequence (λ_n) *in* $C_b^0([0,T] \times \mathbb{R})$, *converging to* λ *in a bounded way,*

(ii) for any sequence (u_n^0) *in* $C_b^0(\mathbb{R})$ *converging in a bounded way to* u^0,

(iii) such there are classical solutions (u_n) *in* $C_b^0([0,T] \times \mathbb{R})$ *of class* $C^{1,2}([0,T[\times \mathbb{R})$ *to* $\mathcal{L}_n u_n = \lambda_n$, $u_n(T,\cdot) = u_n^0$,

then (u_n) *converges in a bounded way to* u.

Remark 7.2 *a) u is said to solve $\mathcal{L}u = \lambda$ if there is $u^0 \in C_b^0(\mathbb{R})$ such that (7.1) holds.*

b) Previous definition depends in principle on the mollifier but it could be easily adapted not to depend.

c) The regularized problem admits a solution: if $u_n^0 \in C_b^3(\mathbb{R})$, $\lambda_n \in C_b^{0,1}([0,T]\times \mathbb{R})$ there is a classical solution u_n in $C^{1,2}([0,T] \times \mathbb{R})$ of

$$\begin{cases} \mathcal{L}_n v &= \lambda_n \\ v(T,\cdot) &= u_n^0 \end{cases}$$

For this it is enough to apply Theorem 5.19 of [16].

We state now a result concerning the case when the operator L is classical. Even if next propsition could be stated when the drift b' is a continuous function, we will suppose it to be zero. In fact it will be later applied to $L = L^0$.

Proposition 7.3 *Let $\varphi, \varphi_n \in C_b^0(\mathbb{R})$, $g, g_n \in C_b^0([0,T] \times \mathbb{R}), n \in \mathbb{N}$, such that $\varphi_n \longrightarrow \varphi$, $g_n \longrightarrow g$ in a bounded way on $\mathbb{R}$ and $[0,T] \times \mathbb{R}$.*
Let σ be a strictly positive real continuous function.

Suppose there exist $u_n \in C^{1,2}([0,T[\times\mathbb{R}) \bigcap C_b^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \mathcal{L}_n u_n &= g_n \\ u_n(T,\cdot) &= \varphi_n \end{cases}$$

Then (u_n) will converge to $u \in C_b^0([0,T] \times \mathbb{R})$ in a bounded way to the function u defined by

$$u(s,x) = \mathbb{E}\left(\varphi(Y_T^{s,x}) + \int_s^T g(r, Y_T^{r,x}) dr\right) \tag{7.2}$$

where $Y = Y^{s,x}$ is the unique solution (in law) to

$$Y_t = x + \int_s^t \sigma(X_r) dW_r. \tag{7.3}$$

and (W_t) is a classical Brownian motion on some suitable filtered probability space.

Remark 7.4 • *Usual Itô calculus implies that*

$$u_n(s,x) = \mathbb{E}\left(\varphi_n(Y_T^{s,x}(n)) + \int_s^T g_n(r, Y_T^{r,x}(n))dr\right) \qquad (7.4)$$

where $Y(n) = Y^{s,x}(n)$ is the unique solution in law to the problem

$$Y_t(n) = x + \int_s^t \sigma_n(Y_r(n))dW_r. \qquad (7.5)$$

- *Theorem 5.4 ch. 5 of [15], affirms it is possible to construct a solution (unique in law) $Y = Y^{s,x}$ to the SDE (7.3), resp. $Y(n) = Y^{s,x}(n)$ to (7.5).*

- *Suppose that L is a classical PDE operator. Let $u \in C^{1,2}([0,T[\times\mathbb{R})$ bounded and continuous on $[0,T]\times\mathbb{R}$. Again Itô calculus shows that u can be represented by (7.2) and (7.3). In particular a classical solution u to $\mathcal{L}u = g$ is also a C_b^0-generalized solution.*

Proof (of Proposition 7.3). We fix $s \in [0,T], x \in \mathbb{R}$. Using the Engelbert-Schmidt construction, see for instance the proof of Theorem 5.4 ch.5 and 5.7 of [15], it is possible to construct a solution $Y = Y^{s,x}$ of the SDE on some fixed probability space which solves (7.3) with respect to some classical Wiener process (W_t). We set $s = 0$ for simplicity. The procedure is based as follows. One fixes a standard Brownian motion (B_t) on some fixed probability space and one sets

$$T_t := \int_0^t \frac{du}{\sigma^2(x+B_u)}.$$

T is a.s. an homeomorphism on $\mathbb{R}_+$ and one defines A as the inverse of T. A solution Y will be then given by $Y_t = x + B_{A_t}$; in fact it is possible to show that the quadratic variation of the local martingale Y is

$$< Y, Y >_t = \int_0^t \sigma^2(Y_s)ds;$$

The Brownian motion W is constructed a posteriori and it is adapted to the natural filtration of Y by setting $W_t = \int_0^t \frac{dY_s}{\sigma(Y_s)}$.

So, on the same probability space we can set $Y_t(n) = x + B_{A_t(n)}$, $A(n)$ being the inverse of $T(n)$ where $T(n)_t ;= \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$.

Consequently, on the same probability space, we construct $Y_t(n) = x + B_{A_t(n)}$ where $A(n)$ is the inverse of $T(n)$ and $T(n)_t := \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$. $Y(n)$ solves equation (7.5) with respect to a Brownian motion depending on n.

By construction, the family $Y_T^{s,x}(n)$ converges a.s. to $Y_T^{s,x}$, Using Lebesgue dominated convergence theorems and the bounded convergence of (φ_n) and (g_n), we can take the limit when $n \to \infty$ in expression (7.4) and obtain the desired result. ■

Remark 7.5 *In particular the corresponding laws of random variables $(Y^{s,x}(n))$ are tight.*

Again for $\varphi \in C^0([0,T] \times \mathbb{R})$ or $C^0(\mathbb{R})$ we set $\tilde{\varphi} = \varphi \circ h^{-1}$ according to the conventions of section 2.
We set $\sigma_h = \sigma h'$ and L^0 is the classical operator defined at (4.9). Let us consider $\mathcal{L}^0 = \partial_t + L^0$ as formal operator.

Corollary 7.6 *Let $g \in C_b^0([0,T] \times \mathbb{R}), \varphi \in C_b^0(\mathbb{R})$. There is a C_b^0-generalized solution u to $\mathcal{L}^0 u = g,\ u(T,\cdot) = \varphi$. This solution is unique and it is given by (7.2).*

We go now back to the original PDE operator $\mathcal{L}$ with distributional drift. We denote again by h the same application defined at section 5 and discuss existence and uniqueness of C_b^0-generalized solutions of related parabolic Cauchy problems.

A useful consequence of Proposition 7.3 is the following.

Theorem 7.7 *Let $\lambda \in C_b^0([0,T] \times \mathbb{R}),\ u^0 \in C_b^0(\mathbb{R})$.*
There is a unique solution $u \in C_b^0([0,T] \times \mathbb{R})$ to

$$\begin{cases} \mathcal{L}u &= \lambda \\ u(T,\cdot) &= u^0 \end{cases} \tag{7.6}$$

Moreover $\tilde{u}$ solves

$$\begin{cases} \mathcal{L}^0 \tilde{u} &= \tilde{\lambda} \\ \tilde{u}(T,\cdot) &= \tilde{u}^0 \end{cases} \tag{7.7}$$

Proof. Conformally to section 4, let $(h_n)_{n\in\mathbb{N}}$ be an approximating sequence which is related to $Lh = 0$. Let us consider the PDE operators $\mathcal{L}_n$ defined at (4.2). Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $C_b^0([0,T) \times \mathbb{R})$, such that $\lambda_n \to \lambda$, $u_n^0 \to u^0$ in a bounded way, for which there are classical solutions u_n of

$$\left\{ \begin{array}{rcl} \mathcal{L}_n u_n & = & \lambda_n \\ u_n(T,\cdot) & = & u_n^0 \end{array} \right.$$

We recall that those sequences always exist because of Remark 7.2 c).

We set

$$g_n = \lambda_n \circ h_n^{-1}, \quad \varphi_n = \varphi \circ h_n^{-1}, \quad v_n = u_n \circ h_n^{-1}.$$

By Lemma 4.14, we have

$$\left\{ \begin{array}{rcl} \mathcal{L}_n^0 v_n & = & g_n \\ v_n(T,\cdot) & = & \varphi_n \end{array} \right.$$

where

$$\mathcal{L}_n^0 \varphi(t,y) = \partial_t \varphi(t,y) + \sigma_{h_n}^2 \circ h_n^{-1}(t,y)\varphi''(t,y)$$

By Proposition 7.3 and Corollary 7.6 $v_n \to \tilde{u}$ in a bounded way, where

$$\left\{ \begin{array}{rcl} \mathcal{L}^0 \tilde{u} & = & \tilde{\lambda} \\ \tilde{u}(T,\cdot) & = & \tilde{u}_0. \end{array} \right.$$

This concludes the proof of the proposition. ■

Now we discuss how C_b^0-generalized solutions transform themself under the action of function k introduced at (4.13). A similar result to Lemma 4.13 for the elliptic case, is the following.

For $\varphi \in C^0([0,T] \times \mathbb{R})$ or $C^0(\mathbb{R})$ we set $\bar{\varphi} = \varphi \circ k^{-1}$.
We set $\sigma_k = \sigma k'$ and consider the formal operator

$$\mathcal{L}^1 f = \partial_t f + \frac{1}{2}\bar{\sigma}_k^2 f'' + \frac{1}{2}(\bar{\sigma}_k^2)' f'.$$

Informally we can write

$$\mathcal{L}^1 f = \partial_t f + \frac{1}{2}(\bar{\sigma}_k^2 f')'.$$

Proposition 7.8 *Let $\lambda \in C_b^0([0,T] \times \mathbb{R})$, $u^0 \in C_b^0(\mathbb{R})$.*
Let u be the unique C_b^0-generalized solution in $C_b^0([0,T] \times \mathbb{R})$ to

$$\begin{cases} \mathcal{L}u &= \lambda \\ u(T,\cdot) &= u^0 \end{cases} \tag{7.8}$$

Then $\bar{u}$ solves

$$\begin{cases} \mathcal{L}^1\bar{u} &= \bar{\lambda} \\ \bar{u}(T,\cdot) &= \bar{u}^0. \end{cases}$$

Proof. Let v be the unique solution to

$$\begin{cases} \mathcal{L}^1 v &= \bar{\lambda} \\ v(T,\cdot) &= \bar{u}^0, \end{cases}$$

which exists because of Theorem 7.7 taking $\mathcal{L} = \mathcal{L}^1$.

We define $H : \mathbb{R} \to \mathbb{R}$ such that

$$H(0) = 0, \quad H'(z) = \frac{1}{\sigma_k^2}(z).$$

Again (4.7) implies that H is bijective on $\mathbb{R}$. This case corresponds to Example a) in Remark 4.4 with $\alpha = 1$.

We set $\tilde{v} = v \circ H^{-1}$. By Theorem 7.7 again, we have

$$\begin{cases} \mathcal{L}^{0,1}\tilde{v} &= \bar{\lambda} \circ H^{-1} \\ \tilde{v}(T,\cdot) &= u^0 \circ (k^{-1} \circ H^{-1}) \end{cases}$$

where $\mathcal{L}^{0,1} f = \frac{a^2}{2} f''$ and

$$a = (\sigma_k H') \circ H^{-1} = \frac{1}{\sigma_k} \circ H^{-1}.$$

Since

$$\sigma_k = (\sigma k') \circ k^{-1} = \frac{e^{\Sigma}}{\sigma} \circ k^{-1},$$

it yields

$$a = (\sigma e^{-\Sigma}) \circ (H \circ k)^{-1}.$$

On the other hand $H \circ k = h$ since

$$\begin{aligned} H \circ k(0) &= 0 = h(0) \\ (H \circ k(x))' &= H'(k(x))k'(x) = \frac{1}{\sigma_k^2} k'(x) = \frac{1}{\sigma^2 k'} = e^{-\Sigma} \\ &= h'. \end{aligned}$$

We can therefore conclude that $\mathcal{L}^{0,1} \equiv \mathcal{L}^0$. Since problem (7.7) has a unique solution then $\tilde{v} = \tilde{u}$ where u solves (7.6) and $\tilde{u} = u \circ h^{-1}$. Finally

$$\begin{aligned} v &= \tilde{v} \circ H = \tilde{u} \circ H = u \circ H \circ h^{-1} \\ &= u \circ k^{-1} = \bar{u}. \end{aligned}$$

■

Proposition 7.9 *The unique C_b^0- generalized solution to (7.6) admits a probabilistic representation in the sense that*

$$u(s,x) = \mathbb{E}\left(u^0(X_T^{s,x}) + \int_s^T \lambda(r, X_T^{r,x}) dr \right) \tag{7.9}$$

where $X^{s,x}$ is the solution to the martingale problem related to L at time s and point x.

Proof. It follows from Theorem 7.7, Corollary 7.6 and Proposition 5.10 which implies the following: if X is a solution to the martingale problem related to L at point x at time s, then $Y = h(X)$ solves the stochastic differential equation (5.3) with initial condition $h(x)$ at time s. ■

8 Density of the associated semigroups

We discuss now the existence of a law density for the solutions $X^{s,x}$ of the martingale problem related to L. First of all we suppose that L is an operator in divergence form with $Lf = (\frac{\sigma^2}{2} f')'$ and there are positive constants such that $c \le \sigma^2 \le C$. We will say in this case that L *has the Aronson form.* This denomination refers to the fundamental paper [1] about exponential estimates of fundamental solutions of non-degenerate parabolic equations. We start with some properties (partly classical) stated in [11]. We just observe that point (ix) is slightly modified with respect to [11] but this new asset can be immediately deduced from the proof in [11]. This preparatory work will be applied for the operator L^1 introduced in (4.14).

Lemma 8.1 *We suppose $0 < c \le \sigma^2 \le C$. Let σ_n, $n \in \mathbb{N}$, be smooth functions such that $0 < c \le \sigma_n^2 \le C$ and $\sigma_n^2 \to \sigma^2$ in C^0 as at the beginning* *of Section 4. We set* $L_n g = (\frac{\sigma_n^2}{2} g')'$. *There exists a family of probability measures* $(\nu_t(dx, y), t \geq 0, y \in \mathbb{R})$, *resp.* $(\nu_t^n(dx, y), t \geq 0, y \in \mathbb{R})$, *enjoying the following properties:*

(i) $\nu_t(dx, y) = p_t(x, y)\, dx$, $\nu_t^n(dx, y) = p_t^n(x, y)\, dy$.

(ii) (Aronson estimates) There exists $M > 0$, *only depending on constants* c, C *with*

$$\frac{1}{M\sqrt{t}} \exp\left(-\frac{M|x-y|^2}{t}\right) \leq p_t(x, y) \leq \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right).$$

(iii) We have

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y \tag{8.1}$$

and

$$\partial_t \nu_t^n(\cdot, y) = L_n \nu_t^n(\cdot, y), \quad \nu_0^n(\cdot, y) = \delta_y.$$

ν *(resp.* ν^n*) is called the fundamental solution related to the previous parabolic linear equation.*

(iv) We have

$$\begin{aligned} \partial_t \nu_t(x, \cdot) &= L\nu(x, \cdot) \\ \partial_t \nu_t^n(x, \cdot) &= L_n \nu^n(x, \cdot) \end{aligned}$$

(v) The map $(t, x, y) \mapsto p_t(x, y)$ *is continuous from* $]0, \infty[\times \mathbb{R}^2$ *to* $\mathbb{R}$.

(vi) The p^n *are smooth on* $]0, \infty[\times \mathbb{R}^2$.

(vii) We have $\lim_{n\to\infty} p_t^n(x, y) = p_t(x, y)$ *uniformly on each compact subset of* $]0, \infty[\times \mathbb{R}^2$.

(viii) $p_t(x, y) = p_t(y, x)$ *holds for every* $t > 0$ *and every* $x, y \in \mathbb{R}$.

(ix) $\int_0^T \sup_y \left(\int_{\mathbb{R}} |\partial_x p_t(x, y)|^2 dx\right)^{\frac{1}{2}} dt < \infty$.

Previous lemma allows to establish the following.

Theorem 8.2 *Let $Z^{s,x}$ be the solution to the martingale problem related to L at time s and point x. Suppose L to be of divergence type having the Aronson form. Then, there is fundamental solution $\nu_t = r_t(x,y)$ of*

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y.$$

with the following properties.

(i) Let $g \in C_b^0([0,T] \times \mathbb{R}), \varphi \in C_b(\mathbb{R})$. The C_b^0-generalized solution u to $\mathcal{L}u = g,\ u(T, \cdot) = \varphi$ is given by

$$u(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) r_{T-r}(x,y)dy. \quad (8.2)$$

(ii) The law of $Z_T^{s,x}$ has $r_{T-s}(x, \cdot)$ as density with respect to Lebesgue.

Proof. Let $(r_t^n(x,y))$ be the fundamental solution corresponding to the parabolic equation associated with the $L_n f(x) = (\frac{\sigma_n^2 f'}{2})'$ as introduced at section 4. We observe that (σ_n^2) converges in a bounded way to σ^2.

(i) We define

$$u_n(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}^n(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) r_{T-r}^n(x,y)dy. \quad (8.3)$$

Point (vi) and (ii) of Lemma 8.1 imply that functions u_n belong to $C^{1,2}([0,T[\times\mathbb{R})$ so they are classical solutions to

$$\begin{cases} \mathcal{L}_n u_n &= g, \\ u_n(T, \cdot) &= u^0. \end{cases}$$

According to points (ii) and (vii) of the same lemma, one can prove that u_n converges in a bounded way to u defined by (8.2). In fact the coefficients σ_n^2 are lower and upper bounded with a common constant, related to c and C. Therefore that u is the C_b^0- generalized solution of the considered Cauchy problem, which is known to exist. By uniqueness, point (i) is established.

(ii) Setting $g = 0$, point (i) implies that $u(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy$ is the C_b^0- generalized solution to $\mathcal{L}u = 0$ with $u(T,x) = \varphi(x)$. By Proposition 7.9, in particular using the probabilistic representation, we get $\mathbb{E}\left(\varphi(Z_T^{s,x})\right) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy$.

■

Remark 8.3 *If L is in the divergence form as before then $\mathcal{D}_L = \{f \in C^1$ such there is $g \in C^1$ with $f' = \frac{g}{\sigma^2}\}$. This is a consequence of Lemma 4.9 and that $e^{-\Sigma} = \frac{1}{\sigma^2}$.*

From now on, we will consider a general PDE operator L with distributional drift, as at section 4, for which the Assumption $(Aronson)$ below

$$(Aronson) \qquad c \le \frac{e^{\Sigma}}{\sigma^2} \le C.$$

We observe that the PDE operator in divergence form of the type $L^1 f = (\frac{\sigma_k^2 f'}{2})'$, where $\sigma_k = (\sigma k') \circ k^{-1}$ has the Aronson form and so previous Theorem can be applied.

Theorem 8.4 *Let $X^{s,x}$ be the solution to the martingale problem related to L at time s and point x. Suppose that L fulfills Assumption $(Aronson)$. Then there is a kernel $p_t(x,y)$ such that*

(i) The law of $X_t^{s,x}$ has $p_{t-s}(x,\cdot)$ as density with respect to Lebesgue for each $t \in]s,T]$.

(ii) Let $g \in C_b^0([0,T]\times\mathbb{R}), \varphi \in C_b^0(\mathbb{R})$. The C_b^0-generalized solution u to $\mathcal{L}u = g,\ u(T,\cdot) = \varphi$ is given by

$$u(s,x) = \int_{\mathbb{R}} \varphi(y) p_{T-s}(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) p_{T-r}(x,y)dy. \quad (8.4)$$

Proof.

(i) Proposition 5.15 says that $Z^{s,x} = k(X^{s,x})$ solves the martingale problem with respect to L^1. Let $r_t(x,y)$ the fundamental solution associated with the parabolic PDE $\mathcal{L}^1 = \partial_t + L^1$. The first point follows then from next observation.

Remark 8.5 *By change of variable it is easy to see that the law density of $X_t^{s,x}$ equals*

$$p_t(x,x_1) = r_t(k(x),k(x_1))k'(x_1) = r_t(k(x),k(x_1))\frac{e^{\Sigma}}{\sigma^2}(x_1).$$

(ii) This is a consequence of point (i), Fubini theorem and Proposition 7.9.

■

At this point we need a lemma which extends to the kernel $p_t(x, x_1)$ the integrability property of the kernel $r_t(x, x_1)$ stated in (8.3) concerning the divergence case.

Lemma 8.6 *Let $p_t(x, x_1)$ be the kernel introduced in Theorem 8.4. Then*

(i) it is continuous in all variables $(t, x, x_1) \in]0, T[\times \mathbb{R}^2$;

(ii) it fulfills Aronson estimates;

(iii) $\int_0^T (\sup_{x_1} \int_{\mathbb{R}} \partial_x p_t(x, x_1)^2 dx)^{\frac{1}{2}} dt < \infty$.

Proof.

We recall by Remark 8.5 that

$$p_t(x, x_1) = r_t(k(x), k(x_1))k'(x_1)$$

where $r_t(z, z_1)$ is the fundamental solution associated with the operator $L^1 f = (\frac{\sigma_k^2}{2} f')'$, $\quad k' = \frac{e^{\Sigma}}{\sigma^2}$. This, and point (v) of Lemma 8.1 directly imply the validity of first point.

Taking in account Assumption (*Aronson*), Aronson estimates for $(r_t(z, z_1))$ and that

$$|k(x) - k(x_1)| = \int_0^1 k'(\alpha x + (1 - \alpha)x_1) d\alpha |x - x_1|,$$

result (ii) follows easily.

With the same conventions as before, we have

$$\partial_x p_t(x, x_1) = \partial_z r_t(k(x), k(x_1))k'(x)k'(x_1).$$

So, for $x \in \mathbb{R}$,

$$\begin{aligned}\left(\int_{\mathbb{R}} (\partial_x p_t(x, x_1))^2 dx\right)^{\frac{1}{2}} &= \left(k'(x_1) \int_{\mathbb{R}} (\partial_z r_t(z, k(x_1)))^2 dz\right)^{\frac{1}{2}} \\ &\leq \sqrt{C} \sup_{z_1} \left(\int_{\mathbb{R}} dz (\partial_z r_t(z, z_1))^2\right)^{\frac{1}{2}}.\end{aligned}$$

(iii) follows after integration with respect to t and because of Lemma 8.1 (ix). ■

Proposition 8.7 *Let $g \in C_b^0([0,T]\times\mathbb{R}) \bigcap L^1([0,T]\times\mathbb{R})$, $\varphi \in C_b^0(\mathbb{R}) \bigcap L^1(\mathbb{R})$ Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the C_b^0-generalized solution to $\mathcal{L}u = g, u(T,\cdot) = \varphi$. Then*

a) $\int_0^T dt \int_{\mathbb{R}} u^2(t,x)dx < \infty$.

b) $x \mapsto u(t,x)$ *is absolutely continuous and*

$$\int_0^T dt \left(\int_{\mathbb{R}} u'^2(t,x)dx \right)^{\frac{1}{2}} < \infty.$$

In particular for a.e. $t \in [0,T]$, $u'(t,\cdot)$ is square integrable.

Remark 8.8 *Previous assumptions imply that g and φ are also square integrable.*

Proof. *We recall the expression given in Theorem 8.4:*

$$u(t,x) = \int_{\mathbb{R}} \varphi(x_1) p_{T-t}(x,x_1)dx_1 + \int_t^T dr \int_{\mathbb{R}} g(r,x_1) p_{T-r}(x,x_1)dx_1.$$

Using Lemma 8.6 and classical integration theorems, we have

$$\begin{aligned} u'(t,x) &= \int_{\mathbb{R}} \varphi(x_1)\partial_x p_{T-t}(x,x_1)dx_1 \\ &+ \int_t^T dr \int_{\mathbb{R}} ds g(s,x_1)\partial_x p_{T-s}(x,x_1)dx_1. \end{aligned} \tag{8.5}$$

Using Jensen inequality we have

$$|u(t,x)|^2 \le \int_{\mathbb{R}} \varphi(x_1)^2 p_{T-t}(x,x_1)dx_1 + (T-t)\int_t^T ds \int_{\mathbb{R}} g^2(s,x_1)p_{T-s}(x,x_1)dx_1.$$

Therefore

$$\int_{\mathbb{R}} u^2(t,x)dx =$$

$$\int_{\mathbb{R}} dx_1 \varphi(x_1)^2 \int_{\mathbb{R}} dx p_{T-t}(x,x_1) + \int_t^T ds(T-t)\int_{\mathbb{R}} dx_1 \int g^2(s,x_1)\int_{\mathbb{R}} dx p_{T-s}(x,x_1).$$

Using Aronson estimates, this quantity is bounded by

$$\text{const}\left(\int_{\mathbb{R}} dx_1\varphi(x_1)^2\int_{\mathbb{R}} dx\frac{1}{\sqrt{T-t}}p\left(\frac{x-x_1}{\sqrt{T-t}}\right)\right.$$

$$\left.+\int_t^T ds\int_{\mathbb{R}} dx_1\int g^2(s,x_1)\int_{\mathbb{R}} dx\frac{1}{\sqrt{T-s}}p\left(\frac{x-x_1}{\sqrt{T-s}}\right)\right),$$

where p is the Gaussian $N(0,1)$ density. This is clearly equal to

$$\text{const}\left(\int_{\mathbb{R}} dx_1\varphi(x_1)^2+\int_0^T ds\int_{\mathbb{R}} dx_1 g^2(s,x_1)\right)$$

This establishes point a).

Concerning point b), in order not to overcharge the notations we will suppose $g=0$. Expression (8.4) implies

$$u'(t,x)=\int_{\mathbb{R}}\varphi(x_1)\partial_x p_{T-t}(x,x_1)dx_1.$$

Jensen inequality implies

$$u'(t,x)^2\le\left(\int_{\mathbb{R}} dx_1|\varphi(x_1)||\partial_x p_{T-t}(x,x_1))|^2\right)\int_{\mathbb{R}} dx_1|\varphi(x_1)|.$$

Integrating with respect to x and taking the squarroot, we get

$$\begin{aligned}\sqrt{\int_{\mathbb{R}} dx u'(t,x)^2} &\le \sqrt{\int_{\mathbb{R}}|\varphi(x_1)|dx_1}\sqrt{\int_{\mathbb{R}} dx_1|\varphi(x_1)|\int_{\mathbb{R}} dx|\partial_x p_{T-t}(x,x_1)|^2}\\ &\le \int_{\mathbb{R}} dx_1|\varphi(x_1)|\sqrt{\sup_{x_1}\int_{\mathbb{R}}|\partial_x p_t(x,x_1)|^2dx}\end{aligned}$$

Integrating with respect to t, it gives

$$\int_0^T dt\|u'(t,\cdot)\|_{L^2(\mathbb{R})}\le\int_{\mathbb{R}} dx_1|\varphi(x_1)|\int_0^T dt\sqrt{\sup_{x_1}\int_{\mathbb{R}}\partial_x p_t(x,x_1)^2dx}$$

This quantity is finite because of Lemma 8.6 (iii).

■

9 Relation with weak solutions of stochastic partial differential equations

As in previous section, we will make assumption (*Aronson*). At this point we wish to investigate the link between C_b^0-generalized solutions and the notion of SPDE's weak solutions for a corresponding Cauchy problem with final condition.

We will use the same conventions as at Section 4. In this section we will suppose that coefficients σ, b are realizations of stochastic processes indexed by $\mathbb{R}$. Let us consider the formal operator $\mathcal{L} = \partial_t + L$ where L acts on the second variable.

We suppose $\sigma = 1$ and we proceed first formally integrating the equation

$$\begin{cases} \mathcal{L}u & = & \lambda \\ u(T,\cdot) & = & u^0 \end{cases} \tag{9.1}$$

in space against a test function $\alpha \in \mathcal{S}(\mathbb{R})$ and in time from t to T. This gives

$$\begin{aligned} -\int_{\mathbb{R}} \alpha(x)u(t,x)dx \quad &+ \quad \int_{\mathbb{R}} \alpha(x)u^0(x)dx - \frac{1}{2}\int_t^T ds \int_{\mathbb{R}} \alpha'(x)u'(s,x)dx \\ &+ \quad \int_t^T ds \int_{\mathbb{R}} \alpha(x)u'(s,x)b(dx) = \int_t^T ds \int_{\mathbb{R}} \alpha(x)\lambda(s,x)dx. \end{aligned} \tag{9.2}$$

It happens that integral $\int_{\mathbb{R}} \alpha(x)u'(s,x)b(dx)$ needs interpretation since b is not of bounded variation; $u'(s,\cdot)b'$ is the product of a distribution and a function which is even in general not continuous. This operation is deterministically undetermined unless one uses a theory of generalized functions. However if b is a frozen realization of a stochastic process η, we can hope to interpret that integral as a stochastic integral. On the other hand, that integral cannot be of Itô type since, even if η were a semimartingale, $u'(s,\cdot)$ is not necessarily adapted to the corresponding filtration.

The aim of this section is to show that a C_b^0-generalized solution to (9.1) provides a solution to a (stochastic) PDE of the type (9.2) where the problematic integral is interpreted in the sense of regularization.

Before we will state and prove a lemma, still supposing σ to be general.

Lemma 9.1 *Let λ (resp. u^0) be a random field with parameter $(t,x) \in [0,T] \times \mathbb{R}$ (resp. $x \in \mathbb{R}$) whose paths are bounded and continuous. Let σ, b be continuous stochastic processes such that Σ is defined a.s. and Assumption (Aronson) is verified. Let u be the random field being a.s. the C_b^0-generalized solution to (9.1). Then it holds*

$$\int_{\mathbb{R}} dx \quad \alpha(x) \left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x) ds \right)$$
$$= \int_{\mathbb{R}} e^{\Sigma(x)} \left(\int_t^T ds u'(s,x) \right) d^0 \left(\alpha \frac{\sigma^2}{2} e^{-\Sigma}(x) \right)$$

Proof. We fix a realisation ω. Theorem 8.4 says that the unique solution to equation (9.1) is given by

$$u(s,x) = \int_{\mathbb{R}} u^0(y) p_{T-s}(x,y) dy + \int_s^T dr \int_{\mathbb{R}} \lambda(r,y) p_{T-r}(x,y) dy. \tag{9.3}$$

where $(p_t(x,y))$ is the law density of the solution to the martingale problem related to L at point x at time s.

Proposition 8.7 b) implies that u' exists and it is integrable on $[0,T] \times \mathbb{R}$.

According to Proposition 7.8 we know that

$$\bar{u}(t,z) = u(t, k^{-1}(z))$$

is a C_b^0-generalized solution to

$$\begin{cases} \mathcal{L}^1 \bar{u} &= \bar{\lambda} \\ \bar{u}(T,\cdot) &= \bar{u}^0 \end{cases} \tag{9.4}$$

where

$$\bar{\lambda}(t,z) = \lambda(t, k^{-1}(z)), \bar{u}^0(z) = u^0(k^{-1}(z)).$$

On the other hand $\bar{u}$ can be represented via (8.2) in Theorem 8.4 via fundamental solutions $(\nu_t) = (r_t(x,y))$ of

$$\partial_t \nu_t(\cdot, y) = L^1 \nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y.$$

Since previous equation holds in Schwartz distribution theory, by inspection, it is not difficult to show that $\bar{u}$ is a solution in the sense of Schwartz distributions to (9.4), which means the following

$$\int_{\mathbb{R}} \alpha(z)(\bar{u}^0(z) - \bar{u}(t,z))dz - \frac{1}{2}\int_t^T ds \int_{\mathbb{R}} \alpha'(z)\bar{u}'(s,z)\sigma_k^2(z) = \int_t^T ds \int_{\mathbb{R}} \alpha(z)\bar{\lambda}(s,z). \tag{9.5}$$

for every test function $\alpha \in \mathcal{S}(\mathbb{R}), t \in [0,T]$. We recall in particular that $\bar{u}'$ is in $L^1([0,T] \times \mathbb{R})$.

We set

$$D(t,z) = \int_t^T \bar{u}'(s,z)ds, \quad \mathcal{D}(t,z) = D(t,z)\frac{\sigma_k^2(z)}{2}.$$

Expression (9.5) shows that

$$\mathcal{D}'(t,\cdot) = -\bar{u}^0 + \bar{u}(t,\cdot) + \int_t^T \bar{\lambda}(s,\cdot)ds. \tag{9.6}$$

in the sense of distributions. Therefore for each $t \in [0,T]$, $\mathcal{D}$ is of class C^1.

For $t \in [0,T]$ and $x \in \mathbb{R}$, we set $A(t,x) = \int_t^T u'(s,x)ds, \mathcal{A}(t,x) = A(t,x)e^{\Sigma(x)}$. We recall that

$$\bar{u}(s,x) = u(s,k(x)), \quad \bar{u}'(s,x) = u'(s,k(x))k'(x).$$

Therefore

$$A(t,x) = D(t,k(x))k'(x)$$

so that

$$\begin{aligned} A(t,x) &= 2\mathcal{D}(t,k(x))\frac{k'(x)}{\sigma_k^2(k(x))} = \mathcal{D}(t,k(x))\frac{2}{\sigma^2(x)k'(x)} \\ &= 2\mathcal{D}(t,k(x))e^{-\Sigma(x)}. \end{aligned}$$

Therefore $\mathcal{A}(t,x) = 2\mathcal{D}(t,k(x))$ and so $\mathcal{A}$ is of class C^1.

Since $\mathcal{A}'(t,x) = 2\mathcal{D}'(t,k(x))k'(x)$, (9.6) gives

$$\mathcal{A}'(t,x) = \left(-u^0(x) + u(t,x) + \int_t^T \lambda(s,x)ds\right) 2\frac{e^{\Sigma}}{\sigma^2}(x). \tag{9.7}$$

Consequently

$$u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds = \left(e^{\Sigma(x)}\int_t^T dsu'(s,x)\right)' e^{-\Sigma(x)}\frac{\sigma^2(x)}{2}.$$

We integrate previous expression against a test function $\alpha \in \mathcal{S}(\mathbb{R})$ to get

$$\int_{\mathbb{R}} dx\alpha(x)\left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds\right) =$$

$$\int_{\mathbb{R}} dx\alpha(x)\left\{\left(e^{\Sigma(x)}\int_t^T dsu'(s,x)\right)e^{-\Sigma(x)}\frac{\sigma^2(x)}{2}\right\}$$

Integration by parts for the symmetric integral provided by Remark 3.2 c) allows to conclude the proof of the lemma. ■

Finally we are able to state the theorem concerning existence of weak solutions for the SPDE.

Theorem 9.2 *Let λ (resp. u^0) be a random field with parameter in $(t,x) \in [0,T] \times \mathbb{R}$ (resp. $x \in \mathbb{R}$) whose paths are bounded and continuous.*

We suppose that $\sigma = 1$ and η being a (two-sided) zero strong cubic variation process such there are two finite and strictly positive random variables Z_1, Z_2 a.s. with $Z_1 \leq e^{\eta(x)} \leq Z_2$ a.s.

Let u be the random field which is a.s. C_b^0-generalized solution to (9.1). Then, u is a (weak) solution of the SPDE

$$\begin{aligned}
-\int_{\mathbb{R}} \alpha(x)u(t,x)dx \quad &+ \quad \int_{\mathbb{R}} \alpha(x)u^0(x)dx - \frac{1}{2}\int_{\mathbb{R}} \alpha'(x)\left(\int_t^T dsu'(s,x)\right)dx \\
&+ \quad \int_{\mathbb{R}} \alpha(x)\left(\int_t^T dsu'(s,x)\right)d^0\eta(x) \\
&= \quad \int_t^T ds\int_{\mathbb{R}} dx\alpha(x)\lambda(s,x).
\end{aligned}$$

Proof. After identification $b = \eta$, previous Lemma 9.1 says

$$\begin{aligned}
\int_{\mathbb{R}} dx \quad \alpha(x)&\left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds\right) \\
&= \quad \int_{\mathbb{R}} e^{2\eta(x)}\left(\int_t^T dsu'(s,x))\right)d^0\left(\frac{\alpha e^{-2\eta}}{2}(x)\right)
\end{aligned}$$

Since η is a zero strong cubic variation process, Proposition 3.8 implies that e^η is also a zero strong cubic variation process. Then Itô chain rule

Proposition 3.9 applied with $F(x,\eta(x)) = \alpha(x)e^{\eta(x)}$, and Remark 3.1 say that the right member of previous expression gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T dsu'(s,x)\right) d^0(\alpha e^{-2\eta(x)}) &= \frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T dsu'(s,x)\right) e^{2\eta(x)} \\ &\quad \left(\alpha'(x)e^{-2\eta}(x)dx + \alpha(x)d^0 e^{-2\eta(x)}\right) \\ &= \frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T dsu'(s,x)\right) \alpha'(x)dx \\ &- \int_{\mathbb{R}}\left(\int_t^T dsu'(s,x)\right) \alpha(x)d^0\eta(x).\end{aligned}$$

This concludes the proof. ■

References

[1] Aronson, D. G., *Bounds on the fundamental solution of a parabolic equation.* Bull. Amer. Math. Soc. **73**, 890-896 (1967).

[2] Bass, R.F., Chen, Z-Q, *Stochastic differential equations for Dirichlet processes.* Probab. Theory Related Fields **1**21 (2001), no. 3, 422-446.

[3] Bertoin, J., *Les processus de Dirichlet en tant qu'espace de Banach.* Stochastics **18**, 155-168 (1986).

[4] Dunford, N., Schwartz, J.T., *Linear Operators, Part I, General Theory.* New York: Wiley (1967).

[5] Errami, M., Russo,F., *n-covariation,generalized Dirichlet processes and calculus with respect to finite cubic variation processes.* Stochastic Processes and their Applications, 104, 259-299 (2003)

[6] Engelbert, H.J., Schmidt, W., *On one-dimensional stochastic differential equations with generalized drift.* Lecture Notes in Control and Information Sciences **69**, 143-155 (1984).

[7] Engelbert, H.J., Schmidt, W., *On solutions of one-dimensional stochastic differential equations without drift.* Z. Wahrscheinlichkeitstheorie verw. Gebiete **68**, 287-314 (1985).

[8] Engelbert, H.J., Wolf, J., *Strong Markov local Dirichlet processes and stochastic differential equations.* Teorija Veroyatnost. i ee Primenen. **43**, 331-348 (1998).

[9] Feyel, D. De la Pradelle, A. *On fractional Brownian processes* Potential Analysis **18**, 273-288 (1999).

[10] Flandoli, F., Russo, F., Wolf, J., *Some SDEs with distributional drift. Part I: General calculus.* Osaka Journal of Mathematics. Vol. 40, No 2, (2003).

[11] Flandoli, F., Russo, F., Wolf, J., *Some SDEs with distributional drift. Part II: Lyons-Zheng structure, Itô formula and semimartingale characterization.* Random Oper. Stochastic Equations **12**, no. 2, 145–184 (2004).

[12] Föllmer, H., *Calcul d'Itô sans probabilités.* Sém. Prob. XV, Lect. Notes Math. **850**, 143-150 (1981).

[13] Föllmer, H., *Dirichlet processes.* In Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980), pp. 476–478, Lecture Notes in Math., **851**, Springer-Verlag 1981.

[14] Gradinaru, M., Russo, F., Vallois, P., (2003). *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$.* Ann. Probab. 31, 1772-1820

[15] Karatzas, I., Shreve, S.E., *Brownian Motion and Stochastic Calculus.* Springer Verlag, Second edition (1991).

[16] Lunardi, A., *Analytic semigroups and optimal regularity in parabolic problems* Birkhäuser (1995).

[17] Mastrangelo, M., Dehen, D., *Opérateurs différentiels paraboliques à coefficients continus par morceaux et admettant un drift généralisé.* Bull. Sci. Math. (2) **116**, no. 1, 67–93 (1992).

[18] Nualart, D., Viens, F., *Evolution equation of a stochastic semigroup with white-noise drift.* Ann. Probab. **28** , no. 1, 36–73 (2000).

[19] Ouknine, Y., *Le "Skew-Brownian motion" et les processus qui en dérivent.* Theory of probability and its applications **35**, no. 1, 173–179 (1990)

[20] Portenko, N.I., *Generalized diffusion processes*, American mathematical society. **Vol. 83**, (1990).

[21] Revuz, D., Yor, M., *Continuous Martingales and Brownian Motion.* Berlin, New York: Springer-Verlag (1994).

[22] Russo, F., Vallois, P., *Forward, backward and symmetric stochastic integration.* Probab. Theory Relat. Fields **97**, 403-421 (1993).

[23] Russo, F., Vallois, P., *The generalized covariation process and Itô formula.* Stochastic Processes and their Applications **59**, 81-104 (1995).

[24] Russo, F., Vallois, P., *Stochastic calculus with respect to a finite variation process.* Stochastics and Stochastic Reports **70**, 1-40 (2000).

[25] Russo, F., Vallois, P., *Elements of stochastic calculus via regularizations.* Preprint 2004-28 LAGA-Paris 13.

[26] Russo, F., Vallois, P., Wolf, J., *A generalized class of Lyons-Zheng processes.* Bernoulli **7 (2)**, 363-379 (2001).

[27] Stroock, D.W., *Diffusion processes corresponding to uniformly elliptic divergence form operators.* Lect. Notes Math. **1321**, 316-347 (1980)

[28] Stroock, D.W., Varadhan, S.R.S., *Multidimensional diffusion processes.* Springer-Verlag (1979).

[29] Young, L.C. *An inequality of Hölder type, connected with Stieltjes integration,* Acta Math. **67**, 251-282 (1936).

[30] Zvonkin, *A transformation of the phase space of a diffusion process that removes the drift.* Math. Sbornik, **T 93 (135)** No 1, (1974).