CONSTRUCTION OF N-PARTICLE LANGEVIN DYNAMICS FOR $H^{1,\infty}$-POTENTIALS VIA GENERALIZED DIRICHLET FORMS

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ABSTRACT. We construct an N-particle Langevin dynamics on a cuboid region in $\mathbb{R}^d$ with periodic boundary condition, i.e., a diffusion process solving the Langevin equation with periodic boundary condition in the sense of the corresponding martingale problem. Our approach works for general $H^{1,\infty}$ potentials allowing N-particle interactions and external forces. Of course, the corresponding forces are not necessarily continuous. Since the generator of the dynamics is non-sectorial, for the construction we use the theory of generalized Dirichlet forms.
Furthermore, for any process constructed by a generalized Dirichlet form, we prove that it is solving the martingale problem for the corresponding generator. Moreover, we give a locality condition for the generator ensuring that a process constructed by a generalized Dirichlet form is a diffusion, i.e., it has continuous sample paths.

1. INTRODUCTION

The Langevin equation (cf. e.g. [Sch04, Section 8.1])

$$\begin{aligned} dX_t &= V_t\, dt \\ dV_t &= -\gamma V_t\, dt - \nabla\Phi(X_t)\, dt + \sqrt{\frac{2\gamma}{\beta}} dB_t \end{aligned} \tag{1.1}$$

is a stochastic differential equation which describes the evolution of the positions $X_t = (X_t^{(1)}, \cdots, X_t^{(N)}) \in (\mathbb{R}^d)^N$ and velocities $V_t = (V_t^{(1)}, \cdots, V_t^{(N)}) \in (\mathbb{R}^d)^N$ of N particles in dimension d. These particles are subject to a stochastic perturbation of the velocities modelled by an $\mathbb{R}^{Nd}$-valued Brownian motion $(B_t)_{t\geq 0}$ and friction, both e.g. caused by a surrounding medium with constant temperature. Furthermore, their motion is affected by an N-particle potential $\Phi : (\mathbb{R}^N)^d \to \mathbb{R}$. This, of course, also covers the case of pair interactions. $\gamma > 0$ describes the (constant) magnitude of the influence of the surrounding medium. $\beta > 0$ is defined by $\beta := \frac{1}{kT}$, where T is the (constant) temperature of the surrounding medium and k is the Boltzmann constant.
Here we consider the case where the motion in "x-direction" is bounded - in particular, we restrict the motion to a cuboid in $\mathbb{R}^d$ with a periodic boundary, which means that if a particle leaves this area "on the right", it enters it at the same time "on the left". Clearly, then also the potential has to be periodic. In order to avoid "jumping" of the particles

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from one side to the other we replace the cuboid in $(\mathbb{R}^d)^N$ by the dN-dimensional manifold which results from glueing the respective opposite (hyper)surfaces together.
We prove (cf. Section 3.4) that for bounded potentials having bounded weak derivatives there exists for many initial distributions ν a weak solution of (1.1), considered as equation on this manifold, in the sense of the corresponding martingale problem. This means that (the law P_ν of) the process $(X_t, V_t)_{t\geq 0}$ we construct solves the martingale problem for the generator

$$L = \frac{\gamma}{\beta}\Delta_v - \gamma v\nabla_v + v\nabla_x - (\nabla_x\Phi)\nabla_v \tag{1.2}$$

corresponding to this equation (via Itô formula). Solving the martingale problem for L on a domain $\mathcal{D}$ means that for all $f \in \mathcal{D}$, the process

$$f(X_t, V_t) - f(X_0, V_0) - \int_0^t Lf(X_s, V_s)\, ds \tag{1.3}$$

is a martingale w.r.t. P_ν. We specify later (cf. Sections 4 and 3.4) how large $\mathcal{D}$ may be chosen. It depends on the initial distribution ν.
Moreover, we show that our solution process is a diffusion, i.e. it has a.s. continuous paths, and that is has infinite life time (both again for many initial distributions/points).

In order to prove the existence of the Langevin dynamics we show that L generates a quasi-regular generalized Dirichlet form. The theory of generalized Dirichlet forms (GDFs) (cf. [Sta99]), then provides us with the existence of a process which is associated with this GDF and hence with L. We emphasize that it is not possible to construct a (sectorial) solution process using the theory of coercive Dirichlet forms (cf. [MR92]) here, since L is non-sectorial, cf. Remark 3.15(ii).
For proving that L generates a GDF it is crucial to find a domain for L on which it is essentially m-dissipative, or equivalently, such that the closure of L generates a C_0 contraction semigroup. We moreover have to show that this semigroup is sub-Markovian, or equivalently, that the closure of L is a Dirichlet operator. Both is done in Section 3.3. The problem at proving essential m-dissipativity of L is that L is not strictly elliptic, which makes a direct application of perturbation theory impossible. We use an idea from [Lei01] to solve this problem. The shape of our domain plays an important role for our proof, since it enables us to find a complete orthonormal system of subspaces of L^2 which is invariant w.r.t. the partial derivatives in the x-directions.
Quasi-regularity of the GDF generated by (the m-dissipative closure of) L is then easily seen, since the domain of essential m-dissipativity we find for L is an algebra of C^∞-functions. Hence we only have to consider the question, whether the martingale problem is solved by the corresponding process and if the process is indeed a diffusion. This is done in a more general setting in Sections 4 and 5.

In literature, one finds many hints and ideas which help to see that a process with state space E constructed from a (generalized) Dirichlet form $\mathcal{E}$ on $L^2(E;m)$ via [Sta99, Theorem IV.2.2] (cf. Theorem 2.9 below) solves the martingale problem for its generator (cf. e.g. [AKR03, Section 5],[AR95, Section 3],[PR02, Theorem 7.4(ii) and Proposition 8.2]). In Section 4 we combine these hints to give a *complete* proof in a general setting. We do not need to restrict to the case of f (cf. (1.3)) being e.g. C_0^∞, but consider any

bounded $\mathcal{E}$-quasi-continuous f in the domain of the generator. (For the notion of $\mathcal{E}$-quasi-continuity, cf. Section 2.)
Moreover, we prove in Section 5 that a process $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ generated by a generalized Dirichlet form $\mathcal{E}$ is a diffusion for many initial points, if the generator $(L, D(L))$ fulfills a locality condition we give in Theorem 5.5. One may find another proof for the diffusion property by reading [Sta04, Proof of Proposition 1.10] carefully.

Let us briefly summarize our core results:

- Given a bounded N-particle potential having bounded weak derivatives, we construct for many initial distributions an N-particle Langevin dynamics on a bounded rectangular area with periodic boundary condition, see Theorem 3.25, Corollary 3.27 and Remark 3.29.
- We prove that a process which is constructed via the theory of GDFs solves the martingale problem for the generator L of the associated GDF for many initial distributions on the subset of bounded functions in the domain $D(L)$ of L, see Corollary 4.9. This is done by proving a weaker result for the case of single initial points, see Theorem 4.7. Moreover, we prove that, if the initial distribution possesses an L^2-integrable density w.r.t. the reference measure m, the martingale problem is solved for any function in $D(L)$, see Corollary 4.11.
- Furthermore, we give a condition for L and its domain $D(L)$ ensuring that such a process is a diffusion for many initial points/distributions, see Theorem 5.5.

In future work we plan to construct (from the process provided here) a Markov process solving the Langevin equation (in the sense of the corresponding martingale problem) for any initial point. This may be done by showing that the associated operator semigroup (resolvent) has strong Feller properties as used in [Doh05], cf. [AKR03] and [FG06]. Another goal is to use the present results to generalize them to a larger class of potentials via an approximation. We are having in mind potentials of Lennard-Jones type as used in the theory of fluids.
The construction of the N-particle Langevin dynamics we consider as a starting point to construct an infinite particle/infinite volume Langevin dynamics, using similar techniques as used in [GKR04]. In this context it is important to show that the constructed process solves the martingale problem, because this property is essential for deriving scaling limits of the Langevin dynamics, see e.g. [OT03], [Spo86], [GKLR03].

At first, let us summarize the most important facts from the theory of generalized Dirichlet forms in Section 2. We then proceed by first considering our application in Section 3 and finally presenting in Sections 4 and 5 the proofs for the martingale property and the diffusion property in the case of generalized Dirichlet forms.

2. Generalized Dirichlet Forms

Throughout this section let $L^2(E; m)$ be the Hilbert space of (classes of) ($\mathcal{B}(E)$-measurable) L^2-integrable functions on a Hausdorff topological space E w.r.t. a σ-finite measure m on the Borel σ-field $\mathcal{B}(E)$ on E. As usual (cf. [MR92, Chapter VI] and [Sta99, Chapter IV]) we assume that $\sigma(C(E)) = \mathcal{B}(E)$. We denote the inner product and the norm of $L^2(E; m)$ by $(\cdot, \cdot)_{L^2(E;m)}$ and $\|\cdot\|_{L^2(E;m)}$, respectively. We will make use of basic knowledge on m-dissipative operators, strongly continuous contraction resolvents and C_0 semigroups (cf. e.g. [MR92, Sections I.1, I.2], [RS75, Section X.8], [Dav80]).

Almost everything presented in this section is taken from [Sta99].

The basic setting of a generalized Dirichlet form consists of a coercive closed form $(\mathcal{A},\mathcal{V})$ on $L^2(E;m)$ (cf. [Sta99][Definition I.1.4] or [MR92, Definition I.2.4]) and an operator $(\Lambda, D(\Lambda))$ fulfilling

(D1) $(\Lambda, D(\Lambda))$ is the generator of a C_0 contraction semigroup $(T_t)_{t\geq 0}$ on $L^2(E;m)$ (i.e. $(\Lambda, D(\Lambda))$ is m-dissipative) and $(T_t)_{t\geq 0}$ can be restricted to a C_0 semigroup on $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$.

Let $\mathcal{V}'$ be the dual space of $\mathcal{V}$. By identifying $L^2(E;m)$ with its dual we have $\mathcal{V}\subset L^2(E;m)\subset \mathcal{V}'$ densely and continuously. By ${}_{\mathcal{V}'}\langle\cdot,\cdot\rangle_{\mathcal{V}}$ we denote the dualization between $\mathcal{V}$ and $\mathcal{V}'$ (i.e. ${}_{\mathcal{V}'}\langle v,w\rangle_{\mathcal{V}} := v(w)$ for $v\in\mathcal{V}'$, $w\in\mathcal{V}$).

[Dav80, Theorem 1.34] tells us that the adjoint operator $(\hat{\Lambda}, D(\hat{\Lambda}))$ of $(\Lambda, D(\Lambda))$ is also m-dissipative and the adjoints $\hat{T}_t$ of T_t, $t\geq 0$, form the corresponding C_0 contraction semigroup. In [Sta99, Lemma I.2.4] it is shown that from (D1) it follows that there are bounded extensions of $\hat{T}_t$ to $\mathcal{V}'$ which form a C_0 semigroup. The generator of this semigroup is clearly an extension of $\hat{\Lambda}$ and it is also denoted by $\hat{\Lambda}$. We denote its domain by $D(\hat{\Lambda},\mathcal{V}')$. [Sta99, Lemma I.2.3] shows that furthermore the operator Λ with domain $\mathcal{V}\cap D(\Lambda)$ is closable as an operator mapping from $\mathcal{V}$ to $\mathcal{V}'$. Denote by $\mathcal{F}$ the domain of its closure, which we denote also by Λ. Clearly $\mathcal{F}$ is a Hilbert space if it is equipped with the graph norm

$$\|\cdot\|_{\mathcal{F}}^2 := \|\cdot\|_{\mathcal{V}}^2 + \|\Lambda\cdot\|_{\mathcal{V}'}^2$$

corresponding to $\Lambda:\mathcal{V}\to\mathcal{V}'$. Define moreover $\hat{\mathcal{F}} := \mathcal{V}\cap D(\hat{\Lambda},\mathcal{V}')$. Then also $\hat{\mathcal{F}}$, endowed with the norm $\|\cdot\|_{\hat{\mathcal{F}}}^2 := \|\cdot\|_{\mathcal{V}}^2 + \|\hat{\Lambda}\cdot\|_{\mathcal{V}'}^2$, is a Hilbert space, since the operator $\hat{\Lambda}: D(\hat{\Lambda},\mathcal{V}')\to\mathcal{V}'$ is (the generator of a C_0 contraction semigroup and hence) closed and $\mathcal{V}$ is a Hilbert space.

Definition 2.1. Let $(\mathcal{A},\mathcal{V})$, $(\Lambda, D(\Lambda))$ be as above, and assume that (D1) holds. Let $\mathcal{F}$ and $\hat{\mathcal{F}}$ be as above. The mapping

$$\begin{aligned}\mathcal{E}:\ \mathcal{F}\times\mathcal{V}\cup\mathcal{V}\times\hat{\mathcal{F}} &\to \mathbb{R}\\ (u,v) &\mapsto \begin{cases}\mathcal{A}(u,v) - {}_{\mathcal{V}'}\langle\Lambda u, v\rangle_{\mathcal{V}} & \text{if } u\in\mathcal{F}, v\in\mathcal{V}\\ \mathcal{A}(u,v) - {}_{\mathcal{V}'}\langle\hat{\Lambda} v, u\rangle_{\mathcal{V}} & \text{if } u\in\mathcal{V}, v\in\hat{\mathcal{F}}\end{cases}\end{aligned}$$

is said to be the *generalized Dirichlet form* (GDF) *associated with* $(\mathcal{A},\mathcal{V})$ *and* $(\Lambda, D(\Lambda))$, if

(D2) for all $u\in\mathcal{F}$ it holds $u^+\wedge 1\in\mathcal{V}$ and $\mathcal{E}(u, u-u^+\wedge 1)\geq 0$

is fulfilled.

For our application we need the following lemma. For the proof see [Sta99, Proposition I.4.7].

Lemma 2.2. *Let $(\Lambda, D(\Lambda))$ be an m-dissipative Dirichlet operator (i.e. Λ is m-dissipative and $(\Lambda u, (u-1)^+)) \leq 0$ for all $u \in D(\Lambda)$). Then*

$$\begin{aligned} \mathcal{E}: \; D(\Lambda) \times \mathcal{H} \cup \mathcal{H} \times D(\hat{\Lambda}) &\to \mathbb{R} \\ (u,v) &\mapsto \begin{cases} -(\Lambda u, v)_{L^2(E;m)} & \text{if } u \in D(\Lambda), v \in \mathcal{H} \\ -(\hat{\Lambda} v, u)_{L^2(E;m)} & \text{if } u \in \mathcal{H}, v \in D(\hat{\Lambda}) \end{cases} \end{aligned}$$

is a generalized Dirichlet form. Here $\mathcal{H} := L^2(E;m)$.

Let $\mathcal{E}$ be a GDF associated with a coercive closed form $(\mathcal{A}, \mathcal{V})$ and an operator $(\Lambda, D(\Lambda))$. By [Sta99, Proposition I.3.4] for each $\alpha > 0$ there exists a mapping $W_\alpha : \mathcal{V}' \to \mathcal{F}$ fulfilling

$$\mathcal{E}_\alpha(W_\alpha v, w) = {}_{\mathcal{V}'}\langle v, w \rangle_{\mathcal{V}} \quad \text{for all } v \in \mathcal{V}', w \in \mathcal{V}, \tag{2.1}$$

where

$$\mathcal{E}_\alpha(f,g) := \alpha(f,g)_{L^2(E;m)} + \mathcal{E}(f,g) \quad \text{ for } (f,g) \in \mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \hat{\mathcal{F}}.$$

The restrictions G_α of W_α to $L^2(E;m)$, $\alpha > 0$, form a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$ (cf. [MR92][Definition I.1.4]). Hence there exists an associated m-dissipative generator $L = \alpha - G_\alpha^{-1}$ with domain $D(L) \subset L^2(E;m)$ and also a C_0 contraction semigroup $(T_t)_{t\geq 0}$ generated by L.

Definition 2.3. $(G_\alpha)_{\alpha>0}$ as above is called the strongly continuous contraction resolvent *associated with* $\mathcal{E}$, and $(T_t)_{t\geq 0}$ and $(L, D(L))$ are said to be the *semigroup and generator associated with* $\mathcal{E}$, respectively.

Remark 2.4. If $\mathcal{A} = 0$, the generator $(L, D(L))$ coincides with $(\Lambda, D(\Lambda))$ (cf. [Sta99, Remark I.4.10]).

It is possible (cf. [Sta99, Section III]) to define the notions of $\mathcal{E}$-nests, $\mathcal{E}$-exceptional sets, properties which hold $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.), $\mathcal{E}$-quasi-uniformly convergent sequences, $\mathcal{E}$-quasi-continuity ($\mathcal{E}$-q.c.) similar to the case of coercive Dirichlet forms (cf. [MR92, Section III]), and there also are Choquet capacities characterizing $\mathcal{E}$-nests and $\mathcal{E}$-exceptional sets as in [MR92, Section III.2]. Again quasi-regularity of a (generalized) Dirichlet form is defined as follows:

Definition 2.5. A generalized Dirichlet form $\mathcal{E}$ is called *quasi-regular*, if it fulfills:

- (q1) There exists an $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact sets.
- (q2) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.
- (q3) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\tilde{u}_n$ and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

We summarize some properties of generalized Dirichlet forms which are also in principle known from/similar as in the theory of coercive Dirichlet forms. They are important for our further considerations, especially in Sections 4 and 5:

Remark 2.6. (i) Countable unions of $\mathcal{E}$-exceptional sets are $\mathcal{E}$-exceptional. Moreover every $\mathcal{E}$-exceptional set is contained in a null set w.r.t. m (cf. [Sta99, Remark III.2.6]).

(ii) A sufficient condition for an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E to be an $\mathcal{E}$-nest is given by

$$\bigcup_{k\in\mathbb{N}} \mathcal{F}_{F_k} \text{ is dense in } \mathcal{F}$$

where we define $\mathcal{F}_A := \{f \in \mathcal{F} | f = 0 \text{ on } E \setminus A\}$ for $\mathcal{F} \subset L^2(E;m)$ and $A \subset E$, A closed. (cf. [Sta99, Remark III.2.11])

(iii) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ and assume that for each $n \in \mathbb{N}$ there exists an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}_n$ of u_n. Assume in addition that $u_n \to u$ in $\mathcal{F}$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an $\mathcal{E}$-q.c. m-version $\tilde{u}$ of u such that $\lim_{k\to\infty} \tilde{u}_{n_k} \to \tilde{u}$ $\mathcal{E}$-quasi-uniformly. (cf. [Sta99, Corollary III.3.8])

(iv) If $\mathcal{E}$ is quasi-regular, then by (iii) and (q2) every $f \in \mathcal{F}$ possesses a quasi-continuous m-version $\tilde{f}$.

(v) Let $\mathcal{E}$ be quasi-regular and let f, g be two $\mathcal{E}$-quasi-continuous functions which coincide m-a.e.. Then they coincide even $\mathcal{E}$-q.e. (cf. [Sta99, Corollary III.3.4 and Lemma III.3.5]). In particular, any two $\mathcal{E}$-q.c. m-versions of the same element in $L^2(E;m)$ coincide $\mathcal{E}$-q.e..

For the definition of an m-tight special standard process (and the definition of a right process etc.) $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with state space E and life time $\zeta : \Omega \to [0,\infty]$ we refer to [Sta99, Section IV.1] or [MR92, Section IV.1]. Here $E_\Delta := E \cup \{\Delta\}$ denotes the extension of E by an isolated point Δ (the cemetery), which is used as the state of the process at times greater or equal ζ. Any function $f : E \to \mathbb{R}$ is extended to E_Δ by setting $f(\Delta) := 0$. As mentioned in [MR92, Remark IV.1.10] (and [Sta99, Remark IV.1.3(i)]) we can choose as corresponding filtration $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of $\mathbb{M}$ for any such process and we may assume that $\mathcal{F}_*$ is the smallest σ-field containing all $\mathcal{F}_t$, $t \geq 0$.

In [MR92, p. 91] the transition semigroup $(p_t)_{t>0}$ of a right process $\mathbb{M}$ as above is defined by

$$p_t f(x) := E_x[f(X_t)] \tag{2.2}$$

for $x \in E$, $t > 0$ and nonnegative $\mathcal{B}(E)$-measurable real-valued functions f. As in [MR92, Section II.4a] we define $p_t f(x) := p_t f^+(x) - p_t f^-(x)$, $x \in E$, $t > 0$, for any $\mathcal{B}(E)$-measurable f for which $p_t f^+(x)$ or $p_t f^-(x)$ is finite. Here f^+, f^- denote the positive and negative part of f, respectively. Moreover (see [MR92, p.91]) the transition resolvent $(R_\alpha)_{\alpha>0}$ of $\mathbb{M}$ is defined by

$$R_\alpha f(x) := E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt\right] \tag{2.3}$$

for $\alpha > 0$, $x \in E$ and $\mathcal{B}(E)$-measurable nonnegative f (or $\mathcal{B}(E)$-measurable f such that $R_\alpha f^+(x)$ or $R_\alpha f^-(x)$ is finite).

Like in the theory of coercive Dirichlet forms a quasi-regular generalized Dirichlet form $\mathcal{E}$ can be used to construct a stochastic process, but an additional condition has to be fulfilled by $\mathcal{E}$ (cf. (D3) in Theorem 2.9 below). Let us first discuss the notion of proper association of stochastic processes with GDFs.

Definition 2.7. Let $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a right process with transition resolvent $(R_\alpha)_{\alpha>0}$. Let $\mathcal{E}$ be a generalized Dirichlet form with associated strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$. $\mathbb{M}$ is said to be *properly associated with $\mathcal{E}$ in the resolvent sense*, if for every $\alpha > 0$, $f \in L^2(E;m)$, with bounded m-version $\hat{f}$, the function $R_\alpha \hat{f}$ is an $\mathcal{E}$-quasi-continuous m-version of $G_\alpha f$.

In the sequel we sometimes need the following lemma.

Lemma 2.8. *Let $\mathbb{M}$, $\mathcal{E}$, $(R_\alpha)_{\alpha>0}$, $(G_\alpha)_{\alpha>0}$ be as in Definition 2.7 and denote by $(p_t)_{t>0}$ the transition semigroup of the right process $\mathbb{M}$ and by $(T_t)_{t\geq 0}$ the C_0 contraction semigroup associated with $\mathcal{E}$. Then it holds (cf.* [MR92, Exercise 2.7 and Exercise 2.9]*)*

(i) For every $t > 0$ and $f \in L^2(E;m)$ with bounded m-version $\hat{f}$, the function $p_t\hat{f}$ is an m-version of $T_t f$.

(ii) For every $t > 0$ ($\alpha > 0$) and for every $f \in L^2(E;m)$ with m-version $\hat{f}$ the function $p_t\hat{f}$ ($R_\alpha\hat{f}$) is an m-version of $T_t f$ ($G_\alpha f$). Moreover, $R_\alpha\hat{f}$ is quasi-continuous.

Proof. Let $f, f_n \in L^2(E;m)$ with nonnegative m-versions $\hat{f}, \hat{f}_n$ ($n \in \mathbb{N}$), such that $\hat{f}_n \uparrow \hat{f}$. Assume that $p_t\hat{f}_n$ is an m-version of $T_t f_n$ for each $n \in \mathbb{N}$ and for some $t > 0$. Then by the monotone convergence theorem it holds $p_t f_n \uparrow p_t f$ pointwise and again by the monotone convergence theorem we find that $p_t f \in L^2(E;m)$ and from Lebesgue's dominated convergence theorem we conclude convergence in $L^2(E;m)$. Moreover, it clearly holds $T_t f_n \to T_t f$ in $L^2(E;m)$, hence $p_t f$ is an m-version of $T_t f$.

Let us now prove (i): By the considerations above and a monotone class argument (and since $\sigma(C(E)) = \mathcal{B}(E)$) we may assume that $\hat{f} \in C(E)$. Moreover, we can clearly restrict our considerations to the case when $\hat{f} \geq 0$.
Note that by continuity and boundedness of $\hat{f}$ and right continuity of $\mathbb{M}$ Lebesgue's dominated convergence theorem implies that for $x \in E$, $t \geq 0$ and for any sequence $t_n \downarrow t$ it holds

$$\lim_{n\to\infty} (p_{t_n}\hat{f})(x) = \lim_{n\to\infty} E_x[\hat{f}(X_{t_n})] = E_x[\hat{f}(X_t)] = (p_t\hat{f})(x)$$

Consequently, the mapping $t \mapsto (p_t\hat{f})(x)$ is right continuous for every $x \in E$.
Let v be a bounded nonnegative measurable function on E fulfilling $m(\{v > 0\}) < \infty$. Then for $\alpha > 0$ it holds by our assumption and by Fubini's theorem

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}(v, p_t\hat{f})_{L^2(E;m)}\,dt &= \left(v, \int_0^\infty e^{-\alpha t}(p_t\hat{f})(\cdot)\,dt\right)_{L^2(E;m)} = (v, R_\alpha\hat{f})_{L^2(E;m)} \\
&= (v, G_\alpha f)_{L^2(E;m)} = \left(v, \int_0^\infty e^{-\alpha t}T_t f\,dt\right)_{L^2(E;m)}
\end{aligned}$$

The integral on the right-hand side is considered as a Riemann integral; clearly the right-hand side is equal to

$$\int_0^\infty e^{-\alpha t}(v, T_t f)_{L^2(E;m)}\,dt$$

By the injectivity of the Laplace transform (cf. [DS58, Lemma VIII.1.15]) and by right continuity of the mappings $t \mapsto (v, p_t\hat{f})_{L^2(E;m)}$ (here we again use Lebesgue's theorem) and $t \mapsto (v, T_t f)_{L^2(E;m)}$ we conclude that for all $t > 0$ it holds $(v, T_t f)_{L^2(E;m)} = (v, p_t\hat{f})_{L^2(E;m)}$.
Since the measure m is σ-finite and since the linear span of the set of functions v as above is dense in $L^2(E;m)$, we easily find that $p_t\hat{f}$ is L^2-integrable and an m-version of $T_t f$, which we desired to prove.

To prove (ii) let $f \in L^2(E;m)$ with m-version $\hat{f}$. W.l.o.g. we may assume that $\hat{f} \geq 0$. We define $\hat{f}_n := \hat{f} \wedge n$. Then by the considerations at the beginning of this proof and by (i) it follows that $p_t\hat{f}$ is an m-version of $T_t f$. In the same way we can prove the corresponding result for R_α and G_α.
Finally, to prove the last assertion we note that $G_\alpha f_n$ converges to $G_\alpha f$ not only in $L^2(E;m)$, but also in $\mathcal{F}$, since $L^2(E;m) \subset \mathcal{V}'$ continuously and G_α is the restriction of the continuous operator $W_\alpha : \mathcal{V}' \to \mathcal{F}$ to $L^2(E;m)$. Hence by Remark 2.6(iii) the pointwise limit $R_\alpha f$ of $(R_\alpha f_n)_{n\in\mathbb{N}}$ is $\mathcal{E}$-quasi-continuous. □

Now we state the existence theorem, which can be found in [Sta99, Theorem IV.2.2].

Theorem 2.9. *Let $\mathcal{E}$ be a quasi-regular generalized Dirichlet form and let $\mathcal{F}$ be defined as above. Assume that it holds*

(D3) There exists a linear subspace $\mathcal{Y} \subset L^2(E;m) \cap L^\infty(E;m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$ and $\lim_{\alpha\to\infty}(\alpha G_\alpha u - u)_E = 0$ in $L^2(E;m)$ for all $u \in \mathcal{Y}$. Moreover, for all $\alpha \geq 0$, it holds $u \wedge \alpha \in \overline{\mathcal{Y}}$, where $\overline{\mathcal{Y}}$ denotes the closure of $\mathcal{Y}$ in $L^\infty(E;m)$.

Then there exists an m-tight special standard process $\mathbb{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$.

Remark 2.10. $(\alpha G_\alpha u - u)_E$ in Theorem 2.9 above denotes the 1-reduced function of $\alpha G_\alpha u - u$ (cf. [Sta99, Definition III.1.8]). We do not need to consider details about this notion here, since we use the following proposition (cf. [Sta99, Proposition 2.1]).

Proposition 2.11. *In the situation of Theorem 2.9 assume that there exists a linear subspace $\mathcal{Y} \subset \mathcal{F} \cap L^\infty(E;m)$, which is dense in $\mathcal{F}$ and closed under multiplication. Then (D3) holds for $\mathcal{Y}$.*

In the sequel we make use of the following result (cf. [Sta99, Lemma IV.3.10]), which tells us that a process $\mathbb{M}$ as in Theorem 2.9 "does not hit" $\mathcal{E}$-exceptional sets. For $U \subset E$, U open, we define $\sigma_U := \inf\{t > 0 | X_t \in U\} = \inf\{t \geq 0 | X_t \in U\}$. σ_U is called the first hitting time of U. We set $\sigma_U := \infty$, if $\{t \geq 0 | X_t \in U\}$ is empty.

Lemma 2.12. *Let $\mathbb{M}$ be an m-tight m-special standard process with life time ζ properly associated in the resolvent sense with a GDF $\mathcal{E}$. Then for any $\mathcal{E}$-nest $(E_n)_{n\in\mathbb{N}}$ it holds*

$$P_x\left(\lim_{n\to\infty} \sigma_{E\setminus E_n} < \zeta\right) = 0 \quad \text{for } \mathcal{E}\text{-q.e. } x \in E$$

Hence if $N \subset E$ is $\mathcal{E}$-exceptional, then $P_x(\exists t \geq 0 : X_t \in N) = 0$ for $\mathcal{E}$-q.e. $x \in E$.

3. An N-particle Langevin dynamics

We now construct a solution to (1.1) in the sense of the corresponding martingale problem, where we consider (1.1), as we mentioned in the Introduction, to be an equation on the manifold resulting from considering the cuboid area of motion in $\mathbb{R}^d$ to have periodic boundary. We first make this setting more precise.

3.1. **The setting.**

3.1.1. *The state space E.* As we mentioned in the Introduction we consider the Langevin equation for N particles moving in a rectangular area in $\mathbb{R}^d$. To simplify notations without losing generality we may assume that the Langevin equation describes the motion of 1 particle moving in $[0, r_1] \times \cdots \times [0, r_{Nd}] \subset \mathbb{R}^{Nd}$. We set $n := Nd$.

In the sequel we often consider functions on the sets $\widetilde{M} := (0, r_1) \times \cdots \times (0, r_n)$, $\mathbb{R}^n$ and $\widetilde{E} := \widetilde{M} \times \mathbb{R}^n$. Throughout the whole section we denote

- an element of $\widetilde{E}$ (or $\mathbb{R}^{2n}$) usually by (x, v), which is to be understood in the sense that $x = (x_1, \cdots, x_n) \in \widetilde{M}$ (or $\mathbb{R}^n$) and $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$.
- by $\partial_{x_1} f, \cdots, \partial_{x_n} f$ the (weak) partial derivatives of $f : \widetilde{M} \to \mathbb{R}$.
- by $\partial_{v_1} f, \cdots, \partial_{v_n} f$ the (weak) partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$.
- by $\partial_{x_1} f, \cdots, \partial_{x_n} f, \partial_{v_1} f, \cdots, \partial_{v_n} f$ the (weak) partial derivatives of $f : \widetilde{E} \to \mathbb{R}$.

We moreover define the formal differential operators ∇_v, ∇_x, Δ_v, $v\nabla_v := v_1\partial_{v_1} + \cdots + v_n\partial_{v_n}$ etc. in the obvious way.
For $x \in \partial\widetilde{M}$, which means $x_i \in \{0, r_i\}$ for some $1 \le i \le n$, we define the opposite point $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$ by

$$\bar{x}_i := \begin{cases} r_i & \text{if } x_i = 0 \\ 0 & \text{if } x_i = r_i \\ x_i & \text{else} \end{cases} \qquad 1 \le i \le n$$

We define

$$C^\infty_{per}(\widetilde{M}) := \{f|_{\widetilde{M}} \,|\, f \in C^\infty(\mathbb{R}^n), f(x) = f(\bar{x}) \,\forall x \in \partial\widetilde{M}, \text{ and the same holds for any derivative of } f\}$$

$$C^\infty_{per,0}(\widetilde{E}) := \{f|_{\widetilde{E}} \,|\, f \in C^\infty_0(\mathbb{R}^{2n}), f(x,v) = f(\bar{x},v) \,\forall (x,v) \in \partial\widetilde{E}, \text{ and the same holds for any derivative of } f\} \tag{3.1}$$

and moreover

$$H^{1,\infty}_{per}(\widetilde{M}) := \{f \in H^{1,\infty}(\widetilde{M}) | f(x) = f(\bar{x}) \,\forall x \in \partial\widetilde{M}\}$$

where $H^{1,\infty}(\widetilde{M})$ denotes the Sobolev space of once weakly differentiably functions $f : \widetilde{M} \to \mathbb{R}$, such that f and its weak partial derivatives are elements of $L^\infty(\widetilde{M}, dx)$ (cf. [Alt02, 1.23]). Note that by [Alt02, Satz 8.5] the elements of $H^{1,\infty}$ have Lipschitz continuous dx-versions, thus $f(x)$ is well-defined for $x \in \partial\widetilde{M}$, $f \in H^{1,\infty}(\widetilde{M})$.
We need to know the following (obvious) facts about $H^{1,\infty}(\widetilde{M})$:

Lemma 3.1. *(i) $H^{1,\infty}(\widetilde{M})$ is an algebra of functions and the product rule holds. The same is true for $H^{1,\infty}_{per}(\widetilde{M})$.*

(ii) Let $f \in H^{1,\infty}(\widetilde{M})$. Then $e^f \in H^{1,\infty}(\widetilde{M})$ and $\nabla_x e^f = (\nabla_x f)e^f$. The same is true for $H^{1,\infty}_{per}(\widetilde{M})$.

We define an n-dimensional manifold M with the help of the equivalence relation $\sim$, given by $x \sim x' :\Leftrightarrow x - x' \in \{(z_1 r_1, \cdots, z_n r_n) | (z_1, \cdots, z_n) \in \mathbb{Z}^n\}$ for $x, x' \in \mathbb{R}^n$. We define $M := \mathbb{R}^n / \sim$. Let $\pi^M : \mathbb{R}^n \to M$ be the mapping which assigns to every $x \in \mathbb{R}^n$ its equivalence class $\pi^M(x) = [x] \in M$ w.r.t. $\sim$ (which we also denote by x in the sequel). If we equip M with the quotient topology w.r.t. π^M, M is a second countable Hausdorff space.
We define for every $x^0 = (x^0_1, \cdots, x^0_n) \in \mathbb{R}^n$ the restriction $\pi^M_{x^0}$ of π^M to $(x^0_1, x^0_1 + r_1) \times \cdots \times (x^0_n, x^0_n + r_n)$. We can use the charts $(\pi^M_{x^0})^{-1}$ to define a (quite natural) differentiable structure on M. We define the global vector fields $\partial^M_{x_1}, \cdots, \partial^M_{x_n}$ to be the images of $\partial_{x_1}, \cdots, \partial_{x_n}$ under the differential mappings of $\pi^M_{x^0}$, $x^0 \in \mathbb{R}^n$. Of course, we denote by $C^\infty(M)$ the space of infinitely often differentiable functions on M.

Moreover, we define the manifold E to be the product manifold $E := M \times \mathbb{R}^n$. This manifold is the state space for our process. We define $\pi := \pi^M \times id^{\mathbb{R}^n} : \mathbb{R}^{2n} \to E$ (i.e. $\pi(x, v) := ([x], v)$, $(x, v) \in \mathbb{R}^{2n}$) and moreover for $x^0 \in \mathbb{R}^n$ we define $\pi_{x^0} := \pi^M_{x^0} \times id^{\mathbb{R}^n}$. π_{x^0} is the restriction of π to $(x^0_1, x^0_1 + r_1) \times \cdots \times (x^0_n, x^0_n + r_n) \times \mathbb{R}^n$. The global vector fields $\partial^E_{x_1}, \cdots, \partial^E_{x_n}, \partial^E_{v_1}, \cdots, \partial^E_{v_n}$ are defined in the same way as the corresponding vector fields on M. We define ∇^E_v, ∇^E_x, ∇^M_x, Δ^E_v etc. in the same way as the notations we fixed above. Let $C^\infty_0(E)$ denote the set of all infinitely often differentiable functions on E having compact support.

We consider any mapping Φ on M also as a mapping on $E = M \times \mathbb{R}^n$ by defining $\Phi(x, v) := \Phi(x)$, $x \in M$, $v \in \mathbb{R}^n$.

Remark 3.2. It is easy to see that there is a countable set $\mathcal{G}$ of nonnegative functions in $C^\infty_0(E)$ such that the open sets $\{x \in E | u(x) > 0\}$, $u \in \mathcal{G}$ form a base of the topology on E. Of course, this implies that $\sigma(C(E)) = \mathcal{B}(E)$.

3.1.2. *L^2-spaces on E and M.* With the help of the mappings $\pi^M_{x^0}$, $x^0 \in \mathbb{R}^n$, it is also possible to transfer the Lebesgue measure dx on $\mathbb{R}^n$ to the manifold M, or, to be precise, to the measurable space $(M, \mathcal{B}(M))$:

Definition 3.3. Let $x^0 \in \mathbb{R}^n$. For $A \in E$ we define $dx^M(A) := dx\left((\pi^M_{x^0})^{-1}(A)\right)$, i.e. we define dx^M to be the image measure of dx under $\pi^M_{x^0}$.

Clearly, this definition is independent of the choice of x^0.

By the definition of dx^M the set $\pi^M(\widetilde{M})$ contains already the total mass. Hence, if we define for any function $f : \widetilde{M} \to \mathbb{R}$ another function $\hat{\Pi}^M f : \pi^M(\widetilde{M}) \to \mathbb{R}$ by $\hat{\Pi}^M f := f \circ (\pi^M_0)^{-1}$, the mapping $\hat{\Pi}^M$ leads to a bijection Π^M between dx-classes of functions on $\widetilde{M}$ and dx^M-classes of functions on M.
Moreover, it is clear that $\Pi^M : L^2(\widetilde{M}; dx) \to L^2(M; dx^M)$ is a unitary transformation.

Note that $\hat{\Pi}^M$ maps $C^\infty_{per}(\widetilde{M})$ bijectively onto $C^\infty(M)$ (in the obvious sense: each function in $C^\infty(M)$ is uniquely determined by its restriction to $\pi^M(\widetilde{M})$), hence Π^M maps $C^\infty_{per}(\widetilde{M}) \subset L^2(\widetilde{M}; dx)$ bijectively onto $C^\infty(M) \subset L^2(M; dx^M)$.

We define on $(E, \mathcal{B}(E))$ the measure $d(x,v)^E$ to be the product measure $d(x,v)^E := dx^M \otimes dv$, where dv denotes the Lebesgue measure on $\mathbb{R}^n$. We define mappings $\hat{\Pi}$, Π for E analogously to $\hat{\Pi}^M$ and Π^M for M, and clearly we get similar results as above. Moreover, note that Π also gives a one-to-one-correspondence between measures μ on $(E, \mathcal{B}(E))$ which are absolutely continuous w.r.t. $d(x,v)^E$ with $m = \frac{d\mu}{d(x,v)^E}$, and measures $\widetilde{\mu}$ on $(\widetilde{E}, \mathcal{B}(\widetilde{E}))$ which are absolutely continuous w.r.t. $d(x,v)$ such that $\widetilde{m} = \frac{d\widetilde{\mu}}{d(x,v)}$, in the sense that $\Pi(\widetilde{m}) = m$. For such a pair $\mu, \widetilde{\mu}$ of measures clearly $\Pi : L^2(\widetilde{E}; \widetilde{\mu}) \to L^2(E; \mu)$ is also a unitary transformation. Clearly Π and Π^M (and their inverses) transform any m-dissipative operator into an m-dissipative operator, the semigroup corresponding to the former one into the semigroup corresponding to the latter one etc. Moreover they transform Dirichlet operators into Dirichlet operators, positivity preserving operators into positivity preserving operators etc.

Remark 3.4. For $\widetilde{\mu}, \mu$ as above any considerations about (differential operators on) the spaces $L^2(\widetilde{E}; \widetilde{\mu})$, $C^\infty_{per,0}(\widetilde{E})$ are also valid for (the corresponding differential operators on) $L^2(E; \mu)$ and $C_0^\infty(E)$, as long as we do not have to include global topological properties of E or $\widetilde{E}$, which is e.g. important to prove quasi-regularity of a generalized Dirichlet form on $L^2(E; \mu)$ (cf. Definition 2.5).

We define

$$H^{1,\infty}(M) := \Pi^M H^{1,\infty}_{per}(\widetilde{M})$$

Remark 3.5. The above definition of $H^{1,\infty}(M)$ is easily verified to be independent of the choice of the natural chart used in the definition of $\hat{\Pi}$ and hence of Π. It consists exactly of those functions $f : M \to \mathbb{R}$ in $L^\infty(M, dx^M)$ which fulfill

$$\int_M f \,\partial^E_{x_i} \psi \, dx^M = - \int_M f_i \psi \, dx^M \quad \text{for all } \psi \in C^\infty(M) \tag{3.2}$$

for some $f_1, \cdots, f_n \in L^\infty(M, dx^M)$.

3.1.3. *The Langevin equation on E.* We consider the stochastic differential equation

$$d(X_t, V_t) = -\gamma V_t \nabla_v dt - (\nabla_x \Phi^M(X_t)) \nabla_v dt + V_t \nabla_x dt + \sum_{k=1}^n \sqrt{\frac{2\gamma}{\beta}} \partial^E_{v_k} dB^k_t \tag{3.3}$$

which has to be understood in the sense of [Swa00]. $(B_t)_{t \geq 0} = (B^1_t, \cdots, B^n_t)_{t \geq 0}$ denotes n-dimensional Brownian motion. We have to choose a connection on E to state this Itô stochastic equation properly. Of course, we use the connection resulting from the natural Riemannian metric d on E defined by $d(\partial_i, \partial_j) = \delta_{ij}$ when $i, j \in \{x_1, \cdots, x_n, v_1, \cdots, v_n\}$. To stay consistent with [Swa00] we would have to use C^∞ potentials. But a solution of (3.3) in the sense of the corresponding martingale problem (see below) can also be defined for more general potentials. We later specify the type of potentials Φ^M we want

to consider.
Let $(X_t, V_t)_{t\geq 0}$ be an E-valued stochastic process with law P defined on a measurable space $(\Omega, \mathcal{M})$, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. P is said to solve (3.3) in the sense of the corresponding martingale problem if it fulfils the martingale problem for the operator $(L^E, C_0^\infty(E))$, which is defined by

$$(L^E f)(x,v) = \frac{\gamma}{\beta}\Delta_v^E f(x,v) - \gamma v \nabla_v^E f(x,v) + v\nabla_x^E f(x,v) - (\nabla_x^M \Phi^M(x))\nabla_v^E f(x,v) \tag{3.4}$$

for $f \in C_0^\infty(E)$. This means, that for any $f \in C_0^\infty(E)$ the process $(M_t^{[f]})_{t\geq 0}$ defined by

$$M_t^{[f]} := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale wrt. P. The image measure $P \circ (X_0, V_0)^{-1}$ of P under (X_0, V_0) is called the *initial distribution* of the solution. Our aim is to find solutions of (3.3) in the sense of the corresponding martingale problem for many initial distributions.

Remark 3.6. Assume that $\Phi : \mathbb{R}^n \to \mathbb{R}$ is such that its (weak) partial derivatives (exist and) are measurable and bounded. If we consider the Langevin equation on $\mathbb{R}^{2n}$ (cf. (1.1)), we find that the operator corresponding to it via the Itô formula (cf. [Dur96, 2.10.2]) is given by

$$L := \frac{\gamma}{\beta}\Delta_v - \gamma v\nabla_v + v\nabla_x - (\nabla_x \Phi)\nabla_v. \tag{3.5}$$

Let us (also in the sequel) consider this operator to be acting on $C_{per,0}^\infty(\widetilde{E})$. Then, if $\Phi = (\hat{\Pi}^M)^{-1}\Phi^M$, it corresponds to $(L^E, C_0^\infty(E))$ in the sense of Remark 3.4. Thus we can assume that (3.3) is a reasonable formulation of the Langevin equation on E.

The type of potentials Φ, Φ^M we want to deal with is described by the following condition:

Condition 3.7. $\Phi \in H_{per}^{1,\infty}(\widetilde{M})$ and $\Phi^M = \Pi^M \Phi (\in H^{1,\infty}(M))$.

Below, Φ, Φ^M always denote functions as in 3.7. Note that when considering the operator L^E (or L) on $L^2(E;\mu)$ (or $L^2(\widetilde{E};\widetilde{\mu})$) such that μ (or $\widetilde{\mu}$) is equivalent to the measure $d(x,v)^E$ (or $d(x,v)$), we do not need to fix versions of Φ^M (or Φ) and its weak partial derivatives to obtain well-definedness of the operator L^E (or L).

We have to choose an appropriate measure to fix the L^2 space on which we consider the generator L^E to be defined. We use the measure μ, defined by

$$\frac{d\mu}{d(x,v)^E}(x,v) = e^{-\beta v^2/2} e^{-\beta \Phi^M(x)}. \tag{3.6}$$

Except for normalization, μ is the canonical Gibbs measure, which is well-known as a candidate for being an invariant measure for the dynamics.

In order to apply Theorem 2.9 to obtain a generalized Dirichlet form corresponding to L^E we have to prove that $(L^E, C_0^\infty(E))$ is an m-dissipative operator on $L^2(E;\mu)$, with μ given as in (3.6), and that its closure is a Dirichlet operator.

Clearly, by Remark 3.4 we can as well consider the operator $(L, C^\infty_{per,0}(\widetilde{E}))$, defined as in (3.5), on $L^2(\widetilde{E};\widetilde{\mu})$, where $\frac{d\widetilde{\mu}}{d(x,v)} = e^{-\beta v^2/2}e^{-\beta\Phi(x)}$.
It is moreover easy to see that we can define $\hat{M} = (0,\hat{r}_1)\times\cdots\times(0,\hat{r}_n)$, $\hat{E} = \hat{M}\times\mathbb{R}^n$ and $\hat{\Phi}: \hat{M}\to\mathbb{R}$ such that the problem of proving essential m-dissipativity of $(L, C^\infty_{per,0}(\widetilde{E}))$ and the Dirichlet property of its closure is equivalent to proving these properties for $(\Delta_v - v\nabla_v + v\nabla_x - (\nabla_x\hat{\Phi})\nabla_v, C^\infty_{per,0}(\hat{E}))$ in $L^2(\hat{E},\hat{\mu})$, where $\frac{d\hat{\mu}}{d(x,v)} := e^{-v^2/2-\hat{\Phi}(x)}$. This shows that we may assume that $\beta = 1$ and $\gamma = 1$.

3.2. **Perturbations of essentially m-dissipative operators.** Before going on we make some considerations about perturbations of essentially m-dissipative operators. Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)_\mathcal{H}$ and corresponding norm $\|\cdot\|_\mathcal{H}$. We call an operator $(L, D(L))$ on $\mathcal{H}$ essentially m-dissipative, if its closure is m-dissipative (and thus generates a C_0 contraction semigroup). This means that its closure $(\overline{L}, D(\overline{L}))$ fulfills $(Lu,u)_\mathcal{H} \le 0$ for all $u \in D(\overline{L})$ (dissipativity) and $Range(1-\overline{L}) = \mathcal{H}$. Essential m-dissipativity of $(L, D(L))$ is equivalent to $(Lu,u)_\mathcal{H} \le 0$ for all $u \in D(L)$ and $\overline{Range(1-L)} = \mathcal{H}$.
The best known result on perturbations of (essentially) m-dissipative operators is Theorem 3.9 below (we slightly changed the "usual" assertion in order to be able to apply it directly below). We need the following definition.

Definition 3.8. Let $(A, D(A))$, $(B, D(B))$ be linear operators on $\mathcal{H}$. B is said to be A-bounded, if $D(B)\supset D(A)$ and there exist real numbers $a, b\ge 0$ such that

$$\|Bf\|_\mathcal{H} \le a\|Af\|_\mathcal{H} + b\|f\|_\mathcal{H} \tag{3.7}$$

for all $f\in D(A)$. The number $\inf\{a\ge 0 | (3.7) \text{ holds for some } b\ge 0\}$ is then called the A-bound of B.

Theorem 3.9. *Let (A,D) be an essentially m-dissipative operator on $\mathcal{H}$ and (B,D) be dissipative. Assume that B is A-bounded with A-bound less than 1. Then $(A+B,D)$ is essentially m-dissipative.*

For the proof of Theorem 3.9 we refer to [Dav80, Corollary 3.8, Lemma 3.9 and Problem 3.10].

A sufficient condition for A-boundedness is given in the following lemma, which is easy to prove.

Lemma 3.10. *Let $(A, D(A)), (B, D(B))$ be linear operators on $\mathcal{H}$ such that $D(B)\supset D(A)$ and for some $M\ge 0$ it holds*

$$\|Bf\|^2_\mathcal{H} \le \langle Af, f\rangle_\mathcal{H} + M\|f\|^2_\mathcal{H} \quad \text{for all } f\in D(A).$$

Then B is A-bounded with A-bound 0.

The idea for Lemma 3.12 below is taken from [Lei01, Lemma 2.1]. The situation we consider is that $\mathcal{H}$ can be represented as the direct sum of orthogonal subspaces in a way which allows A, B to be restricted to these subspaces such that for the restrictions we can apply Theorem 3.9.

Definition 3.11. A sequence $(P_n)_{n\in\mathbb{N}}$ of continuous linear operators on $\mathcal{H}$ is called a *complete orthogonal family*, if every P_n, $n \in \mathbb{N}$, is an orthogonal projection such that for $n, m \in \mathbb{N}$, $n \neq m$, it holds $P_n P_m = 0$ and for every $f \in \mathcal{H}$ it holds $f = \sum_{n=1}^{\infty} P_n f$.

Lemma 3.12. *Let (A, D) be an essentially m-dissipative operator and (B, D) be dissipative. Suppose that there is a complete orthogonal family $(P_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned} P_n D &\subset D, \\ P_n A &= A P_n, \\ P_n B &= B P_n. \end{aligned}$$

Define $A_n := AP_n, B_n := BP_n$, both with domain $D_n := P_n D \subset (P_n\mathcal{H}) \cap D$, as operators in $P_n\mathcal{H}$. Assume that each B_n is A_n-bounded with A_n-bound less than 1. Then $(A+B, D)$ is essentially m-dissipative.

Proof. Let $n \in \mathbb{N}$. Clearly A_n and B_n are dissipative. Define $C_n := A_n + B_n$, and denote in the sequel by I_n the identity operator on $P_n\mathcal{H}$. Let $f \in \mathcal{H}, \varepsilon > 0$, then by essential m-dissipativity of A there is $g \in D$ such that $\|(I - A)g - f\|_{\mathcal{H}} \leq \varepsilon$. Hence $\|(I_n - A_n)P_n g - P_n f\|_{\mathcal{H}} = \|P_n[(I - A)g - f]\|_{\mathcal{H}} \leq \varepsilon$. Thus $Range(I_n - A_n)$ is dense in $P_n\mathcal{H}$, which proves that A_n is essentially m-dissipative. Consequently each (C_n, D_n), $n \in \mathbb{N}$, is an essentially m-dissipative operator on $P_n\mathcal{H}$ by Theorem 3.9.
To show essential m-dissipativity of $(A + B, D)$ let $f \in \mathcal{H}, \varepsilon > 0$, then $f = \sum_{n=1}^{\infty} P_n f$. We have to find $g \in D$ such that $\|(I - (A+B))g - f\|_{\mathcal{H}} \leq \varepsilon$. Choose $N \in \mathbb{N}$ large enough such that

$$\left\| f - \sum_{n=1}^{N} P_n f \right\|_{\mathcal{H}} \leq \varepsilon/2. \tag{3.8}$$

Clearly, by essential m-dissipativity of (C_n, D_n) we find $g_n \in D_n$, $1 \leq n \leq N$, such that

$$\|P_n f - (I_n - C_n)g_n\|_{\mathcal{H}} \leq \frac{\varepsilon}{2N}. \tag{3.9}$$

Let $g := \sum_{i=1}^{N} g_n$, then $g \in D$ and $(I - (A+B))g = \sum_{i=1}^{N}(I - (A+B))P_n g = \sum_{i=1}^{N}(I_n - C_n)g_n$. Hence by (3.8) and (3.9) we obtain $\|(I - (A+B))g - f\|_{\mathcal{H}} \leq \varepsilon$. □

3.3. **The generator for the generalized Dirichlet form.** In this section we show that, if the potential Φ fulfills condition 3.7, the differential operator $L = \Delta_v - v\nabla_v + v\nabla_x - \nabla_x\Phi(x)\nabla_v$ with domain $C^{\infty}_{per,0}(\widetilde{E}) \subset L^2(\widetilde{E}; \widetilde{\mu})$, where $\frac{d\widetilde{\mu}}{d(x,v)} = e^{-v^2/2}e^{-\Phi}$, is essentially m-dissipative and its closure is a Dirichlet operator. Clearly, by the considerations in section 3.1 the same then holds for $(L^E, C_0^{\infty}(E))$ (on $L^2(E; \mu)$, where L^E is defined as in (3.4) and μ is given by (3.6)). We find it considerably easy to prove in Section 3.4 that the associated GDF is quasi-regular and fulfills condition (D3) in Theorem 2.9, since we have that the space $C_0^{\infty}(E)$ is a core for the generator.

Theorem 3.13. *Let Φ be as in condition 3.7, and define the measure $\widetilde{\mu}$ by $\frac{d\widetilde{\mu}}{d(x,v)} = \widetilde{m}$, where $\widetilde{m} : \widetilde{E} \to \mathbb{R}$ is given by $\widetilde{m}(x, v) := e^{-v^2/2}e^{-\Phi(x)}$ for $(x, v) \in \widetilde{E}$.*

Then the operator $L : C^\infty_{per,0}(\widetilde{E}) \to L^2(\widetilde{E};\widetilde{\mu})$*, defined by*

$$L = \Delta_v - v\nabla_v + v\nabla_x - (\nabla_x\Phi)\nabla_v \tag{3.10}$$

is essentially m-dissipative and its closure is a Dirichlet operator.

Theorem 3.13 is shown in the course of this section.

At first we state some basic properties of L and its summands:

Lemma 3.14. *Define* $(L, C^\infty_{per,0}(\widetilde{E}))$ *as in Theorem 3.13. We decompose* L *by* $L = S + A$*, where* $(S, C^\infty_{per,0}(\widetilde{E}))$ *and* $(A, C^\infty_{per,0}(\widetilde{E}))$ *are defined by* $S := \Delta_v - v\nabla_v$ *and* $A := v\nabla_x - (\nabla_x\Phi)\nabla_v$*. It holds*

(i) *S is symmetric and dissipative.*
(ii) *A is antisymmetric.*
(iii) *L is dissipative.*

Proof. Clearly, (iii) follows from (i) and (ii).
To show (i), let $f, g \in C^\infty_{per,0}(\widetilde{E})$. We can use the Gaussian integral formula (cf. [Alt02, A 6.8]) and the fact that f and g have bounded support to obtain $(Sf,g)_{L^2(\widetilde{E};\widetilde{\mu})} = -(\nabla_v f, \nabla_v g)_{L^2(\widetilde{E};\widetilde{\mu})} = (f, Sg)_{L^2(\widetilde{E};\widetilde{\mu})}$. Thus S is symmetric. Moreover, for all for $f \in C^\infty_{per,0}(\widetilde{E})$ it holds $(Sf,f)_{L^2(\widetilde{E};\widetilde{\mu})} = -(\nabla_v f, \nabla_v f)_{L^2(\widetilde{E};\widetilde{\mu})} \leq 0$, which shows that S is dissipative.

Let us now prove (ii). Again by the Gaussian integral formula we find for $f, g \in C^\infty_{per,0}(\widetilde{E})$

$$((\nabla_x\Phi)\nabla_v f, g)_{L^2(\widetilde{E};\widetilde{\mu})} = -(f, (\nabla_x\Phi)\nabla_v g)_{L^2(\widetilde{E};\widetilde{\mu})} + \int_{\widetilde{E}} (v\nabla_x\Phi) f(x,v) g(x,v)\, d\widetilde{\mu} \tag{3.11}$$

and moreover, using Lemma 3.1(i),

$$\begin{aligned}(v\nabla_x f, g)_{L^2(\widetilde{E};\widetilde{\mu})} &= -(f, v\nabla_x g) + \int_{\widetilde{E}} (v\nabla_x\Phi) f(x,v) g(x,v)\, d\widetilde{\mu} \\ &\quad + \int_{\mathbb{R}^n} \left(\int_{\widetilde{M}} v\nabla_x (fge^{-\Phi})(x,v)\, dx \right) e^{-v^2}\, dv \\ &= -(f, v\nabla_x g) + \int_{\widetilde{E}} (v\nabla_x\Phi) f(x,v) g(x,v)\, d\widetilde{\mu},\end{aligned} \tag{3.12}$$

since for fixed $v \in \mathbb{R}^n$ the mapping $x \mapsto f(x,v)g(x,v)e^{-\Phi(x)}$ is periodic. By (3.11) and (3.12) we obtain that $(Af, g) = -(f, Ag)$. □

Remark 3.15. (i) We find by the above proof that neither $v\nabla_x$ nor $(\nabla_x\Phi)\nabla_v$ is antisymmetric, but $v\nabla_x - (1/2)v(\nabla_x\Phi)$ and $-(\nabla_x\Phi)\nabla_v + (1/2)v(\nabla_x\Phi)$ are.

(ii) It is well-known (cf. Lemma 3.17 below) that the closure of the symmetric operator $S = \Delta_v - v\nabla_v$ is an m-dissipative (Dirichlet) operator. Hence we start with the operator S and consider the rest of L as a perturbation. This seems to be easy as far as we think about the last summand in (3.10) (we may use Theorem 3.9 for this). But $v\nabla_x$ is not bounded by S, not even if it would be possible to keep v bounded. This indicates that the lack of strict ellipticity of L

causes difficulties. (One of these difficulties is the fact that L cannot generate a coercive closed form (which is not difficult to prove). This fact forces us to apply the theory of *generalized* Dirichlet forms instead of [MR92].)

(iii) In order to use a strategy as mentioned in (ii), it may be natural first to consider the case of potential free motion and then to add the influence of the potential. Intuitively, if we "fix the positions" and just consider the changes the potential (and of course friction and stochastic perturbation) causes to the velocities, the introduction of motion in x-direction (this direction did not even play any role before) does not seem to be a small perturbation. So the question arises which part of L represents the free motion. Clearly, since we are acting on $L^2(\widetilde{E};\widetilde{\mu})$, and since $\widetilde{\mu}$ depends on Φ, this part can not be the operator $\Delta_v - v\nabla_v + v\nabla_x$.

The above remarks motivate us to define a unitary transformation T which enables us to get rid of $\widetilde{\mu}$. Consider

$$\begin{array}{rrcl} T: & L^2(\widetilde{E};\widetilde{\mu}) & \to & L^2(\widetilde{E};d(x,v)) \\ & f & \mapsto & \sqrt{\widetilde{m}}f \end{array}$$

It is easily seen that it formally holds

$$TLT^{-1} = L' := \Delta_v - \frac{v^2}{4} + \frac{n}{2}I + v\nabla_x - \nabla_x\Phi\nabla_v.$$

(We do not need to check this here, cf. Lemma 3.21.) Instead of thinking about how to make this equation rigorous now, we find a domain of essential m-dissipativity for L'.

Remark 3.16. (i) By Lemma 3.1 the unitary transformation T defined above maps the space $H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n) := \operatorname{span}\{\theta \otimes \varphi | \theta \in H_{per}^{1,\infty}(\widetilde{M}), \varphi \in C_0^\infty(\mathbb{R}^n)\}$ onto itself.
If we can prove essential m-dissipativity of L' on a domain $D \subset H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n)$, essential m-dissipativity of L', defined on $H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n)$, follows directly. In order to show essential m-dissipativity of $(L, C_{per}^\infty(\widetilde{E}))$ we are then left to prove that the domain of its closure $(\overline{L}, D(\overline{L}))$ contains $H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n)$ and that it indeed holds $T\overline{L}T^{-1}f = L'f$ for $f \in H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n)$. This is essentially what we do in Lemma 3.21 below.

(ii) Note that the last two summands of L' are *both* antisymmetric, when we define L' e.g. on a subset of $C_{per}^\infty(\widetilde{E})$ or $H_{per}^{1,\infty}(\widetilde{M}) \otimes C_0^\infty(\mathbb{R}^n)$. This can be seen by arguments as in the proof of Lemma 3.14. Moreover, note that now $\nabla_x\Phi\nabla_v$ is exactly the part corresponding to the potential.

Of course, we follow the strategy explained in Remark 3.15(ii),(iii) to prove essential m-dissipativity of L' on a suitable domain.

The basis functions for invariant subspaces of the operators ∂_{x_i}, $i = 1, \cdots, n$, turn out to be useful. For $z \in \mathbb{Z}$, $1 \le i \le n$, define

$$\psi_z^i : (0, r_i) \to \mathbb{R}$$
$$x \mapsto \begin{cases} \sqrt{\frac{1}{r_i}} & \text{if } z = 0 \\ \sqrt{\frac{2}{r_i}} \cos(z\, c_i x) & \text{if } z > 0 \\ \sqrt{\frac{2}{r_i}} \sin(z\, c_i x) & \text{if } z < 0 \end{cases}$$

where $c_i := 2\pi / r_i$ for $1 \le i \le n$ (here π denotes the Ludolph number). Note that

$$(\psi_z^i)'(x) = c_i z\, \psi_{-z}^i(x) \tag{3.13}$$

for all $z \in \mathbb{Z}, x \in (0, r_i)$. Define for $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{Z}^n$ the function $\psi_{\boldsymbol{z}} : \widetilde{M} \to \mathbb{R}$ by $\psi_{\boldsymbol{z}}(x) := \psi_{z_1}^1(x_1) \cdots \psi_{z_n}^n(x_n)$, $x = (x_1, \cdots, x_n) \in \widetilde{M}$.
Clearly the functions $\psi_{\boldsymbol{z}}$, $\boldsymbol{z} \in \mathbb{Z}^n$, form a complete orthonormal system in $L^2(\widetilde{M}, dx)$ and

$$\begin{aligned} D :&= \operatorname{span}\{\psi_{\boldsymbol{z}} \otimes \varphi | \boldsymbol{z} \in \mathbb{Z}^n, \varphi \in C_0^\infty(\mathbb{R}^n)\} \\ &= \operatorname{span}\{(x, v) \mapsto \psi_{\boldsymbol{z}}(x)\varphi(v) | \boldsymbol{z} \in \mathbb{Z}^n, \varphi \in C_0^\infty(\mathbb{R}^n)\} \end{aligned}$$

forms a dense linear subspace of $L^2(\widetilde{E}; d(x, v))$.

The following is a well-known fact (it may be seen e.g. by [Tri80, §24] and [RS80, Theorem VIII.33]).

Lemma 3.17. *The operator $S' : D \to L^2(\widetilde{E}; d(x, v))$ defined by $S' = \Delta_v - \frac{v^2}{4} + \frac{n}{2} I$, is essentially m-dissipative.*

In the sequel we denote by $(\overline{S'}, D(\overline{S'}))$ the closure of (S', D).

Let us now consider the potential-free case:

Lemma 3.18. *The operator $L_0' : D \to L^2(\widetilde{E}; dx)$ defined by $L_0' = \Delta_v - \frac{v^2}{4} + \frac{n}{2} I + v\nabla_x$, is essentially m-dissipative.*

Proof. By $\|\cdot\|_{(x,v)}$ and $(\cdot,\cdot)_{(x,v)}$ we denote the norm and inner product of $L^2(\widetilde{E}; d(x, v))$ and denote by $\|\cdot\|_v$ and $(\cdot,\cdot)_v$ the norm and inner product of $L^2(\mathbb{R}^n; dv)$.
Here we apply Lemma 3.12 to prove essential m-dissipativity.
For $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{N}_0^n$ we define $\mathcal{I}_{\boldsymbol{z}} := \{\boldsymbol{z}' = (z_1', \cdots, z_n') \in \mathbb{Z}^n | \, |z_i'| = z_i, i = 1, \cdots, n\}$ and

$$D^{\boldsymbol{z}} := \operatorname{span}\{\psi_{\boldsymbol{z}'} \otimes \varphi | \boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}, \varphi \in C_0^\infty(\mathbb{R}^n)\}$$

Clearly $(D^{\boldsymbol{z}})_{\boldsymbol{z} \in \mathbb{N}_0^n}$ is a family of orthogonal subspaces of $L^2(\widetilde{E}; d(x, v))$, such that the linear span of their union is dense. Hence the orthogonal projections $P^{\boldsymbol{z}}$ corresponding to the closures $\overline{D^{\boldsymbol{z}}}$, $\boldsymbol{z} \in \mathbb{N}_0^n$, form a complete orthogonal family. It holds $P^{\boldsymbol{z}} D = D^{\boldsymbol{z}} \subset D$ and every summand of (S', D) commutes with $P^{\boldsymbol{z}}$. Moreover, note that for $\boldsymbol{z}' = (z_1', \cdots, z_n') \in \mathbb{Z}^n$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ it holds by (3.13)

$$v\nabla_x(\psi_{\boldsymbol{z}} \otimes \varphi) = v_1(\partial_{x_1}\psi_{\boldsymbol{z}}) \otimes \varphi + \cdots + v_n(\partial_{x_n}\psi_{\boldsymbol{z}}) \otimes \varphi \tag{3.14}$$

$$= \sum_{i=1}^{n} c_i z_i' v_i \psi_{\boldsymbol{z}_i'} \otimes \varphi$$

where we define $\boldsymbol{z}_i' := (z_1', \cdots, z_{i-1}', -z_i', z_{i+1}', \cdots, z_n')$ for $1 \le i \le n$. (3.14) shows that $P^{\boldsymbol{z}}$ also commutes with $(v\nabla_x, D)$.
Let $\boldsymbol{z} \in \mathbb{N}_0^n$ and choose an arbitrary element $f := \sum_{\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}} \psi_{\boldsymbol{z}'} \otimes \varphi^{\boldsymbol{z}'}$ of $D^{\boldsymbol{z}}$ (with $\varphi^{\boldsymbol{z}'} \in C_0^\infty(\mathbb{R}^n)$, $\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}$). For $\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}$ it holds

$$\sum_{i=1}^{n} \|v_i \varphi^{\boldsymbol{z}'}\|_v^2 = (v^2 \varphi^{\boldsymbol{z}'}, \varphi^{\boldsymbol{z}'})_v \le (v^2 f, f)_{(x,v)}$$

and by (3.14) it follows

$$\begin{aligned}
\|v\nabla_x f\|_{(v,x)}^2 &= \left\| \sum_{\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}} \sum_{i=1}^{n} c_i z_i \psi_{\boldsymbol{z}_i'} \otimes (v_i \varphi^{\boldsymbol{z}'}) \right\|_{(x,v)}^2 \le C \sum_{\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}} \sum_{i=1}^{n} \left\| \psi_{\boldsymbol{z}_i'} \otimes (v_i \varphi^{\boldsymbol{z}'}) \right\|_v^2 \\
&= C \sum_{\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}} \sum_{i=1}^{n} \left\| v_i \varphi^{\boldsymbol{z}'} \right\|_{(x,v)}^2 \le C \sum_{\boldsymbol{z}' \in \mathcal{I}_{\boldsymbol{z}}} (v^2 f, f)_{(x,v)} \\
&\le 2^n C (v^2 f, f)_{(x,v)} \le 4 \cdot 2^n C \left(\left(-S + \frac{n}{2} I \right) f, f \right)_{(x,v)} \\
&= 4 \cdot 2^n C (-Sf, f)_{(x,v)} + 2 \cdot n 2^n C \|f\|_{(x,v)}^2
\end{aligned}$$

where $C := \max\{|z_i c_i|^2 \,|1 \le i \le n\}$. We can now apply Lemma 3.12 (and we use the notations from this lemma): We use the complete orthogonal family $(P^{\boldsymbol{z}})_{\boldsymbol{z} \in \mathbb{N}_0^n}$ given above and define $A := S'$, $B := v\nabla_x$. By Lemma 3.17 (A, D) is essentially m-dissipative and since (B, D) is clearly antisymmetric, it is dissipative. Our above considerations and Lemma 3.10 imply that all the assumptions in Lemma 3.12 are fulfilled. Hence (L_0', D) is essentially m-dissipative. □

Next, we add the potential.

Lemma 3.19. *The operator* $L' : D \to L^2(\widetilde{E}; d(x,v))$ *defined by* $L' = \Delta_v - \frac{v^2}{4} + v\nabla_x - \nabla_x \Phi \nabla_v$*, is essentially m-dissipative.*

Proof. Clearly the operator $((\nabla_x \Phi)\nabla_v, D)$ is dissipative (because it is antisymmetric). Let $C := \max_{1 \le i \le n} \|\partial_{x_i} \Phi\|_{L^\infty(\widetilde{E}, d(x,v))}$. For $f \in D$ it holds

$$\begin{aligned}
\|(\nabla_x \Phi)\nabla_v f\|_{L^2(\widetilde{E}; d(x,v))}^2 &\le C^2 (-\Delta_v f, f)_{L^2(\widetilde{E}; d(x,v))} \\
&\le C^2 \left(\left(-S' + \frac{n}{2} I \right) f, f \right)_{L^2(\widetilde{E}; d(x,v))} \\
&= C^2 \left(\left(-L_0' + \frac{n}{2} I \right) f, f \right)_{L^2(\widetilde{E}; d(x,v))} \\
&= C^2 (-L_0' f, f)_{L^2(\widetilde{E}; d(x,v))} + C^2 \frac{n}{2} \|f\|_{L^2(\widetilde{E}; d(x,v))}^2
\end{aligned}$$

by antisymmetry of $v\nabla_x$. Lemma 3.18, Lemma 3.10 and Theorem 3.9 imply that L' is essentially m-dissipative. □

Hence, the closure $(\overline{L'}, D(\overline{L'}))$ of (L', D) is m-dissipative. Consequently, the same holds for the closure $(\overline{T^{-1}L'T}, D(\overline{T^{-1}L'T}))$ of $(T^{-1}L'T, T^{-1}D)$. We already mentioned (though we did not prove it), that $T^{-1}L'T$ behaves formally like L, but we do not know whether it is an extension of $(L, C^{\infty}_{per,0})$ or not. Lemma 3.21 below answers this question. But first we need to make some further considerations about $H^{1,\infty}_{per}(\widetilde{M})$. We denote by $H^{1,2}(\widetilde{M})$ the space of weakly differentiable functions $f : \widetilde{M} \to \mathbb{R}$, such that f and all its partial derivatives are contained in $L^2(\widetilde{M}; dx)$. $H^{1,2}(\widetilde{M})$ is a Hilbert space w.r.t. the norm $\|\cdot\|_{1,2} := \|\cdot\|_{L^2(\widetilde{M};dx)} + \sum_{i=1}^n \|\partial_{x_i} \cdot\|_{L^2(\widetilde{M};dx)}$ (or the corresponding inner product, denoted by $(\cdot,\cdot)_{1,2}$) and the set $C^{\infty}(\widetilde{M}) \cap H^{1,2}(\widetilde{M})$ is dense in $H^{1,2}(\widetilde{M})$ (cf. [Alt02, Satz 1.24]). Clearly $H^{1,\infty}_{per}(\widetilde{M}) \subset H^{1,\infty}(\widetilde{M}) \subset H^{1,2}(\widetilde{M})$, hence for any $f \in H^{1,\infty}_{per}(\widetilde{M})$ we find an approximating sequence $(f_k)_{k\in\mathbb{N}} \subset C^{\infty}(\widetilde{M})$ w.r.t. this norm. But we need the following (slightly stronger) fact.

Lemma 3.20. *$C^{\infty}_{per}(\widetilde{M})$ is dense in $H^{1,\infty}_{per}(\widetilde{M})$ w.r.t. $\|\cdot\|_{1,2}$.*

Proof. Assume that there is $f \in H^{1,\infty}_{per}(\widetilde{M})$, such that for all $\phi \in C^{\infty}_{per}(\widetilde{M})$ it holds $(\phi, f)_{1,2} = 0$. We have to show that $f = 0$.
It holds for all $\phi \in C^{\infty}_{per}(\widetilde{M})$

$$\begin{aligned} 0 &= (\phi, f)_{1,2} = (\phi, f)_{L^2(\widetilde{M};dx)} + \sum_{i=1}^n (\partial_{x_i}\phi, \partial_{x_i} f)_{L^2(\widetilde{M};dx)} \\ &= (\phi - \Delta_x \phi, f)_{L^2(\widetilde{M};dx)} \end{aligned}$$

by the Gaussian integral formula (cf. [Alt02, A 6.8.2]) and by periodicity of $\partial_i \phi$, $i = 1, \cdots, n$, and f. But the operator $(I - \Delta_x, C^{\infty}_{per}(\widetilde{M}))$ has clearly dense range, since the functions $\psi_{\boldsymbol{z}}$, $\boldsymbol{z} \in \mathbb{Z}^n$, are contained in its domain and form a complete orthonormal system of eigenfunctions of $I - \Delta_x$, such that the corresponding eigenvalues are strictly positive. Thus $f = 0$ and our assertion is shown. □

Lemma 3.21. *$(\overline{T^{-1}L'T}, D(\overline{T^{-1}L'T}))$ is the closure of $(L, C^{\infty}_{per,0}(\widetilde{E}))$.*

Proof. Let $(\overline{L}, D(\overline{L}))$ be the closure of $(L, C^{\infty}_{per,0}(\widetilde{E}))$. Our assertion is shown, if we can prove that the closed dissipative operator $(\overline{L}, D(\overline{L}))$ is an extension of the essentially m-dissipative operator $(T^{-1}L'T, T^{-1}D)$.
It holds $T^{-1}D = \operatorname{span}\{(e^{\Phi/2}\psi_{\boldsymbol{z}}) \otimes \varphi \,|\, \boldsymbol{z} \in \mathbb{Z}^n, \varphi \in C_0^{\infty}(\mathbb{R}^n)\}$. Let $\boldsymbol{z} \in \mathbb{Z}^n$, $\varphi \in C_0^{\infty}(\mathbb{R}^n)$. Since $\psi_{\boldsymbol{z}} \in C^{\infty}_{per}(\widetilde{M})$, we know by Lemma 3.1(i) that it holds $\theta := e^{\Phi/2}\psi_{\boldsymbol{z}} \in H^{1,\infty}_{per}(\widetilde{M})$. Hence by Lemma 3.20 there is a sequence $(\theta_k)_{k\in\mathbb{N}} \subset C^{\infty}_{per}(\widetilde{M})$ approximating θ in $H^{1,2}_{per}(\widetilde{M})$. Thus $\theta_k \to \theta$ and $\partial_{x_i}\theta_k \to \partial_{x_i}\theta$ in $L^2(\widetilde{M}; dx)$ as $k \to \infty$, hence (since Φ is essentially bounded) in $L^2(\widetilde{M}; e^{-\Phi}dx)$. Consequently

$$(\Delta_v - v\nabla_v + v\nabla_x - \nabla_x\Phi\nabla_v)\,\theta_n \otimes \varphi \to (\Delta_v - v\nabla_v + v\nabla_x - \nabla_x\Phi\nabla_v)\,\theta \otimes \varphi$$

in $L^2(\widetilde{E}; \widetilde{\mu})$ when $n \to \infty$. This does not seem to be a surprising result, but it shows that $T^{-1}D \subset D(\overline{L})$ and that $\overline{L}$ looks the same as L on $T^{-1}D$ (but, of course, we are dealing with *weak* derivatives of $\theta\otimes\varphi$ (in x-direction) now). This argumentation is valid for any $\theta\in H^{1,\infty}_{per}(\widetilde{M})$, $\varphi\in C_0^\infty(\mathbb{R}^n)$.

We have to show that $\Delta_v - v\nabla_v + v\nabla_x - (\nabla_x\Phi)\nabla_v = \overline{L}|_{T^{-1}D} \overset{!}{=} T^{-1}L'T$. For $f\in T^{-1}D$ it holds

$$
\begin{aligned}
(3.15)\quad & T^{-1}\left(\Delta_v - \frac{v^2}{4} + \frac{n}{2}I\right)Tf \\
&= T^{-1}\left(\Delta_v - \frac{v^2}{4} + \frac{n}{2}I\right)(e^{-v^2/4}e^{-\Phi/2}f) \\
&= T^{-1}e^{-\Phi/2}\left(\nabla_v\left(e^{-v^2/4}\nabla_v f - \frac{v}{2}e^{-v^2/4}f\right) - \frac{v^2}{4}e^{-v^2/4}f + \frac{n}{2}e^{-v^2/4}f\right) \\
&= T^{-1}e^{-\Phi/4}e^{-v^2/4}\left(\Delta_v f - v\nabla_v f - \frac{n}{2}f + \frac{v^2}{4}f - \frac{v^2}{4}f + \frac{n}{2}f\right) \\
&= \Delta_v f - v\nabla_v f
\end{aligned}
$$

Moreover, it holds by Lemma 3.1(ii)

$$
\begin{aligned}
(3.16)\qquad T^{-1}v\nabla_x Tf &= T^{-1}\left(v\nabla_x(e^{-v^2/4}e^{-\Phi/2}f)\right) \\
&= T^{-1}e^{-v^2/4}e^{-\Phi/2}\left(-\frac{1}{2}v(\nabla_x\Phi)f + v\nabla_x f\right) \\
&= \left(v\nabla_x f - \frac{1}{2}v(\nabla_x\Phi)\right)f
\end{aligned}
$$

and by an analogous calculation we find that

$$
(3.17)\qquad T^{-1}(\nabla_x\Phi)\nabla_v Tf = \left((\nabla_x\Phi)\nabla_v - \frac{1}{2}v(\nabla_x\Phi)\right)f
$$

The equations (3.15), (3.16) and (3.17) complete our proof. □

We still have to show that the m-dissipative closure of $(L, C^\infty_{per,0})$ is a Dirichlet operator. Let us first prove the following lemma.

Lemma 3.22. *For the closure* $(\overline{L}, D(\overline{L}))$ *of* $(L, C^\infty_{per,0}(\widetilde{E}))$ *it holds* $1\in D(\overline{L})$ *and* $\overline{L}1 = 0$.

Proof. Define $1_x : \widetilde{M}\to\mathbb{R}$ by setting $1_x(x) := 1$ for all $x\in\widetilde{M}$. Clearly for all $\varphi\in C_0^\infty(\mathbb{R}^n)$ it holds $1_x\otimes\varphi\in C^\infty_{per,0}(\widetilde{M})$ and

$$
(3.18)\qquad L(1_x\otimes\varphi)(x,v) = \Delta_v\varphi(v) - v\nabla_v\varphi(v) + (\nabla_x\Phi(x))\nabla_v\varphi(v)\quad\text{for } (x,v)\in\widetilde{E}.
$$

For $m\in\mathbb{N}$, let η_m be an element of $C_0^\infty(\mathbb{R})$ such that $\eta_m(t) = 1$ for $t\in[-m,m]$, $\eta_m(t) = 0$ for $t\notin[-m-2, m+2]$ and $|\eta_m(t)|\le 1$, $|\frac{d}{dt}\eta_m(t)|\le 1$ and $|\frac{d^2}{dt^2}\eta_m(t)|\le 1$ for all $t\in\mathbb{R}$. We define $\eta^m:\mathbb{R}^n\to\mathbb{R}$ by $\eta^m(v_1,\cdots,v_n) := \prod_{i=1}^n\eta_m(v_i)$. By Lebesgue's dominated convergence theorem we obtain $1_x\otimes\eta^m\to 1$ in $L^2(\widetilde{E};\widetilde{\mu})$. We choose $C>0$ such that $C\ge\|\partial_{x_i}\Phi\|_{L^\infty(\widetilde{M})}$ for every $i=1,\cdots,n$. Then

$$
|\Delta_v\eta^m(v) - v\nabla_v\eta^m(v) + \nabla_x\Phi(x)\nabla_v\eta^m(v)| \le n + \sum_{i=1}^n |v_i| + Cn
$$

holds for $\widetilde{\mu}$-a.e. $(x,v) \in \widetilde{E}$. Moreover, it holds $\Delta_v \eta^m(v) - v\nabla_v \eta^m(v) + \nabla_x \Phi(x)\nabla_v \eta^m(v) = 0$ for $v \in [-m,m]^n$, $x \in \widetilde{M}$. Consequently, by Lebesgue's dominated convergence theorem and (3.18), $L(1_x \otimes \eta_m) \to 0$ in $L^2(\widetilde{E};\widetilde{\mu})$, and the assertion follows. □

Remark 3.23. Another idea to prove Lemma 3.22 is to consider the operator $(\tilde{L}, D(\tilde{L}))$, where $D(\tilde{L}) := \mathrm{span}(\{1\} \cup C^\infty_{per,0}(\widetilde{E}))$ and $\tilde{L}1 := 0$. If $\tilde{L}$ is dissipative (which is by dissipativity of L equivalent to $(Lg,1) = 0$ for all $g \in C^\infty_{per,0}(\widetilde{E})$), this implies that the closure of $\tilde{L}$ is a closed dissipative extension of the essentially m-dissipative operator L and hence equal to $\overline{L}$. However, in the proof given above we did not use the m-dissipativity of $(\overline{L}, D(\overline{L}))$.

Now we prove the Dirichlet property.

Lemma 3.24. *With the notations of Lemma 3.14 it holds*

(i) $(Su, u^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$ *for all* $u \in C^\infty_{per,0}(\widetilde{E})$

(ii) $(Au, u^+)_{L^2(\widetilde{E};\widetilde{\mu})} = 0$ *for all* $u \in C^\infty_{per,0}(\widetilde{E})$

(iii) $(\overline{L}u, u^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$ *for all* $u \in D(\overline{L})$, *where* $(\overline{L}, D(\overline{L}))$ *again denotes the closure of the operator* $(L, C^\infty_{per,0}(\widetilde{E}))$ *in* $L^2(\widetilde{E};\widetilde{\mu})$.

(iv) $(\overline{L}u, (u-1)^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$ *for all* $u \in D(\overline{L})$

Proof. For each $\varepsilon > 0$ we choose an infinitely often differentiable function $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $\chi_\varepsilon(\xi) = 0$ for $\xi \in (-\infty, 0]$, $0 \le \chi'_\varepsilon(\xi) \le 1$ for all $\xi \in \mathbb{R}$, $\chi'_\varepsilon(x) = 1$ for $\xi \ge \varepsilon$. For $u \in C^\infty_{per,0}(\widetilde{E})$ it clearly holds $\chi_\varepsilon \circ u \in C^\infty_{per,0}(\widetilde{E})$ for all $\varepsilon > 0$. Hence by the proof of Lemma 3.14 we find that

$$(Su, \chi_\varepsilon \circ u)_{L^2(\widetilde{E};\widetilde{\mu})} = -\sum_{i=1}^n (\partial_{v_i} u, (\chi'_\varepsilon \circ u)\partial_{v_i} u)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$$

since $\chi'_\varepsilon \ge 0$. But since clearly $\chi_\varepsilon \circ u \to u^+$ in $L^2(\widetilde{E};\widetilde{\mu})$ as $\varepsilon \to 0$, (i) is shown.
Moreover, Lemma 3.14 shows that

$$\begin{aligned}(Au, \chi_\varepsilon \circ u)_{L^2(\widetilde{E};\widetilde{\mu})} &= -(u, A(\chi_\varepsilon \circ u))_{L^2(\widetilde{E};\widetilde{\mu})} \\ &= -((\chi'_\varepsilon \circ u)u, Au)_{L^2(\widetilde{E};\widetilde{\mu})}\end{aligned}$$

Since $\chi_\varepsilon \circ u \to u^+$ and $(\chi'_\varepsilon \circ u)u \to u^+$ in $L^2(\widetilde{E};\widetilde{\mu})$ as $\varepsilon \to 0$, we obtain

$$(Au, u^+)_{L^2(\widetilde{E};\widetilde{\mu})} = -(u^+, Au)_{L^2(\widetilde{E};\widetilde{\mu})} = -(Au, u^+)_{L^2(\widetilde{E};\widetilde{\mu})}$$

hence $(Au, u^+) = 0$ and (ii) is shown.
By (i) and (ii) it holds $(Lu, u^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$ for $u \in C^\infty_{per,0}(\widetilde{E})$. Clearly this property extends to the closure $(\overline{L}, D(\overline{L}))$. Thus (iii) holds.
To prove (iv) we use (iii) and Lemma 3.22. Let $u \in D(\overline{L})$. Then also $(u-1) \in D(\overline{L})$ and hence by (iii) it holds $(\overline{L}(u-1), (u-1)^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$, and since $\overline{L}1 = 0$, we conclude $(\overline{L}u, (u-1)^+)_{L^2(\widetilde{E};\widetilde{\mu})} \le 0$. □

Proof of Theorem 3.13
Follows by Lemma 3.19, Lemma 3.21 and Lemma 3.24(iv). □

3.4. **An N-particle Langevin dynamics.** As we mentioned at the beginning of Section 3.3, we can now prove the existence of an associated process without much additional effort. For the proof of the following theorem we use notations and definitions from Section 2 and Section 3.1 and we refer to considerations and results from Sections 4 and 5 below. If A is a subset of an L^2-space of real-valued functions, we define $A_b := A \cap L^\infty$.

Theorem 3.25. *Let Φ^M be given as in Condition 3.7 and define μ by (3.6). Then the closure $(\overline{L^E}, D(\overline{L^E}))$ of the essentially m-dissipative operator $(L^E, C_0^\infty(E))$ on $\mathcal{H} = L^2(E;\mu)$, given by $L^E = \frac{\gamma}{\beta}\Delta_v^E - \gamma v\nabla_v^E + v\nabla_x^E - (\nabla_x^M\Phi^M)\nabla_v^E$ is the generator of the quasi-regular GDF*

$$\mathcal{E}: \; D(\overline{L^E}) \times \mathcal{H} \cup \mathcal{H} \times D(\widehat{\overline{L^E}}) \to \mathbb{R}$$
$$(u,v) \mapsto \begin{cases} -(\overline{L^E}u, v)_{\mathcal{H}} & \text{if } u \in D(\overline{L^E}), v \in \mathcal{H} \\ -(\widehat{\overline{L^E}}v, u)_{\mathcal{H}} & \text{if } u \in \mathcal{H}, v \in D(\widehat{\overline{L^E}}) \end{cases}$$

which fulfills (D3) in Theorem 2.9. Hence there exists a μ-tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t, V_t)_{t\geq 0}, (P_{(x,v)})_{(x,v)\in E})$ which is properly associated in the resolvent sense with $\mathcal{E}$ and which has the following properties:

(i) For $\mathcal{E}$-q.e. initial point $(x,v) \in E$ the process $\mathbb{M}$ has $P_{(x,v)}$-a.s. infinite life time.
(ii) For $\mathcal{E}$-q.e. $(x,v) \in E$ the process $\mathbb{M}$ has $P_{(x,v)}$-a.s. continuous paths.
(iii) For all $f \in D(\overline{L^E})_b$ the process $(M_t^{[f]})_{t\geq 0}$ defined by

$$M_t^{[f]} := \left(\tilde{f}(X_t, V_t) - \tilde{f}(X_0, V_0) - \int_0^t L^E f(X_s, V_s)\, ds\right)_{t\geq 0} \tag{3.19}$$

is a martingale w.r.t. $P_{(x,v)}$ for $\mathcal{E}$-q.e. $(x,v) \in E$. Here $\tilde{f}$ denotes an $\mathcal{E}$-quasi-continuous μ-version of f.
(iv) For any measure $\nu \in \mathcal{P}(E)$ whose completion maps every $\mathcal{E}$-exceptional set to 0 P_ν solves the martingale problem for $\overline{L^E}$ on $D(\overline{L^E})_b$, i.e. for all $f \in D(\overline{L^E})_b$ the process $(M_t^{[f]})_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale w.r.t. P_ν. This also holds if $\tilde{f}$ in (3.19) is replaced by any μ-version of f.
(v) For any measure $\nu \in \mathcal{P}(E)$ having an L^2-integrable density w.r.t. μ P_ν solves the martingale problem for $\overline{L^E}$ on $D(\overline{L^E})$. Of course, this also holds if $\tilde{f}$ is replaced by any μ-version of f.
(vi) If ν in (v) is defined by $\nu := \frac{1}{\mu(E)}\mu$, then for all $t \geq 0$ it holds $P_\nu \circ X_t^{-1} = \nu$, i.e. ν is an invariant measure for $\mathbb{M}$.

Proof. We know by Theorem 3.13 and our considerations in Section 3.1 that $(\overline{L^E}, D(\overline{L^E}))$ is an m-dissipative Dirichlet operator and hence $\mathcal{E}$ is a generalized Dirichlet form by Lemma 2.2. By Remark 2.4 $\overline{L^E}$ is the generator of $\mathcal{E}$ (cf. Definition 2.3). Now we prove quasi-regularity (cf. Definition 2.5).
For $k \in \mathbb{N}$ we define the compact subset $F_k := M \times [-k,k]^n \subset E$. If we define $D(\overline{L^E})_{F_k} := \{u \in D(\overline{L^E}) | u(x,v) = 0 \text{ for all } (x,v) \in E \setminus F_k\}$, it holds $C_0^\infty(E) \subset \bigcup_k D(\overline{L^E})_{F_k}$, hence

by Remark 2.6(ii) $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest consisting of compact sets. Hence (q1) is shown. (q2) is clear, since the functions in $C_0^\infty(E)$ are, of course, $\mathcal{E}$-quasi-continuous and form a dense subset of $D(\overline{L^E})$ w.r.t. graph norm. Finally, (q3) follows by Remark 3.2. Condition (D3) in Theorem 2.9 follows from the fact that $C_0^\infty(E)$ is an algebra of bounded functions in $D(\overline{L^E})$ which is dense in $D(\overline{L^E})$ w.r.t. the graph norm and from Proposition 2.11.

The properties (iii)-(v) are shown in Theorem 4.7, Corollary 4.9 and Corollary 4.11 in Section 4 below, and (ii) is seen from Theorem 5.5 in Section 5 below and Remark 3.2. Now we prove (i). By Lemma 3.22 we see that $1 \in D(\overline{L^E})$ and $\overline{L^E}1 = 0$, hence, if $(T_t)_{t\geq 0}$ denotes the C_0 contraction semigroup generated by $\overline{L^E}$, it holds $T_t 1 = 1$ for all $t \geq 0$, hence for the transition semigroup $(p_t)_{t>0}$ of $\mathbb{M}$ it holds by Lemma 2.8(i) $(p_t 1_E)(x,v) = 1$ μ-a.e. for $t > 0$, hence by $\mathcal{E}$-quasi-continuity of $p_t 1$ (cf. Lemma 4.3 below) and Remark 2.6(v) we obtain $P_{(x,v)}(t < \zeta) = P_{(x,v)}(X_t \in E) = (p_t 1_E)(x,v) = 1$ for $\mathcal{E}$-q.e. $(x,v) \in E$. Since countable unions of $\mathcal{E}$-exceptional sets are $\mathcal{E}$-exceptional, we find that $P_{(x,v)}(\zeta = \infty) = 1$ for $\mathcal{E}$-q.e. $(x,v) \in E$.

Finally, we prove (vi). By Lemma 3.14 we find that $C_0^\infty(E) \subset D(\widehat{\overline{L^E}})$ and that it holds $\hat{L}^E = \Delta_v - v\nabla_v - v\nabla_x + (\nabla_x \Phi^M)\nabla_v$, where $\hat{L}^E$ denotes the restriction of $\widehat{\overline{L^E}}$ to $C_0^\infty(E)$. The unitary transformation $T_v : L^2(E;\mu) \to L^2(E;\mu)$, defined by $T_v f(x,v) := f(x,-v)$, transforms $(L^E, C_0^\infty(E))$ into $(\hat{L}^E, C_0^\infty(E))$ and maps the constant 1-function to itself. By Lemma 3.22 we conclude that $1 \in D(\widehat{\overline{L^E}})$ and $\widehat{\overline{L^E}}1 = 0$. If $(\hat{T}_t)_{t\geq 0}$ denotes the adjoint semigroup to $(T_t)_{t\geq 0}$ (which is the C_0 contraction semigroup generated by $\widehat{\overline{L^E}}$), it follows that $\hat{T}_t 1 = 1$ for all $t \geq 0$. Consequently $(T_t f, 1)_{L^2(E;\mu)} = (f, \hat{T}_t 1)_{L^2(E;\mu)} = (f,1)_{L^2(E;\mu)}$ for all $f \in L^2(E;\mu)$. Thus for $A \in \mathcal{B}(E), t \geq 0$ it holds

$$\begin{aligned} P_\nu \circ X_t^{-1}(A) &= E_\nu\left[1_A(X_t)\right] = \int_E p_t 1_A d\nu \\ &= (T_t 1_A, 1)_{L^2(E;\nu)} = (1_A, 1)_{L^2(E;\nu)} = \nu(A), \end{aligned}$$

where 1_A denotes the indicator function for A. This proves (vi). □

Remark 3.26. Of course the unitary transformation T_v given in the proof of (vi) in the above theorem enables us to find that also $(\hat{L}^E, C_0^\infty)$ is essentially m-dissipative and its closure is a Dirichlet operator. Moreover, clearly the arguments given in the above proof are also valid for $\widehat{\overline{L^E}}$ (which, being an m-dissipative operator extending $(\hat{L}^E, C_0^\infty)$, is equal to the closure of this operator) and consequently Theorem 3.25 holds also with L^E replaced by $\hat{L}^E$.

From the above theorem we obtain:

Corollary 3.27. *Consider the situation of Theorem 3.25. We assume that bounded versions of $\partial_i \Phi^M$, $i \in \mathcal{I} := \{x_1, \cdots, x_n, v_1, \cdots, v_n\}$, are fixed and consider $\overline{L^E}$ as an operator on $C_0^2(E)$ (acting pointwise). Then for $\mathcal{E}$-q.e. initial point $(x,v) \in E$ the law $P_{(x,v)}$ solves the martingale problem for $\overline{L^E}$ on $C_0^2(E)$.*

Proof. Let $K \subset E$ be open and relatively compact, i.e. bounded in v-direction. We define

$$\|\cdot\|_{C^2(\overline{K})} := \|\cdot\|_\infty + \sum_{i\in\mathcal{I}} \|\partial_i \cdot\|_\infty + \sum_{i,j\in\mathcal{I}} \|\partial_i\partial_j \cdot\|_\infty$$

where $\|g\|_\infty := \max_{x\in\overline{K}} |g(x)|$ for $g \in C(\overline{K})$. The space $C^2(\overline{K})$, equipped with the norm $\|\cdot\|_{C^2(\overline{K})}$, is a separable Banach space. Let $K_l \subset E$, $l \in \mathbb{N}$, be an increasing sequence of open and relatively compact subsets such that $E = \bigcup_{l\in\mathbb{N}} K_l$. Then we can choose a countable set $\mathcal{X}$ of functions in $C_0^2(E)$ such that $\mathcal{X}$ contains a countable dense subset of $C^2(\overline{K_l})$ for each $l \in \mathbb{N}$ (in the sense of taking restrictions). Since countable unions of $\mathcal{E}$-exceptional sets are $\mathcal{E}$-exceptional, there is an $\mathcal{E}$-exceptional set $N \subset E$ such that $P_{(x,v)}$ solves the martingale problem for $\overline{L^E}$ on $\mathcal{X}$ for all $(x,v) \in E \setminus N$, i.e., $(M_t^{[g]})_{t\geq 0}$ is a martingale w.r.t. $P_{(x,v)}$ for all $(x,v) \in E \setminus N, g \in \mathcal{X}$.
Now, let $f \in C_0^2(E)$. There exists $l \in \mathbb{N}$ such that $f \in C^2(\overline{K_l})$. Thus we find a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{X} \cap C^2(\overline{K_l})$ such that $f_k \to f$ w.r.t. $\|\cdot\|_{C^2(\overline{K_l})}$ as $k \to \infty$. Then it holds $|f_k - f| \to 0$ and $|\overline{L^E} f_k - \overline{L^E} f| \to 0$ uniformly as $k \to \infty$. Since the kernels p_t, $t > 0$, are sub-Markovian, also $p_t|f - f_k| \to 0$ and $\int_0^t p_s|\overline{L^E} f - \overline{L^E} f_k|(\cdot)\, ds \to 0$ uniformly as $k \to \infty$ for any $t > 0$. By considerations as in the proof of Corollary 4.11 below we conclude that also $(M_t^{[f]})_{t\geq 0}$ is a martingale w.r.t. $P_{(x,v)}$ for $(x,v) \in E \setminus N$. Thus our assertion is shown. □

Remark 3.28. We should consider the question why we do not stay on $\widetilde{E}$ to construct the Langevin dynamics with periodic boundary condition. Of course, the closure of $(L, C_{per,0}^\infty(\widetilde{E}))$ (cf. Theorem 3.13) can also be used to construct a generalized Dirichlet form $\mathcal{E}'$ as in Theorem 3.25. Let us assume that $\mathcal{E}'$ is associated with a special standard process on $\widetilde{E}$, $[0,r_1) \times \cdots \times [0,r_n) \times \mathbb{R}^n$ or $[0,r_1] \times \cdots \times [0,r_n] \times \mathbb{R}^n$. We may replace $\widetilde{E}$ by one of the latter domains, since this does not affect the corresponding L^2-spaces. It is reasonable to assume that (for most initial points) with probability 1 our process hits the periodic boundary, and, moreover, that it even crosses it, when hitting it e.g. for the first time. But then the process cannot be both right continuous and quasi-left continuous. Hence it is not special standard (cf. [MR92, Definition IV.1.13]). Therefore we cannot expect that it is possible to construct the Langevin dynamics directly from $\mathcal{E}'$ using Theorem 2.9.

Remark 3.29. We note that the restriction π_* of π (given in Section 3.1.1) to $\widetilde{E}' := [0,r_1) \times \cdots \times [0,r_n) \times \mathbb{R}^n$ is a measurable bijection, and also its inverse is measurable. Hence, using the process $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t,V_t)_{t\geq 0}, (P_{(x,v)})_{(x,v)\in E_\Delta})$ from Theorem 3.25 we can define another process $\tilde{\mathbb{M}} := (\Omega, \mathcal{F}_*, (\widetilde{X}_t, V_t)_{t\geq 0}, (\widetilde{P}_{(x,v)})_{(x,v)\in\widetilde{E}'})$ on $\widetilde{E}'$ by setting $\widetilde{P}_{(x,v)} := P_{\pi(x,v)}$ and $(\widetilde{X}_t, V_t) = \pi_*^{-1}(X_t, V_t)$. Note that as in Section 3.1 we denote for $x \in \widetilde{M}$ the element $[x] = \pi^M(x) \in M$ also by x.
Let the measure $\widetilde{\mu}$ on $\widetilde{E}$ be defined by $\frac{d\widetilde{\mu}}{d(x,v)} := \hat{\Pi}^{-1}\left(\frac{d\mu}{d(x,v)^E}\right)$ (cf. Section 3.1.2). We can assume $\widetilde{\mu}$ to be extended to $\widetilde{E}'$ by continuous extension of its density w.r.t. Lebesgue measure. Since the boundary of $\widetilde{E}'$ has $\widetilde{\mu}$-measure 0, we may identify $L^2(\widetilde{E}, \widetilde{\mu})$ and $L^2(\widetilde{E}', \widetilde{\mu})$.

Then for any $f \in D(\overline{L})_b \subset L^2(\widetilde{E}', \widetilde{\mu})$ (where $(\overline{L}, D(\overline{L}))$ is the closure of $(L, C^\infty_{per,0})$, defined by $L := \frac{\gamma}{\beta}\Delta_v - \gamma v\nabla_v + v\nabla_x - (\nabla_x\Phi)\nabla_v)$ and for any measure $\widetilde{\nu}$ on $(\widetilde{E}', \mathcal{B}(\widetilde{E}'))$ which is absolutely continuous w.r.t. $\widetilde{\mu}$ the process

$$f(\widetilde{X}_t, V_t) - f(\widetilde{X}_0, V_0) - \int_0^t \overline{L}f(\widetilde{X}_s, V_s)\, ds$$

is a martingale w.r.t. the measure $\widetilde{P}_{\widetilde{\nu}}$, which is defined by

$$\begin{aligned} \widetilde{P}_{\widetilde{\nu}}(A) &:= \int_{\widetilde{E}'} \widetilde{P}_{(x,v)}(A)\, d\widetilde{\nu}(x,v) \\ &= \int_E P_{(x,v)}(A)\, d\nu(x,v) = P_\nu(A) \end{aligned}$$

for $A \in \mathcal{F}_*$, where ν is defined by $\frac{d\nu}{d\mu} := \hat{\Pi}\frac{d\widetilde{\nu}}{d\widetilde{\mu}}$. The martingale property is seen by the facts that the expectations $\widetilde{E}_{\widetilde{\nu}}$ and E_ν (corresponding to $\widetilde{P}_{\widetilde{\nu}}$ and P_ν, respectively) coincide and that $f \circ \pi_*^{-1} = (\Pi f)$ (μ-a.e.). It follows that our process is a solution of the Langevin equation (1.1) on $\widetilde{E}'$ in the sense of the corresponding martingale problem on $D(\overline{L})_b \supset C^\infty_{per,0}(\widetilde{E})$. Here, of course, we have to extend the functions in the latter set to $\widetilde{E}'$. Note that if $\widetilde{\nu}$ posseses an L^2-integrable density w.r.t. $\widetilde{\mu}$, we may replace $D(\overline{L})_b$ by $D(\overline{L})$.

4. The martingale problem

Let E and m be as in Section 2. Let $\mathcal{E}$ be a quasi-regular GDF on $L^2(E;m)$ (in the sense of Definition 2.1) associated with a coercive closed form $(\mathcal{A}, \mathcal{V})$ and an operator $(\Lambda, D(\Lambda))$. Let $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be an m-tight special standard process with life time ζ, properly associated with $\mathcal{E}$ in the resolvent sense (cf. Definition 2.7). By $(\mathcal{F}_t)_{t\geq 0}$ we denote the natural filtration for $\mathbb{M}$. Let L, $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha>0}$ denote the C_0 contraction semigroup, the strongly continuous contraction resolvent and the generator associated with $\mathcal{E}$, respectively (cf. Definition 2.3). Moreover, we denote by $(p_t)_{t>0}$ and $(R_\alpha)_{\alpha>0}$ the transition semigroup and resolvent of $\mathbb{M}$, respectively (cf. (2.2) and (2.3)). We remember that proper association of $\mathbb{M}$ with $\mathcal{E}$ implies that for any $f \in L^2(E;m)$ with m-version $\hat{f}$ the function $p_t\hat{f}$ is an m-version of T_tf and $R_\alpha\hat{f}$ is an $\mathcal{E}$-quasi-continuous m-version of $(G_\alpha f)_{\alpha>0}$ by Lemma 2.8.
For any probability measure μ on $(E, \mathcal{B}(E))$ the probability measure P_μ on $(\Omega, \mathcal{F}_*)$ is defined by

$$P_\mu(A) := \int_E P_x(A)\, d\mu(x) \tag{4.1}$$

for $A \in \mathcal{F}_*$.
Throughout this section we fix for every $f \in D(L)$ an $\mathcal{E}$-quasi-continuous m-version $\tilde{f}$ of f (which exists by Remark 2.6(iv)).

In this section we consider the question whether $\mathbb{M}$ solves the martingale problem for the generator $(L, D(L))$ of $\mathcal{E}$, i.e., we want to know, if for $f \in D(L)$ the process $(M_t^{[f]})_{t\geq 0}$

defined by

$$M_t^{[f]} := \tilde{f}(X_t) - \tilde{f}(X_0) - \int_0^t Lf(X_s)\, ds, \quad t \geq 0, \tag{4.2}$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. Clearly, the answer to this question may depend on the initial distribution we choose. The answer might be 'yes' if we consider $(M_t^{[f]})_{t\geq 0}$ w.r.t. to the probability measures P_μ with probability measures μ on $(E, \mathcal{B}(E))$, which are absolutely continuous w.r.t. m, but 'no' if we choose one of the probability measures P_x, $x \in E$. In fact, we cannot expect that $(M_t^{[f]})_{t\geq 0}$ is a martingale w.r.t. *all* P_x, $x \in E$, even if $\tilde{f}$ is continuous, since the process $\mathcal{M}$ is constructed only $\mathcal{E}$-quasi-everywhere and trivially extended (cf. [Sta99, p.88]).
We prove that the martingale problem is solved in the P_μ-case (see Corollary 4.11 below) if μ has an L^2-integrable density w.r.t. m. Moreover, we prove that in the P_x-case for any bounded $f \in D(L)$ (i.e. f possesses a bounded m-version) the process $(M_t^{[f]})_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale w.r.t. P_x for $\mathcal{E}$-q.e. $x \in E$ (cf. Theorem 4.7 below; this section consists mainly of its proof).

Note that by now we cannot even be sure that $(M_t^{[f]})_{t\geq 0}$ is well defined, since Lf is only determined up to a set of m-measure zero (cf. [AR95, Definition 3.1]). We need to show that the third summand in (4.2) exists a.s. and is moreover a.s. (w.r.t. to a probability measure P_μ or P_x) independent of the m-version we choose.
The argument we use to prove this is taken from [PR02, Thm 7.4(ii)].

Lemma 4.1. *Let $g \in L^2(E; m)$, with m-version $\hat{g}$, $t \geq 0$. Then*

$$\int_0^t p_s|\hat{g}|(x)\, ds \leq e^t R_1|\hat{g}|(x), \quad \textit{for all } x \in E,$$

In particular, $\int_0^t p_s\hat{g}(\cdot)\, ds$ exists $\mathcal{E}$-q.e. and is independent of the m-version $\hat{g}$ we choose. More precisely, the integrals of two different m-versions differ at most on an $\mathcal{E}$-exceptional set. Moreover, it is an element of $L^2(E; m)$ which continuously depends on $g \in L^2(E; m)$.

Proof. The inequality follows directly from Fubini's theorem and the definition of R_1 and p_s, $s > 0$.
Since $R_1|\hat{g}|$ is by Lemma 2.8(ii) $\mathcal{E}$-quasi-continous, it is finite $\mathcal{E}$-q.e.. Thus, the $\mathcal{E}$-q.e. existence of the integral is proven.
For two m-versions $\hat{g}_1, \hat{g}_2$ of g it holds

$$\left| \int_0^t p_s\hat{g}_1(x) - p_s\hat{g}_2(x)\, ds \right| \leq e^t R_1|\hat{g}_1 - \hat{g}_2|(x), \quad \text{for all } x \in E. \tag{4.3}$$

This is equal to 0 m-a.e., since $R_1|\hat{g}_1 - \hat{g}_2|(x)$ is an m-version of $G_1|g_1 - g_2| = 0 \in L^2(E; m)$. Thus by Remark 2.6(v) it is equal to 0 $\mathcal{E}$-q.e.. This proves the $\mathcal{E}$-q.e. independence of the m-version $\hat{g}$.
For (different) $g_1, g_2 \in L^2(E; m)$ with m-versions $\hat{g}_1, \hat{g}_2$ it also holds (4.3). By squaring and integrating w.r.t. m we obtain the last assertion, since G_1 is a continuous linear operator on $L^2(E; m)$. □

Remark 4.2. Let g, $\hat{g}$ be as in Lemma 4.1.

(i) Clearly the mapping $(t,\omega) \mapsto X_t(\omega)$, $\omega \in \Omega$, $t \geq 0$, is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_*$ measurable, where $\mathcal{B}([0,\infty))$ denotes the Borel-σ-field on $[0,\infty)$. Since by Fubini's theorem

$$E_x \int_0^t \hat{g}(X_s)\, ds = \int_0^t p_s \hat{g}(x)\, ds$$

whenever the integral on the right-hand side exists, the above lemma shows that for $\mathcal{E}$-q.e. $x \in E$ the integral $\int_0^t \hat{g}(X_s)\, ds$ exists P_x-a.s. and is P_x-a.s. independent of the m-version we choose (which has to be understood in a similar way as in Lemma 4.1).

(ii) Note that the $\mathcal{E}$-q.e. existence of $\int_0^t p_s\hat{g}(\cdot)\, ds$ for any $t \geq 0$ implies that all the integrals $\int_0^t p_s\hat{g}(x)\, ds$, $t \geq 0$, exist for x outside an $\mathcal{E}$-exceptional set (i.e., this $\mathcal{E}$-exceptional set can be chosen independently of $t \geq 0$), since countable unions of $\mathcal{E}$-exceptional sets are $\mathcal{E}$-exceptional.

(iii) Clearly, by (ii) the integrals $\int_0^t \hat{g}(X_s)\, ds$, $t \geq 0$, exist P_μ-a.s. for all $\mu \in \mathcal{P}(E_\Delta)$ such that any $\mathcal{E}$-exceptional set is contained in a null set w.r.t. μ (which holds e.g. if μ is absolutely continuous w.r.t. m, see Remark 2.6(i)).

We need some more "technical information" to proceed.

Lemma 4.3. *Let $f \in D(L)$. Then*

$$p_s\tilde{f} \text{ is } \mathcal{E}\text{-quasi-continuous for all } s > 0,$$

and if f is bounded (i.e. f has a bounded m-version), the function $\tilde{f}$ is bounded $\mathcal{E}$-q.e. and the mapping

$$s \mapsto p_s\tilde{f}(x)$$

is for $\mathcal{E}$-q.e. $x \in E$ right continuous and bounded on $[0,\infty)$.

Proof. Let $h := (I-L)f$ and choose an m-version $\hat{h}$ of h. Clearly $f = G_1 h$ and hence $\tilde{f} = R_1\hat{h}$ holds m-a.e.. Applying Lemma 2.8(ii) and Remark 2.6(v) we see that this holds even $\mathcal{E}$-q.e.. Lemma 2.12 implies that for $\mathcal{E}$-q.e. $x \in E$ the paths $(X_t)_{t\geq 0}$ do P_x-a.s. not hit the $\mathcal{E}$-exceptional set where $\tilde{f} \neq R_1\hat{h}$, hence for these x it holds $(p_s\tilde{f})(x) = E_x[\tilde{f}(X_t)] = E_x[R_1\hat{h}(X_t)] = (p_sR_1\hat{h})(x)$. Consequently, for $s > 0$, by Fubini's theorem it holds $p_s\tilde{f}(x) = p_sR_1\hat{h}(x) = R_1p_s\hat{h}(x)$ for those $x \in E$, for which additionally $R_1p_s|\hat{h}|(x) < \infty$. But since this function is $\mathcal{E}$-quasi-continuous by Lemma 2.8(ii), it is finite $\mathcal{E}$-q.e., consequently $p_s\tilde{f}$ and the $\mathcal{E}$-q.c. function $R_1p_s\hat{h}$ coincide $\mathcal{E}$-q.e.. Thus, $p_s\tilde{f}$ is $\mathcal{E}$-quasi-continuous.

To prove the second assertion, let f have an m-version which is bounded in absolute value by $C > 0$. Then clearly $(\tilde{f} \wedge C) \vee (-C)$ is a bounded $\mathcal{E}$-quasi-continuous m-version of f, hence by Remark 2.6(v) it differs from $\tilde{f}$ only on an $\mathcal{E}$-exceptional set. Consequently $\tilde{f} = (\tilde{f} \wedge C) \vee (-C)$ $\mathcal{E}$-q.e. proving that $|\tilde{f}| \leq C$ holds $\mathcal{E}$-q.e.

Together with quasi-continuity of $\tilde{f}$ this enables us to find an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $\tilde{f}$ is continuous on each F_k, $k \in \mathbb{N}$, and $|\tilde{f}(z)| \leq C$ for all $z \in \bigcup_{k\in\mathbb{N}} F_k$. By Lemma 2.12 we know that for $\mathcal{E}$-q.e. $x \in E$ it holds

$$P_x(\lim_{k\to\infty} \sigma_{F_k^c} \geq \zeta) = 1. \tag{4.4}$$

Now let $\omega \in \Omega$ be such that $\lim_{k\to\infty} \sigma_{F_k^c}(\omega) \geq \zeta(\omega)$. Then for any $0 \leq r < \zeta(\omega)$ we find $k \in \mathbb{N}$ such that $r < \sigma_{F_k^c}(\omega)$, consequently $X_s(\omega) \in F_k$ for all $s \in [0, r)$. But this implies that $s \mapsto \tilde{f}(X_s(\omega))$ is right continuous and bounded by C on $[0, r)$, hence on $[0, \zeta)$, since $r < \zeta$ was chosen arbitrarily. Since $X_s(\omega) = \Delta$ for all $s \geq \zeta(\omega)$ (remember that every function $f : E \to \mathbb{R}$ is extended to E_Δ by $f(\Delta) = 0$) we obtain right continuity on $[0, \infty)$. By this and (4.4) we have shown that for $\mathcal{E}$-q.e. $x \in E$ the process $(\tilde{f}(X_s))_{s\geq 0}$ is right continuous and bounded by C P_x-a.s.. Hence Lebesgue's dominated convergence theorem implies that for those x the function $s \mapsto E_x\tilde{f}(X_s) = p_s\tilde{f}(x)$ is right continuous. □

To prove the martingale property of M_t we first observe that for $\omega \in \Omega$

$$M^{[f]}_{t+s}(\omega) - M^{[f]}_t(\omega) = f(X_{t+s}(\omega)) - f(X_t(\omega)) - \int_t^{t+s} Lf(X_r(\omega))\, dr = M^{[f]}_s \circ \theta_t(\omega) \tag{4.5}$$

if the integral exists. By Remark 4.2(ii),(i) this is true (for all $t, s \geq 0$) P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. $\theta_t : \Omega \to \Omega$ denotes the time shift operator (cf. [Sta99, Definition IV.1.1]).

Equation (4.5) together with the Markov property of $\mathbb{M}$ leads to the following useful lemma.

Lemma 4.4. *Let $f \in D(L)$ be bounded. Suppose that for all $t > 0$ it holds*

$$E_x M^{[f]}_t = 0 \quad \mathcal{E}\text{-q.e. } x \in E \tag{4.6}$$

Then $(M^{[f]}_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale w.r.t. P_z for $\mathcal{E}$-q.e. $z \in E$.

Proof. From (4.6) we obtain for $\mathcal{E}$-q.e. $x \in E$

$$E_x M^{[f]}_t = 0 \quad \forall t \in \mathbb{Q} \cap [0, \infty) \tag{4.7}$$

Note that for all $t > 0$ and for $\mathcal{E}$-q.e. $x \in E$ it holds $E_x M^{[f]}_t = p_t\tilde{f}(x) - \tilde{f}(x) - \int_0^t p_s Lf(x) ds$. This is seen from the definition of $(p_t)_{t>0}$ (cf. (2.2)), Remark 4.2(ii) (implying the existence of the third summand $\mathcal{E}$-q.e.) and Fubini's theorem.
The mapping $t \mapsto p_t\tilde{f}(x) - \tilde{f}(x) - \int_0^t p_s Lf(x) ds$ is right continuous for $\mathcal{E}$-q.e. $x \in E$: The first summand is right continuous by Lemma 4.3, the second is constant and the integral function in the third summand is, of course, continuous.
Consequently, we obtain from (4.7) that for $\mathcal{E}$-q.e. $x \in E$ it holds

$$E_x M^{[f]}_t = 0 \quad \forall t \in [0, \infty) \tag{4.8}$$

Now, as mentioned before, by (4.5) and the Markov property of $\mathcal{M}$, it holds for $\mathcal{E}$-q.e. $z \in E$

$$E_z[M^{[f]}_{t+s} - M^{[f]}_t | \mathcal{F}_t] = E_{X_t}[M^{[f]}_s] \quad P_z\text{-a.s. for all } t, s \geq 0$$

If this is shown to be 0 P_z-a.s. for $\mathcal{E}$-q.e. $z \in E$ and all $t, s \geq 0$ we are done.
But this becomes clear, when we again as in the proof of Lemma 4.3 apply Lemma 2.12: Since (4.8) holds for $\mathcal{E}$-q.e. $x \in E$, the exceptional set, where it is not fulfilled, is for $\mathcal{E}$-q.e. initial point $z \in E$ not hit by the process. □

Now, for a bounded $f \in D(L)$ and $t > 0$ it remains to show that for $\mathcal{E}$-q.e. $x \in E$ it holds

$$E_x M_t^{[f]} = p_t\tilde{f}(x) - \tilde{f}(x) - \int_0^t (p_s Lf)(x)\,ds \overset{!}{=} 0 \tag{4.9}$$

We know that this is true in the sense of L^2-functions, if we replace "p" by "T":

$$T_t f - f = \int_0^t (T_s Lf)\,ds.$$

We also know that $p_t\tilde{f}$ is an m-version of $T_t f$, so it is reasonable to prove

Lemma 4.5. *Let $g \in L^2(E;m)$ with m-version $\hat{g}$. It holds*

$$\int_0^t (p_s\hat{g})(x)ds = \left(\int_0^t T_s g\,ds\right)(x)$$

for m-a.e. $x \in E$.

Proof. We use a similar argument as in the proof of Lemma 2.8(i) (taken from [FOT94, Proof of Theorem 4.2.3]). Let $v \in L^2(E;m)$, $v \geq 0$. We already know by Lemma 4.1 that $\int_0^t (p_s\hat{g})(\cdot)\,ds \in L^2(E;m)$, and this still remains true with $p_s\hat{g}$ replaced by $|p_s\hat{g}|$. Since consequently $\left(\int_0^t (|p_s\hat{g}|)(\cdot)\,ds, v\right)_{L^2(E;m)} < \infty$, we can apply Fubini's theorem to obtain

$$\left(\int_0^t p_s\hat{g}(\cdot)\,ds, v\right)_{L^2(E;m)} = \int_0^t (p_s\hat{g}, v)_{L^2(E;m)}\,ds = \int_0^t (T_s g, v)_{L^2(E;m)}\,ds,$$

where we used the fact that $p_s\hat{g}$ is an m-version of $T_s g$. But since $\int_0^t T_s g\,ds$ exists as a Riemann integral and the mapping $(\cdot, v)_{L^2(E;m)}$ from $L^2(E;m)$ to $\mathbb{R}$ is a continuous linear functional, we obtain

$$\left(\int_0^t p_s\hat{g}(\cdot)\,ds, v\right)_{L^2(E;m)} = \left(\int_0^t T_s g\,ds, v\right)_{L^2(E;m)}$$

for any nonnegative $v \in L^2(E;m)$, hence for any $v \in L^2(E;m)$. This implies our assertion. □

By now we only know that (4.9) is fulfilled for m-a.e. $x \in E$. But since the first two summands are $\mathcal{E}$-q.c. (see Lemma 4.3), we are in view of Remark 2.6(v) finally left to prove the following lemma. The proof is mainly taken from [AKR03, Lemma 5.1(iii)].

Lemma 4.6. *Let $f \in L^2(E;m)$ with m-version $\hat{f}$. Then*

$$\int_0^t p_s\hat{f}(\cdot)\,ds$$

is $\mathcal{E}$-quasi-continuous.

Proof. We may assume at first for convenience that $\hat{f} \geq 0$, such that throughout this proof we only integrate over nonnegative functions.
By Remark 4.2(ii) (and $\mathcal{E}$-quasi-continuity of $R_1\hat{f}$), we know that $p_s\hat{f}(x) \in L^1_{loc}([0,\infty))$ and $R_1\hat{f}(x) < \infty$ for $\mathcal{E}$-q.e. $x \in E$. Let $x \in E$ be such that both holds, then by

[Wer02, Satz A.1.10] we find that the function $t \mapsto e^t \int_0^t e^{-s} p_s \hat{f}(x)\, ds$ is locally absolutely continuous and, moreover, that we can apply the product rule and the fundamental theorem of calculus (and thus integration by parts) to obtain

$$\begin{aligned}\int_0^t p_s \hat{f}(x)\, ds &= \int_0^t e^s e^{-s} p_s \hat{f}(x) ds \\ &= \left(e^t \int_0^t e^{-s} p_s \hat{f}(x)\, ds - 0\right) - \int_0^t e^s \int_0^s e^{-r} p_r \hat{f}(x)\, dr\, ds.\end{aligned}$$

Since for any $s \geq 0$

$$\begin{aligned}\int_0^s e^{-r} p_r \hat{f}(x)\, dr &= (R_1 \hat{f})(x) - \int_s^\infty e^{-r} p_r \hat{f}(x)\, dr \\ &= (R_1 \hat{f})(x) - \int_0^\infty e^{-(r+s)} p_{r+s} \hat{f}(x)\, dr \\ &= (R_1 \hat{f})(x) - e^{-s} (R_1 p_s \hat{f})(x),\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^t p_s \hat{f}(x)\, ds &= e^t (R_1 \hat{f})(x) - (R_1 p_t \hat{f})(x) - \int_0^t e^s (R_1 \hat{f})(x)\, ds \\ &\quad + \int_0^t (R_1 p_s \hat{f})(x)\, ds \\ &= (R_1 \hat{f})(x) - (R_1 p_t \hat{f})(x) + \int_0^t (R_1 p_s \hat{f})(x)\, ds.\end{aligned} \tag{4.10}$$

Clearly, by Lemma 2.8(ii), $R_1 \hat{f}$ and $R_1 p_t \hat{f}$ are $\mathcal{E}$-quasi-continuous. Fubini's theorem implies

$$\int_0^t (R_1 p_s \hat{f})(x)\, ds = R_1 \left(\int_0^t p_s \hat{f}(\cdot)\, ds\right)(x)$$

Consequently, since by Lemma 4.1 $\int_0^t p_s \hat{f}(\cdot)\, ds \in L^2(E; m)$, we can again apply Lemma 2.8(ii) to find that also the last summand in (4.10) is $\mathcal{E}$-quasi-continuous.
Hence we have shown the assertion for $\hat{f} \geq 0$, which immediately extends to the case of general $\hat{f}$. □

This completes the proof of the following theorem.

Theorem 4.7. *Let $f \in D(L)$ be bounded and denote by $\tilde{f}$ an $\mathcal{E}$-quasi-continuous m-version of f. Then for $\mathcal{E}$-q.e. $x \in E$ the process $(M_t^{[f]})_{t \geq 0}$ defined by (4.2) is an $(\mathcal{F}_t)_{t \geq 0}$-martingale w.r.t. the probability measure P_x.*

Remark 4.8. Note that the only fact keeping us away from extending Theorem 4.7 to general $f \in D(L)$ is that we did not prove right continuity of $t \mapsto p_t f(x)$ $\mathcal{E}$-q.e. for those f (cf. Lemma 4.3).

This result also yields (in view of (4.1) and Remark 4.2(iii)) a similar result for the P_μ-case for bounded $f \in D(L)$, but quite general $\mu \in \mathcal{P}(E)$:

Corollary 4.9. *Let $\mu \in \mathcal{P}(E)$ be such that any $\mathcal{E}$-exceptional set is contained in a null set w.r.t. μ. Let $f \in D(L)$ be bounded and denote again by $\tilde{f}$ an $\mathcal{E}$-quasi-continuous m-version of f. Then the process $(M_t^{[f]})_{t\geq 0}$ defined by (4.2) is an $(\mathcal{F}_t)_{t\geq 0}$-martingale w.r.t. P_μ.*

Before we state the result for the P_μ-case we announced at the beginning of this section, we make the following remark.

Remark 4.10. Let μ be a probability measure on E such that μ has a density w.r.t. m which is L^2-integrable w.r.t. m. Then by the Cauchy-Schwarz inequality in $L^2(E;m)$ it holds $L^2(E;m) \subset L^1(E;\mu)$ continuously.

Corollary 4.11. *Let $\mu \in \mathcal{P}(E)$ have an L^2-integrable density w.r.t. m and let $f \in D(L)$ with $\mathcal{E}$-quasi-continuous m-version $\tilde{f}$. Then w.r.t. the probability measure P_μ given as in (4.1) the process $(M_t^{[f]})_{t\geq 0}$ defined by (4.2) is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.*

Proof. Let $s,t \geq 0$. We have to show that

$$E_\mu[M_{t+s}^{[f]} - M_t^{[f]} | \mathcal{F}_t] = 0 \quad P_\mu\text{-a.s.}. \tag{4.11}$$

If $f \in D(L)$ is bounded this is true by Corollary 4.9.
So, let $f \in D(L)$ be unbounded. Define $g := (I-L)f$, then $f = G_1 g$, and setting $g_n := (g \wedge n) \vee (-n)$, $n \in \mathbb{N}$, the property that G_1 is sub-Markovian implies that $f_n := G_1 g_n$ is m-a.e. bounded (in absolute value) for all $n \in \mathbb{N}$. Moreover, since $g_n \to g \in L^2(E;m)$ it follows $f_n = G_1 g_n \to G_1 g = f$ in $D(L)$ w.r.t. the graph norm as $n \to \infty$. (This shows that the bounded $D(L)$-functions form a dense subset of $D(L)$.)
It holds for $n \in \mathbb{N}$

$$\begin{aligned} &E_\mu[|E_\mu(M_{t+s}^{[f]} - M_t^{[f]} | \mathcal{F}_t)|] \\ &\quad = E_\mu\left[\left|E_\mu(M_{t+s}^{[f]} - M_{t+s}^{[f_n]} - (M_t^{[f]} - M_t^{[f_n]}) | \mathcal{F}_t)\right|\right] \\ &\quad \leq E_\mu[|M_{t+s}^{[f]} - M_{t+s}^{[f_n]}|] + E_\mu[|M_t^{[f]} - M_t^{[f_n]}|], \end{aligned}$$

where we applied (4.11) for f_n. Let $\widehat{Lf}$, $\widehat{Lf_n}$ denote m-versions of $Lf, Lf_n \in L^2(E;m)$, $n \in \mathbb{N}$. For any $r \geq 0$ we obtain

$$\begin{aligned} E_\mu[|M_r^{[f]} - M_r^{[f_n]}|] &= \|E_\bullet[|M_r^{[f]} - M_r^{[f_n]}|]\|_{L^1(E;\mu)} \\ &\leq \|p_r|\tilde{f} - \tilde{f}_n|\|_{L^1(E;\mu)} + \|\tilde{f} - \tilde{f}_n\|_{L^1(E;\mu)} \\ &\quad + \left\|E_\bullet \int_0^r |\widehat{Lf}(X_s) - \widehat{Lf_n}(X_s)|\, ds\right\|_{L^1(E;\mu)} \\ &= \|p_r|\tilde{f} - \tilde{f}_n|\|_{L^1(E;\mu)} + \|\tilde{f} - \tilde{f}_n\|_{L^1(E;\mu)} \\ &\quad + \int_0^r \|p_s|\widehat{Lf} - \widehat{Lf_n}|\|_{L^1(E;\mu)}\, ds \\ &\leq 2C\|f - f_n\|_{L^2(E;m)} + Cr\|Lf - Lf_n\|_{L^2(E;m)} \stackrel{n\to\infty}{\longrightarrow} 0 \end{aligned} \tag{4.12}$$

for some $C > 0$. Here we used Fubini's theorem, Remark 4.10, and the facts that $p_s\hat{g}$ is an m-version of $T_s g$ for all $g \in L^2(E;m)$ with m-version $\hat{g}$ and that T_s is a contraction for $s \geq 0$. We conclude that (4.11) holds for general $f \in D(L)$. □

Remark 4.12. Note that in the P_μ-case (with μ being absolutely continuous w.r.t. m) we could choose *any* m-version $\hat{f}$ of $f \in D(L)$ and define

$$M_t^{[\hat{f}]} := \hat{f}(X_t) - \hat{f}(X_0) - \int_0^t Lf(X_s)\, ds$$

for $t \geq 0$, to obtain a martingale, since

$$\begin{aligned} & E_\mu[|E_\mu(M_{t+s}^{[\hat{f}]} - M_t^{[\hat{f}]}|\mathcal{F}_t)|] \\ = \;& E_\mu\left[\left|E_\mu(M_{t+s}^{[\hat{f}]} - M_{t+s}^{[f]} - (M_t^{[\hat{f}]} - M_t^{[f]})|\mathcal{F}_t)\right|\right] \\ \leq \;& E_\mu[|M_{t+s}^{[\hat{f}]} - M_{t+s}^{[f]}|] + E_\mu[|M_t^{[\hat{f}]} - M_t^{[f]}|] \end{aligned}$$

and having another look at (4.12) we see that this is equal to 0.

5. The diffusion property for local generators

In this section we give a condition for the generator $(L, D(L))$ of a quasi-regular generalized Dirichlet form $\mathcal{E}$ on $L^2(E; m)$ ensuring that an m-tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}_*, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with life time ζ, which is properly associated with $\mathcal{E}$ in the resolvent sense (cf. Definition 2.7), is a diffusion in the sense that

$$P_x(\,(X_t)_{t\geq 0} \text{ is continuous on } [0,\zeta)\,) = 1 \quad \text{for } \mathcal{E}\text{-q.e. } x \in E. \tag{5.1}$$

Let again $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha>0}$ be the C_0 contraction semigroup and the strongly continuous contraction resolvent associated with L, and let $(p_t)_{t>0}$ and $(R_\alpha)_{\alpha>0}$ be the transition semigroup and resolvent of $\mathbb{M}$. $(\mathcal{F}_t)_{t\geq 0}$ denotes again the natural filtration for $\mathbb{M}$. For technical reasons it makes sense to set $X_\infty := \Delta$ (as in [MR92, p.89]).

To prove continuity of $(X_t)_{t\geq 0}$ w.r.t. P_x for an $x \in E$, we use the following lemma. Except of one argument its proof is the same as the proof of [MR92, Theorem V.1.5 (p.153)].

Lemma 5.1. *Let $\mathcal{U}$ be a base of the topology of E. Suppose that for every $U \in \mathcal{U}$ it holds*

$$P_z(X_{\sigma_U} \in U) = 0 \quad \text{for } \mathcal{E}\text{-q.e. } z \in E \setminus U. \tag{5.2}$$

Then (5.1) is valid.

Proof. Let $K \subset E$ be compact and metrizable. Then $\mathcal{U} \cap K := \{U \cap K | U \in \mathcal{U}\}$ forms a base of the topology of K. Since K is second countable, it is strongly Lindelöf (cf. [Sch73, p.104]) and thus any element of a countable base of K is a countable union of elements in $\mathcal{U} \cap K$. Thus there exists a countable subset $\mathcal{U}_K \subset \mathcal{U}$ such that also $\mathcal{U}_K \cap K$ is a base of the topology of K. Hence, if $(K_j)_{j\in\mathbb{N}}$ is an $\mathcal{E}$-nest of compact metrizable subsets of E, which exists by quasi-regularity of $\mathcal{E}$ (cf. [MR92, Proof of Theorem V.1.5 and Remark IV.3.2(iii)]), the set $\hat{\mathcal{U}} := \bigcup_{j\in\mathbb{N}} \mathcal{U}_{K_j}$ is such that $\hat{\mathcal{U}} \cap K_j$ is a base of the topology of K_j for every $j \in \mathbb{N}$ and (5.2) holds for every $U \in \hat{\mathcal{U}}$. The rest follows as in [MR92, Theorem V.1.5]. □

Remark 5.2. Note that, of course, the argument in the above proof is not necessary in the situation of Section 3, where we know that (5.2) holds for a countable base of the topology of the second countable manifold E. It is included to avoid a case differentiation when stating Theorem 5.5 below.

The idea to prove the condition given in 5.1 is also taken from [MR92, Lemma V.1.8]. Since we are dealing with a special case here (cf. the assumptions in Theorem 5.5 and see also Remark 5.7), we do not need to transfer the complete argumentation from [MR92, Section V.1]. We first prove the following lemma (cf. [MR92, p. 129]).

Lemma 5.3. *Let $f \in L^2(E;m)$ with m-version $\hat{f}$, and let σ be an $(\mathcal{F}_t)_{t\geq 0}$-stopping time, then it holds for $\mathcal{E}$-q.e. $z \in E$*

$$E_z[e^{-\sigma} R_1 \hat{f}(X_\sigma)] = E_z\left[\int_\sigma^\infty e^{-t}\hat{f}(X_t)\,dt\right]. \tag{5.3}$$

Proof. If $\hat{f}$ is bounded, we can use the strong Markov property (cf. [MR92, Definition IV.1.8, Exercise IV.1.9]) to derive

$$\begin{aligned} E_z\left[e^{-\sigma} R_1 \hat{f}(X_\sigma)\right] &= E_z\left[e^{-\sigma} E_{X_\sigma}\left[\int_0^\infty e^{-t}\hat{f}(X_t)\,dt\right]\right] = E_z\left[e^{-\sigma}\int_0^\infty e^{-t}\hat{f}(X_{t+\sigma})\,dt\right] \\ &= E_z\left[e^{-\sigma}\int_\sigma^\infty e^{-(t-\sigma)}\hat{f}(X_t)\,dt\right] = E_z\left[\int_\sigma^\infty e^{-t}\hat{f}(X_t)\,dt\right] \end{aligned}$$

Consequently, by the monotone convergence theorem, (5.3) holds also for functions $\hat{f} \in \mathcal{B}^+$. Since for L^2-integrable positive Borel functions $\hat{f}$ we know that

$$E_z\int_\sigma^\infty e^{-t}\hat{f}(X_t)\,dt \leq R_1\hat{f}(z) < \infty \quad \mathcal{E}\text{-q.e. } z \in E$$

by $\mathcal{E}$-quasi-continuity of $R_\alpha f$ (cf. Lemma 2.8), for any $f \in L^2(E;m)$ with m-version $\hat{f}$ the integrals/expectations in (5.3) exist $\mathcal{E}$-q.e. and (5.3) holds. □

Lemma 5.4. *Let $U \subset E$, U open, and assume that there exists $u \in D(L)$ with $\mathcal{E}$-q.c. m-version $\tilde{u}$, such that $\tilde{u} = 0$ $\mathcal{E}$-q.e. on $E \setminus U$, $\tilde{u} > 0$ $\mathcal{E}$-q.e on U and $Lu = 0$ m-a.e. on $E \setminus U$.*
Then it holds

$$P_z(X_{\sigma_U} \in U) = 0 \quad \mathcal{E}\text{-q.e. on } E \setminus U$$

Proof. Let $f := (I - L)u$. By our assumptions we can choose an m-version $\hat{f}$ of f such that $\hat{f}(x) = 0$ for all $x \in E\setminus U$. According to Remark 2.6(v) we can assume that $\tilde{u} = R_1\hat{f}$, since $R_1\hat{f}$ is $\mathcal{E}$-quasi-continuous by Lemma 2.8(ii). Then by Lemma 5.3 it holds

$$\begin{aligned} E_z[e^{-\sigma_U}\tilde{u}(X_{\sigma_U})] &= E_z[e^{-\sigma_U} R_1\hat{f}(X_{\sigma_U})] \\ &= E_z\int_{\sigma_U}^\infty e^{-t}\hat{f}(X_t)\,dt \\ &= E_z\int_0^\infty e^{-t}\hat{f}(X_t)\,dt = R_1\hat{f}(z) = \tilde{u}(z) = 0 \end{aligned} \tag{5.4}$$

for $\mathcal{E}$-q.e. $z \in E \setminus U$. Here we used the fact that $\hat{f} = 0$ on $E \setminus U$.
By our assumptions the set $N := \{x \in E | \tilde{u}(x) < 0\} \cup \{x \in U | \tilde{u}(x) = 0\}$ is $\mathcal{E}$-exceptional. Thus by Lemma 2.12 and (5.4) we know that for $\mathcal{E}$-q.e. $z \in E \setminus U$ it holds

$$P_z(\exists s : X_s \in N) = 0$$

and

$$E_z[e^{-\sigma_U} \tilde{u}(X_{\sigma_U})] = 0.$$

Hence for those z we obtain $e^{-\sigma_U} \tilde{u}(X_{\sigma_U}) > 0$ P_z-a.s. on $\{X_{\sigma_U} \in U\}$, but

$$0 \leq E_z[1_{\{X_{\sigma_U} \in U\}} e^{-\sigma_U} \tilde{u}(X_{\sigma_U})] \leq E_z[e^{-\sigma_U} \tilde{u}(X_{\sigma_U})] = 0$$

proving that P_z-a.s. it holds $X_{\sigma_U} \notin U$. □

The following theorem is just a combination of Lemma 5.4 and Lemma 5.1.

Theorem 5.5. *Suppose that $\mathcal{U}$ is a base of the topology of E and that for any $U \in \mathcal{U}$ there exists $u \in D(L)$ with $\mathcal{E}$-q.c. m-version $\tilde{u}$ such that*

(i) $\tilde{u} = 0$ *$\mathcal{E}$-q.e. on* $E \setminus U$,
(ii) $\tilde{u} > 0$ *$\mathcal{E}$-q.e. on* U,
(iii) $Lu = 0$ *m-a.e. on* $E \setminus U$.

Then $P_x((X_t)_{t \geq 0}$ is continuous$) = 1$ for $\mathcal{E}$-q.e. $x \in E$.

Remark 5.6. Clearly, if the assumptions of Theorem 5.5 are fulfilled, then $(X_t)_{t \geq 0}$ is also P_μ-a.s. continuous for every $\mu \in \mathcal{P}(E_\Delta)$ whose completion assigns 0 to every $\mathcal{E}$-exceptional set. In particular this holds for μ being absolutely continuous w.r.t. m.

Remark 5.7. The conditions in Theorem 5.5 are e.g. fulfilled if L is a differential operator without terms of order 0 on an open subset of $\mathbb{R}^n$ and its domain contains the infinitly often differentiable functions with compact support.

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