Large Deviations for Stochastic Generalized Porous Media Equations *

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Abstract

The large deviation principle is established for the distributions of a class of generalized stochastic porous media equations for both small noise and short time.

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1 Introduction and Main Results

We first recall the existence and uniqueness results on strong solutions to the stochastic generalized porous media equations obtained recently in [9]. Let $(E, \mathscr{M}, \mathbf{m})$ be a separable probability space and $(L, \mathscr{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ with spectrum contained in $(-\infty, -\lambda_0]$ for some $\lambda_0 > 0$.

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We assume that, for a fixed number $r > 1$, L^{-1} is bounded in $L^{r+1}(\mathbf{m})$, which is e.g. the case if L is a Dirichlet operator (cf. e.g. [16]) since in this case the interpolation theorem or simply Jensen's inequality implies $\|e^{tL}\|_{r+1} \leq e^{-\lambda_0 t 2/(r+1)}$ for all $t \geq 0$, where and in what follows, $\|\cdot\|_p$ denotes the norm in $L^p(\mathbf{m})$ for $p \geq 1$. A classical example of L is the Laplace operator on a smooth bounded domain in a complete Riemannian manifold with Dirichlet boundary condition.

Let $H^1 := \mathscr{D}(\sqrt{-L})$ be the real Hilbert space with inner product

$$\langle f, g\rangle_{H^1} := \langle \sqrt{-L}f, \sqrt{-L}g\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product in $L^2(\mathbf{m})$. Then the embedding $H^1 \subset L^2(\mathbf{m})$ is dense and continuous. Let $H := H^{-1}$ be the dual Hilbert space of H^1 realized through this embedding.

The existence and uniqueness of strong solutions to the following stochastic differential equation has been proved in [9]:

$$\mathrm{d}X_t = (L\Psi(t, X_t) + \Phi(t, X_t))\mathrm{d}t + Q\mathrm{d}W_t, \tag{1.1}$$

where $Q : L^2(\mathbf{m}) \to H$ is a Hilbert-Schmidt operator with $q := q(Q)$ the square of its Hilbert-Schmidt norm, W_t is a cylindrical Brownian motion on $L^2(\mathbf{m})$ w.r.t. a complete filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$,

$$\Psi, \Phi : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$$

are progressively measurable functions, i.e. for any $t \geq 0$, restricted on $[0, t] \times \mathbb{R} \times \Omega$ they are measurable w.r.t. $\mathscr{B}([0, t]) \times \mathscr{B}(\mathbb{R}) \times \mathscr{F}_t$, and for any $(t, \omega) \in [0, \infty) \times \Omega$, $\Psi(t, \cdot)(\omega)$ and $\Phi(t, \cdot)(\omega)$ are continuous on $\mathbb{R}$ and satisfy certain monotonicity conditions. See [1, 2] for an account of the classical (deterministic) porous media equations and [3, 4, 7, 8] for the study of weak solutions and invariant measures for some stochastic generalized porous media equations.

To explain what is meant by strong solutions to (1.1), let us introduce the embeddings

$$V \subset H \subset V^*$$

as follows. Consider the reflexive separable Banach space $V := L^{r+1}(\mathbf{m})$. Then we can obtain a presentation of its dual space V^* through the embeddings $V \subset H \equiv H' \subset V^*$, where H is identified with its dual through the Riesz-isomorphism. In other words V^* is just the completion of H with respect to the norm

$$\|f\|_{V^*} := \sup_{\|g\|_{r+1} \leq 1} \langle f, g\rangle_H, \quad f \in H.$$

Since H is separable, so is V^*. We note that this is different from the usual representation of $V = L^{r+1}(\mathbf{m})$ through the embedding

$$V \subset L^2(\mathbf{m}) \equiv L^2(\mathbf{m})',$$

which, of course, gives $L^{(r+1)/r}(\mathbf{m})$ as dual. But it is easy to identify the isomorphism between $L^{(r+1)/r}(\mathbf{m})$ and V^*. Below we simply use $\langle\,,\,\rangle_H$ to denote ${}_{V^*}\langle\,,\,\rangle_V$, i.e. the duality between V and V^*, since ${}_{V^*}\langle\,,\,\rangle_V = \langle\,,\,\rangle_H$ holds on $H \times V$. It is explained in [9] that $L : L^{(r+1)/r}(\mathbf{m}) \to V^*$ is a densely defined bounded operator, so that it extends uniquely to a fully defined bounded operator, denoted once again by L. Likewise, the natural embedding $L^2(\mathbf{m}) \subset H \subset V^*$ extends uniquely to a one-to-one map from $L^{(r+1)/r}(\mathbf{m})$ to V^* (cf. [9, Corollary 1.2]). Since $\Psi(t,v)(\omega), \Phi(t,v)(\omega) \in L^{(r+1)/r}(\mathbf{m})$, by condition (1.2) below, the map $b := L\Psi + \Phi : [0,\infty) \times V \times \Omega \to V^*$ is well-defined.

We assume that there exist two constants $c, \alpha > 0$ such that

$$\begin{aligned} &|\Psi'(\cdot,s)| + |\Phi'(\cdot,s)| \le c(1+|s|^{r-1}), \\ &{}_V\langle u-v, b(\cdot,u) - b(\cdot,v)\rangle_{V^*} \le -\alpha \|u-v\|_{r+1}^{r+1} + c\|u-v\|_H^2, \quad u,v \in L^{r+1}(\mathbf{m}) \end{aligned} \tag{1.2}$$

holds on $[0,T] \times \Omega$. In particular, according to [9], the second inequality in (1.2) holds for some $\alpha, c > 0$ if there exist constants $\theta_1 > \theta_2/\|L^{-1}\|_{r+1} \ge 0$ and $\sigma \in \mathbb{R}$ such that

$$\begin{aligned} &(s-t)(\Psi(\cdot,s) - \Psi(\cdot,t)) \ge \theta_1 |s-t|^{r+1}, \\ &|\Phi(\cdot,s) - \Phi(\cdot,t)| \le \theta_2 |s-t|^r + \sigma|s-t|, \quad s,t \in \mathbb{R} \end{aligned} \tag{1.3}$$

holds on $[0,T] \times \Omega$. According to [9] (see also [15, Theorems II.2.1 and II.2.2] for more general situations), condition (1.2) implies that equation (1.1) has a unique strong solution; that is, there is a unique H-valued continuous $(\mathscr{F}_t)$-adapted process X_t with $X \in L^{r+1}([0,T] \times \Omega \times E, \mathrm{d}t \times P \times \mathbf{m})$ such that for any $e \in L^{r+1}(\mathbf{m})$,

$$\langle X_t, e\rangle_H = \langle X_0, e\rangle_H - \int_0^t \mathbf{m}\big(\Psi(s,X_s)e + \Phi(s,X_s)L^{-1}e\big)\mathrm{d}s + \langle QW_t, e\rangle_H, \quad t \in [0,T]. \tag{1.4}$$

To see that the solution defined above satisfies the equation

$$X_t = x + \int_0^t (L\Psi + \Phi)(s, X_s)\mathrm{d}s + QW_t, \quad t \in [0,T] \tag{1.5}$$

in H, we first observe that by (1.2), the right hand side of (1.5) exists in V^* for any $t > 0$ since $X \in L^{r+1}([0,T] \times \Omega \times E, \mathrm{d}t \times P \times \mathbf{m})$. Since both $X_t - x$ and QW_t take values in H, (1.5) indeed holds in H.

Remark 1.1. In order to imply the large deviation principle, our assumptions are indeed stronger than those used in [15] to prove existence and uniqueness of strong solutions. On the other hand, in [9] we present a direct proof for existence, uniqueness and ergodicity of strong solutions for (1.1) under the extra assumption that the spectrum of L is discrete.

Since this assumption was not really used in the proofs, it can be dropped from that paper. Furthermore, in the recent work [17], the existence and uniqueness of strong solutions have been obtained for a much more general framework so that one may take Orlicz norms in place of $L^{r+1}(\mathbf{m})$ in applications. Our arguments for the large deviation principle presented below are, however, difficult to be extended to the general situation of [17].

In this paper we study the large deviation property of the above stochastic generalized porous medium equation for both small noise and short time. Recall ([11]) that a sequence of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ on some Polish space E satisfies, as $\varepsilon \to 0$, the *large deviation principle (LDP in short) with speed* $\lambda(\varepsilon) \to +\infty$ *(as* $\varepsilon \to 0$*) and rate function* $I : E \to [0, +\infty]$, if I is a good rate function, i.e., the level sets $\{I \le r\}$, $r \in \mathbb{R}^+$ are compact, and for any Borel subset A of E,

$$-\inf_{x\in A^o} I(x) \le \liminf_{\varepsilon\to 0} \frac{1}{\lambda(\varepsilon)} \log \mu_\varepsilon(A) \le \limsup_{\varepsilon\to 0} \frac{1}{\lambda(\varepsilon)} \log \mu_\varepsilon(A) \le -\inf_{x\in \bar{A}} I(x),$$

where A^o and $\bar{A}$ are respectively the closure and the interior of A in E. In that case we shall simply say that (μ_ε) satisfies the $LDP(\lambda(\varepsilon), I)$ on E, or even more simply write $(\mu_\varepsilon) \in LDP(\lambda(\varepsilon), I)$ on E. We say that the family of E-valued random variables X^ε satisfies the $LDP(\lambda(\varepsilon), I)$ if the family of their laws does.

Let us first consider the following stochastic differential equation with small noise:

$$\text{(1.6)} \qquad \mathrm{d}X_t^\varepsilon = (L\Psi(t, X_t^\varepsilon) + \Phi(t, X_t^\varepsilon))\mathrm{d}t + \varepsilon Q \mathrm{d}W_t, \quad \varepsilon > 0, X_0^\varepsilon = x \in H.$$

From now on, let $T > 0$ and $x \in H$ be fixed. To state our main results, let us first introduce the skeleton equation associated to (1.6):

$$\text{(1.7)} \qquad \frac{\mathrm{d}z_t^\phi}{\mathrm{d}t} = L\Psi(t, z_t^\phi) + \Phi(t, z_t^\phi) + \phi_t, \quad z_0^\phi := x,$$

where $\phi \in L^2([0,T]; H)$. An element $z^\phi \in C([0,T]; H) \cap L^{r+1}([0,T] \times E, \mathrm{d}t \times \mathbf{m})$ is called a solution to (1.7) if for any $e \in L^{r+1}(\mathbf{m})$,

$$\text{(1.8)} \qquad \langle z_t^\phi, e\rangle_H = \langle x, e\rangle_H - \int_0^t \big\{\langle L^{-1}e, \phi_t + \Phi(s, z_s^\phi)\rangle + \langle e, \Psi(s, z_s^\phi)\rangle\big\}\mathrm{d}s, \quad t \in [0,T].$$

We shall prove that (1.2) and (1.3) imply the existence and the uniqueness of the solution to (1.7) for any $\phi \in L^2([0,T]; H)$, and thus, as explained above for the solution to (1.1), the solution satisfies the corresponding integral equation of (1.7) in H.

Now, we introduce the rate function. For any $\phi \in L^2([0,T]\times E, \mathrm{d}t \times \mathbf{m})$, let $\|\phi\|_{L^2}^2 := \int_0^T \mathrm{d}t \int_E \phi_t^2 \mathrm{d}\mathbf{m}$. Define

$$(1.9) \quad I(z) := \frac{1}{2} \inf\{\|\phi\|_{L^2}^2 : \ z = z^{Q\phi}, \ \phi \in L^2([0,T]\times E, \mathrm{d}t \times \mathbf{m})\}, \quad z \in C([0,T]; H),$$

where we set $\inf \emptyset = \infty$ by convention. The following result is of a Freidlin-Wentzell type estimate:

Theorem 1.1. *Assume* (1.2). *For each* $\varepsilon > 0$, *let* $X^\varepsilon = (X_t^\varepsilon)_{t\in[0,T]}$ *be the solution to* (1.6). *Then as* $\varepsilon \to 0$, (X^ε) *satisfies the LDP*(ε^{-2}, I) *on* $C([0,T]; H)$(*equipped with the sup-norm topology*), *where the rate function* I *is given by* (1.9).

Next, we consider the LDP of the solution X_t to (1.1) for short time, which in the classical finite dimensional case is the famous Varadhan's large deviation estimate. Since $X_{\varepsilon^2 t}$ solves the equation

$$(1.10) \qquad \mathrm{d}\tilde{X}_t^\varepsilon = \varepsilon^2 (L\Psi(\varepsilon^2 t, \tilde{X}_t^\varepsilon) + \Phi(\varepsilon^2 t, \tilde{X}_t^\varepsilon))\mathrm{d}t + \varepsilon Q \mathrm{d}\tilde{W}_t, \quad \varepsilon > 0, \tilde{X}_0^\varepsilon = x,$$

where $(\tilde{W}_t := (1/\varepsilon) W_{\varepsilon^2 t})$ is a BM of the same law as (W_t), it suffices to establish the LDP for the law of $\tilde{X}^\varepsilon$.

Theorem 1.2. *Assume* (1.2). *If* $x \in L^{r+1}(\mathbf{m})$ *then* $\tilde{X}^\varepsilon = (X_{\varepsilon^2 t})$ *satisfies the LDP*$(\varepsilon^{-2}, \tilde{I})$ *where*

$$\tilde{I}(z) := \frac{1}{2} \inf\left\{\|\phi\|_{L^2}^2 : \ z_t = x + Q\int_0^t \phi_s \mathrm{d}s\right\}, \quad z \in C([0,T]; H).$$

Let us make some historical comments. In the finite dimensional case, under the Lipschitzian condition, the LDP of $X_{\varepsilon^2 \cdot}$ is the famous Varadhan's estimate [18], and Theorem 1.1 is the well known Freidlin-Wentzell's LDP ([12]). For the extensions to infinite dimensional diffusions or stochastic PDE under global Lipschitz condition on the nonlinear term, we refer the reader to Da Prato and Zabczyk [10](also for the literature until 1992). For the case of local Lipschitz conditions we refer to [6] where also multiplicative and degenerate noise is handled. Unlike in our situation, in [6] the drift still contains a nontrivial (therefore smoothing) linear part. In many examples of SPDE, however, (local) Lipschitz conditions are rarely satisfied (such as the porous equation in this work). Without Lipschitz conditions, each type of stochastic non-linear PDE requires specific techniques and adapted estimates. So the situation becomes much more dispersive. Here we mention only the work of Cardon-Weber [5] on the LDP for stochastic Burgers equations with small noise and the important work of Hino and Ramirez [13] for the Varadhan's small time estimate of large deviations for general symmetric Markov processes, where the reader may also find other recent references.

Here are some remarks on Theorem 1.2 related with the general work of Hino and Ramirez [13]: 1) As our process (X_t) is highly non-symmetric, the result in [13] can not be applied. 2) The extra condition on $x \in L^{r+1}(\mathbf{m})$ (not all $x \in H$) in Theorem 1.2 is also a quite general phenomenon in infinite dimension because the result of [13] holds only for $\mu - a.e.x$ where μ is the invariant measure, and in our case, the invariant measure is supported in $L^{r+1}(\mathbf{m})$ ([9]). 3) Furthermore the LDP in Theorem 1.2 is pathwise, unlike that in [13] which is only for the marginal law.

This paper is organized as follows. The next section is devoted to the study of the skeleton process z^ϕ, which is crucial for identifying the rate function of our LDP. In §3 we give an *a priori* exponential estimate and recall the generalized contraction principle. The proof of Theorem 1.1 is presented in §4, and our strategy is based on two procedures of approximation: first for finite dimensional noise (i.e., only a finite number of directions are stochastically perturbed) we approximate the path of QW piecewise linear; second, we approximate the whole noise QW by the finite dimensional noises. This strategy can be easily adapted for the proof of Theorem 1.2 in §5.

2 The skeleton process

Proposition 2.1. *Assume* (1.2)*. Let* $\|z\| := \sup_{t\in[0,T]} \|z_t\|_H$ *for* $z \in C([0,T]; H)$*. For any* $x \in H$ *and any* $\phi \in L^2([0,T]; H)$ *there exists a unique solution* z^ϕ *to* (1.7) *and*

$$\int_0^T \mathbf{m}(|z_t^\phi - z_t^\psi|^{r+1})\mathrm{d}t \le C \int_0^T \|\phi_t - \psi_t\|_H^2 \mathrm{d}t, \tag{2.1}$$

$$\|z^\phi - z^\psi\| \le C \int_0^T \|\phi_t - \psi_t\|_H \mathrm{d}t \tag{2.2}$$

hold for some constant $C > 0$ *and all* $x \in H,\ \phi, \psi \in L^2([0,T]; H)$.

Proof. To verify the existence of the solution, we make use of [15, Theorem II.2.1]. Let $V := L^{r+1}(\mathbf{m})$ and V^* the duality of V w.r.t. H, and let $B := 0$ and

$$A(s,v) := L\Psi(s,v) + \Phi(s,v) + \phi_s.$$

Then, due to (1.2), it is trivial to verify Assumptions $A_i)(i = 1, .., 5)$ on page 1252 of [15] for some $K, \alpha > 0, p := r+1, q := \frac{r+1}{r}$, and $f(t) := c(1 + \|\phi_t\|_H^q)$ for some constant $c > 0$. Then, by [15, Theorems II.2.1 and II.2.2] (see also [20, Theorem 30.A]) (1.7) has a unique solution. Let z^ϕ be the unique solution to (1.7) for $\phi \in L^2([0,T]; H)$.

By Itô's formula due to [15, Theorem I.3.2] and (1.2), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_t^\phi - z_t^\psi\|_H^2 = &- 2\langle z_t^\phi - z_t^\psi, \Psi(t, z_t^\phi) - \Psi(t, z_t^\psi)\rangle \\
&- 2\langle L^{-1}(z_t^\phi - z_t^\psi), \Phi(t, z_t^\phi) - \Phi(t, z_t^\psi) + \phi_t - \psi_t\rangle \\
\le &- 2\alpha\mathbf{m}(|z_t^\phi - z_t^\psi|^{r+1}) + 2c\|z_t^\phi - z_t^\psi\|_H^2 + 2\|z_t^\phi - z_t^\psi\|_H\|\phi_t - \psi_t\|_H.
\end{aligned} \tag{2.3}
$$

Since (2.3) implies, for any $\varepsilon > 0$, that

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon + \|z_t^\phi - z_t^\psi\|_H^2)^{1/2} \le \|\phi_t - \psi_t\|_H + c\|z_t^\phi - z_t^\psi\|_H,
$$

by Gronwall's lemma we have

$$
\mathrm{e}^{-cT}\sqrt{\varepsilon + \|z^\phi - z^\psi\|^2} \le \varepsilon + \int_0^T \|\phi_t - \psi_t\|_H \mathrm{d}t.
$$

This implies (2.2) for $C := \mathrm{e}^{cT}$ by letting $\varepsilon \to 0$. Finally, (2.1) follows by combining (2.2) with (2.3). □

3 Exponential estimates and a generalized contraction principle

The following *a priori* estimate will be crucial for the proof of Theorem 1.1.

Lemma 3.1. *Assume* (1.2). *Then for any $\gamma > 0, q_0 > 0$ and $\varepsilon_0 > 0$ there exits a constant $c > 0$ such that for all Q with $q(Q) \le q_0$ and all $\varepsilon \in (0, \varepsilon_0)$,*

$$
\mathbb{E}\exp\left(\gamma\varepsilon^{-2}\int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1}\mathrm{d}t\right) \le e^{c\varepsilon^{-2}}. \tag{3.1}
$$

Throughout this paper we adopt the following notation: for two continuous real semimartingales (x_t) and (y_t), $dx_t \le dy_t$ means that their martingale parts are the same and $x_t - x_s \le y_t - y_s$ for all $t > s \ge 0$.

Proof. By (1.2), (1.3) with $\theta_2 < \theta_1$ and using Itô's formula due to [15, Theorem I.3.2], there exist constants $c_0, c_1, c_2 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$
\begin{aligned}
d\|X_t^\varepsilon\|_H^2 &\le -2\langle X_t^\varepsilon, (\Psi + L^{-1}\Phi)(t, X_t^\varepsilon)\rangle\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon, Q\mathrm{d}W_t\rangle + q\varepsilon^2\mathrm{d}t \\
&\le -\mathbf{m}\left(\alpha|X_t^\varepsilon|^{r+1} - c_0[|X_t^\varepsilon|^2 + 1]\right)\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon, Q\mathrm{d}W_t\rangle + (q\varepsilon^2 + c_0\|X_t^\varepsilon\|_H^2)\mathrm{d}t \\
&\le -c_1\mathbf{m}(|X_t^\varepsilon|^{r+1})\mathrm{d}t + c_2\mathrm{d}t + 2\varepsilon\langle X_t^\varepsilon, Q\mathrm{d}W_t\rangle_H.
\end{aligned}
$$

Then

$$\|X_T^\varepsilon\|_H^2 - \|x\|_H^2 + c_1 \int_0^T \mathbf{m}(|X_t^\varepsilon|^{r+1})\mathrm{d}t \le 2\varepsilon \int_0^T \langle X_t^\varepsilon, Q\mathrm{d}W_t\rangle_H + c_2 T. \tag{3.2}$$

Letting $dM_t := \langle X_t^\varepsilon, Q\mathrm{d}W_t\rangle_H$ (with $M_0 = 0$), since $\forall \lambda \in \mathbb{R},\ \xi_t := \exp(\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t)$ is a martingale and the quadratic variational process $\langle M\rangle_t$ satisfies $d\langle M\rangle_t \le q_0\|X_t^\varepsilon\|_H^2 \mathrm{d}t$, we obtain from (3.2) that, for $\lambda := 8\gamma/c_1\varepsilon$,

$$\begin{aligned}
&\mathbb{E}\exp\left(\gamma\varepsilon^{-2}\int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1}\mathrm{d}t\right) = \mathbb{E}\exp\left(2\gamma\varepsilon^{-2}\int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1}\mathrm{d}t - \gamma\varepsilon^{-2}\int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1}\mathrm{d}t\right)\\
&\le \mathbb{E}\exp\left(\frac{4\gamma}{c_1\varepsilon}M_T + \frac{2c_2\gamma}{c_1\varepsilon^2}T + \frac{2\gamma\|x\|_H^2}{c_1\varepsilon^2} - \gamma\varepsilon^2\int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1}\mathrm{d}t\right)\\
&\le \mathbb{E}\exp\left(\frac{\lambda}{2}M_T - \frac{\lambda^2}{4}\langle M\rangle_T + \int_0^T \left(\frac{q_0\lambda^2}{4}\|X_t^\varepsilon\|_H^2 - \gamma\varepsilon^{-2}\|X_t^\varepsilon\|_{r+1}^{r+1}\right)\mathrm{d}t + \frac{2c_2\gamma}{c_1\varepsilon^2}T + \frac{2\gamma\|x\|_H^2}{c_1\varepsilon^2}\right)\\
&\le \left\{\mathbb{E}\xi_T\right\}^{1/2}\left\{\mathbb{E}\exp\left(\int_0^T \left(\frac{q_0\lambda^2}{2}\|X_t^\varepsilon\|_H^2 - 2\gamma\varepsilon^{-2}\|X_t^\varepsilon\|_{r+1}^{r+1}\right)\mathrm{d}t + \frac{4c_2\gamma}{c_1\varepsilon^2}T + \frac{4\gamma\|x\|_H^2}{c_1\varepsilon^2}\right)\right\}^{1/2}\\
&\le \exp(c\varepsilon^{-2}T)
\end{aligned}$$

for some constant $c > 0$ and all $\varepsilon \in (0, \varepsilon_0)$, where the last step is due to the martingale property of ξ_t and that $\|\cdot\|_{r+1} \ge c\|\cdot\|_H$ for some $c > 0$ and that $r > 1$. □

In large deviation theory, when (μ_ε) satisfies the $LDP(\lambda(\varepsilon), I)$ on a Polish space $\mathbf{E}$ and if $f : \mathbf{E} \to \mathbf{F}$ is continuous where $\mathbf{F}$ is another Polish space, then $(\mu_\varepsilon \circ f^{-1}) \in LDP(\lambda(\varepsilon), I_f)$, where

$$I_f(z) := \inf_{f^{-1}(z)} I, \qquad z \in \mathbf{F}.$$

That is the so called contraction principle. The following generalization is taken from [19] (some preceding weaker versions can be found in [11, Theorems 4.2.16 and 4.2.23]).

Theorem 3.2. (Generalized Contraction Principle) *Let $\mathbf{E}, \mathbf{F}$ be two Polish spaces and (μ_ε) a family of probability measures on E. If $(\mu_\varepsilon) \in LDP(\lambda(\varepsilon), I)$ and there exists a sequence of continuous mappings $f^N : \mathbf{E} \to \mathbf{F}$ such that*

$$\lim_{N\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{\lambda(\varepsilon)}\log\mu_\varepsilon\left(\rho_F\left(f^N, f\right) > \delta\right) = -\infty, \quad \delta > 0, \tag{3.3}$$

where ρ_F is some compatible metric on $\mathbf{F}$ and $f : \mathbf{E} \to \mathbf{F}$ is a measurable mapping, then there exists a continuous function $\tilde{f} : \{I < +\infty\} \to \mathbf{F}$ such that

$$\lim_{N\to\infty}\sup_{I\le r}\rho_F(f^N, \tilde{f}) = 0, \qquad r > 0; \tag{3.4}$$

and $(\mu_\varepsilon(f\in\cdot))\in LDP(\lambda(\varepsilon),I_{\tilde f})$, *where*

$$I_{\tilde f}(z):=\inf_{\tilde f^{-1}(z)} I,\qquad z\in\mathbf{F}. \tag{3.5}$$

4 Proof of Theorem 1.1

We shall prove Theorem 1.1 by two procedures of approximation. Let $\{e_i : i\geq 1\}$ be dense in L^{r+1} and hence, also dense in H. For any fixed $n\geq 1$, let $H_n:=\mathrm{span}\{e_i : 1\leq i\leq n\}$ and $P_n : H\to H_n$ be the orthogonal projection. Let $X_t^{\varepsilon,n}$ be the solution of

$$\mathrm{d}X_t^{\varepsilon,n}=(L\Psi+\Phi)(t,X_t^{\varepsilon,n})\mathrm{d}t+\varepsilon P_nQ\mathrm{d}W_t,\quad X_0^{\varepsilon,n}=x. \tag{4.1}$$

Next for each $N\in\mathbb{N}$ and for any path $w\in C([0,T];H)$, let $t_i:=iT/N$ for $0\leq i\leq N$ and define the ($N-$times) piecewise linear approximation of w by

$$w_t^{(N)}:=\frac{N}{T}\sum_{i=0}^{N-1}1_{(t_i,t_{i+1}]}(t)\big((t-t_i)w_{t_{i+1}}+(t_{i+1}-t)w_{t_i}\big),\quad t\in[0,T].$$

By Proposition 2.1, the following equation has a unique solution $X_{t,N}^{\varepsilon,n}$ in H:

$$\dot X_{t,N}^{\varepsilon,n}:=\frac{\mathrm{d}X_{t,N}^{\varepsilon,n}}{\mathrm{d}t}=L\Psi(t,X_{t,N}^{\varepsilon,n})+\Phi(t,X_{t,N}^{\varepsilon,n})+\varepsilon\frac{d}{\mathrm{d}t}(P_nQW)_t^{(N)},\quad X_{0,N}^{\varepsilon,n}=x. \tag{4.2}$$

We claim that it is enough to establish

$$\limsup_{N\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(\|X^{\varepsilon,n}-X_{\cdot,N}^{\varepsilon,n}\|>\delta)=-\infty,\quad\forall\delta>0 \tag{4.3}$$

and

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(\|X^{\varepsilon,n}-X^{\varepsilon}\|>\delta)=-\infty,\quad\forall\delta>0. \tag{4.4}$$

In fact, by Schilder's theorem, the law of εQW satisfies the LDP on $C([0,T];H)$ with speed $\lambda(\varepsilon)=\varepsilon^{-2}$ and with rate function given by

$$J(\tilde\phi)=\inf\left\{\frac{1}{2}\|\phi\|_{L^2}^2:\ \tilde Q\phi=\tilde\phi(\cdot)\right\}$$

where $\tilde Q:\phi\mapsto\int_0^\cdot Q\phi(s)\mathrm{d}s$ is a continuous linear mapping from $L^2([0,T]\times E,\mathrm{d}t\times\mathbf{m})$ to $C([0,T];H)$ (with the convention that $\inf\emptyset:=+\infty$). Next, let $f_{n,N}$ denote the map which associates each path $\omega \in C([0,T];H)$ of εQW to the solution $X_{t,N}^{\varepsilon,n}$ of (4.2), i.e., $\gamma := f_{n,N}(\omega)$ is the the unique solution of

$$\gamma_t = x + \int_0^t [L\Psi(s,\gamma_s) + \Phi(s,\gamma_s)]\mathrm{d}s + (P_n w)_t^{(N)},$$

where $(P_n w)_t^{(N)}$ is the ($N-$times) piecewise linear approximation of $P_n w$. Applying Proposition 2.1 with Q replaced by $P_n Q$ and noting that $(P_n w)_t^{(N)} = P_n (P_n w)_t^{(N)}$ and that all norms on H_n are equivalent, we see that $f_{n,N} : C([0,T];H) \to C([0,T];H)$ is continuous. Furthermore, by (4.3) and (4.4), for each n, there is some $N(n)$ such that

$$\limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\|X^\varepsilon - f_n(\varepsilon QW)\| > \delta) = -\infty, \quad \forall \delta > 0$$

where $f_n := f_{n,N(n)}$. Hence by Theorem 3.2, X^ε satisfies the LDP on $C([0,T];H)$ with rate function given by

$$I(z) = \inf\{J(\tilde{\phi}) : \ \tilde{f}(\tilde{\phi}) = z\} = \inf\{\frac{1}{2}\|\phi\|_{L^2}^2 : \ \tilde{f}(\tilde{Q}\phi) = z\}$$

and $\tilde{f}(\tilde{Q}\phi) = \lim_{n\to\infty} f_{n,N(n)}(\tilde{Q}\phi)$ by (3.4). But by Proposition 2.1 and the following Lemma 4.1, $f_{n,N}(\tilde{Q}\phi) \to z^\phi$ as n, N goes to infinity. Thus $\tilde{f}(\tilde{Q}\phi) = z^\phi$, which yields the claimed rate function.

Lemma 4.1. *For any $\phi \in L^2([0,T]\times E, \mathrm{d}t \times \mathbf{m})$, let $h_t := \int_0^t Q\phi_s \mathrm{d}s, t \in [0,T]$. For any sequence $N(n) \to \infty$ as $n \to \infty$ we have*

$$\lim_{n\to\infty} \int_0^T \left\| \frac{\mathrm{d}}{\mathrm{d}t}(P_n h)_t^{N(n)} - Q\phi_t \right\|_H^2 \mathrm{d}t = 0.$$

Proof. Let $t_i := Ti/N(n)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} h^{(N(n))} = \sum_{i=1}^{N(n)} 1_{[t_{i-1},t_i)} \frac{N(n)}{T} \int_{t_{i-1}}^{t_i} Q\phi_s \mathrm{d}s$$

which converges to $Q\phi$ in $L^2([0,T];H)$ as $n \to \infty$, it suffices to prove that

$$I_n := \int_0^T \left\| \frac{\mathrm{d}}{\mathrm{d}t}(P_n h)_t^{(N(n)} - \frac{\mathrm{d}}{\mathrm{d}t} h_t^{(N(n))} \right\|_H^2 \mathrm{d}t \to 0 \tag{4.5}$$

as $n \to \infty$. Note that for any $\psi \in L^2([0,T];H)$ we have

$$\int_0^T \Big\|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t \psi_s \mathrm{d}s\Big)^{(N(n))}\Big\|_H^2 \mathrm{d}t = \sum_{i=1}^{N(n)} \frac{T}{N(n)}\Big\|\frac{N(n)}{T}\int_{t_{i-1}}^{t_i} \psi_t \mathrm{d}t\Big\|_H^2$$
$$\le \sum_{i=1}^{N(n)} \int_{t_{i-1}}^{t_i} \|\psi_t\|_H^2 \mathrm{d}t = \int_0^T \|\psi_t\|_H^2 \mathrm{d}t.$$

Then

$$\lim_{n\to\infty} I_n \le \lim_{n\to\infty} \int_0^T \|P_n Q\phi_t - Q\phi_t\|_H^2 \mathrm{d}t = 0.$$

□

So, to finish the proof of Theorem 1.1, we have to prove (4.3) and (4.4) which will be done in the following two subsections.

4.1 Proof of (4.3)

Let $b := L\Psi + \Phi$, and

$$\hat{X}_t := X_t^{\varepsilon,n} - X_{t,N}^{\varepsilon,n}, \quad \hat{\gamma}_t := \varepsilon P_n Q(W_t - W_t^{(N)}),$$

By (4.1) and (4.2) we have

$$\frac{\mathrm{d}\|\hat{X}_t - \hat{\gamma}_t\|_H^2}{\mathrm{d}t} = 2\langle \hat{X}_t - \hat{\gamma}_t, b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t) - b(t, X_{t,N}^{\varepsilon,n})\rangle_H + 2\langle \hat{X}_t - \hat{\gamma}_t, b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t)\rangle_H.$$

Combining this with (1.2) and (1.3) with $\theta_2 < \theta_1$, and using Young's inequality $xy \le x^{r+1}/(r+1) + [r/(r+1)]y^{(r+1)/r}, \forall x, y \ge 0$, we conclude that there exist $\lambda, c > 0$ and $c(\lambda) > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}\|\hat{X}_t - \hat{\gamma}_t\|_H^2 \mathrm{e}^{-ct}}{\mathrm{d}t} \\ &\le -\lambda\|\hat{X}_t - \hat{\gamma}_t\|_{r+1}^{r+1} + 2\|\hat{X}_t - \hat{\gamma}_t\|_{r+1}\|L^{-1}(b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t))\|_{(r+1)/r} \\ &\le c(\lambda)\|L^{-1}(b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t))\|_{(r+1)/r}^{(r+1)/r} + c\|\hat{X}_t - \hat{\gamma}_t\|_H^2. \end{aligned} \tag{4.6}$$

Since $|\Psi'(s)| + |\Phi'(s)| \le c(1 + |s|^{r-1})$ and L^{-1} is bounded in $L^{(r+1)/r}(\mathbf{m})$, there exist $c_1, c_2 > 0$ such that

$$
\begin{aligned}
&\|L^{-1}(b(t,X_t^{\varepsilon,n})-b(t,X_t^{\varepsilon,n}-\hat\gamma_t))\|_{(r+1)/r}^{(r+1)/r}\le c_1\||\hat\gamma_t|(1+|\hat\gamma_t|^{r-1}+|X_t^{\varepsilon,n}|^{r-1})\|_{(r+1)/r}^{(r+1)/r}\\
&\le c_2\int_E(|\hat\gamma_t|^{r+1}+|\hat\gamma_t|^{(r+1)/r}+|\hat\gamma_t|^{(r+1)/r}|X_t^{\varepsilon,n}|^{(r^2-1)/r})\mathrm{d}\mathbf{m}.
\end{aligned}
\tag{4.7}
$$

From (4.6) and (4.7) and Young's inequality we obtain that for each $R>1$,

$$
\frac{\mathrm{d}\|\hat X_t-\hat\gamma_t\|_H^2\mathrm{e}^{-ct}}{\mathrm{d}t}\le c_2\big\{(1+R)\|\hat\gamma_t\|_{r+1}^{r+1}+\|\hat\gamma_t\|_{(r+1)/r}^{(r+1)/r}+c(r)R^{-1/(r-1)}\|X_t^{\varepsilon,n}\|_{r+1}^{r+1}\big\}
\tag{4.8}
$$

for some $c(r)>0$. Since all L^p-norms $(1\le p\le r+1)$ on H_n are equivalent, for any norm $\|\cdot\|_p$ on H_n, by the LDP of εP_nQW_t on $C([0,T];H_n)$, whose good rate function is denoted by I_n, and recalling that $\hat\gamma_t:=\varepsilon P_n[(QW)_t-(QW)_t^{(N)}]$, we have

$$
\begin{aligned}
&\limsup_{N\to\infty}\limsup_{\varepsilon\to0}\varepsilon^{-2}\log\mathbb{P}\Big(\sup_{t\in[0,T]}\|\hat\gamma_t\|_p>\delta\Big)\\
&\le\limsup_{N\to\infty}-\inf\{I_n(w):\ w\in C([0,T];H_n),\sup_{t\in[0,T]}\|w_t-w_{t,N}\|_p\ge\delta\}=-\infty,\quad\forall\delta>0,
\end{aligned}
$$

where the equality follows from the fact that $\inf_{F_N}I_n\to+\infty$ $(N\to\infty)$ for any sequence of closed subsets decreasing to $\emptyset$ (an elementary property of a good rate function).

Combining this with (4.8), we see that for any $\delta\in(0,1)$, there exists $c_3>0$ such that the l.h.s. of (4.3) is less than

$$
\limsup_{n\to\infty}\limsup_{\varepsilon\to0}\varepsilon^2\log\mathbb{P}\left(c_3R^{-1/(r-1)}\int_0^T\|X_t^{\varepsilon,n}\|_{r+1}^{r+1}\mathrm{d}t>\delta\right)
$$

which goes to $-\infty$ when $R\to+\infty$ by Chebychev's inequality and the a priori exponential estimate in Lemma 3.1.

4.2 Proof of (4.4)

By (1.2)and using Itô's formula in [15, Theorem I.3.2], we have

$$
\mathrm{d}\|X_t^\varepsilon-X_t^{\varepsilon,n}\|_H^2\le(\delta(n)\varepsilon^2+c\|X_t^\varepsilon-X_t^{\varepsilon,n}\|_H^2)\mathrm{d}t+2\varepsilon\mathrm{d}M_t^{(n)},
$$

where $c>0$ is a constant, $\delta(n):=q(P_nQ-Q)$ is the square of the Hilbert-Schmidt norm of P_nQ-Q from $L^2(\mathbf{m})$ to H, and $\mathrm{d}M_t^{(n)}:=\langle X_t^\varepsilon-X_t^{\varepsilon,n},(I-P_n)Q\mathrm{d}W_t)\rangle_H$. The quadratic variation process of the local martingale $M^{(n)}$ verifies

$$
d\langle M^{(n)}\rangle_t\le\|X_t^\varepsilon-X_t^{\varepsilon,n}\|_H^2\delta(n)\mathrm{d}t.
$$

For any constant $\alpha > 0$, let $\xi_t := \exp[\alpha\varepsilon^{-2}\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 \mathrm{e}^{-(1+c)t}] =: \exp[\alpha\varepsilon^{-2}Y_t]$. We have by Itô's formula in [15, Theorem I.3.2] that

$$
\begin{aligned}
\mathrm{d}\xi_t &\le 2\alpha\varepsilon^{-1}\mathrm{e}^{-(1+c)t}\xi_t \mathrm{d}M_t^{(n)} \\
&\qquad + \varepsilon^{-2}\alpha\mathrm{e}^{-(1+c)t}\xi_t\big\{\delta(n)\varepsilon^2 - \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 + 2\alpha\mathrm{e}^{-t}\delta(n)\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2\big\}\mathrm{d}t \\
&\le 2\alpha\varepsilon^{-1}\mathrm{e}^{-(1+c)t}\xi_t\mathrm{d}M_t^{(n)} + \alpha\delta(n)\xi_t\mathrm{d}t,
\end{aligned}
$$

once $1 \ge 2\alpha\delta(n)$ which holds for all sufficiently large n for $\delta(n) \to 0$ as $n \to \infty$. So $N_t := \xi_t \exp[-\alpha\delta(n)t]$ is a supermartingale. Therefore, for all n large enough,

$$
\begin{aligned}
\mathbb{P}(\|X^\varepsilon - X^{\varepsilon,n}\| > \delta) &\le \mathbb{P}\Big(\sup_{t\in[0,T]} N_t > \exp[\delta^2\alpha\varepsilon^{-2}\mathrm{e}^{-(1+c)T} - \alpha\delta(n)T]\Big) \\
&\le \exp[-\alpha\delta^2\varepsilon^{-2}\mathrm{e}^{-(1+c)T} + \alpha\delta(n)T].
\end{aligned}
$$

This implies (4.4) since $\alpha > 0$ was arbitrary.

5 Proof of Theorem 1.2

Proof of Theorem 1.2. (a) We first assume that there exists $n \in \mathbb{N}$ such that $q_{ij} = 0$ for $i > n$. In this case the law of $\varepsilon QW_t + x$ satisfies the large deviation principle with the given rate function of compact level sets. Thus by the approximation lemma in large deviations (see [11, Theorem 4.2.13]), it suffices to show that

$$
\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\|\tilde{X}^\varepsilon - x - \varepsilon Q\tilde{W}\| > \delta) = -\infty, \quad \delta > 0. \tag{5.1}
$$

By (1.2) and (1.3) with $\theta_2 < \theta_1$, there exists $\lambda, c, c_0 > 0$ such that

$$
\begin{aligned}
\frac{\mathrm{d}\|\tilde{X}_t^\varepsilon - \varepsilon Q\tilde{W}_t - x\|_H^2\mathrm{e}^{-ct}}{\mathrm{d}t} &\le -\lambda\varepsilon^2\|\tilde{X}_t^\varepsilon - \varepsilon Q\tilde{W}_t - x\|_{r+1}^{r+1} \\
&\qquad + 2\varepsilon^2\mathrm{e}^{-ct}\langle\tilde{X}_t^\varepsilon - \varepsilon Q\tilde{W}_t - x, (L\Psi + \Phi)(\varepsilon Q\tilde{W}_t + x)\rangle_H \\
&\le c_0\varepsilon^2\|(\Psi + L^{-1}\Phi)(x + \varepsilon Q\tilde{W}_t)\|_{(r+1)/r}^{(r+1)/r}, \quad t \in [0,T].
\end{aligned}
$$

Since L^{-1} is bounded in $L^{(r+1)/r}(E, \mathbf{m})$, $|\Psi'(s)| + |\Phi'(s)| \le c(1 + |s|^{r-1})$ for some $c > 0$, and $x \in L^{r+1}$, there exists $c_1 > 0$ such that

$$
\frac{\mathrm{d}\|\tilde{X}_t^\varepsilon - \varepsilon Q\tilde{W}_t - x\|_H^2\mathrm{e}^{-ct}}{\mathrm{d}t} \le c_1\varepsilon^2(\|\varepsilon Q\tilde{W}_t\|_{r+1}^{r+1} + 1).
$$

This immediately implies (5.1) by the LDP of $\varepsilon Q\tilde{W}$ in $C([0,T];H_n)$. Note that on H_n the norms $\|\cdot\|_H$ and $\|\cdot\|_{r+1}$ are equivalent.

(b) In general, for any $n\geq 1$ let $Q^{(n)}:=P_nQ$. By (a), the law of $\tilde{X}_t^{\varepsilon,n}$, the solution to (1.10) for $Q^{(n)}$ in place of Q, satisfies the LDP with the good rate function

$$\tilde{I}_n(z):=\frac{1}{2}\inf\left\{\|\phi\|_2^2:\ z_t=x+\int_0^t Q^{(n)}\phi_s\mathrm{d}s\right\},\quad z\in C([0,T];H).$$

Similarly to the proof of (4.3) in §4.2 we have

$$\limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2\log\mathbb{P}(\|\tilde{X}^{\varepsilon,n}-\tilde{X}^{\varepsilon}\|>\delta)=-\infty,\quad \delta>0.$$

Moreover, since $\delta_n:=\|L^{-1/2}(Q-Q^{(n)})\|_{2\to 2}\to 0$ as $n\to\infty$ and since

$$\int_0^T\|(Q-Q^{(n)})\phi_t\|_H\mathrm{d}t\leq\delta(n)\int_0^T\|\phi_t\|_2\mathrm{d}t\leq\delta(n)\sqrt{T}\|\phi\|_2,$$

we conclude that the law of $\tilde{X}^{\varepsilon}$ satisfies the LDP with the claimed rate function I by the approximation lemma (see [11, Theorem 4.2.13]). $\square$

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