Measures not charging polar sets and Schrödinger equations in L^p

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Abstract. We study the Schrödinger equation $(q - \mathcal{L})u + \mu u = f$, where $\mathcal{L}$ is the generator of a Borel right process and μ is a measure on the state space. We prove existence and uniqueness results in L^p, $1 \le p < \infty$. We improve essentially results of R. K. Getoor.

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Introduction

In a series of articles ([Ge 95], [Ge 99a], [Ge 99b]), R. K. Getoor studied the Schrödinger equation

$$(*) \qquad (q - \mathcal{L})u + \mu u = f$$

where $\mathcal{L}$ is the generator of the (sub-Markovian) resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ associated with a Borel right process X with state space E, a Lusin topological space, μ is a signed measure charging no m-semipolar set (m is a fixed $\mathcal{U}$-excessive measure) and satisfying a smooth property, and q is a strictly positive real number. The problem which arises is the following: If $p \ge 1$, then under what conditions does $\mathcal{L} - \mu$, in some sense, generate a strongly continuous resolvent of contractions on $L^p(E, m)$? Such a problem was considered in many other papers, using different approaches related to harmonic spaces, Dirichlet forms or Markov processes, see e.g., [BoHaHu 87], [FeLa 88], [AlBlMa 89], [Ma 91], [MaRö 92], [StVo 96], [ChZa 95].

In this paper we investigate the equation $(*)$ for the essentially larger class of measures μ charging no m-polar set, but possible charging some m-semipolar set (even being carried by such a set). An example is given by the heat operator

$$\mathcal{L} = \Delta - \frac{\partial}{\partial t}$$

in $\mathbb{R}^{n+1}$, where Δ is the Laplacean in $\mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure on a horizontal hyperplane in $\mathbb{R}^{n+1}$, which is a semipolar set for the process in $\mathbb{R}^{n+1}$ having the generator $\Delta - \frac{\partial}{\partial t}$. The consideration of the wider class of measures charging no set that is m-polar, imposed us to use a class of functions (namely the strongly supermedian functions; see e.g. [BeBo 04]) which is larger than the set of all excessive functions, usually taken into account.

A main tool in our approach is given by the so called *Revuz formula*, and the *Revuz correspondence*, which associates to every positive σ-finite measure charging no set that is m-polar a strongly supermedian kernel (i.e. a kernel taking values in the set of strongly supermedian functions and satisfying some domination principle), uniquely determined up to m-polar sets. In classical situations, when a Green function exists, the Revuz correspondence is precisely the association of the Green potential to a positive measure. In this case the Revuz formula reduces to the Green one.

If in addition the positive measure μ has a smoothness (that is, a kind of finiteness) property, then there exists a m-inessential set N in E and a sub-Markovian resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ on $E \setminus N$ which is subordinate to $\mathcal{U}$ (i.e., $W_\alpha \le U_\alpha$ for all $\alpha > 0$), such that for every $p \in [1, \infty)$, each positive measurable function f lying in $L^p(E, m)$ and $q > 0$, the function $W_q f$ is a *weak solution* of the equation $(*)$ and the family $\mathcal{W} = (W_\alpha)_{\alpha>0}$ becomes a strongly continuous resolvent of contractions on $L^p(E, m)$.

We consider also the Schrödinger equation $(*)$ with μ a signed measure, $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive σ-finite measures charging no m-polar set and in addition μ^+ is satisfying the smoothness property. We construct a new resolvent of kernels $(W'_\alpha)_{\alpha>0}$ on $E\setminus N$, such that if certain conditions concerning q and f are fulfilled, then $W'_q f$ will be the weak solution of the equation $(*)$.

We complete the introduction with some probabilistic interpretations. If the measure μ is positive and charges no m-semipolar set, then the above resolvent $\mathcal{W}$ is associated with a right Markov process with state space $E\setminus N$, which is obtained by killing the initial proces X with a (right) continuous multiplicative functional $(M_t)_{t\geq 0}$, i.e.

$$(**)\qquad W_\alpha f(x) = E^x \int_0^\infty M_t e^{-\alpha t} f\circ X_t \mathsf{d}t.$$

If the measure μ charges some m-semipolar set, then there is no such a multiplicative functional. Generally, one can construct a left continuous multiplicative functional (in the sense of Fitzsimmons-Getoor, see [FiGe 03]) such that the Feynman-Kac formula $(**)$ holds. However, there exists a right process with state spece $E\setminus N$ as above such that its resolvent equals $\mathcal{W}$ m-a.e., for detailes we refer to Chapter 5 in [BeBo 04].

The paper is organized as follows: In Section 1, after introducing our setup, we present the necessary results on the subordinate resolvents. Particularly, we study the subordination induced by some potential kernels, more precisely, by the regular strongly supermedian kernels generating sub-Markovian resolvents. Section 2 is devoted to the Revuz formula and correspondence, the proofs are presented in Appendix. The main results of the paper, on the existence and uniqueness of the weak solutions for the equation $(*)$, are established in Section 3. Finally we give an example, including some computations.

1 Subordination induced by regular strongly supermedian kernels

Let $\mathcal{U}=(U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on a Lusin measurable space $(E,\mathcal{B})$. We shall denote by $p\mathcal{B}$ (resp. $bp\mathcal{B}$ the set of all numerical positive (resp. bounded positive) $\mathcal{B}$-measurable functions on E.

A function $s\in p\mathcal{B}^u(\mathcal{B}^u$ is the universally completion of $\mathcal{B})$ is termed *$\mathcal{U}$-supermedian* if $\alpha\mathcal{U}_\alpha s\leq s$ for all $\alpha>0$. A $\mathcal{U}$-supermedian function s is named *$\mathcal{U}$-excessive* if in addition $\sup_{\alpha>0}\alpha U_\alpha s = s$. We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive functions on E. If s is $\mathcal{U}$- supermedian then the function $\widehat{s}$ defined by $\widehat{s}(x)=\sup_{\alpha>0}\alpha U_\alpha s, x\in E$ is $\mathcal{U}$-excessive.

Recall that a σ-finite measure ξ on $(E,\mathcal{B})$ is called $\mathcal{U}$-excessive if $\xi\circ\alpha U_\alpha\leq\xi$ for all $\alpha>0$. We denote by $\mathsf{Exc}_{\mathcal{U}}$ the set of all $\mathcal{U}$-excessive measures.

We assume further in this section that $\mathcal{U}$ is proper, (i.e. there exists a strictly positive function $f_o\in b\rho\mathcal{B}$ such that $Uf_o\leq 1$, where $U=U_0=\sup_{\alpha>0}U_\alpha$ is the initial kernel of $\mathcal{U}$) and that $\mathcal{E}(\mathcal{U})$ is min-stable, contains the positive constant functions and $p\mathcal{B}\cap\mathcal{E}(\mathcal{U})$ generates the σ-algebra $\mathcal{B}$.

A $\mathcal{U}$-excessive measure of the form $\mu\circ U$ (where μ is a σ-finite measure) is called *potential*. The set E is called *semisaturated with respect to $\mathcal{U}$* provided that every $\mathcal{U}$-excessive measure dominated by a potential is also a potential. Recall that the set E is semisaturated with respect to $\mathcal{U}$ if and only if there exists a Lusin topology on E such that $\mathcal{B}$ is the σ-algebra of all Borel sets on E and there exists a transient right process with state space E, having $\mathcal{U}$ as associated resolvent.

We suppose that E is semisaturated with respect to $\mathcal{U}$.

For each $s\in\mathcal{E}(\mathcal{U})$ and every subset A of E we consider as usual the function $R^A s=\inf\{t\in\mathcal{E}(\mathcal{U})/\ t\geq s$ on $A\}$, called the *reduced function* of s on A. It is known that if $A\in\mathcal{B}$ then R^As is a universally $\mathcal{B}$-measurable $\mathcal{U}$-supermedian function and we put $B^As=\widehat{R^As}$. Let μ be a σ-finite on E.

A subset M of E is called *μ-polar* (with respect to $\mathcal{U}$) provided that there exists $M_0\in\mathcal{B}$, $M_0\supset M$, such that $B^{M_0}1=0$ μ-a.e. The set M is named *nearly $\mathcal{B}$-measurable* if for every finite measure μ on E there exists a set $M_1\subset M$, $M_1\in\mathcal{B}$, such that the set $M\setminus M_1$ is μ-polar and μ-negligible. We denote by $\mathcal{B}^n$ the σ-algebra of all nearly $\mathcal{B}$-measurable subsets of E and clearly we have $\mathcal{B}\subset\mathcal{B}^n\subset\mathcal{B}^u$. A function $s\in p\mathcal{B}^n$ is called *strongly supermedian* (with respect to $\mathcal{U}$) if for every two finite measures μ,ν on E such that $\mu\circ U\leq\nu\circ U$ we have $\mu(s)\leq\nu(s)$. Obviously each strongly supermedian function is $\mathcal{U}$-supermedian and since $\mathcal{E}(\mathcal{U})\subset p\mathcal{B}^n$, it follows that every $\mathcal{U}$-excessive function is strongly supermedian. Notice that if $(s_n)_n$ is a sequence of strongly supermedian functions then $\liminf_n s_n$ is also a strongly supermedian function.

Let further $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent on $(E, \mathcal{B}^u)$. A kernel P on $(E, \mathcal{B}^u)$ is called *weak subordination operator* with respect to $\mathcal{V}$ provided that $Pu \leq u$ and the function $\inf(u, Pu+v-Pv+Pf)$ is $\mathcal{V}$-supermedian for all u, $v \in \mathcal{E}(\mathcal{V})$ with $v < \infty$ and $f \in p\mathcal{B}$. If in addition the above function is $\mathcal{V}$-excessive, then P is named *exact subordination operator*.

If P is a weak subordination operator with respect to $\mathcal{V}$ we denote by E_P the set defined by

$$E_P = \{x \in E / \text{ there exists } s \in \mathcal{E}(\mathcal{V}) \text{ with } Ps(x) < s(x)\}.$$

A second sub-Markovian resolvent of kernels $\mathcal{W} = (W_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^n)$ is called *subordinate* to $\mathcal{V}$ provided that $W_\alpha \leq V_\alpha$ for all $\alpha > 0$.

We collect now some results on the subordination operators and subordinate resolvents; cf. [BeBo 04].

i) Let P be a weak subordination operator with respect to $\mathcal{V}$. Then there exists a sub-Markovian resolvent $\mathcal{W} = (\mathcal{W}_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^u)$ such that $\mathcal{W}$ is subordinate to $\mathcal{V}$ and $Wf = Vf - PVf$ for all $f \in p\mathcal{B}^u$ with $Vf < \infty$. The sub-Markovian resolvent $\mathcal{W}$ is called *generated* by P.

ii) Let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent on $(E, \mathcal{B}^u)$ which is subordinate to $\mathcal{U}$. Then there exists a uniquely determined weak subordination operator with respect to $\mathcal{U}$ such that $\mathcal{W}$ is generated by P, particularly we have $PUf = Uf - Wf$ for all $f \in p\mathcal{B}$ with $Uf < \infty$.

iii) If $q > 0$ then the kernel qU_q is an exact subordination operator with respect to $\mathcal{U}$. The resolvent generated by qU_q is $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$. One can show that a set $M \subset E$ is μ-polar with respect to $\mathcal{U}$ if and only if it is μ-polar with respect to $\mathcal{U}_q$.

iv) Let $q > 0, f \in p\mathcal{B}$ and s, t two strongly supermedian functions, $t < \infty$. Then the function $\inf(s, qU_q s + t - qU_q t + qU_q f)$ is also strongly supermedian.

v) Let $\mathcal{E}_\mathcal{U}$ be the set of all $\mathcal{U}$-excessive functions which are finite $\mathcal{U}$-a.e. and let us denote by $\preceq$ the specific order in $\mathcal{E}_\mathcal{U} : s \prec t$ means that there exists $u \in \mathcal{E}_\mathcal{U}$ such that $t = s + u$. Let further $P : \mathcal{E}_\mathcal{U} \longrightarrow \mathcal{E}_\mathcal{U}$ be a map which is additive, increasing and contractive (i.e. $Ps \leq s$ for all $s \in \mathcal{E}_\mathcal{U}$). Then for every $s \in \mathcal{E}_\mathcal{U}$ there exists $s' \in \mathcal{E}_\mathcal{U}$, $s' \prec s$, such that $s' - Ps' = s - Ps$ and $s' \leq u$ for all $u \in \mathcal{E}_\mathcal{U}$ with $s - Ps \leq u - Pu$. This assertion follows from Theorem 1.2.6 and Lemma 2.1.3 in [BeBo 04].

A kernel P on $(E, \mathcal{B}^u)$ is called *subordination operator* with respect to $\mathcal{U}$ if the following properties hold:

a) P is a weak subordination operator with respect to every proper sub-Markovian resolvent $\mathcal{U}'$ on $(E, \mathcal{B})$ such that $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{U}')$.

b) If $\mathcal{U}'$ is as above, then $\mathcal{E}(\mathcal{W}) = \mathcal{E}(\mathcal{W}')$, where $\mathcal{W}, \mathcal{W}'$ are the subordinate resolvents to $\mathcal{U}$ and respectively to $\mathcal{U}'$ generated by P.

vi) If P is kernel satisfying condition a) and in addition Ps is a strongly supermedian function for all $s \in \mathcal{E}(\mathcal{U})$, then P is a subordination operator.

Recall that a proper kernel V on $(E, \mathcal{B}^n)$ is called *regular strongly supermedian* (with respect to $\mathcal{U}$) provided that Vf is a strongly supermedian function for every $f \in p\mathcal{B}$ and if $s \in \mathcal{E}(\mathcal{U})$ is such that $Vf \leq s$ on the set $[f > 0]$ then the inequality holds on E.

Proposition 1.1. *Let V be a regular strongly supermedian kernel on $(E, \mathcal{B}^n)$. Then for every $q > 0$ the kernel Q^q on $(E, \mathcal{B}^n)$ given by*

$$Q^q f = Vf - qU_q Vf$$

for all $f \in p\mathcal{B}$ with $Vf < \infty$, satisfies the complete maximum principle. Moreover for every $f, g \in bp\mathcal{B}$ and $w \in \mathcal{E}(\mathcal{U}_q)$ we have $Q^q f \leq Q^q g + w$ provided that the inequality holds on the set $[f > 0]$.

Proof. We may assume that V is a bounded kernel. Let $w = U_q h$ with $h \in p\mathcal{B}, Uh < \infty$, and $f, g \in bp\mathcal{B}$ be such that $Q^q f \leq Q^q g + w$ on the set $[f > 0]$. We get $Vf \leq t$ on $[f > 0]$, where $t = qU_q Vf + Vg - qU_q Vg + w$. By *iv*) we deduce that the function $\inf(Vf, t)$ is strongly supermedian and therefore $Vf \leq t$ on E or equivalently $Q^q f \leq Q^q g + w$. ∎

Proposition 1.2. *Let V be a regular strongly supermedian kernel such that there exists a sub-Markovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^n)$ having V as initial kernel. If $q > 0$ then there exists a sub-Markovian resolvent of kernels $(Q^q_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^n)$ having $Q^q = V - qU_qV$ as initial kernel and moreover $Q^q_\alpha \le V_\alpha$ for all $\alpha > 0$.*

Proof. By Hunt approximation theorem it follows that every $\mathcal{V}$-excessive function is strongly supermedian and therefore from assertion $iv)$ we get that the kernel qU_q is a weak subordination operator with respect to $\mathcal{V}$. By $i)$ there exists a resolvent $(Q^q_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^n)$ which is subordinate to $\mathcal{V}$ and having $Q^q = V - qU_qV$ as initial kernel. □

Theorem 1.3. *Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a proper sub-Markovian resolvent of kernels on $(E, \mathcal{B}^n)$ such that its initial kernel V is regular strongly supermedian and there exists a strictly positive functin $f_o \in bp\mathcal{B}$ with Uf_o bounded and $\inf_\alpha \alpha V_\alpha U f_o = 0$. Then the kernel V_1 is a subordination operator with respect to $\mathcal{U}$ such that $E_{V_1} = E$ and $U = W + VW$, where $W = U - V_1U$. Moreover the sub-Markovian resolvent of kernels on $(E, \mathcal{B}^n)$ having W as initial kernel is given by*

$$W_\alpha = U_\alpha - Q^\alpha_1 U_\alpha$$

where $(Q^\alpha_\beta)_{\beta>0}$ is the resolvent of kernels associated with V in Proposition 1.2.

Proof. Since every $s \in \mathcal{E}(\mathcal{U})$ is $\mathcal{V}$-supermedian it follows that $V_1 s$ is $\mathcal{V}$-excessive and thus it is strongly supermedian. By assertion $vi)$ we only have to show that V_1 is a weak subordination operator with respect to every proper sub-Markovian resolvent $\mathcal{U}'$ on (E, B) such that $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{U}')$. We may assume that $\mathcal{U} = \mathcal{U}'$. By Proposition 1.2 for every $q > 0$ there exists a sub-Markovian resolvent of kernels $\mathcal{Q}^q = (Q^q_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}^n)$ having $Q^q = V - qU_qV$ as initial kernel and $Q^q_\alpha \le V_\alpha$ for all $\alpha > 0$. From Proposition 1.1 we deduce that every $\mathcal{U}_\alpha$-excessive function is $\mathcal{Q}^\alpha$-supermedian and therefore we may define a kernel W_α on $(E, \mathcal{B}^n)$ by

$$W_\alpha f = U_\alpha - Q^\alpha_1 U_\alpha f,\ f \in bp\mathcal{B}, Uf < \infty.$$

We get $U_\alpha f = W_\alpha f + Q^\alpha W_\alpha f + v$, where $v = \inf_n (Q^\alpha_1)^n U_\alpha f$. It follows that if $f \le f_0$ then $Q^\alpha_1 v = v \le Uf_o$ and so $qQ^\alpha_q v = v$ for all $q > 0$, $v = \inf_q qV_q v \le \inf_q qV_q Uf_o = 0$. We deduce that $U_\alpha f = W_\alpha f + Q^\alpha W_\alpha f$ for all $f \in bp\mathcal{B}$ with $Uf < \infty$ and we obtain $Uf = (I + \alpha U)U_\alpha f = W_\alpha f + \alpha U W_\alpha f + VW_\alpha f$. Since $\alpha U W_\alpha f \le Uf < \infty$, it results that $(I - V_1)Uf = (I - V_1)(I + V)W_\alpha f + (I - V_1)U(\alpha W_\alpha f) = W_\alpha f + (I - V_1)U(\alpha W_\alpha f), Wf = W_\alpha f + W(\alpha W_\alpha f)$ and therefore $W = W_\alpha + \alpha W W_\alpha$.

It remains to prove that $WW_\alpha = W_\alpha W$. We define the kernel T on (E, B^n) by $Tf = f + \sum_{n=1}(\alpha W_\alpha)^n f$ for all $f \in p\mathcal{B}$. Because $W_\alpha f \le U_\alpha f$ and $(I + \alpha U)f_o = f_o + \sum_{n=1}^\infty (\alpha U_\alpha)^n f_o$ we conclude that for all $f \in p\mathcal{B}, f \le f_o$ we have: $Tf \le (I + \alpha U)f < \infty, (I - \alpha W_\alpha)Tf = f, Wf = W(I - \alpha W_\alpha)Tf = W_\alpha Tf = \sum_{n=1}^\infty \alpha^{n-1} W^n_\alpha f$ and thus $WW_\alpha f = W_\alpha W f$.

We show now that $E_{V_1} = E$. Assume that there exists $a \in E \backslash E_{V_1}$. We get $V_1 s(a) = s(a)$ for all $s \in \mathcal{E}(\mathcal{U})$ and consequently $V_1 g(a) = 0$ for all $g \in p\mathcal{B}$ with $Vg(a) < \infty$, contradicting the relation $0 \ne Uf_o(a) = V_1 U f_o(a)$. □

Remark. 1. Theorem 1.3 details and completes Theorem 5.1.8 from [BeBo 04]. In the case when V is a bounded kernel, the result has been obtained essentially by G. Mokobodzki in [Mo 83].
2. One can show that the set $\mathcal{E}(\mathcal{W})$ of all $\mathcal{W}$-excessive functions is min-stable and genearally $1 \notin \mathcal{E}(\mathcal{W})$. The constant function 1 is $\mathcal{W}$-excessive if and only if $\mathcal{W}$ is exactly subordinate to $\mathcal{U}$ or equivalently, V is a regular $\mathcal{U}$-excessive kernel (cf. [BeBo 04], Remark following Theorem 5.1.20).

2 The Revuz correspondence

In this section $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be a proper sub-Markovian resolvent as in Section 1. Notice that if $q > 0$ then the resolvent $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$ possesses the same properties as $\mathcal{U}$, namely $\mathcal{E}(\mathcal{U}_q)$ is min-stable, $1 \in \mathcal{E}(\mathcal{U}_q)$, $p\mathcal{B} \cap \mathcal{E}(\mathcal{U}_q)$ generates $\mathcal{B}$ and E is semisaturated with respect to $\mathcal{U}_q$; see also [BeBoRö 05].

If s is a strongly supermedian function with respect to $\mathcal{U}$ and $\xi \in Exc_{\mathcal{U}}$, we define $L(\xi, s)$ by

$$L(\xi, s) = \sup\{\mu(s)/\mu \circ U \leq \xi\}.$$

The functional $(\xi, s) \mapsto L(\xi, s)$ is called the *energy functional* associated with $\mathcal{U}$. We shall denote by L_q the energy functional with respect to $\mathcal{U}_q$.

Let $\mathcal{S}$ be the convex cone of all finite strongly supermedian functions. Recall that $\mathcal{S}$ is a cone of potentials, particularly if $s, t \in \mathcal{S}, s \leq t$, then the *reduced function* of $t-s$, $R(t-s) = \inf\{u \in \mathcal{S}/u \geq t-s\}$, belongs to $\mathcal{S}$ and $R(t-s) \prec t$, where $\prec$ denotes here the specific order in $\mathcal{S}$. A function $s \in \mathcal{S}$ is called *regular* (in $\mathcal{S}$) if for every sequence $(s_n)_n \subset \mathcal{S}, s_n \nearrow s$, we have $\inf_n R(s - s_n) = 0$. It is known that a function $s \in \mathcal{S}$ is regular if and only if there exists a regular strongly supermedian kernel V such that $s = V1$. Recall that the following *Mertens decomposition* holds for every finite strongly supermedian function $s : s = s_0 + s_1$, where $s_0 \in \mathcal{S}$ is regular and s_1 is $\mathcal{U}$-excessive (cf. [BeBo 04]). We shall denote by $\mathcal{S}^q$ the convex cone of all finite strongly supermedian functions with respect to $\mathcal{U}_q$ and notice that $\mathcal{S} \subset \mathcal{S}^q$.

The proofs of the next two results are presented in Appendix.

Proposition 2.1. *Let $q > 0$ be fixed and $t \in \mathcal{B}^n$ a finite function on E. Then the following assertions hold.*

1. *The function t is strongly supermedian with respect to $\mathcal{U}$ if and only if $t - qU_qt$ is strongly supermedian with respect to $\mathcal{U}_q$. Particularly we have: $t \in \mathcal{E}(\mathcal{U})$ if and only if $t - qU_qt \in \mathcal{E}(\mathcal{U}_q)$.*

2. *Assume that $t \in \mathcal{S}$ and let $m \in Exc_{\mathcal{U}}$. Then $L(m, t_0) = L_q(m, t - qU_qt)$, where $t_0 = t - \inf_\alpha \alpha U_\alpha t$. The function t is regular in $\mathcal{S}$ if and only if $\inf_\alpha \alpha U_\alpha t = 0$ and $t - qU_qt$ is regular in $\mathcal{S}^q$.*

In the sequel m will be a fixed $\mathcal{U}$-excessive measure. Clearly for all $q > 0$ the measure m is $\mathcal{U}_q$-excessive. We denote by $\mathcal{N}(m)$ the set of all nearly $\mathcal{B}$-measurable sets which are m-polar. Let σ_m be the set of positive σ-finite measures charging no set from $\mathcal{N}(m)$.

A set $N \in \mathcal{B}^n$ is called *m-inessential* (with respect to $\mathcal{U}$) if it belongs to $\mathcal{N}(m)$ and $R^N 1 = 0$ on $E \backslash N$.

We remark that every element from $\mathcal{N}(m)$ is the subset of a m-inessential set. Indeed, let $A \in \mathcal{N}(m)$ and $A_0 \in \mathcal{B}$, $A \subset A_0$, such that $R^{A_0}1 = 0$ m-a.e. We consider the set $N = [R^{A_0}1 > 0]$ and we get $A_0 \subset N \in \mathcal{B}^n$, $m(N) = 0$ and $R^N 1 = 0$ on $E \setminus N$, hence N is m-inessential.

A property depending on $x \in E$ is said to hold *m-quasi everywhere* (abbreviated m-q.e.) if the set of all $x \in E$ for which it does not hold is m-polar.

Recall that the *fine topology* is the topology on E generated by all $\mathcal{U}$-excessive functions. A function $f \in p\mathcal{B}^n$ is named *m-finely continuous* if it is finely continuous outside a set from $\mathcal{N}(m)$.

If $g \in p\mathcal{B}^n$ then a *m-fine version* of g is a function f which is m-finely continuous and $f = g$ m-a.e.

By Theorem 4.4.2 in [BeBo 04] it follows that if $\xi \in \mathsf{Exc}_{\mathcal{U}}$ and $\xi \ll m$ then there exists a m-fine version of the Radon-Nikodym derivative $\mathsf{d}\xi/\mathsf{d}m$.

If $\mu \in \sigma_m$ and $q \geq 0$ then by Theorem 6.1.2 in [BeBo 04] there exists a kernel V^q_μ on $(E, \mathcal{B}^n)$ which is regular strongly supermedian with respect to $\mathcal{U}_q$, such that $\mu(f) = L_q(m, V^q_\mu f)$, for all $f \in p\mathcal{B}$.

The kernel V^q_μ as above is uniquely determined m-q.e. and for every $\xi \in Exc_{\mathcal{U}_q}$ such that $\xi \ll m$ the following *Revuz formula* holds:

$$L_q(\xi, V^q_\mu f) = L_q(m, V^q_\mu(\widetilde{t}f)).$$

where $\widetilde{t}$ is a m-fine version of the Radon-Nikodym derivative $\mathsf{d}\xi/\mathsf{d}m$. The map $\mu \mapsto V^q_\mu$ is called the *Revuz correspondence*. We shall write V_μ instead of V^0_μ.

A measure $\mu \in \sigma_m$ is termed *smooth* (with respect to $\mathcal{U}$) provided that there exists an increasing sequence $(A_k)_k, A_k \in \mathcal{B}^n, \mu(A_k) < \infty$ for all k, such that the set $[\inf_k R^{E \setminus A_k} p_o > 0]$ belongs to $\mathcal{N}(m)$; p_o is a bounded $\mathcal{U}$-excessive function of the form $p_o = Uf_o$ with $f_o > 0$, $f_o \in bp\mathcal{B}$. Notice that the notion of smoothness we considere here is precisely the "m-smoothness" from [BeBo 04]. Clearly if $\mu \in \sigma_m$ is a finite measure then it is smooth.

Proposition 2.2. *The following assertions hold for every $q > 0$.*

1. *If $\mu \in \sigma_m$ and $f \in p\mathcal{B}^n$ is such that $V_\mu f < \infty$ then we have m-q.e.*

$$V_\mu^q f = V_\mu f - qU_q V_\mu f.$$

2. *A measure from σ_m is smooth with respect to $\mathcal{U}$ if and only if it is smooth with respect to $\mathcal{U}_q$.*
3. *A set will be m-inessential with respect to $\mathcal{U}$ if and only if it has the same property with respect to $\mathcal{U}_q$.*

3 Weak solution for the Schrödinger equation

In this section $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be the sub-Markovian resolvent of a right process X with state space E, a Lusin topological space,

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f \circ X_t \mathrm{d}t, \quad f \in p\mathcal{B}, \alpha > 0$$

($\mathcal{B}$ is the Borel σ-algebra on E). We do not assume that the process is transient (i.e. U is not necessary a proper kernel). However the result from the previous sections, obtained in the transient case, will be applyed further to the bounded resolvent $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$ for $q > 0$. The results on the semisaturation of E extend to the non-transient case as follows [cf. [BeBoRö 05]): If $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on a Lusin measurable space $(E, \mathcal{B})$, then there exists a Lusin topology on E such that $\mathcal{B}$ is the Borel σ-algebra on E and a right process with state space E having $\mathcal{U}$ as associated resolvent if and only if for one $q > 0$ (or equivalently for all $q > 0$) the set $\mathcal{E}(\mathcal{U}_q)$ is min-stable, $1 \in \mathcal{E}(\mathcal{U}_q), p\mathcal{B} \cap \mathcal{E}(\mathcal{U}_q)$ generates $\mathcal{B}$ and E is semisaturated with respect to $\mathcal{U}_q$.

Let m be a fixed $\mathcal{U}$-excessive measure and $\mathcal{N}(m)$ be the family of all sets from $\mathcal{B}^n$ which are m-polar (with respect to $\mathcal{U}_q$) for one $q > 0$ (and therefore for all $q > 0$).

We shall maintain the notation σ_m for the set of all positive σ-finite measures charging no set from $\mathcal{N}(m)$.

Let $\mu \in \sigma_m$ be a smooth measure (with respect to $\mathcal{U}_q$ for one $q > 0$, and therefore for all $q > 0$; cf. Proposition 2.2) and V_μ^q the regular strongly supermedian kernel with respect to $\mathcal{U}_q$, associated with μ by the Revuz correspondence.

Theorem 3.1. *If $\mu \in \sigma_m$ is a smooth measure, then there exists a m-inessential set N and a sub-Markovian of kernels $(\mathcal{W}_q)_{q>0}$ on $(E\backslash N, \mathcal{B}^n|_{E\backslash N})$ such that for all $q > 0$ we have*

$$U_q|_{E\backslash N} = W_q + V_\mu^q|_{E\backslash N}(W_q).$$

More precisely for all $q, q' > 0$, $q' > q$, we have $V_\mu^{q'}|_{E\backslash N} = V_\mu^q|_{E\backslash N} - (q'-q)U_{q'}V^q|_{E\backslash N}$ and $V_\mu^q|_{E\backslash N}$ is the initial kernel of a sub-Markovian resolvent of kernels $\mathcal{V}^q = (V_\alpha^q)_{\alpha>0}$ on $(E\backslash N, \mathcal{B}^n|_{E\backslash N})$,

$$W_q = U_q|_{E\backslash N} - V_1^q(U_q|_{E\backslash N}) \text{ and } \inf_\alpha \alpha V_\alpha^q(U_q 1|_{E\backslash N}) = 0.$$

Proof. By Theorems 6.3.1 and 6.3.2 in [BeBo 04] for every $q > 0$ there exists a m-inessential set N_q and a sub-Markovian resolvent of kernels $(V_\alpha^q)_{\alpha>0}$ on $(E\backslash N_q, \mathcal{B}^n|_{E\backslash N_q})$ having $V^q = V_\mu^q|_{E\backslash N_q}$ as initial kernel and such that $\inf_\alpha \alpha V_\alpha^q(U_q 1|_{E\backslash N_q}) = 0$. If $q' > q$ then from Proposition 2.2 we get $V_\mu^{q'} = V_\mu^q - (q'-q)U_{q'}|_{E\backslash N_q}V^q$.

The set $N = \bigcap_{n=1}^\infty N_{\frac{1}{n}}$ is a m-innesential set and considering the restriction of all the kernels to $E\backslash N$, we may consider in the sequel that $N = \emptyset$. The claimed resolvent $\mathcal{V}^q$, $q > 0$, is that one with the initial kernel V^q as follows: $V^{\frac{1}{n+1}} = V^{\frac{1}{n}} + \frac{1}{n(n+1)}U_{\frac{1}{n}+1}V^{\frac{1}{n}}$ for all $n \geq 1$ and $V^q = V^{\frac{1}{n}} - (q - \frac{1}{n})U_q V^{\frac{1}{n}}$ if $q \in [\frac{1}{n}, \frac{1}{n-1})$, with the convention $\frac{1}{0} = \infty$. If we define $W_q = U_q - V_1^q U_q$ for all $q > 0$, one can check, using Theorem 1.3, that the family $\mathcal{W} = (W_q)_{q>0}$ is a resolvent of kernels. ☐

Let $\alpha > 0$. Since m is a $\mathcal{U}$-excessive measure it follows that $U_\alpha f = 0$ m-a.e. provided that $f = 0$ m-a.e. and U_α induces a bounded operator on $L^p(E,m)$ for all $p \in [1,\infty)$: if $f \in L^p(E,m)$ then $U_\alpha f \in L^p(E,m)$, $\|\alpha U_\alpha f\|_{L^p(E,m)} \le \|f\|_{L^p(E,m)}$ (see e.g. Proposition 7.5.1 in [BeBo 04]) and $\lim_{\alpha\to\infty}\alpha U_\alpha f = f$ in $L^p(E,m)$. We shall denote by $\mathcal{L}$ the infinitesimal operator of $\mathcal{U}$, that is the map defined on $D(\mathcal{L}) = U_\alpha(L^p(E,m))$ with values in $L^p(E,m)$ by

$$\mathcal{L}(U_\alpha f) = \alpha U_\alpha f - f,\ f \in L^p(E,m).$$

Let $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ be a second sub-Markovian resolvent of kernels on $(E,\mathcal{B})$ such that $\sigma(p\mathcal{B}\cap\mathcal{E}(\mathcal{U}^*_q)) = \mathcal{B}$, $\mathcal{E}(\mathcal{U}^*_q)$ is min-stable, $1 \in \mathcal{E}(\mathcal{U}^*_q)$ for one $q > 0$ and $\int fU_\alpha g\mathrm{d}m = \int gU^*_\alpha f\mathrm{d}m$ for all $f, g \in p\mathcal{B}$ and $\alpha > 0$; such a resolvent $\mathcal{U}^*$ always exists by Corollary 2.4 in [BeBoRö 05].

Definition. Let $q > 0$ and $p \in [1,\infty)$. A *test function* (with respect to $\mathcal{U}, q, p$ and m) is a m-finely continuous positive function φ such that $\varphi\cdot m$ is a $\mathcal{U}_q$-excessive measure and there exists $g \in L^{p'}_+(E,m)$ such that $\varphi \le U^*_q g$ m-a.e., where $1/p + 1/p' = 1$.

Remark 3.2. 1. If $g \in L^{p'}_+(E,m)$ then $U^*_q g \in L^{p'}_+(E,m)$ and the measure $U^*_q g\cdot m$ is a potential $\mathcal{U}_q$-excessive measure, $U^*_q g\cdot m = (g\cdot m)\circ U_q$. Therefore the m-fine version of $U^*_q g$ will be a test function.
2. Since E is semisaturated with respect to $\mathcal{U}_q$, it follows that $\varphi\cdot m$ will be a potential $\mathcal{U}_q$-excessive measure, hence for every test function φ there exists a unique σ-finite measure λ_φ on $(E,\mathcal{B})$ such that $\varphi\cdot m = \lambda_\varphi\circ U_q$.
3. Let λ be a σ-finite measure on $(E,\mathcal{B})$. By Theorem 3.4.2 in [BeBo 04] it follows that $\lambda \in \sigma_m$ if and only if there exists a sequence $(\varphi_n)_n$ of test functions (with respect to $\mathcal{U}, q, p$ and m) such that $\lambda = \sum_n \lambda_{\varphi_n}$. Consequently a set will be in $\mathcal{N}(m)$ if and only if it is λ_φ-negligible for every test function φ (with respect to $\mathcal{U}, q, p$ and m).

Following [Ge 99a] we consider signed measures: let $\mu^+, \mu^- \in \sigma_m$ be such that $\mu^+ \perp \mu^-$. We write $\mu = (\mu^+,\mu^-)$ and think of $\mu = \mu^+ - \mu^-$. Let also $|\mu| = \mu^+ + \mu^-$.

Definition. Let $\mu = (\mu^+,\mu^-)$ and $q > 0$. A numerical $\mathcal{B}^n$-measurable function u on E is called *weak solution* (with respect to m) *of the Schrödinger equation*

$$(*)\qquad (q - \mathcal{L})u + \mu u = f$$

where f is a given function from $L^p(E,m)$, provided that the following two conditions are satisfied for every test function φ (with respect to $\mathcal{U}, q, p$ and m):
a) $u \in L^1(E, \lambda_\varphi + \varphi\cdot|\mu|)$;
b) $\int u\mathrm{d}\lambda_\varphi + \int u\varphi\mathrm{d}\mu^+ - \int u\varphi\mathrm{d}\mu^- = \int f\varphi\mathrm{d}m$.

Remark. 1 (The case $\mu^+ = \mu^- = 0$). It is easy to see that $U_q f$ is the weak solution of the Schrödinger equation $(q - \mathcal{L})u = f$.
2. If u_i, $i = 1, 2$, is a weak solution of the Schrödinger equation $(q-\mathcal{L})u + \mu u = f_i$ then for all $\alpha \in \mathbb{R}$ the function $u_1 + \alpha u_2$ is a weak solution of the equation $(q - \mathcal{L})u + \mu u = (f_1 + \alpha f_2)$.
3. R. K. Getoor has considered in [Ge 99a] the functions $U^*_q f$ with $f \in L^{p'}(E,m)$ as being the test functions, where $(U^*_\alpha)_{\alpha>0}$ is the "moderate dual resolvent". Since the equation $(*)$ has been considered for the particular measures μ charging no m-semipolar set, and $U^*_q f$ differs from its m-fine version outside a m-semipolar set, we conclude that in this case our class of test functions is a larger one. However the weak solution in the sense of [Ge 99a] coincides with that from this paper.

Proposition 3.3. *Let $u \in b\mathcal{B}^n$, $f \in L^p_+(E,m)$ and $q > 0$. Then u will be a weak solution of the Schrödinger equation $(*)$ if and only if the following two conditions hold:*
a′) $V^q_{|\mu|}(|u|) \in L^p(E,m)$;
b′) $u + V^q_{\mu^+}u = V^q_{\mu^-}u + U_q f$ m-q.e.
Particularly every weak solution of the equation $(*)$ possesses a m-fine version.

Proof. Notice firstly that by the Revuz formula, for every test function φ, each $\nu \in \sigma_m$ and $h \in p\mathcal{B}^n$ we get $\int V_\nu^q h \mathsf{d}\lambda_\varphi = L_q(\varphi \cdot m, V_\nu^q h) = L_q(m, V_\nu^q(\varphi h)) = \int \varphi h \mathsf{d}\nu$. Applying this to $\nu = |\mu|$ and $h = |u|$, we deduce that for every $g \in L^{p'}(E,m)$ we have $\int g V_{|\mu|}^q(|u|)\mathsf{d}m = \int \varphi |u| \mathsf{d}|\mu|$, where $\varphi = \widetilde{U_q^* g}$, and we conclude that assertion $a)$ from the definition of a test function is equivalent with $a')$. Applying again the above equality for $\nu = \mu^+, \mu^-$, we obtain that condition $b)$ is equivalent with $\int u \mathsf{d}\lambda_\varphi + \int V_{\mu^+}^q u \mathsf{d}\lambda_\varphi = \int V_{\mu^-}^q u \mathsf{d}\lambda_\varphi + \int U_q f \mathsf{d}\lambda_\varphi$ for every test function φ. By assertion 3 of Remark 3.2, the last equality is equivalent with condition $b')$. Since the functions $V_{\mu^-}^q u, V_{\mu^+}^q u$ and $U_q f$ possess m-fine versions, it follows that u has the same property. □

The case of positive measures

For the next two results we assume that $\mu^- = 0$ and we shall write μ instead of μ^+.

Proposition 3.4. *Let $\mu \in \sigma_m$ and $f \in L_+^p(E,m)$. Then u will be a weak solution of the Schrödinger equation $(*)$ if and only $u \geq 0$ and $u + V_\mu^q u = U_q f$ m-q.e.*

Proof. If u is a weak solutin of the equation $(*)$, then by Proposition 3.3 we get $u + V_\mu^q = U_q f$ m-q.e. If we take a m-inessential set N such that $[u + V_\mu^q u \neq U_q f] \subset N$ and we consider the restrictions of u, V_μ^q and U_q to $E \setminus N$, we may assume that $u + V_\mu^q = U_q f$ everywhere and thus $V_\mu^q u \leq U_q f$ on the set $[u > 0]$. The kernel V_μ^q being regular stronlgy supermedian, we conclude that $V_\mu^q u \leq U_q f$, or equivalently, $u \geq 0$ m-q.e. Converesely, if $u \geq 0$ and $u + V_\mu^q = U_q f$ m-q.e., then clearly u and $V_\mu^q u$ belong to $L_+^p(E,m)$ and again by Proposition 3.3 it follows that u is a weak solution of the equation $(*)$. □

We state now the main result of this paper.

Theorem 3.5. *If $\mu \in \sigma_m$ is a smooth measure then the following assertions hold.*

1. *For every $q > 0$ and $f \in L^p(E,m)$ the Schrödinger equation $(*)$ has a unique weak solution. Moreover if N is the m-inessential set and $\mathcal{W} = (W_q)_{q>0}$ the sub-Markovian resolvent on $(E \backslash N, \mathcal{B}^n|_{E \backslash N})$ associated with μ in Theorem 3.1, then the function from $p\mathcal{B}^n$ which equals $W_q f$ on $E \backslash N$ is a weak solution of the equation $(*)$.*

2. *If $f \in L^p(E,m)$, let $W_q f$ denote the element of $L^p(E,m)$ given by the weak solution of the equation $(*)$. Then the family $(W_q)_{q>0}$ is a strongly continuous resolvent of contractions on $L^p(E,m)$.*

Proof. 1. The uniqueness follows by Proposition 3.4. For the existence we may assume that $f \geq 0$ and let $u \in p\mathcal{B}$ be such that $u = W_q f$ on $E \setminus N$. We get $u \in L^p(E,m)$ since clearly $u \leq U_q f$. From $W_q f + V_\mu^q|_{E \backslash N}(W_q f) = U_q f$ on $E \backslash N$ we deduce that $u + V_\mu^q u = U_q$ m-q.e. on E. By assertion 3 in Remark 3.2 it follows that for every test function φ with respect to $\mathcal{U}, q, p$ and m we have $\int u \mathrm{d}\lambda_\varphi + \int u\varphi \mathrm{d}\mu = \int (u + V_\mu^q u)\mathrm{d}\lambda_\varphi = \int U_q f \mathsf{d}\lambda_\varphi = \int f\varphi \mathrm{d}m < \infty$ and we conclude that u is a weak solution of the equation $(*)$.

2. Let $\mathcal{V}^q = (V_\alpha^q)_{\alpha>0}$ be the sub-Markovian resolvent of kernels on $(E \backslash N, \mathcal{B}^n|_{E \backslash N})$ having $V_\mu^q|_{E \backslash N}$ as initial kernel.

In the sequel we may suppose that $N = \emptyset$. Because $W_q f \leq U_q f$ for all $f \in p\mathcal{B}^n$, we get $\|qW_q\|_{L^p(E,m)} \leq 1$ and from $V_1^q U_q f \leq U_q f$ we obtain $\|V_1^q U_q f\|_{L^p(E,m)} \leq \|U_q f\|_{L^p(E,m)}$. The set $U_q(L^p(E,m))$ being dense in $L^p(E,m)$, we can extend V_1^q by continuity to a bounded linear operator $\widetilde{V}_1^q$ on $L^p(E,m)$ with $\|\widetilde{V}_1^q\|_{L^p(E,m)} \leq 1$. Notice that if $h \in L^p(E,m) \cap \mathcal{E}(\mathcal{U}_q)$ then the function $V_1^q h$ is a version of $\widetilde{V}_1^q h$. We fix now $q_o > 0$ and let $g_o \in p\mathcal{B} \cap L^p(E,m)$, $0 < g_o \leq 1$. Then there exists $f_o \in p\mathcal{B}, 0 < f_o \leq 1$ such that $V^{q_o} f_o \leq U_{q_o} g_o$ and thus $V^{q_o} f_o \in L^p(E,m)$. From $V^{q_o+\alpha} = V^{q_o} - \alpha U_{q_o+\alpha} V^{q_o}$ it follows that $\lim_{\alpha\to\infty} V^{q_o+\alpha} f_o = 0$ and then $\lim_{q\to\infty} V_1^q f_o = 0$ in $L^p(E,m)$. The set $\{f \in L^p(E,m) \ / \text{ there exists } \theta > 0 \text{ with } |f| \leq \theta f_o\}$ is dense in $L^p(E,m)$ and as a consequence $\lim_{q\to\infty} \widetilde{V}_1^q g = 0$ for all $g \in L^p(E,m)$. We conclude that for all $f \in p\mathcal{B} \cap L^p(E,m)$ we have $\lim_{q\to\infty} V_1^q(qV_q f) = 0$ and so $\lim_{q\to\infty} qW_q f = f$ in $L^p(E,m)$. □

The case of L^p-resolvents

Let $(\mathsf{U}_\alpha)_{\alpha>0}$ be a strongly continuous sub-Markovian resolvent of contractions on $L^p(E,m)$, where m is a σ-finite measure on the Lusin measurable space $(E,\mathcal{B})$. Then by Theorem 2.2 in [BeBoRö 05] there exists a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ such that U_α regarded as operator on $L^p(E, m)$ coincides with U_α for all $\alpha > 0$ and for some $\beta > 0$ we have $1 \in \mathcal{E}(\mathcal{U}_\beta)$, $p\mathcal{B} \cap \mathcal{E}(\mathcal{U}_\beta)$ in min-stable and generates $\mathcal{B}$. Moreover there exists a Lusin topological space E_1 such that $E \subset E_1, E \in \mathcal{B}_1$ (the σ-algebra of Borel sets on E_1) and a sub-Markovian resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1, \mathcal{B}_1)$ which is an extension of $\mathcal{U}$ to E_1 (i.e. $U^1_\alpha(1_{E_1\setminus E}) = 0$ for all $\alpha > 0$) and $\mathcal{U}^1$ is associated with a right process with state space E_1. We assume that $m \in \mathrm{Exc}_{\mathcal{U}}$ and let σ_m be the family of all σ-finite measure on $(E, \mathcal{B})$ such that there exists a resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ as above and for all $m' \in \mathrm{Exc}_{\mathcal{U}_q}$, $m \ll m'$, $m' \ll m$, there exists a sequence $(\mu_k)_k$ of finite measures on $(E, \mathcal{B})$ with $\mu = \sum_k \mu_k$ and $\mu_k \circ U_q \leq m'$ for all k and one $q > 0$. We remark that by Theorem 3.4.2 in [BeBo 04] this notation agrees with that already considered. Let $\overline{m}$ be the measure on $(E_1, \mathcal{B}_1)$ extending m by zero on $E_1\setminus E$. In this way $(\mathsf{U}_\alpha)_{\alpha>0}$ may be viewed as a resolvent on $L^p(E_1, \overline{m})$ and on E_1 one can consider the test functions with respect to $\mathcal{U}^1, p, q$ and $\overline{m}$.

Taking into account the above considerations, we can establish the following result on the Schrödinger equation for L^p-resolvents, as a consequence of Theorem 3.5.

Corollary 3.6. *Let $p \in [1, \infty)$, $(\mathsf{U}_\alpha)_{\alpha>0}$ be a strongly continuous sub-Markovian resolvent of contractions on $L^p(E, m)$ and $\mu \in \sigma_m$ be a smooth measure. Then for every $q > 0$ and $f \in L^p(E, m)$ the Schrödinger equation $(*)$ (verified with the test functions on E_1) has a unique weak solution $W_q f$. The family $(W_q)_{q>0}$ is a strongly continuous resolvent of contractions on $L^p(E, m)$ and*

$$W_q = U_q - V^q_1 U_q,$$

for all $q > 0$, where V^q_1 is a sub-Markovian contraction on $L^p(E, m)$.

Signed measures

We return now to the general case: $\mu = (\mu^+, \mu^-)$, with μ^+, $\mu^- \in \sigma_m$, and assume that μ^+ is a smooth measure. For every $q > 0$ let $V^q = V^q_{\mu^+}$ (resp. $V^q_- = V^q_{\mu^-}$) be the regular strongly supermedian kernel associated with μ^+ (resp. μ^-) by the Revuz correspondence. Theorem 3.1 implies the existence of a m-inessential set N such that, considering the restrictions of the kernels U_q, V^q and V^q_- to $E \setminus N$, for all α, $q > 0$, $\alpha < q$, we have $V^q_- = V^\alpha_- - (q - \alpha)U_q V^\alpha_-$, $U_q V^\alpha_- = U_\alpha V^q_-$ and V^α is the initial kernel of a sub-Markovian resolvent of kernels $\mathcal{V}^\alpha = (V^\alpha_\beta)_{\beta>0}$ on $(E\setminus N, \mathcal{B}^n|_{E\setminus N})$, such that $\inf_\beta \beta V^\alpha_\beta(U_\alpha 1) = 0$. The family $\mathcal{W} = (W_\alpha)_{\alpha>0}$, where $W_\alpha = U_\alpha - V^\alpha_1 U_\alpha$, is a sub-Markovian resolvent of kernels on $(E\setminus N, \mathcal{B}^n|_{E\setminus N})$.

Since the function $V^\alpha_- f$ is $\mathcal{U}_\alpha$-strongly supermedian for every $f \in p\mathcal{B}^n$, it follows that $V^\alpha_1 V^\alpha_- f \leq V^\alpha_- f$ and therefore there exists a kernel T_α on $(E \setminus N, \mathcal{B}^n|_{E\setminus N})$ such that

$$T_\alpha f = V^\alpha_- f - V^\alpha_1 V^\alpha_- f$$

for every $f \in p\mathcal{B}^n$ with $V^\alpha_- f < \infty$. Because there exists a function $g_o \in p\mathcal{B}$, $0 < g_o \leq 1$, such that $V^\alpha_- g_o \leq U_\alpha 1$, we deduce that $(I + V^\alpha)T_\alpha = V^\alpha_-$.

Proposition 3.7. *If α, $\beta > 0$, $\alpha < \beta$, then the following assertions hold.*

1. *$T_\beta = T_\alpha - (\beta - \alpha)W_\beta T_\alpha$ and $W_\beta T_\alpha = W_\alpha T_\beta$.*
2. *For every natural number n we have: $T^n_\alpha W_\alpha = T^n_\beta W_\beta + (\beta - \alpha)\sum_{i+j=n} T^i_\beta W_\beta T^j_\alpha W_\alpha$ and $\sum_{i+j=n} T^i_\beta W_\beta T^j_\alpha W_\alpha = \sum_{i+j=n} T^i_\alpha W_\alpha T^j_\beta W_\beta$.*

Proof. Let us put $S = T_\alpha - (\beta - \alpha)W_\beta T_\alpha$. Because T_α is a $\mathcal{W}_\alpha$-supermedian kernel, it follows that S is a kernel and we get $V^\beta T_\alpha \leq V^\alpha T_\alpha \leq V^\alpha_-$. We have to show that $S = T_\beta$, or equivalently, $(I + V^\beta)S = (I + V^\beta)T_\beta$. Indeed, we have $(I + V^\beta)S = (I + V^\beta)(I - (\beta - \alpha)W_\beta)T_\alpha = (I + V^\beta - (\beta - \alpha)U_\beta)T_\alpha = (I - (\beta - \alpha)U_\beta)(I + V^\alpha)T_\alpha = (I - (\beta - \alpha)U_\beta)V^\alpha_- = V^\beta_- = (I + V^\beta)T_\beta$. Further we have $W_\alpha T_\beta = W_\alpha(I - (\beta - \alpha)W_\beta)T_\alpha = W_\beta T_\alpha$.

The second assertion follows by an induction procedure, similar to the proof of Lemma 5.2.1 in [BeBo 04]. □

For every $\alpha > 0$ we define the kernel W'_α on $(E \setminus N, \mathcal{B}^n|_{E\setminus N})$ by

$$W'_\alpha = \sum_{n=0}^{\infty} T^n_\alpha W_\alpha.$$

We can state now the existence result for the equation $(*)$, in the general case of the signed measures.

Theorem 3.8. *Let $\mu = (\mu^+, \mu^-)$, with μ^+, $\mu^- \in \sigma_m$, and assume that μ^+ is a smooth measure. Let further $\alpha > 0$ and $f \in L^p(E,m)$ such that $V^\alpha_{\mu_-} W'_\alpha(|f|) \in L^p(E,m)$. Then for every $q \geq \alpha$ the Schrödinger equation $(*)$ has a weak solution. More precisely a function from $p\mathcal{B}^n$ which equals $W'_q f$ on $E \setminus N$ is a weak solution of the equation $(*)$. The family $(W'_\alpha)_{\alpha>0}$ is a resolvent of kernels on $(E \setminus N, \mathcal{B}^n|_{E\setminus N})$.*

Proof. We may assume that $f \geq 0$. On $E \setminus N$ we have $(I + V^q)W'_q f = V^q_- W'_q f + U_q f$, hence $u + V^q_{\mu^+} u = V^q_{\mu^-} u + U_q f$ m-q.e., where $u \in p\mathcal{B}^n$, $u = W'_q f$ on $E \setminus N$. By Proposition 3.3, in order to prove that u is a weak solution of the equation $(*)$, it remains to show that $V^q_{|\mu|} u \in L^p(E,m)$, which holds since by hypothesis $V^q_{\mu^-} u \in L^p(E,m)$ and $V^q_{\mu^+} \leq V^q_{\mu^-} u + U_q f$.

Let us check now that the family of kernels $(W'_\alpha)_{\alpha>0}$ is a resolvent. By Proposition 3.7, for all α, $\beta > 0$, $\alpha < \beta$, we have $W'_\beta W'_\alpha = \sum_{i,j=0}^\infty T^i_\beta W_\beta T^j_\alpha W_\alpha = \sum_{n=0}^\infty \sum_{i+j=n} T^i_\beta W_\beta T^j_\alpha W_\alpha = \sum_{n=0}^\infty \sum_{i+j=n} T^i_\alpha W_\alpha T^j_\beta W_\beta = W'_\alpha W'_\beta$. We get also $W'_\beta + (\beta-\alpha) W'_\beta W'_\alpha = \sum_{n=0}^\infty (T^n_\beta W_\beta + (\beta-\alpha) \sum_{i+j=n} T^i_\beta W_\beta T^j_\alpha W_\alpha) = \sum_{n=0}^\infty T^n_\alpha W_\alpha = W'_\alpha$. □

Remark. In a forthcoming paper, using probabilistic methods (the stochastic calculus for the positive left additive functionals), we shall give suplementary L^p-properties for the solution $W'_\alpha f$ of the Schrödinger equation $(*)$, under suitable Kato type hypothesis (as in [Ge 99a]) on the measures μ^+ and μ^-.

Example. We consider the heat operator in $E = \mathbb{R}^n \times \mathbb{R}$,

$$\mathcal{L} = \Delta - \frac{\partial}{\partial t}.$$

Let $a \in \mathbb{R}$ and μ be the n-dimensional Lebesgue measure on the horizontal hyperplane $\mathbb{R}^n \times \{a\}$. We denote by $\mathcal{U} = (U_\alpha)_{\alpha>0}$ the proper sub-Markovian resolvent generated by $\mathcal{L}$ and let m be the $(n+1)$-dimensional Lebesgue measure on E. Notice that m is a *reference* measure for $\mathcal{U}$ (i.e. $U(1_M) = 0$ provided that $m(M) = 0$) and therefore a subset M of E will be m-polar if and only if it is polar (i.e. $B^M 1 = 0$). Clearly the measure μ charges no polar set and it is smooth will respect to $\mathcal{U}$ since there exists an increasing sequence $(K_n)_n$ of compact sets in E such that $\bigcup_n K_n = E$ and $\inf_n B^{E\setminus K_n} p_o = 0$ where $p_o = U f_o$ is bounded, $f_o \in p\mathcal{B}, 0 < f_o \leq 1$. By Theorem 3.4 for every $q > 0$, $p \in [1,\infty)$ and $f \in L^p(E,m)$ the Schrödinger equation $(*)$ has a unique weak solution $W_q f$ in $p\mathcal{B}^n \cap L^p(E,m)$ and the family of operators $(W_q)_{q>0}$ on $L^p(E,m)$ is a strongly continuous resolvent of contractions on $L^p(E,m)$.

Notice that the results of R. K. Getoor from [Ge 95] and [Ge 99a] are not applicable to this situation, since the measure μ is carried by a semipolar set.

We shall complete this example with computations in the one dimensional case. In fact we take $E = [0,\infty)$ and the proper sub-Markovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on E associated with the uniform motion to the right:

$$U_\alpha f(x) = \int_0^\infty e^{-\alpha t} f(x+t)\mathrm{d}t, \quad f \in p\mathcal{B}(E).$$

Let m be the Lebesgue measure on $(E, \mathcal{B}(E))$ and $\mu = \varepsilon_{x_0}$, where $x_0 \in (0,\infty)$. Let V be the regular strongly supermedian kernel on $(E, \mathcal{B}(E))$ given by

$$Vf = f(x_0) R^{\{x_0\}} 1.$$

It is easy to see that $VV = V$ and therefore the family $\mathcal{V} = (\frac{1}{1+\alpha} V)_{\alpha>0}$ is a sub-Markovian resolvent on $(E, \mathcal{B}(E))$ having V as initial kernel. Since $m = \varepsilon_0 \circ U$ we get $L(m, Vf) = Vf(0) = f(x_0) R^{\{x_0\}} 1(0) = \mu(f)$, and so V is associates with μ by the Revuz correspondence (with respect to $\mathcal{U}$ and m), $V = V^0_\mu$. By Proposition 2.2, for every $q > 0$ we have $V^q_\mu f = V - qU_q V f = f(x_0) h_q$, where $h_q = R^{\{x_0\}} 1 - qU_q R^{\{x_0\}} 1$. We deduce that $(V^q_\mu)^n = \theta_q^{n-1} V$, with $\theta_q = h_q(x_0), 0 < \theta_q < 1$. If $\mathcal{V}^q = (V^q_\alpha)_{\alpha>0}$ is the sub-Markovian resolvent of kernels having V^q_μ as initial kernel (notice that $N = \phi$) it is easy to see that

$$V^q_1 = \sum_{n=1}^\infty (-1)^{n-1} (V^q_\mu)^n = \frac{1}{1+\theta q} V.$$

Theorem 3.5 implies that the Schrödinger equation

$$(q - \frac{\mathrm{d}}{\mathrm{d}t})u + \mu u = f,$$

where $f \in p\mathcal{B} \cap L^p(E, m)$, has as weak solution the function on E given by

$$W_q f = U_q f - V_1^q U_q f = U_q f - \frac{1}{1+\theta q} U_q f(x_0) R^{\{x_0\}} 1.$$

Appendix

Proof of Proposition 2.1.

1. Let $t \in \mathcal{S}$ and μ, ν be two finite measures on $(E, \mathcal{B})$ such that $\mu \circ U_q \leq \nu \circ U_q$. We may suppose that t is bounded. We get $(\mu + \nu \circ qU_q) \circ U \leq (\nu + \mu \circ qU_q) \circ U$ and therefore $(\mu + \nu \circ qU_q)(t) \leq (\nu + \mu \circ qU_q)(t)$ or equivalently $\mu(t - qU_q t) \leq \nu(t - qU_q t)$. Consequently $t - qU_q t \in \mathcal{S}^q$. Conversely, assume that $t - qU_q t \in \mathcal{S}^q$. Since $t - qU_q t$ is $\mathcal{U}_q$-supermedian we get that t is $\mathcal{U}$-supermedian. Let $u = \inf_\alpha \alpha U_\alpha t$ and $t' = t - u$. Then $t' \in p\mathcal{B}^n, \alpha U_\alpha u = u$ for all $\alpha > 0, t' - qU_q t' = t - qU_q t$ and $\inf \alpha U_\alpha t' = 0$. Because clearly $u \in \mathcal{E}(\mathcal{U})$, it remains to show that $t' \in \mathcal{S}$. Let μ, ν be two finite measures such that $\mu \circ U \leq \nu U$. We have $\mu \circ (I + qU) \circ U_q \leq \nu \circ (I + qU) \circ U_q$ and from $U(t' - qU_q t') = \sup_\alpha U_\alpha (t' - qU_q t') = \sup_\alpha U_q(t' - \alpha U_\alpha t') = U_q t'$ we conclude that $\mu(t') = \mu \circ (I + qU)(t' - qU_q t') \leq \nu \circ (I + qU)(t' - qU_q t') = \nu(t')$, $t' \in \mathcal{S}$.

2. Let $t \in \mathcal{S}$ and μ be a finite measure on $(E, \mathcal{B}), \mu \circ U \leq m$. We may assume that $t = t'$, or equivalently $\inf_\alpha \alpha U_\alpha t = 0$. In this case we have $t = (I + qU)(t - qU_q t)$ and since $\mu \circ (I + qU) \circ U_q = \mu \circ U \leq m$ we get: $L(\mu \circ U, t) = \mu(t) = \mu \circ (I + qU)(t - qU_q t) = L_q(\mu \circ (I + qU) \circ U_q, t - qU_q t) \leq L_q(m, t - qU_q t)$. To prove the converse inequality let μ be a finite measure on $(E, \mathcal{B})$ such that $\mu \circ U_q \leq m$. We get $L_q(\mu \circ U_q, t - qU_q t) = \mu(t) - \mu \circ qU_q(t) = L(\mu \circ U, t) - L((\mu \circ qU_q) \circ U, t) \leq L(m, t)$, $L_q(m, t - qU_q t) \leq L(m, t)$.

Assume that t is regular in $\mathcal{S}$ then there exists a sequence $(t_n)_n \subset \mathcal{S}$ such that $t = \sum_n t_n$ and $t_n \leq U f_0$ for all n. Therefore $\inf_\alpha \alpha U_\alpha t_n = 0$ for all n, $\inf_\alpha \alpha U_\alpha t = 0$. Let $t' = t - qU_q t$ and $(s'_n)_n \subset \mathcal{S}^q$, $s'_n \nearrow t'$. If we put $s_n = s'_n + qU s'_n$ then $(s_n)_n \subset \mathcal{S}$ and $s_n \nearrow t$. The function t being regular in $\mathcal{S}$ we get $R_q(t' - s'_n) \leq R_q(t - s_n) \leq R(t - s_n) \searrow 0$, where R_q denotes the reduction operator in $\mathcal{S}^q$.

Suppose now that $t \in \mathcal{S}$ is such that $\inf_\alpha \alpha U_\alpha t = 0$ and t' is regular in $\mathcal{S}^q$. By Mertens decomposition, to show that t is regular we may assume that t is $\mathcal{U}$-excessive. We consider a sequence $(f_n)_n \subset bp\mathcal{B}$ such that $U_q f_n \nearrow t'$. It follows that $U f_n \nearrow t = (I + qU)t'$ and $R(t - U f_n) \leq (I + qU) R_q(t' - U_q f_n) \leq t < \infty$. Since $\inf_n R_q(t' - U_q f_n) = 0$ we get $\inf_n R(t - U f_n) = 0$ and by Proposition 2.4.6 in [BeBo 04] we conclude that t is regular, completing the proof.

Proof of Proposition 2.2.

1. Since $V_\mu f$ is a regular function from $\mathcal{S}$, by Proposition 2.1 we have $\mu(f) = L(m, V_\mu f - qU_q V_\mu f)$. The assertion follows now because by Proposition 1.1 the kernel $V_\mu - qU_q V_\mu$ is regular strongly supermedian with respect to $\mathcal{U}_q$.

2. If μ is smooth with respect to $\mathcal{U}$, then we consider an increasing sequence $(A_k)_k \subset \mathcal{B}^n$ such that $\mu(A_k) < \infty$ for all k and the set $[\inf_k R^{E \setminus A_k} p_o > 0]$ belongs to $\mathcal{N}(m)$. Since $R_q^A p_o \leq R^A p_o$ (where $R_q^A p_o$ denotes the reduced function of p_o on $A \in \mathcal{B}^n$, with respect to $\mathcal{E}(\mathcal{U}_q)$) we get $[\inf_k R_q^{E \setminus A_k} p_o > 0] \in \mathcal{N}(m)$. Therefore the measure μ is smooth with respect to $\mathcal{U}_q$. Conversely, assume that μ is smooth with respect to $\mathcal{U}_q$ and let $(A'_k)_k \subset \mathcal{B}^n$ be such that $\mu(A'_k) < \infty$ for all k and $[\inf_k R_q^{E \setminus A'_k} p'_o > 0] \in \mathcal{N}(m)$, where $p'_o = U_q f_0$. From $R^{E \setminus A'_k} p_o \leq (I + qU) R_q^{E \setminus A'_k} p'_o \leq p_o < \infty$ we get $[\inf R^{E \setminus A'_k} p_o > 0] \in \mathcal{N}(m)$.

3. Let $N \in \mathcal{B}^n$. Since $R_q^N 1 \leq R^N 1$ we have to prove that if $R_q^N 1 = 0$ on $E \setminus N$ then $R^N 1 = 0$ on $E \setminus N$. Indeed, if $R_q^N 1 = 0$ on $E \setminus N$, we get $1_N \in \mathcal{S}^q$, $\alpha U_{q+\alpha}(1_N) \leq 1_N$ and therefore $\varepsilon_x \circ U_q(1_N) = 0$, for all $x \in E \setminus N$, $U(1_N) = 0$ on $E \setminus N$. From $R^N 1 \leq (I + qU) R_q^N 1$ we deduce now that $R^N 1 = 0$ on $E \setminus N$.

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