Sobolev-Orlicz inequalities, Ultracontractivity and spectra of time changed Dirichlet forms

Ali Ben Amor*
Universität Osnabrück. FB Informatik/Mathematik.
49069. Osnabrück. Germany.
E-mail: abamor@mathematik.uni-osnabrueck.de

Abstract

Let $\mathcal{E}$ be a regular Dirichlet form on $L^2(X,m)$, μ a positive Radon measure charging no sets of zero capacity and Φ an N-function. We prove that the Sobolev-Orlicz inequality $\|f^2\|_{L^\Phi(X,\mu)} \leq C\mathcal{E}_1[f]$ for every $f \in D(\mathcal{E})$ is equivalent to a capacitary-type inequality. Further we show that if $D(\mathcal{E})$ is continuously embedded into $L^2(X,\mu)$, the latter one is equivalent to some integrability condition, which is nothing else but the classical uniform integrability condition if μ is finite. We also prove that a Sobolev-Orlicz inequality for $\mathcal{E}$ yields a Nash-type inequality and if further $\mu = m$ it yields the ultracontractivity of the corresponding semigroup. After, in the spirit of Sobolev-Orlicz inequalities, we derive criteria for $D(\mathcal{E})$ to be compactly embedded into $L^2(\mu)$, provided μ is finite. As an illustration of the theory, we shall relate the compactness of the latter embedding to the discreteness of the spectrum of the time changed Dirichlet form and shall derive lower bounds for its eigenvalues in term of Φ.

Key words: Sobolev-Orlicz inequality, ultracontractivity, measure, spectrum.

1 Introduction

In this paper we continue our investigations which we began in [BA04b, BA05] devoted to give necessary and sufficient conditions for the validity of Sobolev-type inequalities (or more generally trace inequalities) for regular Dirichlet forms and give criteria for their domain to be compactly embedded into some Lebesgue space.
Here is a short description of our program: Let X be a locally compact separable metric space and m a positive Radon measure on X whose support is X. Let $\mathcal{E}$ be a regular Dirichlet form on $L^2 := L^2(X,m)$ and μ a Borel measure on X charging no subsets of zero capacity. It is well known that the Sobolev's inequality (SI for short)

$$(SI): \ \Big(\int_X |f|^{\frac{2}{\kappa}}\, d\mu\Big)^\kappa \leq C\mathcal{E}_1(f,f), \text{ for every } f \in D(\mathcal{E}),\ 0 < \kappa \leq 1,$$

*This work has been supported by the Deutsche Forschungsgemeinschaft.

may be extended to a much more general inequality, namely Sobolev-Orlicz inequality. On the other hand the Sobolev's inequality (SI) is equivalent to (See [AH96, BA04b, BA05, FU03a, Maz85]) the capacitary inequality (CI'):

$$(CI'):\ \big(\mu(K)\big)^{\kappa} \le C\mathrm{Cap}(K), \text{for every compact } K.$$

Now substituting (SI) by the Sobolev-Orlicz inequality (SOI for short):

$$(SOI):\ \|f^2\|_{L^{\Phi}(\mu)} \le C\mathcal{E}_1(f,f) \text{ for every } f \in D(\mathcal{E}),$$

suggests to replace (CI') by an appropriate capacitary inequality involving the conjugate function of Φ, Ψ. This was done, first, by Mazja [Maz85, Satz 4.2] for the gradient energy form on $\mathbb{R}^d$ and by Carron for the energy form on a nice Riemannian manifold [Car97] with $\mu = m$.
Also Kaimanovich [Kai92] extends the result to the case where the transition kernel of the Dirichlet norm is a probability measure and $\mu = m$.

One of our aims in this work is to establish an equivalence between (SOI) and a capacitary-type inequality for general Dirichlet forms. This step is an extension of [BA04b, Theorem 3.1],[FU03a, Theorem 3.1] on one hand and of [Maz85, Satz 4.2] and [BCLSC95, Theorem 10.4] on the other hand (where the case $\mu = m$ is considered). Using this equivalence we show that if I_μ is bounded (defined below), then both (SOI) and a capacitary inequality are equivalent to some uniform integrability condition (see Theorem 3.2) which reduces to the usual one if $\mu(X) < \infty$.

Further we prove that a (SOI) with an N-function Φ leads to a Nash-type inequality and if moreover $\mu = m$ the latter one leads to the ultracontractivity of the semigroup e^{-tH} for every $t > 0$, i.e. e^{-tH} maps continuously L^1 into L^∞ for every $t > 0$. This result will play a central role for deriving lower bounds for the eigenvalues in case where H has a discrete spectrum.

Then we derive necessary and sufficient conditions for the domain of the Dirichlet form, $D(\mathcal{E})$, to be compactly embedded into some $L^2(\mu)$ for finite measures μ. Precisely, relying on establishing a relationship between the compactness of I_μ and the compactness of the *trace* of $\mathcal{E}_1$ on the support of μ, we shall prove that the compactness of the embedding

$$I_\mu := (D(\mathcal{E}), \mathcal{E}_1) \to L^2(\mu),\ \ f \mapsto f, \tag{1.1}$$

is equivalent to a (SOI) with some N-function Φ, provided μ is finite.
The scope of (SOI) in the special case $\mu = m$ was extensively studied and its importance is well known. However, we stress that the general case is still of great importance, especially for the study of the perturbed Dirichlet form $\mathcal{E} + \mu^+ - \mu^-$ [Bra01, BA04a, Sto94] (localization of the essential spectrum, estimate of the number of the negative eigenvalues). It is of great interest as well, for the study of the trace of the Dirichlet form $\mathcal{E}$ on the support of the measure μ as done by Fukushima-Uemura [FU03a] and as we shall illustrate in the fourth and last paragraphs.

2 Preliminaries

For a positive Radon measure μ on X and $1 \leq p \leq \infty$, we shall denote by $L^p(\mu)$ the usual (real) Lebesgue space of Borel measurable (equivalence classes) of functions on X equipped with the usual norm $\|\cdot\|_{L^p(\mu)}$. If $\mu = m$, the space $L^p(m)$ will be denoted simply by L^2 further the notation a.e. means m-a.e.. The space of continuous functions with compact support in X will be denoted by $C_c(X)$.
Let $\mathcal{E}$ be a regular Dirichlet form defined in L^2. We denote by $D(\mathcal{E})$ the domain of $\mathcal{E}$. By the regularity we mean that $D(\mathcal{E}) \cap C_c(X)$ is dense in $D(\mathcal{E})$ with respect to the norm $\mathcal{E}_\alpha := \mathcal{E} + \alpha(\cdot,\cdot)_{L^2}$ for some $\alpha > 0$ and dense in $C_c(X)$ with respect to the uniform norm. The positive selfadjoint operator associated to $\mathcal{E}$ via the representation theorem is denoted by H and is defined as follows:

$$D(H^{\frac{1}{2}}) = D(\mathcal{E}),\ \ \mathcal{E}[f] := \mathcal{E}(f,f) = (H^{\frac{1}{2}}f, H^{\frac{1}{2}}f)_{L^2}\cdot \tag{2.1}$$

For every open subset $\Omega \subset X$ and every $\alpha > 0$ we define [FŌT94, p. 61]

$$\mathcal{L}_\Omega := \{f \in D(\mathcal{E}),\ \ f \geq 1 \text{ a.e. on } \Omega\} \tag{2.2}$$

and

$$\mathrm{Cap}_\alpha(\Omega) := \begin{cases} \inf_{f\in\mathcal{L}_\Omega} \mathcal{E}_\alpha[f] & \text{if} \quad \mathcal{L}_\Omega \neq \emptyset \\ +\infty & \text{if} \quad \mathcal{L}_\Omega = \emptyset \end{cases}$$

For an arbitrary subset $\Omega \subset X$ we define

$$\mathrm{Cap}_\alpha(\Omega) = \inf_{\Omega\subset\omega,\ \omega \text{ open}} \mathrm{Cap}_\alpha(\omega)\cdot \tag{2.3}$$

It is known that Cap_α defines a capacity on X. For $\alpha = 1$ we shall denote the corresponding capacity by Cap.
A property is said to be satisfied quasi-everywhere (q.e. for short) if it holds true up to a set of zero capacity. A function f is said to be quasi-continuous (q.c.) if for every $\epsilon > 0$ there is an open subset O such that $\mathrm{Cap}(O) < \epsilon$ and the restriction of f to $X \setminus O$ is continuous.
We recall (cf. [FŌT94, p. 65]) that the regularity property of Dirichlet forms implies that every element of $D(\mathcal{E})$ can be corrected so as to become q.c.. In the sequel we shall implicitly assume that the elements of $D(\mathcal{E})$ has been corrected in this way. Let us observe that together with Urysohn's lemma the regularity property implies that every compact subset has finite capacity.
From now on we assume that all measures under consideration are Radon measures and they do not charge subsets of zero capacity, we also denote by I_μ the mapping:

$$I_\mu :=\ (D(\mathcal{E}), \mathcal{E}_1) \longrightarrow L^2(\mu),\ \ f \mapsto\ f. \tag{2.4}$$

For the convenience of the reader we also recall some facts about Orlicz spaces. A function $\Phi : [0,\infty) \to [0,\infty)$ is called a *Young's function* if it is convex, $\Phi(0) = 0$ and

$\lim_{x\to\infty}\Phi(x)=\infty$. A Young's function is called an N-function if moreover $\Phi(x)=0$ if and only if $x=0, \lim_{x\to 0}\Phi(x)/x=0$ and $\lim_{x\to\infty}\Phi(x)/x=\infty$.
To every Young's function Φ, one can associate an other Young's function Ψ, which is defined by

$$\Psi(x)=\sup\big\{yx-\Phi(y) : y\geq 0\big\},\ x\geq 0.$$

The function Ψ is called the *complementary* function of Φ. From the very definition the following Young's inequality holds true

$$xy\leq\Phi(x)+\Psi(y),\ x,y\geq 0.$$

We also note that if Φ is an N-function then Ψ also is.
The following important inequality, valid for N-functions, (see [RR91, p.14]) will be used later:

$$x<\Phi^{-1}(x)\Psi^{-1}(x)\leq 2x,\ x>0. \tag{2.5}$$

For a positive Radon measure μ and a Young's function Φ the Orlicz space $L^\Phi(\mu):=L^\Phi(X,\mu)$ is the space of μ-measurable (equivalence classes) of functions f on X such that

$$\|f\|_{L^\Phi(\mu)}:=\sup\big\{\big|\int_X fg\,d\mu\big| :\ \int_X\Psi(|g|)\,d\mu\leq 1\big\}<\infty \tag{2.6}$$

It is well known that $\big(L^\Phi(\mu),\|\cdot\|_{L^\Phi(\mu)}\big)$ is a Banach space whose norm is equivalent to the Luxembourg norm:

$$\|f\|_{(\Phi)}:=\inf\big\{\lambda>0 :\ \int_X\Phi(|f|/\lambda)\,d\mu\leq 1\big\}. \tag{2.7}$$

Precisely we have

$$\|f\|_{(\Phi)}\leq\|f\|_{L^\Phi(\mu)}\leq 2\|f\|_{(\Phi)},\ \forall\, f\in L^\Phi(\mu). \tag{2.8}$$

We also recall that if Φ is an N-function and B is a Borel subset then (see [RR91, p.79])

$$\|1_B\|_{L^\Phi(\mu)}=\mu(B)\Psi^{-1}(1/\mu(B)). \tag{2.9}$$

For Orlicz spaces the Hölder's inequality reads as follows:

$$\int_X|fg|\,d\mu\leq\|f\|_{L^\Phi(\mu)}\|g\|_{L^\Psi(\mu)}. \tag{2.10}$$

Let us emphasize that in some places (and only in some!) we would pay no attention to the constants appearing in the inequalities. So that we shall denote them all by C.

3 Sobolev-Orlicz inequalities and their consequences

In this paragraph we shall be concerned with two questions: First, show how a Sobolev-Orlicz inequality, an isocapacitary-type inequality and some uniform integrability condition are equivalent to each other.
Second, which consequences can one get from the (SOI)? Namely we shall prove that a (SOI) with an N-function yields a Nash-type inequality which, in turns, implies the ultracontractivity of the related semigroup.

3.1 Sobolev-Orlicz inequalities: necessary and sufficient conditions

The following generalizes both [Maz81, Satz 3.1], [Kai92, Theorem 3.1] and [BA05, Theorem 3.1].

Theorem 3.1. *Let Φ be an N-function. Then the following assertions are equivalent:*
i) There is a constant C_1 such that

$$\|f^2\|_{L^\Phi(\mu)} \le C_1\mathcal{E}_1[f],\ \forall f \in D(\mathcal{E}). \tag{3.1}$$

ii) There is a constant C_2 such that

$$(\mathrm{CI}):\ \mu(K)\Psi^{-1}(1/\mu(K)) \le C_2\mathrm{Cap}(K),\ \forall K \text{ compact}.$$

Moreover the constants may be chosen so that $C_2 \le C_1 \le 4C_2$.

Let us stress that if the Dirichlet form $\mathcal{E}$ is transient then the same result holds true if on changes $\mathcal{E}$ by $\mathcal{E}$ and the 1-capacity by the corresponding one.

Proof. i)$\Rightarrow$ii): Follows directly from (2.9) and the definition of the capacity.
ii)$\Rightarrow$i): The main input in the proof of this implication is the *'strong capacitary inequality'* (see [BA05, FU03a]):

$$\int_X \mathrm{Cap}\{|f| \ge t\}\,d(t^2) \le 4\mathcal{E}_1[f],\ \forall f \in D(\mathcal{E}). \tag{3.2}$$

Observe that by the regularity property of $\mathcal{E}$, it suffices to prove inequality (3.1) on $D(\mathcal{E})\cap C_c(X)$. Further by Markov property for $\mathcal{E}$ i.e. $\mathcal{E}[|f|] \le \mathcal{E}[f]$, $\forall f \in D(\mathcal{E})$, it suffices to prove it for positive elements from $D(\mathcal{E}) \cap C_c(X)$.
Let f be such a function. Then by definition and use of Fubini's theorem we get

$$\begin{aligned}\|f^2\|_{L^\Phi(\mu)} &= \sup\big\{\big|\int_X f^2 g\,d\mu\big| : \int_X \Psi(|g|)\,d\mu \le 1\big\}\\ &= \sup\big\{\big|\int_0^\infty \big(\int_{\{f\ge t\}} g\,d\mu\big)\,d(t^2)\big| : \int_X \Psi(|g|)\,d\mu \le 1\big\}.\end{aligned}$$

For $t \ge 0$, set $E_t := \{f \ge t\}$. From the latter inequality we derive

$$\begin{aligned}\|f^2\|_{L^\Phi(\mu)} &\le \int_0^\infty \big(\sup\big\{\big|\int_X 1_{E_t} g\,d\mu\big| : \int_X \Psi(|g|)\,d\mu \le 1\big\}\big)\,d(t^2) = \int_0^\infty \|1_{E_t}\|_{L^\Phi(\mu)}\,d(t^2)\\ &= \int_0^\infty \mu(E_t)\Psi^{-1}(1/\mu(E_t))\,d(t^2) \le C_2\int_0^\infty \mathrm{Cap}(E_t)\,d(t^2)\\ &\le 4C_2\mathcal{E}_1[f],\end{aligned}$$

where the latter inequality is obtained from inequality (3.2).

□

Next we shall use Theorem 3.1 to prove that for measures μ such that I_μ is bounded, the (SOI) and (CI) are equivalent to the fact that the unit ball of $D(\mathcal{E})$ is nearly $L^2(\mu)$-uniformly integrable. Here we say that a family $\mathcal{F}$ of Borel measurable functions is $L^p(\mu)$-nearly uniformly integrable $(1 \le p < \infty)$ if

$$\text{(NUI)}: \ \lim_{\lambda\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|\ge\lambda\}} |f|^p \, d\mu = 0. \tag{3.3}$$

If the measure μ is finite, the latter definition reduces to the classical definition of uniform integrability [DS58].

For this goal we need:

Lemma 3.1. *Assume that a family $\mathcal{F}$ is $L^1(\mu)$-nearly uniformly integrable. Then there is a Young's function Φ such that*

$$\sup_{f\in\mathcal{F}} \int_X \Phi(f)\, d\mu < \infty. \tag{3.4}$$

If further μ is finite, then Φ may be chosen so as to be an N-function.

For $\mu(X) < \infty$ the latter lemma is known as 'la Vallée Poussin' criterion.

Proof. The proof runs as in the case $\mu(X) < \infty$ [RR91, DM75]. We shall reproduce it just for the convenience of the reader.
Let (λ_n) be a sequence of real numbers such that $0 < \lambda_n < \lambda_{n+1} \to \infty$ and $\lambda_n^{-1}\lambda_{n+1} \le C < \infty$. By the assumption (NUI), we may assume that

$$\sup_{f\in\mathcal{F}} \Big(\sum_{n=0}^{\infty} 2^n \int_{\{|f|\ge\lambda_n\}} f\, d\mu \Big) < \infty.$$

For if not there is always a subsequence of (λ_n) which satisfies the latter condition.
Set $\phi(t) := \sum_{n\ge0} 2^n 1_{[\lambda_n,\lambda_{n+1})}(t)$ and

$$\Phi(t) := \int_0^t \phi(s)\, ds.$$

Then Φ is a Young's function. We shall show that Φ is the sought function. Observe that it suffices to prove (3.4) for positive $f \in \mathcal{F}$. Let f be such a function. Then

$$\begin{aligned} \int_X \Phi(f)\, d\mu &= \sum_{n=0}^{\infty} \int_{\{\lambda_n\le f<\lambda_{n+1}\}} \Phi(f)\, d\mu \le \sum_{n=0}^{\infty} \Phi(\lambda_{n+1})\mu\{\lambda_n \le f < \lambda_{n+1}\} \\ &\le \sum_{n=0}^{\infty} \big(\Phi(\lambda_{n+1}) - \Phi(\lambda_n)\big)\mu\{f \ge \lambda_n\} = \sum_{n=0}^{\infty} 2^n(\lambda_{n+1} - \lambda_n)\mu\{f \ge \lambda_n\} \\ &\le C \sum_{n=0}^{\infty} 2^n \int_{\{f\ge\lambda_n\}} f\, d\mu < \infty, \text{ uniformly in } f \end{aligned}$$

If μ is finite, we modify Φ on the interval $[0, \lambda_0]$ so as to get an N-function. Whence the norm of the Orlicz space associated to the modified function is equivalent to the first one. and the proof is completed. □

From now on we denote by $\mathbb{B}$ the unit ball of $(D(\mathcal{E}), \mathcal{E}_1)$.

Theorem 3.2. *Assume that I_μ is bounded. If (SOI) holds true with some N-function then $\mathbb{B}$ is $L^2(\mu)$-nearly uniformly integrable.*
Conversely if $\mathbb{B}$ is $L^2(\mu)$-nearly uniformly integrable, then there is a Young's function Φ for which (SOI) is satisfied.

Proof. i)⇒ii): Since I_μ is bounded by [BA04b, Theorem3.1] there is a constant C such that

$$\mu(K) \quad \leq \quad C\text{Cap}(K), \text{ for all } K \text{ compact.} \tag{3.5}$$

Now let $f \in C_c(X) \cap D(\mathcal{E})$ be positive such that $\mathcal{E}_1[f] \leq 1$ and $\lambda > 0$. Then by Hölder inequality we have

$$\begin{aligned}\int_{\{f\geq\lambda\}} f^2\, d\mu \quad &\leq \quad \|f^2\|_{L^\Phi(\mu)} \|1_{\{f\geq\lambda\}}\|_{L^\Psi(\mu)} \\ &\leq C\mathcal{E}_1[f]\mu\{f \geq \lambda\}\Phi^{-1}\big(\frac{1}{\mu\{f\geq\lambda\}}\big) \\ &\leq 2C\big(\Psi^{-1}\big(\frac{1}{\mu\{f\geq\lambda\}}\big)\big)^{-1},\end{aligned} \tag{3.6}$$

where the latter inequality follows from (2.5). By the 'weak capacitary inequality'

$$\text{Cap}\{f \geq \lambda\} \leq \lambda^{-2}\mathcal{E}_1[f],$$

and inequality (3.5) we obtain $\mu\{f \geq \lambda\} \leq C/\lambda^2$. Since by the properties of Ψ, $\lim_{\lambda\to\infty} \Psi(t) = \infty$, we achieve

$$\int_{\{f\geq\lambda\}} f^2 \leq C\big(\Psi^{-1}(\lambda^2/C)\big)^{-1} \to 0 \text{ as } \lambda \to \infty, \tag{3.7}$$

uniformly in f, implying that $\mathbb{B}$ is $L^2(\mu)$-nearly uniformly integrable.
ii)⇒i): follows from Lemma 3.1.

□

3.2 Some consequences of Sobolev-Orlicz inequalities

For $\Phi(t) = 1/pt^p$, $p > 1$, inequality (3.1) yields the Sobolev's inequality. Further, Sobolev's inequality leads to Nash inequality which in turns leads to the ultracontractivity of the related semi group (see [Var85]). Our next task is to prove that such implication still holds true in our general setting: Namely a (SOI) with an N-function leads to a Nash-type inequality.

First we shall fix some notations. For an N-function Φ we set

$$\Lambda(s) := \frac{1}{s\Phi^{-1}(1/s)},\ s > 0 \text{ and } \kappa_\mu := \sup\{(\Lambda(\mu(K))\text{Cap}(K))^{-1},\ K \text{ compact}\}, \tag{3.8}$$

where the ratio is understood to be zero if $\mathrm{Cap}(K) = 0$.
We observe that Λ satisfies the following property:

$$0 < s_1 \le s_2 \Rightarrow \Lambda(s_2) \le 2\Lambda(s_1). \tag{3.9}$$

Indeed, from inequality (2.5) we infer

$$\Lambda(s_2) < \Psi^{-1}(1/s_2) \le \Psi^{-1}(1/s_1) \le 2\Lambda(s_1).$$

Theorem 3.3. *Assume that (SOI) holds true with an N-function Φ. Then for every $f \in D(\mathcal{E}) \setminus \{0\}$ and every $\epsilon \in (0,1)$ we have*

$$\mathcal{E}_1[f] \ge (2\kappa_\mu)^{-1}(1-\epsilon)\Big(\int_X f^2\,d\mu\Big)\Lambda\Big(\frac{2\|f\|^2_{L^1(\mu)}}{\epsilon\int_X f^2\,d\mu}\Big). \tag{3.10}$$

For $\Phi(t) = \frac{d-2}{d}t^{d/d-2}$, $d > 2$ then $\Lambda(s) = cs^{-2/d}$. Whence if moreover $\mu = m$, then (SOI) is the classical (SI) and (3.10) is the usual Nash inequality:

$$\Big(\int_X f^2\,dm\Big)^{(1+2/d)} \le C\|f\|^{4/d}_{L^1}\mathcal{E}_1[f],\ \forall f \in D(\mathcal{E}).$$

For the gradient energy form on 'nice' manifolds, Grigor'yan proved in [Gri99, Lemma 6.3] a similar inequality.

Proof. We follow an idea of Grigor'yan with some modifications.
By Markov property we learn that it suffices to make the proof for positive elements from $D(\mathcal{E}) \setminus \{0\}$. Let f be such an element and $\epsilon \in (0,1)$. We first suppose that $\mu(X) < \infty$. Clearly

$$f^2 \le (f-s)_+^2 + 2sf,\ \forall s \ge 0. \tag{3.11}$$

Setting $\Omega_s : \{f > s\}$, $A := \int_X f\,d\mu$, $B := \int_X f^2\,d\mu$ and using Hölder inequality, we get

$$\begin{aligned} B &\le \int_{\Omega_s}(f-s)^2\,d\mu + 2sA \le \int_{\Omega_s} f^2\,d\mu + 2sA \\ &\le \|1_{\Omega_s}\|_{L^\Psi(\mu)}\|f^2\|_{L^\Phi(\mu)} + 2sA = \frac{1}{\Lambda(\mu(\Omega_s))}\|f^2\|_{L_\Phi(\mu)} + 2sA \\ &\le \frac{1}{\Lambda(\mu(\Omega_s))}\kappa_\mu\mathcal{E}_1[f] + 2sA. \end{aligned} \tag{3.12}$$

On the other hand we have $\mu(\Omega_s) \le s^{-1}A$. So that using (3.9) and choosing $s = \frac{\epsilon B}{2A}$ we get the desired result for finite μ. Now for arbitrary μ, set B_k the open ball of X centered at x_0 with radius k and $\mu_k : 1_{B_k}\mu$. Then

$$\begin{aligned} \mathcal{E}_1[f] &\ge (2\kappa_{\mu_k})^{-1}(1-\epsilon)\Big(\int_X f^2\,d\mu_k\Big)\Lambda\Big(\frac{2\|f\|^2_{L^1(\mu_k)}}{\epsilon\int_X f^2\,d\mu_k}\Big) \\ &\ge (2\kappa_\mu)^{-1}(1-\epsilon)\Big(\int_X f^2\,d\mu_k\Big)\Lambda\Big(\frac{2\|f\|^2_{L^1(\mu_k)}}{\epsilon\int_X f^2\,d\mu_k}\Big) \\ &\to (2\kappa_\mu)^{-1}(1-\epsilon)\Big(\int_X f^2\,d\mu\Big)\Lambda\Big(\frac{2\|f\|^2_{L^1(\mu)}}{\epsilon\int_X f^2\,d\mu}\Big) \text{ as } k \to \infty, \end{aligned} \tag{3.13}$$

which was to be proved. □

The latter result is the main input to show that (SOI) implies the ultracontractivity property of the related semigroup $T_t := e^{-tH}$, for $t > 0$.
Let Φ be an N-function and Λ defined as before. Set $\kappa := \kappa_m$. Following Grigor'yan, we set γ the function defined via

$$t := \kappa \int_0^{\gamma(t)} \frac{1}{s\Lambda(s)}\, ds,$$

and for every $\epsilon \in (0,1)$

$$\beta_\epsilon(t) := \frac{2}{\epsilon\gamma(2(1-\epsilon)t)}.$$

Observe that owing to the behavior of Φ at infinity the integral equation defining γ is well posed.

Theorem 3.4. *Assume (SOI) with an N-function Φ and $\mu = m$. Then for every $t > 0$, T_t is ultracontractive and*

$$\|T_t\|_{L^1,L^\infty} \le \beta_\epsilon(t/2)e^t,\ \forall \epsilon \in (0,1). \tag{3.14}$$

It follows that for every $t > 0$, T_t is an integral operator, with kernel $p_t(x,y)$ defined everywhere and such that

$$\sup_{x,y\in X} p_t(x,y) \le \beta_\epsilon(t/2)e^t,\ \forall t > 0. \tag{3.15}$$

We note that, in an other context, Coulhon [Cou96, Proposition II.1] established a similar result with a bound which is different from ours.

Proof. We are going to prove the boundedness of $H_t := e^{-t(H+1)}$ from L^1 into L^2 with the bound $\beta_\epsilon^{1/2}$, which is by the symmetry of T_t and duality equivalent to the claim of the theorem.
For this purpose, thank to the Markov property for H_t:

$$|H_t f| \le H_t|f|,\ a.e.\ \forall\ f \in L^2,$$

it suffices to prove that for every $t > 0$ and every $f \in L^2 \cap L^1$, $f \ge 0$ and $\|f\|_{L^1} = 1$, we have

$$\|H_t f\|_{L^2}^2 \le \beta_\epsilon(t).$$

Let f be such a function. Set

$$f_t(x) := H_t f(x),\ J(t) := \int_X f_t(x)^2\, dm(x) = \|H_t f\|_{L^2}^2. \tag{3.16}$$

Observing that for every $t > 0$, $f_t \in D(H)\setminus\{0\}$, differentiating with respect to t, applying Nash inequality (3.10) and using inequality (3.9) , we achieve

$$\begin{aligned} J'(t) &= -2\big((H+1)H_t f, H_t f\big) = -2\mathcal{E}_1[f_t] \\ &\leq -2(1-\epsilon)\kappa^{-1}\Big(\int_X f_t^2\, d\mu\Big)\Lambda\Big(\frac{2\|f_t\|_{L^1(m)}^2}{\epsilon\int_X f_t^2\, dm}\Big) \\ &\leq -2(1-\epsilon)\kappa^{-1}\Big(\int_X f_t^2\, dm\Big)\Lambda\Big(\frac{2}{\epsilon\int_X f_t^2\, dm}\Big) \\ &= -2(1-\epsilon)\kappa^{-1}J(t)\Lambda\Big(\frac{2}{\epsilon J(t)}\Big),\ \forall t > 0. \end{aligned} \tag{3.17}$$

The latter inequality leads to

$$\frac{J'(t)}{J(t)\Lambda\big(\frac{2}{\epsilon J(t)}\big)} \leq -2(1-\epsilon)\kappa^{-1}. \tag{3.18}$$

Integrating between 0 and t and making the change of variable $s = 2\epsilon^{-1}J(t)^{-1}$ we obtain

$$\kappa\int_0^{2\epsilon^{-1}J(t)^{-1}} \frac{ds}{s\Lambda(s)} \geq \kappa\int_{2\epsilon^{-1}J(0)^{-1}}^{2\epsilon^{-1}J(t)^{-1}} \frac{ds}{s\Lambda(s)} \geq 2(1-\epsilon)t, \tag{3.19}$$

which by the definition of β_ϵ yields

$$J(t) \leq \beta_\epsilon(t), \tag{3.20}$$

which gives the first part of the theorem.

The rest of the proof follows from Dunford-Pettis theorem (see [Tre67, Theorem 46.1 p.471]).

□

Let us observe that for $\mu = m$, inequality (3.1), with $\mathcal{E}_1$ replaced by $\mathcal{E}$, leads to the following Faber-Krahn-type inequality

$$\big(m(B)\Phi^{-1}(1/m(B))\big)^{-1} \leq \lambda_1(B),\ \forall\, B \text{ open bounded}. \tag{3.21}$$

Here $\lambda_1(B) := \inf\{\mathcal{E}[f] : \int_X f^2\, dm = 1,\ f \in D(\mathcal{E}),\ f = 0\ q.e. \text{ on } X \setminus B\}$ is a sort of a 'Dirichlet eigenvalue'.
It is well known that in some special cases inequality (3.1) and (3.21) are equivalent to each other. For example if $\Phi(t) = 1/pt^p$, $p > 1$ (see [GY03]) or if X is a nice Riemannian manifold and $\mathcal{E}$ is the gradient energy form (see [Car97]).
However it is still an open question whether the both inequalities are equivalent in general or not!

4 Sobolev-Orlicz inequalities, compactness of I_μ and relationship to time changed Dirichlet forms

In this section we shall discuss to which extend is a (SOI) equivalent to the compactness of I_μ, provided μ is finite. Further, to illustrate the importance of such results we shall relate them to the compactness of a new Dirichlet form, namely a *time changed Dirichlet form* or the *trace* of the Dirichlet form $\mathcal{E}_1$ on the support of μ. For $\mu = m$, a direct use of Theorem 3.4 yields the following

Proposition 4.1. *Assume that m is finite. Then a (SOI) implies the compactness of I_m.*

We mention that the latter result was obtained by Cipriani [Cip00, Theorem 4.2] under the additional assumption that H is a Persson's operator. However, we now know that this assumption is not useful!

4.1 Relationship to time changed Dirichlet forms

To illustrate the relevance of the compactness property of I_μ (and hence the existence of a (SOI)), we shall relate it to the compactness of the resolvent of a new Dirichlet form.

We start with the definition of the trace of the Dirichlet form $\mathcal{E}$ on some subsets. To this end we denote by $\mathcal{F}_e$ the extended Dirichlet space of $\mathcal{E}$ (see [FŌT94, p.35]).
Let μ be a positive Radon measure charging no set of zero capacity and A the positive continuous functional whose Revuz measure is μ. We denote by F the support of μ and by $\tilde{F}$ the support of A. It is known [FŌT94, p.265] that $\tilde{F}$ is a quasi-support for μ, $\mu(F \setminus \tilde{F}) = 0$ and that by [FŌT94, theorem 4.6.2] if two elements from $\mathcal{F}_e$ coincide μ-a.e. then they coincide q.e. as well.
We introduce the subspaces

$$\mathcal{F}_{X-\tilde{F}} := \{f \in \mathcal{F}_e \colon f = 0 \text{ q.e. on } \tilde{F}\}, \tag{4.1}$$

and $\mathcal{H}^{\tilde{F}}$ the $\mathcal{E}$-orthogonal complement of $\mathcal{F}_{X-\tilde{F}}$ in the space $\mathcal{F}_e$ so that the following decomposition holds true [FŌT94, p.265]:

$$\mathcal{F}_e = \mathcal{F}_{X-\tilde{F}} \oplus \mathcal{H}^{\tilde{F}}. \tag{4.2}$$

Let P be the orthogonal projection onto $\mathcal{H}^{\tilde{F}}$. We define the trace of $\mathcal{E}$ on $L^2(F,\mu)$ as follows (see [FŌT94, p.266 and Theorem 4.6.5]):

$$\begin{aligned} D(\check{\mathcal{E}}) &:= \{f \in L^2(F,\mu) \colon f = u - \mu \text{ a.e. on } F, \text{ for some } u \in \mathcal{F}_e\}, \\ \check{\mathcal{E}}[f] &= \mathcal{E}[Pu],\ f = u - \mu \text{ a.e. on } F \\ &= \inf\{\mathcal{E}[u] : u \in D(\mathcal{F}_e),\ u = f - \mu \text{a.e. on } F\}. \end{aligned}$$

It is known that $\check{\mathcal{E}}$ is also a Dirichlet form in $L^2(F,\mu)$ [FŌT94, p.266] and the Dirichlet space $(\check{\mathcal{E}}, D(\check{\mathcal{E}}))$ is called the *time changed Dirichlet space* or the *trace* of the space $(\mathcal{E}, D(\mathcal{E}))$ on F relative to μ.
We denote by $\check{H}$ the operator associated to $\check{\mathcal{E}}$ via the representation theorem. To give an explicit formula for the resolvent of $\check{H}$ we give some auxiliary notations: Set A_t^μ the PCAF associated to μ and and for $\alpha, p \geq 0$ the operators $\check{R}$, $U^{\alpha,\mu}$ defined respectively on the spaces $L^2(F,\mu)$ and $L^2(\mu)$ by

$$\check{R}f(x) := E_x\big(\int_0^\infty e^{-A_t^\mu} f(X_t)\, dA_t^\mu \text{ and } U_p^{\alpha,\mu}f(x) := E_x\big(\int_0^\infty e^{-\alpha t - pA_t^\mu} f(X_t)\, dA_t^\mu.$$

Here X_t designate the process associated to $\mathcal{E}$. Clearly for every α, $p > 0$ and every $f \in L^2(\mu)$, $U_p^{\alpha,\mu} f$ is the α-potential of the signed measure $f\mu$ with respect to the Dirichlet form $\mathcal{E} + p\mu$. Further by [BA04b, Theorem 3.1], for every α, $p > 0$, $U_p^{\alpha,\mu}$ is bounded on $L^2(\mu)$ and by [FŌT94, p.274]

$$(\check{H} + 1)^{-1} = \check{R}. \tag{4.3}$$

We further introduce the auxiliary operators

$$K_\alpha^\mu := L^2(\mu) \to L^2(\mu),\ f \mapsto K_\alpha^\mu f := U^{\alpha,\mu} f\ \mu - a.e.\ \forall \alpha \geq 0 \tag{4.4}$$

Then by [FŌT94, p.266], $\check{R} = K_0^\mu$ on $L^2(F,\mu)$.

Now set $\check{\mathcal{T}}$ the trace of $\mathcal{E}_1$ on the support of μ and T the positive selfadjoint operator related to $\check{\mathcal{T}}$ via the representation theorem.
Assume further that I_μ is bounded. Then for every $f \in L^2(\mu)$ the signed measure has finite energy integral. Set K_1 its 1-potential w.r.t. to $\mathcal{E}$. From the latter discussion (change $\mathcal{E}$ by $\mathcal{E}_1$) we get that if I_μ is bounded, then the resolvent of T is given by

$$T^{-1} = K_1|_{L^2(F,\mu)} = \big(I_\mu I_\mu^*\big)|_{L^2(F,\mu)}. \tag{4.5}$$

Indeed: Let $f \in L^2$ and $g \in D(\check{\mathcal{T}})$. Then there is $\tilde{g} \in D(\mathcal{E})$ such that $g = \tilde{g}$ μ-a.e. on F. From the definition of K_1 we get

$$\begin{aligned} \check{\mathcal{T}}(K_1|_{L^2(F,\mu)} f, g) \quad &= \mathcal{E}_1(K_1 f, \tilde{g}) = \int_X f\tilde{g}\, d\mu \\ &= \int_F fg\, d\mu, \end{aligned} \tag{4.6}$$

which gives the latter identity, whose use yields the following

Lemma 4.1. *Assume that I_μ is bounded. Then The operator I_μ is compact if and only if the time changed Dirichlet form $\check{\mathcal{T}}$ has discrete spectrum.*

4.2 Sobolev-Orlicz inequalities and compactness of I_μ

Theorem 4.1. *Assume that the measure μ under consideration is finite. If (SOI) holds true with an N-function Φ, then I_μ is compact.*

Proof. First, we claim that if a (SOI) inequality holds for $\mathcal{E}$ with an N-function Φ then I_μ is bounded. Indeed, by [BA04b, Theorem 3.1] showing this claim amounts to show the following capacitary inequality

$$\mu(K) \leq C\mathrm{Cap}(K),\ \forall K \subset X \text{ compact}, \tag{4.7}$$

where C is a positive constant.
Now by Theorem 3.1, for every compact subset $K \subset X$ we have

$$\mu(K) \leq C\big(\Psi^{-1}(1/\mu(K))\big)^{-1}\mathrm{Cap}(K)$$

Further since μ is finite, there is $\rho > 0$ such that $\mu(X \setminus B_\rho) \leq 1$. So that by the latter inequality and that fact that Ψ^{-1} is increasing, we get

$$\mu(K) \leq C\big(\Psi^{-1}(1/\mu(B_\rho))\big)^{-1}\mathrm{Cap}(K),\ \forall K \subset B_\rho$$

and

$$\mu(K) \leq C\big(\Psi^{-1}(1))\big)^{-1}\mathrm{Cap}(K),\ \forall K \subset X \setminus B_\rho,$$

which yields (4.7) and therefore the boundedness of I_μ.
On the other hand, if a (SOI) inequality holds for $\mathcal{E}$ with an N-function, then it also holds true for $\check{\mathcal{T}}$ with the same function. Indeed from the very definition of $\check{\mathcal{T}}$, we get

$$\|f^2\|_{L^\Phi(F,\mu)} \leq C\check{\mathcal{T}}[f],\ \forall f \in D(\check{\mathcal{T}}). \tag{4.8}$$

Now we get the result from Proposition 4.1 and Lemma 4.1.

□

We proceed now to prove that some converse to Theorem 4.1 holds. This was already observed by Cipriani, for $\mu = m$, and the following result extends [Cip00, Theorem 5.1].

Theorem 4.2. *Assume that μ is finite and that I_μ is compact. Then there is an N-function Φ and a constant C such that*

$$\|f^2\|_{L^\Phi(\mu)} \leq C\mathcal{E}_1[f],\ \forall f \in D(\mathcal{E}). \tag{4.9}$$

It follows that if μ is finite then (SOI)$\Leftrightarrow$ compactness of I_μ $\Leftrightarrow$ the $L^2(\mu)$-uniform integrability of the unit ball of $D(\mathcal{E})$. This indeed generalizes partly [GW02, Theorem 1.2].

Proof. The proof is inspired from Cipriani [Cip00].
The key idea is to show that the family

$$\mathcal{F}' := \{f^2 : f \in D(\mathcal{E}),\ \mathcal{E}_1[f] \leq 1\},$$

is uniformly integrable then use Lemma 3.1.
From the assumption, $\mathcal{F}'$ is relatively compact in $L^1(\mu)$. Hence it is $\sigma(L^1(\mu), L^\infty(\mu))$-sequentially compact. It follows, by Dunford-Pettis criterion (see [DS58, Corollary 11 p.294]), that

$$\lim_{\mu(B)\to 0} \int_B f^2\, d\mu = 0, \tag{4.10}$$

uniformly in $f \in \mathcal{F}'$. Let $\lambda > 0$ and $f \in C_c(X) \cap D(\mathcal{E})$ such that $f \geq 0$ and $\mathcal{E}_1[f] \leq 1$. Since I_μ is bounded, making use of [BA04b, Theorem 3.1] together with the 'weak capacitary inequality', we achieve

$$\mu\{f \geq \lambda\} \leq C\,\mathrm{Cap}\{f \geq \lambda\} \leq \frac{C}{\lambda^2}, \tag{4.11}$$

which implies $\lim_{\lambda\to\infty} \mu\{|f| \geq \lambda\} = 0$ uniformly in $f \in \mathbb{B}$, by the regularity of $\mathcal{E}$. Hence by (4.10), we get

$$\lim_{\lambda\to\infty} \int_{\{|f|\geq\lambda\}} f^2\, d\mu = 0, \tag{4.12}$$

uniformly in $f \in \mathbb{B}$. Now using Lemma 3.1 we conclude that there is an N-function Φ such that

$$\sup_{f\in\{D(\mathcal{E}),\ \mathcal{E}_1[f]\leq 1\}} \int_X \Phi(f^2)\, d\mu := C < \infty.$$

Hence by the very definition of $\|\cdot\|_{(\Phi)}$ we get

$$\|f\|_{(\Phi)} \leq C\mathcal{E}_1[f],\ \forall\, f \in D(\mathcal{E}).$$

We finally get the result by recalling that $\|\cdot\|_{(\Phi)}$ is equivalent to $\|\cdot\|_{L^\Phi(\mu)}$.

□

5 Lower bounds for eigenvalues of time changed Dirichlet forms

Let $\check{\mathcal{T}}$ be as in the latter section and assume that a (SOI) inequality holds true for $D(\mathcal{E})$. Then it holds also true for $D(\check{\mathcal{T}})$ (the trace of $\mathcal{E}_1$!). Hence if moreover μ is finite, from Lemma 4.5, we learn that $\check{\mathcal{T}}$ has discrete spectrum. Further the first eigenvalue satisfies the following Faber-Krahn-type inequality

$$\check{\lambda}_1 \geq C(\mu(F)\Phi^{-1}(1/\mu(F)))^{-1} = C\Lambda(\mu(F)). \tag{5.1}$$

We are going to prove that the same type of isoperimetric inequality for the higher eigenvalues also holds true.
We denote by $(\check{\lambda}_k)_{k\geq}1$ the eigenvalues of $\check{\mathcal{T}}$, ordered in an increasing way and by $\check{H}$ the operator associated to $\check{\mathcal{T}}$.

Theorem 5.1. *Under the above assumptions we have*

$$\check{\lambda}_k \geq \frac{1-\epsilon}{2}\Lambda\big(\gamma(2\gamma^{-1}(\frac{2\mu(F)}{k}))\big),\ \forall\, 0<\epsilon<1. \tag{5.2}$$

Proof. By Theorem 3.4 (changing $\mathcal{E}$ by $\mathcal{E}_1$), we conclude that for every $t>0$ the operator $\check{H}_t := e^{-t\check{H}}$ has a kernel p_t^μ which is bounded by $\beta_\epsilon(t/2)$. Whence $\check{H}_t$ is a Hilbert-Schmidt operator and

$$\begin{aligned}\textstyle\sum_{k=1}^\infty e^{-t\lambda_k} &= \|\check{H}_t\|_{HS} = \int_F\int_F \big(p_{t/2}^\mu(x,y)\big)^2\, d\mu(x)\, d\mu(y)\\ &= \int_F p_t^\mu(x,x)\, d\mu(x) \leq \mu(F)\beta_\epsilon(t/2).\end{aligned}$$

Thereby

$$ke^{-t\check{\lambda}_k} \leq \mu(F)\beta_\epsilon(t/2), \tag{5.3}$$

which implies

$$\check{\lambda}_k \geq \frac{1}{t}\log(\frac{k}{\mu(F)\beta_\epsilon(t/2)}). \tag{5.4}$$

Whence changing t by $(1-\epsilon)t$ we get

$$\check{\lambda}_k \geq \frac{1-\epsilon}{t}\log\big(\frac{k\epsilon\gamma(t)}{2\mu(F)}\big)\ \forall\, t>0. \tag{5.5}$$

Choosing t such that $2\mu(F) = k\epsilon\gamma(t/2)$, we conclude that there is $\theta\in(t/2,t)$ such that

$$\check{\lambda}_k \geq \frac{(1-\epsilon)}{2}\Lambda(\gamma(\theta)). \tag{5.6}$$

Now we get the result by observing that γ increases and using property (3.9).

□

6 Two applications

6.1 Ultracontractivity and spectra for traces of s-stable processes on semi (d,Γ)-sets

As an illustration, we shall use the above obtained results to derive some spectral properties of the generator of the s-stable process over *d-sets*. The novelty here (compared to

Triebel [Tri01, Theorem 23.2 p.349]) is that we do no more assume that the d-measure related to the d-set under consideration has compact support. We shall only assume that it is finite.

Here are some basic notions that we shall use in the sequel (see [Tri01, p.334]): A closed subset $F \subset \mathbb{R}^d$ is called a *semi* (d,Γ)*-set*, where $0 < d \le n$, if there is a positive measure μ supported by F, a real-valued function Γ on $(0,1]$, positive constants $C > 0$, $C_1 \ge 0$ and $0 < C_2 < 1$ and a real number b such that:

i)

$$\mu\big(B_r(x)\big) \le C r^d \Gamma(r),\ \forall\, x \in F,\ r \in (0,1]. \tag{6.1}$$

ii) The function Γ is such that

$$\Gamma(r) \le C_1 |\log(C_2 r)|^b,\ \forall\, r \in (0,1]. \tag{6.2}$$

The measure μ is called a *semi* (d,Γ)*-measure*.
If $b = 0$ we say that μ is a *semi* d*-measure* and if moreover

$$\mu\big(B_r(x)\big) \ge C^{-1} r^d,\ \forall\, x \in F,\ r \in (0,1], \tag{6.3}$$

we say that μ is a d*-measure*.

By [Tri01, Proposition22.8], for every pair (d,Γ) satisfying the above conditions there is a semi (d,Γ)-set.

Example 6.1. 1) The restriction of Lebesgue measure on a closed subset of $\mathbb{R}^n$ is a semi n-measure.
2) The d-dimensional Hausdorff measure of a semi d-set is a semi d-measure.

From now on, we fix s such that $0 < s \le 1$ and we designate by $\mathcal{E}^{(s)}$ the following Dirichlet form:

$$D(\mathcal{E}^{(s)}) = \big\{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\hat f(x)|^2 |x|^{2s}\, dx < \infty\big\},\ \mathcal{E}^{(s)}(f,g) = \int_{\mathbb{R}^d} \hat f(x)\overline{\hat g(x)}|x|^{2s}\, dx.$$

It is known that $\mathcal{E}^{(s)}$ is a Dirichlet form whose domain the space of Bessel potentials $L^{s,2}(\mathbb{R}^d)$ and is related to the operator $(-\Delta)^s$:

$$D((-\Delta)^s) = L^{2s,2}(\mathbb{R}^d),\ \mathcal{E}^{(s)}(f,g) = ((-\Delta)^s f, g), \forall\, f \in L^{2s,2}(\mathbb{R}^d),\, g \in D(\mathcal{E}^{(s)}) = L^{s,2}(\mathbb{R}^d).$$

From now on we assume that μ is a semi (d,Γ)-measure such that $n - 2s < d \le n$.
For semi d-measures the following result is due to D.R. Adams (see [JW84, p.214].

Lemma 6.1. *Let μ be a semi (d,Γ)-measure. Then*

i) For $n > 2s$ and $2 < p < \frac{2d}{n-2s}$ ($p \le \frac{2d}{n-2s}$ if $b = 0$), the following inequality holds true

$$\Big(\int_X |f|^p\, d\mu\Big)^{2/p} \le C\mathcal{E}_1^{(s)}[f],\ \forall\, f \in D(\mathcal{E}^{(s)}). \tag{6.4}$$

ii) For $n \le 2s$ and $2 < p < \infty$ ($p = \infty$ if $n < 2s$) the following inequality holds true

$$\Big(\int_X |f|^p\,d\mu\Big)^{2/p} \le C\,\mathcal{E}_1^{(s)}[f],\ \forall\, f \in D(\mathcal{E}^{(s)}). \tag{6.5}$$

Proof. We give the proof only for $b \neq 0$. For the other case it runs in the same way.
For $2 < p < \infty$, by Theorem 3.1, we have to show that there is a constant C such that

$$(\mu(K))^{1/p} \le C(\mathrm{Cap}(K))^{1/2},\ \text{for all } K \text{ compact}. \tag{6.6}$$

Let G be the kernel of $((-\Delta)^s + 1)^{-1}$. Then for every compact subset having positive capacity we have

$$\frac{\mu(K)}{\mathrm{Cap}(K)} \le \sup_{x\in\mathbb{R}^n} \int_K G(x,y)\,d\mu(y). \tag{6.7}$$

We shall consider the different cases separately.
i) The case $n < 2s$.
For $p = \infty$, it is known that the embedding I_{dx}, where dx refers to Lebesgue measure, is bounded. Since dx-essentially bounded functions are bounded q.e. and since μ does not charge sets having zero capacity, we conclude that I_μ is bounded as well.
Now assume that $p < \infty$. In this situation it is known that the kernel G is continuous and bounded and that points have positive capacity. Hence there is $A > 0$ such that for every nonempty compact subset K we have $\mathrm{Cap}(K) \ge A$.
Let K be a compact subset with positive capacity. Thanks to the latter observations and inequality (6.7), we get

$$\frac{(\mu(K))^{1/p}}{(\mathrm{Cap}(K))^{1/2}} \le A^{1/p-1/2}\Big(\frac{\mu(K)}{\mathrm{Cap}(K)}\Big)^{1/p} \le A^{1/p-1/2}\Big(\sup_{x\in\mathbb{R}^n}\int_K G(x,y)\,d\mu(y)\Big)^{1/p}. \tag{6.8}$$

On the other hand, it is not difficult to realize (see [BA04a, p.14]) that there is a constant $C(n)$, depending only on n, such that

$$\sup_{x\in\mathbb{R}^n}\int_{B_x(1)^c} G(x,y)\,d\mu(y) \le C(n)\sup_{x\in\mathbb{R}^n}\int_{B_x(1)} G(x,y)\,d\mu(y). \tag{6.9}$$

Thus

$$\begin{aligned}\frac{\mu(K)}{\mathrm{Cap}(K)} \ &\le\ (1 + C(n))\sup_{x\in\mathbb{R}^n}\int_{B_x(1)} G(x,y)\,d\mu(y)\\ &\le \sup_{x,y} G(x,y)CC_1\Gamma(1) < \infty,\end{aligned}$$

yielding inequality (6.6).
The case $n > 2s$. The proof runs substantially as the latter case. Recalling that $G(x,y) \le A|x-y|^{2s-n}$, we are lead an other time to show that

$$\sup_{x\in\mathbb{R}^n}\int_{B_x(1)} |x-y|^{q(2s-n)}\,d\mu(y) < \infty,$$

for $1 < q < \frac{d}{n-2s}$.
Let q be such a number. Observe that for $0 < t < 1$ we have

$$B(t) := t^{q(2s-n)}\mu(B_x(t)) \le CC_1 t^{d-q(n-2)}\big(\log(C_2 t)\big)^b,$$

so that by assumption on q, $\lim_{t\to 0} B(t) = 0$. Hence a routine computation yields

$$\begin{aligned}\int_{B_x(1)} |x-y|^{q(2s-n)}\,d\mu(y) &= \mu(B_x(1)) + q(n-2s)\int_0^1 t^{q(2s-n)-1}\mu(B_x(t))\,dt\\ &\le C\Gamma(1) + CC_1\int_0^1 t^{d-1-q(n-2s)}|\log(C_2 t)|^b\,dt.\end{aligned}$$

As a final step we have to prove that the latter integral is finite. Clearly it is, if $b \le 0$. On the other hand iterating the integration by parts an other time, we realize that we are lead to prove the claim for $0 < b \le 1$. For such b, $|\log C_2 t|^b \le |\log C_2 t|$. So that it suffices to make the proof for $b = 1$.

$$\int_0^1 t^{d-1-q(n-2s)}|\log(C_2 t)|\,dt = C'|\log(C_2)| + C''\int_0^1 t^{d-1-q(n-2s)}\,dt < \infty, \tag{6.10}$$

by assumption on q.
The case $n = 2s$. Let $q > 1$ and q' its conjugate exponent. Making use of (6.6) together with Hölder inequality and replacing G by G^q in the inequality (6.9), we obtain

$$\begin{aligned}\frac{(\mu(K))^{1/2q}}{(\mathrm{Cap}(K))^{1/2}} &\le \sup_{x\in\mathbb{R}^n}\Big(\int_K G^q(x,y)\,d\mu(y)\Big)^{1/2q}\\ &\le (1+C(n))^{1/2q}\sup_{x\in\mathbb{R}^n}\Big(\int_{B_x(1)} G^q(x,y)\,d\mu(y)\Big)^{1/2q}.\end{aligned}$$

So that we are lead to show the finiteness of the latter integral. We recall that there is a constant A such that

$$G(x,y) \le A\log(2/|x-y|),\ \text{for } |x-y| \le 1. \tag{6.11}$$

Now using Fubini theorem, integrating by parts and taking the the properties of the measure μ into account we achieve

$$\int_{B_x(1)} (\log(2/|x-y|))^q\,d\mu(y) = \log(2)^q\mu(B_x(1)) + qC'\int_0^1 t^{d-1}|\log t|^{q-1}|\log(C_2 t)|^b\,dt.$$

The rest of the proof runs now in a similar way as in final step of the latter case, getting the sought result, which finishes the proof. □

Theorem 6.1. *Let μ be a semi (d,Γ)-measure with support F such that $n-2s < d \le n$. Set $\check{\mathcal{T}}$ the trace of $\mathcal{E}_1^{(s)}$ on $L^2(F,\mu)$ and $\check{p}_t$ the kernel of the related semigroup. Let $p > 2$ be as given in the latter lemma. Then*

$$\check{p}_t \le C\frac{1}{t^q},\ \forall\, t > 0, \tag{6.12}$$

where q is the conjugate of $p/2$.
If further μ is finite, then $\check{\mathcal{T}}$ has discrete spectrum. Set $(\check{\lambda}_k)_k$ its eigenvalues, ordered in an increasing way. Then the following inequalities hold true:

$$\check{\lambda}_k \geq Ck^{\frac{p-2}{p}},\ \forall\, k \geq 1. \tag{6.13}$$

Proof. By Lemma 6.1, we have a (SOI) with $\Psi(t) = p^{-1}t^p,\ 2 < p < \infty$. Further by Lemma 4.1, I_μ is compact so that by Lemma 4.1, $\check{\mathcal{T}}$ has discrete spectrum.
For $p < \infty$, the rest of the proof is obtained by a straightforward application of Theorem 3.4 and Theorem 5.1.
For $n < 2s$ and $p = \infty$ we get the proof by passing to the limit ($p \to \infty$) and by observing that the resulting constant is finite (which follows from the first step in the proof of Lemma 6.1).

□

6.2 Spectra of certain Dirichlet forms of jump-type on Besov spaces over d-sets

We pursue to work in the same setting. We assume that $n = 2s$, and that μ is a d-measure. Set

$$\Phi(t) := e^t - 1.$$

Then, by a result due to Mazja [Maz81, Folgerung6.2, p.46], the space $L^{s,2}(\mathbb{R}^n)$ is embedded into the Orlicz space $L_\Phi(\mu)$: There is a constant C such that

$$\|f^2\|_{L_\Phi(\mu)} \leq C\mathcal{E}_1[f],\ \forall f \in L^{s,2}(\mathbb{R}^n). \tag{6.14}$$

Let F be the d-set related to μ and Δ its diagonal. Define δ by $\delta := s + \frac{d}{2} = \frac{n+d}{2}$.
Let $\mathcal{B}$ be the form defined by

$$\mathcal{B}[f] := \int_{F\times F\setminus\Delta} \frac{(f(x)-f(y))^2}{|x-y|^{d+2\delta}}\, d\mu(x)\, d\mu(y),\ \ D(\mathcal{B}) := \{f \in L^2(F,\mu) : \mathcal{B}[f] < \infty\}.$$

Then $\mathcal{B}$ is a regular Dirichlet form on $L^2(F,\mu)$ (see [FU03b]) and its domain is the Besov space $B_\delta^{2,2}(F)$. Combining inequality (6.14) with the Jonsson-Wallin trace theorem [JW84, Theorem 1, p.141] which asserts that

$$B_\delta^{2,2}(F) = L^{s,2}(\mathbb{R}^n)_{|F}$$

and that both the restriction and extension operators are continuous, we obtain

$$\|f^2\|_{L_\Phi(\mu)} \leq C\mathcal{B}_1[f],\ \forall f \in B_\delta^{2,2}(F), \tag{6.15}$$

where C is a positive constant.
We would like to mention that Fukushima-Uemura [FU03a] proved that for $n > 2s$ and $\delta < \frac{d}{2}$ then a Sobolev-type inequality holds true for $B_\delta^{2,2}(F)$. So that the latter result

completes the small gap in the latter mentioned work.

Whence making use of Theorem 3.4, we conclude that the semigroup of the operator related to $\mathcal{B}$ has a kernel p_t^μ and by Proposition 4.1 it has a discrete spectrum provided μ is finite.
The Λ corresponding to Φ is given by

$$\Lambda(s) = \frac{1}{s\log(1/s+1)}, \tag{6.16}$$

so that the corresponding γ is given by the identity

$$t = \kappa_\mu \gamma(t) \log\big(\frac{1+\gamma(t)}{\gamma(t)}\big) + \kappa_\mu \log(1+\gamma(t)),\ t>0. \tag{6.17}$$

The lower bounds for the eigenvalues are described in the following

Theorem 6.2. *Let μ be a finite d-measure. Then $\mathcal{B}$ has discrete spectrum. Set $\lambda_1 < \lambda_2 ... \leq \lambda_k \leq ...$ the eigenvalues of $\mathcal{B}_1$. Then*

$$\lambda_k \geq C\frac{(1-\epsilon)}{2\mu(F)}\frac{k}{\log(\frac{k}{2\mu(F)}+1)},\ \forall\, 0<\epsilon<1,$$

where C is positive constant.

Proof. Making use of Theorem 5.1 we get

$$\lambda_k \geq \frac{1-\epsilon}{2}\Lambda\big(\gamma(2\gamma^{-1}(\frac{2\mu(F)}{k}))\big). \tag{6.18}$$

Hence, owing to an idea of Grigor'yan, we get the result if we show that the function $\Lambda o \gamma$ has at most polynomial decay i.e.,

$$\frac{\Lambda(\gamma(2t))}{\Lambda(\gamma(t))} \geq \delta > 0,\ \forall t>0. \tag{6.19}$$

We proceed now to prove the latter claim. For every $t>0$, set $Q(t) := \frac{\Lambda(\gamma(2t))}{\Lambda(\gamma(t))}$. By the properties of λ and γ, we have $Q(t)>0$ for every $t>0$. On the other hand we have

$$Q(t) = \frac{\gamma(t)\log(1+1/\gamma(t))}{\gamma(2t)\log(1+1/\gamma(2t))}.$$

Since $\gamma^{-1}(t)\to\infty$ as $t\to\infty$, γ also does. So that $\lim_{t\to\infty} Q(t) = 1$. Further we have

$$\frac{\gamma^{-1}(t)}{t} = \kappa_\mu \log(1+1/t) + \kappa_\mu \frac{\log(1+t)}{t},$$

which implies $\lim_{t\to 0}\gamma(t)/t = 0$. Thereby

$$\lim_{t\to 0} Q(t) = \lim_{t\to 0}\frac{t-\kappa_\mu\log(1+\gamma(t))}{2t-\kappa_\mu\log(1+\gamma(2t))} = 1/2,$$

which yields (6.19) and the proof is finished.

□

References

[AH96] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory.* Springer-Verlag, Berlin, 1996.

[BA04a] Ali Ben Amor. Invariance of essential spectra for generalized Schrödinger operators. *Math. Phys. Electron. J.*, 10(7):18 pp., 2004.

[BA04b] Ali Ben Amor. Trace inequalities for operators associated to regular Dirichlet forms. *Forum Math.*, 16(3):417–429, 2004.

[BA05] Ali Ben Amor. On the equivalence between trace and capacitary inequalities for the abstract contractive space of Bessel potentials. *Osaka J. Math.*, 42(1), 2005.

[BCLSC95] D. Bakry, T. Coulhon, M. Ledoux, and L. Saloff-Coste. Sobolev inequalities in disguise. *Indiana Univ. Math. J.*, 44(4):1032–1074, 1995.

[Bra01] J. F. Brasche. Upper bounds for Neumann-Schatten norms. *Potential Anal.*, 14(2):175–205, 2001.

[Car97] Gilles Carron. Inégalités de Faber-Krahn et inclusion de Sobolev-Orlicz. *Potential Anal.*, 7(2):555–575, 1997.

[Cip00] Fabio Cipriani. Sobolev-Orlicz imbeddings, weak compactness, and spectrum. *J. Funct. Anal.*, 177(1):89–106, 2000.

[Cou96] Thierry Coulhon. Ultracontractivity and Nash type inequalities. *J. Funct. Anal.*, 141(2):510–539, 1996.

[DM75] Claude Dellacherie and Paul-Andre Meyer. *Probabilités et potentiel. Chap. I à IV.* Publications de l'Institut de Mathematique de l'Universite de Strasbourg. XV. Actualites scientifiques et industrielles. 1372. Paris: Herman., 1975.

[DS58] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General theory.* (Pure and Applied Mathematics. Vol. 6) New York and London: Interscience Publishers., 1958.

[FŌT94] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes.* Walter de Gruyter & Co., Berlin, 1994.

[FU03a] Masatoshi Fukushima and Toshihiro Uemura. Capacitary bounds of measures and ultracontractivity of time changed processes. *J.Math.Pures Appl.*, 82:553–572, 2003.

[FU03b] Masatoshi Fukushima and Toshihiro Uemura. On Sobolev and capacitary inequalities for contractive Besov spaces over d-sets. *Potential Anal.*, 18(1):59–77, 2003.

[Gri99] Alexander Grigor'yan. Estimates of heat kernels on Riemannian manifolds. In *Davies, Brian (ed.) et al., Spectral theory and geometry. Proceedings of the ICMS instructional conference, Edinburgh, UK, 30 March-9 April, 1998. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 273, 140-225* . 1999.

[GW02] Fuzhou Gong and Fengyu Wang. Functional inequalities for uniformly integrable semigroups and application to essential spectrums. *Forum Math.*, 14(2):293–313, 2002.

[GY03] Alexander Grigor'yan and Shing-Tung Yau. Isoperimetric properties of higher eigenvalues of elliptic operators. *Am. J. Math.*, 125(4):893–940, 2003.

[JW84] Alf Jonsson and Hans Wallin. Function spaces on subsets of R^n. Technical report, Math. Rep. Ser. 2, No.1. , 1984.

[Kai92] Vadim A. Kaimanovich. Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators. *Potential Anal.*, 1(1):61–82, 1992.

[Maz81] Wladimir Mazja. *Zur Theorie Sobolewscher Räume.* Teubner-Texte zur Mathematik, 38. Leipzig: BSB B. G. Teubner Verlagsgesellschaft., 1981.

[Maz85] Vladimir G. Maz'ja. *Sobolev spaces.* Springer-Verlag, Berlin, 1985.

[RR91] M.M. Rao and Z.D. Ren. *Theory of Orlicz spaces.* Pure and Applied Mathematics, 146. New York etc.: Marcel Dekker, Inc., 1991.

[Sto94] P. Stollmann. Störungstheorie von Dirichletformen mit Anwendung auf Schrödingeroperatoren. Habilitationsschrift. Frankfurt, 1994.

[Tre67] François Treves. *Topological vector spaces, distributions and kernels.* New York-London: Academic Press, 1967.

[Tri01] Hans Triebel. *The structure of functions.* Monographs in Mathematics. 97. Basel: Birkhäuser, 2001.

[Var85] N.Th. Varopoulos. Hardy-Littlewood theory for semigroups. *J. Funct. Anal.*, 63:240–260, 1985.