INTEGRABILITY OF OPTIMAL MAPPINGS

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Abstract. We study integrability properties of the optimal transportation mapping T which pushes forward a probability measure μ to another measure $g \cdot \mu$. We assume that T minimizes some cost function c and μ is satisfied some special properties related to c (the infimum-convolution inequality or the logarithmic c-Sobolev inequality). We apply our results for measures of the type $\exp(-|x|^{\alpha})$.

Keywords: optimal transportation, logarithmic Sobolev inequality, transportation inequalities.

1. Introduction

In this paper we consider a probability measure μ and an optimal mapping

$$T(x) = x + F(x)$$

which pushes forward μ to another probability measure $g \cdot \mu$ and minimizes some cost function c. The latter means that T minimizes the following integral:

$$K(\mu, g \cdot \mu, c, T) := \int_X c(F(x))\, \mu(dx).$$

We assume that μ satisfies some special inequalities related to the cost c such as the infimum-convolution inequality or the logarithmic c-Sobolev inequality. We recall that a measure μ on $\mathbb{R}^d$ is said to satisfy the logarithmic Sobolev inequality if for every smooth function f one has

$$\mathrm{Ent}_{\mu} f^2 \le 2C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu, \tag{1.1}$$

where

$$\mathrm{Ent}_{\mu} g := \int_{\mathbb{R}^d} g \log g \, d\mu - \Bigl(\int_{\mathbb{R}^d} g \, d\mu\Bigr) \log \int_{\mathbb{R}^d} g \, d\mu.$$

Applying (1.1) to $1 + \varepsilon f$ one obtains in the limit $\varepsilon \to 0$ the Poincaré inequality

$$\int_{\mathbb{R}^d} f^2 \, d\mu - \Bigl(\int_{\mathbb{R}^d} f \, d\mu\Bigr)^2 \le C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu. \tag{1.2}$$

It is well known that every measure satisfying (1.1) satisfies the infimum-convolution inequality

$$\int_{\mathbb{R}^d} \exp Q_C f \, d\mu \le \exp \int_{\mathbb{R}^d} f \, d\mu, \quad Q_C f(x) = \inf_y \Bigl[f(y) + \frac{|x-y|^2}{2C} \Bigr] \tag{1.3}$$

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and the transportation inequality

$$\int_{\mathbb{R}^d} |F|^2 \, d\mu \le 2C \mathrm{Ent}_\mu g, \tag{1.4}$$

where $T = Id + F$ is the optimal transportation mapping sending μ to $g \cdot \mu$ and corresponding to the cost $c = |x|^2$ (see [1], [2]). According to the results from [3], [4] (see also recent books [5], [6], [7]), T has the form $T = \nabla V$, where V is a convex function.

Inequality (1.4) (proved in [8] for the standard Gaussian measure and extended in [9] to every measure satisfying (1.1)) gives a very simple integral estimation of $|F(x)|$ by some quantity depending only on g and μ. A further step in this direction due to Fernique [10] who considered a Gaussian measure γ on a separable Fréchet space and a probability measure $g \cdot \gamma$ such that $g \in L^p(\gamma)$ for some $p > 1$. It was shown in [10] that there exists a mapping $T = U + S$, where the mapping U preserves the measure γ and S is a mapping with values in the Cameron–Martin space H of γ, such that the function $\exp(\omega |S|_H^2)$ is integrable for sufficiently small ω (however, the mapping T is not necessarily the optimal transportation). This result was generalized in [11]. In particular, the following theorem was obtained there.

Theorem 1. *Suppose that μ satisfies the logarithmic Sobolev inequality (1.1). Consider the optimal transportation mapping $T(x) = x + F(x)$ which pushes forward μ to $g \cdot \mu$ and minimizes $c(x) = \frac{x^2}{2}$. If $g |\log g|^p \in L^1(\mu)$, then $|F|^{2p} \in L^1(\gamma)$. Moreover, if $g \in L^p(\mu)$, then $\exp(\omega |F|^2) \in L^1(\mu)$ for some sufficiently small $\omega = \omega(C, p)$.*

In addition, certain precise estimates in the Gaussian case for different types of mappings (also non-optimal) and some similar estimates for measures satisfying the Poincaré inequality were obtained.

In this paper, we give a generalization of Theorem 1 for non-quadratic costs. As a main example we consider the probability measure

$$\mu_\alpha = \frac{1}{Z_\alpha^d} \prod_{i=1}^d e^{-|x_i|^\alpha} dx_i$$

where $Z_\alpha = \int_{\mathbb{R}} e^{-|x|^\alpha} dx$ and $1 < \alpha \le 2$. In the proof we use recent results from [12].

2. Main results

We consider a cost function $c(x)$ on $\mathbb{R}^d$. Throughout the paper c satisfies the following assumptions:

A1) c is non-negative, even and $c(0) = 0$.

A2) c is strictly convex. This means that c is convex and, in addition, the equality

$$c(tx + (1-t)y) = tc(x) + (1-t)c(y)$$

implies that $x = y$, or $t = 0$, or $t = 1$.

A3) c is superlinear, i.e., one has

$$\lim_{x \to \infty} \frac{c(x)}{|x|} = \infty$$

A4) Given $r > 0$ and $\theta \in (0, \pi)$, whenever $p \in \mathbb{R}^d$ is far enough from the origin, there exists a direction $z \in \mathbb{R}^d$ such that on the truncated cone K defined by

$$K = \big\{ x \in \mathbb{R}^d, |x-p||z| \cos(\theta/2) \le \big\langle z, x-p \big\rangle \le r|z| \big\},$$

the function c assumes its maximum at p.

Assumptions **A2) - A4)** were introduced in [13]. Below we use some existence and regularity results from [13].

For every c one defines the corresponding conjugated function

$$c^*(x) = \sup_{y\in\mathbb{R}^d}\bigl(\langle x, y\rangle - c(y)\bigr).$$

Let μ and $g\cdot\mu$ be absolutely continuous probability measures such that

$$W_c(\mu, g\cdot\mu) = \inf_m \int_{(\mathbb{R}^d)^2} c(x-y)\, dm < \infty,$$

where the infimum is taken among the measures on $(\mathbb{R}^d)^2$ with the projections μ and $g\cdot\mu$. It is known that under **A2) - A4)** there exists an optimal mapping $T = Id + F$ such that T pushes forward μ to $g\cdot\mu$ and

$$\int_{\mathbb{R}^d} c(F)\, d\mu = W_c(\mu, g\cdot\mu).$$

It is known (see [13] or Theorem 2.44 in [7]) that T has the form

$$T(x) = x + \nabla c^*(\nabla\Phi) \quad \mu - \text{a.e.} \tag{2.5}$$

for some c-convex function Φ. This means that

$$-\Phi = Q_c\Psi$$

for some Ψ. Here

$$Q_c f(x) = \inf_y \bigl[f(y) + c(x-y)\bigr]$$

is the infimum-convolution $Q_c f$ of f. For the theory of convex convolutions and the structure of c-convex potentials, see [5], [6], [7]. We set $\Phi^c = -Q_c\Phi$ and note that $\Psi = \Phi^c$. By a result in [13], there exists a convex set K such that Φ is locally Lipschitz on $\Omega = \mathrm{Int}(K)$ and

$$\Omega \subseteq \mathrm{Dom}(\Phi) \subseteq K,$$

hence (2.5) is well-defined. Note that unlike the standard way we do not use the representation $T(x) = x - \nabla c^*(\nabla\tilde{\Phi})$, where $\tilde{\Phi}$ is a c-*concave* function. However, since c is assumed to be even, our representation (2.5) is equivalent to the standard one.

We will use the following well-known formulas:

$$\Phi(x) + \Phi^c(y) \ge -c(x-y),$$
$$\Phi(x) + \Phi^c(T(x)) = -c(T(x) - x) \quad \text{for} \quad \mu - \text{a.e. } x.$$

A measure μ is said to satisfy the infimum-convolution inequality for c if for every bounded measurable function f one has

$$\int_{\mathbb{R}^d} e^{Q_c f}\, d\mu \int_{\mathbb{R}^d} e^{-f}\, d\mu \le 1. \tag{2.6}$$

Note that (2.6) and the Jensen inequality imply

$$\int_{\mathbb{R}^d} e^{Q_c f}\, d\mu \le e^{\int_{\mathbb{R}^d} f\, d\mu}. \tag{2.7}$$

It is easy to verify (see, for example, [11], where the case of the quadratic cost was considered) that (2.7) holds under (2.6) for every μ-integrable f.

It is well-known (see [1], [2]) that the Talagrand inequality

$$\int_{\mathbb{R}^d} c(F)\, d\mu \le \mathrm{Ent}_\mu g$$

for the convex cost c is equivalent to (2.7).

We say that a probability measure μ on $\mathbb{R}^d$ satisfies the logarithmic c-Sobolev inequality with the cost c if for some $\Lambda > 0$ the following estimate holds for every locally Lipschitz function f:

$$\mathrm{Ent}_\mu f^2 \le \Lambda \int_{\mathbb{R}^d} c^*\Bigl(\frac{\nabla f}{f}\Bigr) f^2\, d\mu. \tag{2.8}$$

We note that for many cost functions (2.8) implies (2.7). It was proved in [12] that the product of the one-dimensional measures of the type $e^{-|x|^\alpha}\, dx$ satisfies (2.8) for a cost function c which is quadratic for small x and equals to $A\sum_{i=1}^d |x_i|^\alpha$ for large x. We consider this example in the next section.

In the lemma below we generalize a result from [14]. The proof is very similar to the original one, however, it is given for the reader's convenience.

Lemma 1. *Suppose that for every locally Lipschitz function φ the probability measure μ satisfies the following inequality:*

$$Ent_\mu e^\varphi \le \frac{1}{2}\int_X H(\nabla\varphi) e^\varphi\, d\mu, \tag{2.9}$$

where $H : \mathbb{R}^d \to \mathbb{R}^+$ has the following property: the function $\lambda \to H(\sqrt{\lambda} x)$ is non-decreasing and convex on $[0,\infty)$ for every x and $H(0) = 0$. Then

$$\int_{\mathbb{R}^d} \exp\Bigl[\varphi - \int_{\mathbb{R}^d} \varphi\, d\mu\Bigr]\, d\mu \le \int_{\mathbb{R}^d} e^{H(\nabla\varphi)}\, d\mu.$$

In particular, if $t \to c^*(\sqrt{t} x)$ is convex and non-decreasing on $[0,\infty)$ and μ satisfies (2.8), then for every locally Lipschitz φ one has

$$\int_{\mathbb{R}^d} \exp\Bigl[\varphi - \int_{\mathbb{R}^d} \varphi\, d\mu\Bigr]\, d\mu \le \int_{\mathbb{R}^d} \exp\Bigl(2\Lambda c^*\Bigl[\frac{\nabla\varphi}{2}\Bigr]\Bigr)\, d\mu. \tag{2.10}$$

Proof. Set

$$g = H(\nabla\varphi) - \log \int_{\mathbb{R}^d} e^{H(\nabla\varphi)}\, d\mu,$$

so that $\int_{\mathbb{R}^d} e^g\, d\mu = 1$. By a well known property of the entropy

$$\mathrm{Ent}_\mu f = \sup\Bigl\{\int_{\mathbb{R}^d} fg\, d\mu\colon\ g \text{ is such that } \int_{\mathbb{R}^d} \exp(g)\, d\mu \le 1\Bigr\}$$

one obtains

$$\int_{\mathbb{R}^d} e^\varphi g\, d\mu \le \mathrm{Ent}_\mu e^\varphi.$$

Hence by (2.9) we have

$$2\mathrm{Ent}_\mu e^\varphi \le \int_{\mathbb{R}^d} e^\varphi H(\nabla\varphi)\, d\mu \le \Bigl(\int_{\mathbb{R}^d} e^\varphi\, d\mu\Bigr)\log\Bigl(\int_{\mathbb{R}^d} e^{H(\nabla\varphi)}\, d\mu\Bigr) + \mathrm{Ent}_\mu e^\varphi.$$

Hence

$$\mathrm{Ent}_\mu e^\varphi \le \Bigl(\int_{\mathbb{R}^d} e^\varphi\, d\mu\Bigr)\log\Bigl(\int_{\mathbb{R}^d} e^{H(\nabla\varphi)}\, d\mu\Bigr).$$

Applying the latter to $\lambda\varphi$, we obtain

$$\mathrm{Ent}_{\mu} e^{\lambda\varphi} \le \Bigl(\int_{\mathbb{R}^d} e^{\lambda\varphi}\, d\mu\Bigr) \log\Bigl(\int_{\mathbb{R}^d} e^{H(\lambda\nabla\varphi)}\, d\mu\Bigr). \tag{2.11}$$

Set

$$K(\lambda) = \frac{1}{\lambda} \log \int_{\mathbb{R}^d} e^{\lambda\varphi}\, d\mu.$$

Let us calculate the following derivative:

$$K^{'}(\lambda) = \frac{\mathrm{Ent}_{\mu} e^{\lambda\varphi}}{\lambda^2 \int_{\mathbb{R}^d} e^{\lambda\varphi}\, d\mu} \le \frac{\log\Bigl(\int_{\mathbb{R}^d} e^{H(\lambda\nabla\varphi)}\, d\mu\Bigr)}{\lambda^2}.$$

Let

$$F(t) = \log\Bigl(\int_{\mathbb{R}^d} e^{H(\sqrt{t}\nabla\varphi)}\, d\mu\Bigr).$$

Obviously, F is non-negative and non-decreasing. Let us show that F is convex. Indeed, it follows from the fact (which is easily verified) that any function of the type $\sum_{i=1}^{d} e^{W_i}$, where every W_i is convex, has the form e^W for some convex W. We get our claim by approximating the integral by finite sums.

Taking into account that $F(0) = 0$, we obtain that $\frac{F(t)}{t}$ is non-decreasing. Hence

$$\lambda \to \frac{\log\Bigl(\int_{\mathbb{R}^d} e^{H(\lambda\nabla\varphi)}\, d\mu\Bigr)}{\lambda^2}$$

is non-decreasing. Consequently, one has

$$\log \int_{\mathbb{R}^d} e^{\varphi}\, d\mu = K(1) \le K(0) + \int_0^1 K^{'}(\lambda)\, d\lambda \le \int_{\mathbb{R}^d} \varphi\, d\mu + \log\Bigl(\int_{\mathbb{R}^d} e^{H(\nabla\varphi)}\, d\mu\Bigr)$$

and we obtain our claim. Finally, (2.10) follows from the Lemma and (2.8) by setting $f^2 = e^g$. □

Theorem 2. *Suppose that c satisfies assumptions* **A1)** - **A4)** *and μ satisfies infimum-convolution inequality (2.7) and Poincaré inequality (1.2). Suppose in addition that*

1) *Inequality (2.10) holds*
2) *$t \to c^*(\sqrt{t}x)$ is convex and non-decreasing for every x on $[0, \infty)$*
3) *There exists a function $N(\tau) > 0$ such that*

$$c^*\Bigl(\frac{\tau x}{2}\Bigr) \le N(\tau) c(\nabla c^*(x))$$

and $\lim_{\tau\to 0} \frac{N(\tau)}{\tau} = 0$.

Then for every $p > 1$ there exist positive numbers $\omega = \omega(p, \Lambda, N(\tau))$, $M = M(p, \Lambda, N(\tau))$ such that

$$M \int_{\mathbb{R}^d} c(F) e^{\omega c(F)}\, d\mu \le \|g\|_{L^p(\mu)}.$$

Proof. First we note that $\Phi \in L^2(\mu)$. Indeed, in follows from assumption 2) that $c^*(x) \ge a|x|^2 + b$ for some $a > 0$. Then it follows from assumption 3) that for some $B > 0$ one has

$$|x|^2 \le A + Bc(\nabla c^*(x)).$$

Hence

$$|\nabla\Phi|^2 \le A + Bc(\nabla c^*(\nabla\Phi)) = A + Bc(T(x) - x) = A + Bc(F).$$

Since (2.7) implies the Talagrand inequality for the cost c and $\mathrm{Ent}_\mu g < \infty$, we obtain that

$$\int_{\mathbb{R}^d} c(F)\,d\mu < \infty, \quad |\nabla\Phi| \in L^2(\mu).$$

Hence by the Poincaré inequality $\Phi \in L^2(\mu)$. Let us choose Φ in such a way that $\int_{\mathbb{R}^d} \Phi\,d\mu = 0$.

Let $\tau > 2\omega > 0$. By the Young inequality $xy \le x\log x - x + e^y$, where $x \ge 0$, $y \in \mathbb{R}$, one has

$$\tau c(F)e^{\omega c(F)} = \tau(-\Phi - \Phi^c(T))e^{\omega c(F)} \le e^{-\tau\Phi} + e^{-\tau\Phi^c(T)} + 2\omega c(F)e^{\omega c(F)} - 2e^{\omega c(F)}.$$

Hence for every $A \subset \mathbb{R}^d$ we have

$$(\tau - 2\omega)\int_A c(F)e^{\omega c(F)}\,d\mu + 2\int_A e^{\omega c(F)}\,d\mu \le \int_A e^{-\tau\Phi}\,d\mu + \int_A e^{-\tau\Phi^c(T)}\,d\mu.$$

We estimate the right-hand side as follows:

$$\int_{\mathbb{R}^d} e^{-\tau\Phi^c(T)}\,d\mu = \int_{\mathbb{R}^d} e^{-\tau\Phi^c} g\,d\mu \le \Bigl(\int_{\mathbb{R}^d} g^p\,d\mu\Bigr)^{\frac{1}{p}} \Bigl(\int_{\mathbb{R}^d} e^{-q\tau\Phi^c}\,d\mu\Bigr)^{\frac{1}{q}}.$$

Suppose that $\tau \le \frac{1}{q}$. We find by (2.7) that

$$\int_{\mathbb{R}^d} e^{-q\tau\Phi^c}\,d\mu \le \Bigl(\int_{\mathbb{R}^d} e^{-\Phi^c}\,d\mu\Bigr)^{\frac{1}{q\tau}} \le e^{\frac{1}{\tau q}\int_{\mathbb{R}^d}\Phi\,d\mu} = 1$$

and

$$\int_{\mathbb{R}^d} e^{-\tau\Phi^c(T)}\,d\mu = \int_{\mathbb{R}^d} e^{-\tau\Phi^c} g\,d\mu \le \Bigl(\int_{\mathbb{R}^d} g^p\,d\mu\Bigr)^{\frac{1}{p}}.$$

Further, set $A := A_N = \bigl\{x : -\Phi(x) \le N\bigr\}$. One has $\nabla\bigl[\max(\Phi, -N)\bigr] = \nabla\Phi\chi_{A_N}$ μ-a.e. Therefore, 3) and (2.10) yield that

$$\int_{A_N} e^{-\tau\Phi}\,d\mu \le \int_{\mathbb{R}^d} \exp\bigl(-\tau\max(\Phi,-N)\bigr)\,d\mu \le \int_{\mathbb{R}^d} \exp\Bigl(2\Lambda c^*\Bigl(\frac{\tau}{2}\nabla\Phi\chi_{A_N}\Bigr)\Bigr)\,d\mu$$

$$\le \int_{\mathbb{R}^d} \exp\Bigl(2\Lambda N(\tau) c\bigl[\nabla c^*(\nabla\Phi\chi_{A_N})\bigr]\Bigr)\,d\mu = \int_{A_N} e^{2\Lambda N(\tau)c\bigl(F\bigr)}\,d\mu + 1 - \mu(A_N).$$

We note that for all $x \in A_N$ one has

$$c(T(x) - x) = -\Phi(x) - \Phi^c(T(x)) \le N - \Phi^c(T(x)).$$

Choosing τ in such a way that $2\Lambda N(\tau) \le \tau \le \frac{1}{q}$, we obtain that

$$\chi_{A_N} e^{2\Lambda N(\tau)c\bigl(F\bigr)} \le e^{\tau N} e^{-\tau\Phi^c(T)}$$

and by the above estimate $\chi_{A_N} e^{2\Lambda N(\tau)c\bigl(F\bigr)}$ is integrable and

$$(\tau - 2\omega)\int_{A_N} c(F)e^{\omega c(F)}\,d\mu + 2\int_{A_N} e^{\omega c(F)}\,d\mu \le$$

$$\Bigl(\int_{\mathbb{R}^d} g^p\,d\mu\Bigr)^{\frac{1}{p}} + \int_{A_N} e^{2\Lambda N(\tau)c\bigl(F\bigr)}\,d\mu + 1 - \mu(A_N).$$

Setting $\omega := 2\Lambda N(\tau)$, $M := \tau - 8\Lambda N(\tau)$ and choosing sufficiently small τ we obtain

$$(\tau - 2\omega)\int_{A_N} c(F)e^{\omega c(F)}\,d\mu + \int_{A_N} e^{\omega c(F)}\,d\mu \le \Bigl(\int_{\mathbb{R}^d} g^p\,d\mu\Bigr)^{\frac{1}{p}} + 1 - \mu(A_N).$$

We obtain our claim letting $N \to \infty$. $\square$

Theorem 3. *Let c satisfy assumptions* **A1) - A4)** *and let μ satisfy infimum-convolution inequality (2.7) and Poincaré inequality (1.2). Suppose in addition that*

1) $\int_{\mathbb{R}^d} g|\log g|^p \, d\mu < \infty$ *for some* $p \ge 1$.
2) *There exists* $p' \ge 1$ *and* $N_{p'} > 0$ $M_{p'} > 0$ *such that* $p' \ge p$ *and* $|\nabla c|^{2p'} \le N_{p'} c^p + M_{p'}$.
3) *There exists some* $B > 0$ *such that*

$$|x|^2 \le A + Bc(\nabla c^*(x)).$$

Then $c^p(F) \in L^1(\mu)$.

Proof. The idea of the proof is essentially the same as in Theorem 2. We just give below the formal estimates which imply the result. A more detailed proof can be given exactly in the same way as in Theorem 2.

It follows from the identity $c(x - T(x)) = -\Phi - \Phi^c(T)$ that

$$\int_{\mathbb{R}^d} c^p(x - T(x)) \, d\mu = -\int_{\mathbb{R}^d} c^{p-1}(x - T(x))\big(\Phi + \Phi^c(T)\big) \, d\mu.$$

By using 3) and the Poincaré inequality we show as in Theorem 2 that $\Phi \in L^2(\mu)$. We choose Φ in such a way that $\int_{\mathbb{R}^d} \Phi \, d\mu = 0$. Then by the Young inequality one has

$$-\int_{\mathbb{R}^d} \Phi^c(T(x)) c^{p-1}(x - T(x)) \, d\mu = -\int_{\mathbb{R}^d} \Phi^c(x) c^{p-1}(x - T^{-1}(x)) g \, d\mu \le \int_{\mathbb{R}^d} e^{-\Phi^c(x)} \, d\mu$$
$$+ \int_{\mathbb{R}^d} c^{p-1}(x - T^{-1}(x)) g \log\big[c^{p-1}(x - T^{-1}(x)) g\big] \, d\mu - \int_{\mathbb{R}^d} c^{p-1}(x - T^{-1}(x)) g \, d\mu.$$

By using (2.7) we obtain

$$\int_{\mathbb{R}^d} e^{-\Phi^c} \, d\mu \le e^{\int_{\mathbb{R}^d} \Phi \, d\mu} = 1.$$

Hence

$$-\int_{\mathbb{R}^d} \Phi^c(T(x)) c^{p-1}(x - T(x)) \, d\mu \le 1 + \int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \log\big[c^{p-1}(x - T(x)) g(T)\big] \, d\mu$$
$$-\int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \, d\mu \le 1 + \int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \log\big[c^{p-1}(x - T(x))\big] \, d\mu$$
$$+ \int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \log g(T) \, d\mu - \int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \, d\mu.$$

By the Hölder inequality and the change of variables formula

$$\int_{\mathbb{R}^d} c^{p-1}(x - T(x)) \log g(T) \, d\mu \le \Big[\int_{\mathbb{R}^d} c^p(x - T(x)) \, d\mu\Big]^{1 - \frac{1}{p}} \Big[\int_{\mathbb{R}^d} |\log g|^p g \, d\mu\Big]^{\frac{1}{p}}.$$

In addition,

$$\Big| -\int_{\mathbb{R}^d} \Phi(x) c^{p-1}(x - T(x)) \, d\mu\Big| \le \big[\int_{\mathbb{R}^d} c^p(x - T(x)) \, d\mu\big]^{1 - \frac{1}{p}} \big[\int_{\mathbb{R}^d} |\Phi(x)|^p \, d\mu\big]^{\frac{1}{p}}.$$

We note that every measure that satisfies the Poincaré inequality satisfies also the following inequality for every $p' \ge 1$:

$$\int_{\mathbb{R}^d} \Big|\varphi - \int_{\mathbb{R}^d} \varphi \, d\mu\Big|^{2p'} \, d\mu \le C_{2p'} \int_{\mathbb{R}^d} \big|\nabla \varphi\big|^{2p'} \, d\mu$$

(see, for example, [11]). Hence by the Hölder inequality we obtain

$$\int_{\mathbb{R}^d} |\Phi|^p \, d\mu \le \Bigl(\int_{\mathbb{R}^d} |\Phi|^{2p'} \, d\mu\Bigr)^{\frac{p}{2p'}} \le \Bigl(\int_{\mathbb{R}^d} C_{2p'} |\nabla \Phi|^{2p'} \, d\mu\Bigr)^{\frac{p}{2p'}}$$

$$= C_{2p'}^{p/2p'} \Bigl(\int_{\mathbb{R}^d} |\nabla c(F)|^{2p'} \, d\mu\Bigr)^{\frac{p}{2p'}} \le C_{2p'}^{p/2p'} \Bigl(\int_{\mathbb{R}^d} N_{p'} c^p(F) \, d\mu + M_{p'}\Bigr)^{\frac{p}{2p'}}.$$

Finally, for an appropriate choice of $A' > 0$ and $B' > 0$ we obtain

$$-\int_{\mathbb{R}^d} \Phi c^{p-1}(F) \, d\mu \le A' \Bigl[\int_{\mathbb{R}^d} c^p(F) \, d\mu\Bigr]^{1-\frac{1}{p}+\frac{1}{2p'}} + B'$$

and

$$-\int_{\mathbb{R}^d} \Phi^c(T) c^{p-1}(F) \, d\mu \le 1 + \int_{\mathbb{R}^d} c^{p-1}(F) \log\bigl[c^{p-1}(F)\bigr] \, d\mu$$

$$+ \Bigl[\int_{\mathbb{R}^d} c^p(F) \, d\mu\Bigr]^{1-\frac{1}{p}} \Bigl[\int_{\mathbb{R}^d} |\log g|^p g \, d\mu\Bigr]^{\frac{1}{p}}.$$

Hence $\int_{\mathbb{R}^d} c^p(F) \, d\mu$ does not exceed the sum of the right-hand sides of these inequalities. This estimate easily implies the result. $\square$

Remark 1. 1) *Examples of costs satisfying conditions of Theorem 2 and Theorem 3 are functions of the type*

$$c(x) = \frac{1}{p}|x|^p,$$

where $1 < p \le 2$.

2) *Let* $c = \frac{x^2}{2}$. *Then Theorem 1 follows from Theorem 2, Theorem 3 and Lemma 1.*

3. Examples

Let $\alpha \ge 1$. We define the following probability measure on $\mathbb{R}^d$:

$$\mu_\alpha = \frac{1}{Z_\alpha^d} \prod_{i=1}^d e^{-|x_i|^\alpha} dx_i$$

where $Z_\alpha = \int_{\mathbb{R}} e^{-|x|^\alpha} dx$.

The spectral properties of this measure were studied first in [15]. In particular, it was shown that μ_α satisfies a family of inequalities which can be considered as an interpolation between log-Sobolev and Poincaré. In our paper we use another result obtained recently by Gentil, Guillin and Miclo in [12].

They have shown that μ_α satisfies the transportation inequality for the cost functions of the following type:

$$L_{A,\alpha}^d(x) = \sum_{i=1}^d L_{A,\alpha}(x_i),$$

where $2 \ge \alpha > 1$, $A > 0$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and

$$L_{A,\alpha}(x) = \begin{cases} \frac{x^2}{2} & \text{if } |x| \le A \\ A^{2-\alpha}\frac{|x|^\alpha}{\alpha} + A^2\frac{\alpha-2}{2\alpha} & \text{if } |x| \ge A. \end{cases}$$

One can verify that $\bigl(L_{A,\alpha}^d\bigr)^* = H_{A,\alpha}^d$, $\bigl(H_{A,\alpha}^d\bigr)^* = L_{A,\alpha}^d$, where

$$H^d_{A,\alpha}(x)=\sum_{i=1}^d H_{A,\alpha}(x_i),\quad H_{A,\alpha}(x)=\left\{\begin{array}{ll}\frac{x^2}{2} & \text{if } |x|\le A\\ A^{2-\beta}\frac{|x|^\beta}{\beta}+A^2\frac{\beta-2}{2\beta} & \text{if } |x|\ge A.\end{array}\right.$$

In what follows we suppress the index d and write $L_{A,\alpha},\ H_{A,\alpha}$.

Now let us formulate the main results of [12].

Theorem. *The following inequalities hold for μ_α:*

1) *(Logarithmic c-Sobolev ineqiality) There exists a constant $C_\alpha>0$ such that for every $f\in C_0^\infty(\mathbb{R}^d)$ one has*

$$Ent_{\mu_\alpha}f^2\le C_\alpha\int_{\mathbb{R}^d}H_{A,\alpha}\Bigl(\frac{\nabla f}{f}\Bigr)f^2\,d\mu_\alpha.$$

2) *(Transportation inequality) For every probability measure $g\cdot\mu_\alpha$ one has*

$$T_{L_{\frac{AC_\alpha}{2},\alpha}}(\mu_\alpha,g\cdot\mu_\alpha)\le\frac{C_\alpha}{4}Ent_{\mu_\alpha}g,$$

where

$$T_{L_{\frac{AC_\alpha}{2},\alpha}}(\mu_\alpha,g\cdot\mu_\alpha)=\inf\Bigl\{\int_{\mathbb{R}^{2d}}L_{\frac{AC_\alpha}{2},\alpha}(x-y)\,d\pi(x,y)\Bigr\},$$

where the infimum is taken over the set of probability measures π on $\mathbb{R}^d\times\mathbb{R}^d$ such that π has the marginals $g\cdot\mu_\alpha$ and μ_α.

3) *(Infimum-convolution inequality) For every bounded measurable φ one has*

$$\int_{\mathbb{R}^d}e^{Q\varphi}\,d\mu_\alpha\le e^{\int_{\mathbb{R}^d}\varphi\,d\mu_\alpha},$$

where

$$Q\varphi=\inf_y\bigl\{\varphi(y)+\frac{4}{C_\alpha}L_{\frac{TC_\alpha}{2},\alpha}(x-y)\bigr\}.$$

In fact, items 2) and 3) follow from 1). If $\alpha=2$, we arrive at the classical log-Sobolev and transportation inequalities for Gaussian measures. It is worth noting that this result also holds in the case $\alpha=1$ for the following cost function:

$$L^d_{A,1}(x)=\sum_{i=1}^d L_{A,1}(x_i),$$

where

$$L_{A,1}(x)=\left\{\begin{array}{ll}\frac{x^2}{2} & \text{if } |x|\le A\\ A|x|-\frac{A^2}{2} & \text{if } |x|\ge A,\end{array}\right.\qquad H_{A,1}(x)=L^*_{A,1}(x)=\left\{\begin{array}{ll}\frac{x^2}{2} & \text{if } |x|\le A\\ \infty & \text{if } |x|\ge A.\end{array}\right.$$

Proposition 1. *Let $g\cdot\mu_\alpha$ be a probability measure and $T(x)=x+F(x)$ be the optimal transportation mapping pushing forward μ_α to $g\cdot\mu_\alpha$ and corresponding to the cost $c=L_{\frac{AC_\alpha}{2},\alpha}$.*

1) *If $g\in L^p(\mu_\alpha)$ for some $p>1$ then $e^{\varepsilon c(F)}\in L^1(\gamma)$ for some $\varepsilon=\varepsilon(\alpha,p)>0$.*
2) *If $g|\log g|^p\in L^1(\mu_\alpha)$ for some $p>1$, then $c(F)\in L^p(\mu_\alpha)$.*

Proof. For the proof of 1) let us apply Theorem 2. Let us check that all the requirements for c are fulfilled. Indeed, **A1) - A4)** and assumption 2) of Theorem 2 are easily verified. We note that for $d=1$ one has

$$H_{A,\alpha}(x)=\max\Bigl\{f_1(x),f_2(x)\Bigr\},\ \text{where } f_1(x)=\frac{x^2}{2},\ f_2(x)=A^{2-\beta}\frac{|x|^\beta}{\beta}+A^2\frac{\beta-2}{\beta}.$$

Since the functions $t \to f_1(\sqrt{t}x)$, $t \to f_2(\sqrt{t}x)$ are convex, we obtain that $t \to c^*(\sqrt{t}x)$ is convex and increasing. Let us show 3). Indeed, it is readily verified that for every A one has

$$L_{A,\alpha}\bigl(\nabla H_{A,\alpha}(x)\bigr) = \begin{cases} \frac{x^2}{2} & \text{if } |x| \le A \\ A^{2-\beta}\frac{|x|^\beta}{\alpha} + A^2\bigl(\frac{\alpha-2}{2\alpha}\bigr) & \text{if } |x| \ge A. \end{cases}$$

By using this formula one easily verifies that for small enough τ there holds the estimate

$$H\Bigl(\frac{\tau x}{2}\Bigr) \le \Bigl(\frac{\tau}{2}\Bigr)^2 L_{A,\alpha}\Bigl(\nabla H_{A,\alpha}(x)\Bigr).$$

It is well-known that μ_α satisfies Poincaré inequality (see [12]). Now let us show inequality (2.10). By Theorem 3 for every nice function f we have

$$\int_{\mathbb{R}^d} \exp\Bigl(f - \int_{\mathbb{R}^d} f\, d\mu_\alpha\Bigr)\, d\mu_\alpha \le \int_{\mathbb{R}^d} \exp\Bigl(2C_\alpha H_{A,\alpha}\Bigl(\frac{\nabla f}{2}\Bigr)\Bigr)\, d\mu_\alpha.$$

We note that for $A \le A'$ and some appropriate $M(A, A' \ge 1$ one has $H_{A,\alpha} \le H_{A',\alpha} \le M(A, A' H_{A,\alpha}$. Hence (2.10) holds also for the function $H_{\frac{AC_\alpha}{2},\alpha} = c^*$ and an appropriate number $\Lambda > 0$. Inequality (2.7) for the cost function $\frac{4}{C_\alpha} L_{\frac{TC_\alpha}{2},\alpha}$ follows from Theorem 3. Hence it holds also for the cost $L_{\frac{TC_\alpha}{2},\alpha}$ up to the constant $\frac{4}{C_\alpha}$. The reader can easily verify that the conclusion of Theorem 2 is true also in this case. The proof of 1) is complete.

Item 2) easily follows from Theorem 3, the main result of [12] and the assumption $\alpha \le 2$. In order to verify that assumption 2) of Theorem 3 is satisfied we set $p' = \frac{p}{2(1-\frac{1}{\alpha})}$. The verification of assumption 3) of Theorem 3 follows the same line as in the verification of assumption 3) of Theorem 2. □

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