

ELLIPTIC EQUATIONS FOR MEASURES: REGULARITY AND GLOBAL BOUNDS OF DENSITIES

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ABSTRACT. We consider elliptic equations of the form $L^*\mu = \nu$ for measures on \mathbb{R}^n . The membership of solutions in the Sobolev classes $W^{p,1}(\mathbb{R}^n)$ is established under appropriate conditions on the coefficients of L . Bounds of the form $\varrho(x) \leq C\Phi(x)^{-1}$ for the corresponding densities are obtained.

1. INTRODUCTION

In recent years there has been a growing interest in the elliptic equations

$$L^*\mu = 0 \tag{1.1}$$

for measures (see [1]–[12], [16], [17], where one can find additional references). The interpretation of such an equation is the following: given an elliptic operator L of the form

$$Lf = L_{A,b}f := \sum_{i,j \geq 1} a^{ij} \partial_{x_i} \partial_{x_j} f + \sum_{i=1}^n b^i \partial_{x_i} f, \tag{1.2}$$

where $x \mapsto A(x) = (a^{ij}(x))$ is a Borel mapping with values in the space of positive symmetric matrices and $x \mapsto b(x) = (b^i(x))$ is a Borel measurable vector field, we say that a measure μ on \mathbb{R}^n is a solution to (1.1) if $L\varphi \in L^1(|\mu|)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and one has

$$\int L\varphi d\mu = 0. \tag{1.3}$$

One can also consider divergence form operators

$$Lf = \mathcal{L}_{A,b}f := \sum_{i,j \geq 1} \partial_{x_i} (a^{ij} \partial_{x_j} f) + \sum_{i=1}^n b^i \partial_{x_i} f. \tag{1.4}$$

For divergence form operators with $a^{ij} \in W_{loc}^{1,1}$ equation (1.1) is equivalent to the equation

$$\operatorname{div} (A\nabla\mu - b\mu) = 0$$

in the sense of distributions. We write

$$L_A := L_{A,0}, \quad \mathcal{L}_A := \mathcal{L}_{A,0}.$$

Let $W^{p,r}(\mathbb{R}^n)$ denote the Sobolev class of functions that belong to $L^p(\mathbb{R}^n)$ along with their generalized partial derivatives of order up to r . The class $W_{loc}^{p,r}$ consists of all functions f such that $\zeta f \in W^{p,r}(\mathbb{R}^n)$ for every $\zeta \in C_0^\infty(\mathbb{R}^n)$. The dual to $W^{p,1}(\mathbb{R}^n)$ is denoted by $W^{p',-1}(\mathbb{R}^n)$, $p' := p(p-1)^{-1}$. Analogous notation is used for classes on a domain.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the class of all bounded Borel measures on \mathbb{R}^n (possibly signed) and let $\mathcal{P}(\mathbb{R}^n)$ be the subclass of all probability measures.

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Throughout $B(z, r)$ denotes the open ball of radius r centered at z and $\text{Tr}A$ is the trace of the matrix A .

It has been shown in [4], among other things, that if A is locally Hölder continuous and nondegenerate, then μ has a density $\varrho \in L^r_{loc}(\mathbb{R}^n)$ for any $r < \frac{n}{n-1}$. If, also the local condition

$$a^{ij} \in W^{1,\alpha}_{loc} \quad \text{and either } |b| \in L^\alpha_{loc}(dx) \text{ or } |b| \in L^\alpha_{loc}(|\mu|) \text{ where } \alpha > n, \quad (1.5)$$

is fulfilled, then μ has a continuous density $\varrho \in W^{1,\alpha}_{loc}$. Under suitable global assumptions, it has recently been shown in [16] that $\varrho \in W^{\alpha,1}(\mathbb{R}^n)$, in particular, the continuous version of ϱ is uniformly bounded. Furthermore, combining this result with certain estimates from [5], based on Lyapunov functions techniques, useful global bounds for ϱ have been obtained in [16]. In this paper, we extend these important results from [16] by refining the methods of Metafuno, Pallara, and Rhandi and developing some new tools. There are two main ingredients in these extensions: we derive a new elliptic regularity result for the operators $L_{A,0}$ and $\mathcal{L}_{A,0}$ and consider non-homogeneous equations

$$L^* \mu = \nu \quad (1.6)$$

with the right-hand side in $W^{p,-1}$ and $L = L_{A,b}$ or $L = \mathcal{L}_{A,b}$. By definition (1.6) means that $L\varphi \in L^1(|\mu|)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and

$$\int L\varphi d\mu = \nu(\varphi).$$

As compared to [16], we require lower local regularity of A and no boundedness of A is needed. Furthermore, we consider signed measures. The main results are Theorems 2.1, 2.7, and 3.1.

Our global elliptic regularity result employs the following uniform local condition on A . If $a^{ij} \in W^{1,1}_{loc}$, we set

$$\Theta_A(x) := \sum_{j=1}^n \left| \sum_{i=1}^n \partial_{x_i} a^{ij}(x) \right|.$$

Given $p > 1$ and $\gamma > 0$, we set

$$q = q(n, p, \gamma) = \begin{cases} n & \text{if } p > n/(n-1), \\ n + \gamma & \text{if } p = n/(n-1), \\ p' = p/(p-1) (> n) & \text{if } p < n/(n-1). \end{cases}$$

We say that A satisfies condition (C1) for $p > 1$ if $a^{ij} \in W^{1,1}_{loc}$ and

$$\limsup_{r \rightarrow 0} \sup_{z \in \mathbb{R}^n} \int_{B(z,r)} \Theta_A^q(x) dx = 0, \quad (C1)$$

where q is as defined above (in the case $p = n/(n-1)$ equality (C1) must be fulfilled with $q = n + \gamma$ for some $\gamma > 0$).

It is clear that if there is $p_0 > n > 1$ such that

$$\sup_{z \in \mathbb{R}^n} \int_{B(z,1)} \sum_{i,j} |\nabla a^{ij}(x)|^{p_0} dx < \infty, \quad (1.7)$$

then A satisfies condition (C1) for $p < p_0$ and is uniformly continuous (even uniformly Hölder continuous) on all of \mathbb{R}^n . In particular, both properties hold if A is uniformly Lipschitzian.

It is worth noting that although in most of our results we assume that $a^{ij} \in W_{loc}^{p,1}$, hence one can write $\mathcal{L}_{A,b}$ as L_{A,b_0} with $b_0^i := b^i + \partial_{x_j} a^{ij}$, the case of $\mathcal{L}_{A,b}$ does not always reduce to that of A , because the global integrability assumptions on $|b|$ and $|\nabla a^{ij}|$ are different. In some situations, it is easier to deal with divergence form operators, in others the standard form is more convenient. In the manifold case, usually divergence form operators lead to more natural geometric objects. Apparently, the most natural setting for most of the problems discussed would appeal to the geometry related to A and weighted Sobolev spaces. However, the corresponding techniques, in particular, embedding theorems, is less developed than the classical Sobolev theory. Furthermore, similarly to our work [4], analogous parabolic problems can be considered. Finally, we would like to mention that one of the motivations of this paper is to contribute to the development of a theory to analyze partial differential operators L on L^p -spaces with respect to a measure satisfying (1.1). Such measures are intrinsically related to L and, therefore, more appropriate than Lebesgue measure for the analysis of L .

2. GLOBAL SOBOLEV CLASS MEMBERSHIP OF DENSITIES

We recall that according to the Sobolev embedding theorem, for any $p > \frac{n}{n-1}$, there is a constant $C(n, p)$ such that

$$\|\varphi\|_{p,-1} \leq C(n, p) \|\varphi\|_{L^s(\mathbb{R}^n)}, \quad s = \frac{pn}{p+n} > 1, \quad (2.1)$$

for all $\varphi \in L^s(\mathbb{R}^n)$. If $1 < p < \frac{n}{n-1}$, then

$$\|\varphi\|_{p,-1} \leq C(n, p) \|\varphi\|_{L^1(\mathbb{R}^n)}. \quad (2.2)$$

In the case $p = \frac{n}{n-1}$, that is $p' = n$, for any $r > n$ one has $\kappa := n/r \in (0, 1)$, i.e.,

$$1 - \frac{n}{p'} = \kappa - \frac{n}{r},$$

and by the Sobolev embedding theorem for fractional classes one obtains

$$W^{p',1}(\mathbb{R}^n) \subset H^{r,\kappa}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n).$$

Therefore, $L^{r'}(\mathbb{R}^n) \subset W^{p,-1}(\mathbb{R}^n)$ with

$$\|\varphi\|_{p,-1} \leq C(n, n', r') \|\varphi\|_{L^{r'}(\mathbb{R}^n)}. \quad (2.3)$$

Theorem 2.1. *Let $\varepsilon \cdot I \leq A(x)$ for some $\varepsilon > 0$. Let $p > 1$ and let $u \in W_{loc}^{p,1}(\mathbb{R}^n)$ be such that $u \operatorname{Tr} A \in L^p(\mathbb{R}^n)$ and $L_A u \in W^{p,-1}(\mathbb{R}^n)$. Suppose that A satisfies condition (C1) for p with $q = q(n, p)$ as defined above and that the functions $a^{ij} \in W_{loc}^{q,1}(\mathbb{R}^n)$ are uniformly continuous. Then $u \in W^{p,1}(\mathbb{R}^n)$. The same is true in the case of \mathcal{L}_A .*

Proof. It is worth noting that under our assumptions $L_A u$ is well-defined as an element of $W_{loc}^{p,-1}(\mathbb{R}^n)$, hence our hypothesis that $L_A u \in W^{p,-1}(\mathbb{R}^n)$ makes sense. Indeed, one has $\partial_{x_i} \partial_{x_j} u \in W_{loc}^{p,-1}(\mathbb{R}^n)$. In addition, a^{ij} is a multiplier of the class $W_{loc}^{p,-1}(\mathbb{R}^n)$, since, given $\varphi \in W_0^{p',1}(\mathbb{R}^n)$, one has $a^{ij} |\nabla \varphi| \in L^{p'}(\mathbb{R}^n)$ and $\varphi |\nabla a^{ij}| \in L^{p'}(\mathbb{R}^n)$. The latter inclusion follows by the Sobolev embedding theorem. Indeed, if $p > n/(n-1)$, then $p' < n$ and $\varphi \in L^{p'n/(n-p')}(\mathbb{R}^n)$, hence by Hölder's inequality with exponent $r = n/(n-p')$ (so that $r' = n/p'$) and our choice of q one obtains the integrability of $|\varphi|^{p'} |\nabla a^{ij}|^{p'}$. The cases $p = n/(n-1)$ and $p < n/(n-1)$ are similar.

Without loss of generality we may assume that $2\varepsilon < C_0$, where C_0 is chosen as follows: if A_0 is a symmetric matrix with $2^{-1}\varepsilon \cdot I \leq A_0 \leq 2\varepsilon \cdot I$, then

$$\|\psi\|_{p,1} \leq C_0 \|L_{A_0} \psi\|_{p,-1}$$

for every $\psi \in W_0^{p,1}(\mathbb{R}^n)$ having support in a ball of radius 1.

Let $\delta \in (0, \varepsilon/2)$ be so small that $1 - (C(n, p) + 1)C_0\delta > 1/2$. Let $r \in (0, 1/2)$ be such that $\|A(x) - A(y)\| < \delta$ whenever $|x - y| < 2r$ and

$$\int_{B(z, 2r)} \Theta_A(x)^q dx \leq \delta^q \quad \text{for all } z \in \mathbb{R}^n.$$

Let us show that there is $C > 0$ such that whenever $v \in W_0^{p,1}(\mathbb{R}^n)$ has support in a cube B of diameter $2r$, then

$$\|v\|_{p,1} \leq C\|L_A v\|_{p,-1}. \quad (2.4)$$

Let $A_0 = A(x_0)$, where x_0 is the center of B . Then $L_{A_0}v = L_A v + L_{A_0-A}v$, hence

$$\|v\|_{p,1} \leq C_0\|L_A v\|_{p,-1} + C_0\|L_{A_0-A}v\|_{p,-1}.$$

Let us estimate the last term on the right. Given $\varphi \in W^{p',1}(\mathbb{R}^n)$, we have

$$\int \varphi L_{A_0-A}v dx = - \int ((A_0 - A)\nabla v, \nabla \varphi) dx - \int \partial_{x_i}(a_0^{ij} - a^{ij})\partial_{x_j}v \varphi dx.$$

The first integral on the right is majorized by $\delta\|\nabla v\|_{L^p(B)}\|\nabla \varphi\|_{L^{p'}(B)}$. The second one is estimated by

$$\|\varphi\|_{p',1}\|\Theta_A|\nabla v|\|_{p,-1}.$$

By (2.1) and Hölder's inequality for $p > n/(n-1)$ one has

$$\|\nabla v|\Theta_A|\|_{p,-1} \leq C(n, p)\|\nabla v|\Theta_A|\|_{L^s(\mathbb{R}^n)} \leq C(n, p)\|\Theta_A\|_{L^n(B)}\|v\|_{p,1} \leq \delta C(n, p)\|v\|_{p,1}.$$

Hence

$$\|L_{A_0-A}v\|_{p,-1} \leq \delta(1 + C(n, p))\|v\|_{p,1}.$$

This proves (2.4) with $C = 2C_0$ for $p > n/(n-1)$.

If $p < n/(n-1)$, then $p' > n$ and by (2.2) and Hölder's inequality

$$\|\Theta_A|\nabla v|\|_{p,-1} \leq C(n, p)\|\Theta_A|\nabla v|\|_{L^1(\mathbb{R}^n)} \leq C(n, p)\|v\|_{p,1}\|\Theta_A\|_{L^{p'}(B)} \leq \delta C(n, p)\|v\|_{p,1},$$

which again leads to (2.4). Finally, if $p = n/(n-1)$, then we apply (2.3) with any $r > n$. For r' sufficiently close to 1 we can use Hölder's inequality and obtain (2.4).

It is easily verified that \mathbb{R}^n can be covered by finitely many collections $\mathcal{F}_1, \dots, \mathcal{F}_N$ of cubes with the following property: each \mathcal{F}_i consists of a sequence of cubes K_j of diameter r such that the cubes B_j with the same centers and twice bigger diameters are disjoint. Let us fix one of these collections and take functions $\zeta_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_j \leq 1, \quad \sup_{j,x} \left[|\nabla \zeta_j(x)| + \|D^2 \zeta_j(x)\| \right] := M < \infty, \quad \text{supp} \zeta_j \subset B_j, \quad \zeta_j|_{K_j} = 1.$$

We have

$$L_A(\zeta_j u) = \zeta_j L_A u + 2\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j u) - 2u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j) - uL_A \zeta_j.$$

Therefore,

$$\begin{aligned} \|\zeta_j u\|_{p,1} \leq C & \left[\|\zeta_j L_A u\|_{p,-1} \right. \\ & \left. + 2M \sum_{i,k=1}^n \|a^{ik}u\|_{L^p(B_j)} + 2\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{p,-1} + \|uL_A \zeta_j\|_{p,-1} \right]. \quad (2.5) \end{aligned}$$

Let us estimate separately each term on the right. The second term is estimated as follows:

$$2M \sum_{i,k=1}^n \|a^{ik}u\|_{L^p(B_j)} \leq 2Mn^2 \|u \operatorname{Tr} A\|_{L^p(B_j)}.$$

For the last term one has

$$\|uL_A\zeta_j\|_{p,-1} \leq \|uL_A\zeta_j\|_{L^p(B_j)} \leq n^2M \|u \operatorname{Tr} A\|_{L^p(B_j)}.$$

Let us consider the third term. If $n/(n-1) < p$, one has

$$2\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{p,-1} \leq 2C(p,n)\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{L^s(B_j)}$$

with $s = pn/(p+n)$. By Hölder's inequality with the exponents $(p+n)/n$ and $(p+n)/p$ and condition (C1) we find

$$\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{L^s(B_j)} \leq \|u\|_{L^p(B_j)} \|\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{L^n(B_j)} \leq \text{const}\|u\|_{L^p(B_j)}.$$

If $1 < p < n/(n-1)$, we have $p' > n$, hence $W^{p',1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. In addition, $q = p'$, so that $\|\Theta_A\|_{L^{p'}(B_j)} \leq M_1 < \infty$. Note also that

$$\|u\|_{L^p(B_j)} \leq \varepsilon^{-1} \|u \operatorname{Tr} A\|_{L^p(B_j)}.$$

Therefore,

$$\begin{aligned} \|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{p,-1} &\leq C(n,p)\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{L^1(B_j)} \\ &\leq C(n,p)\|uL_A\zeta_j\|_{L^1(B_j)} + C(n,p)\|u\partial_{x_k}a^{ik}\partial_{x_i}\zeta_j\|_{L^1(B_j)} \\ &\leq Mn^2C(n,p)\|u \operatorname{Tr} A\|_{L^p(B_j)} + MC(n,p)\|u\|_{L^p(B_j)}\|\Theta_A\|_{L^{p'}(B_j)} \\ &\leq (Mn^2C(n,p) + MM_1\varepsilon^{-1})\|u \operatorname{Tr} A\|_{L^p(B_j)}. \end{aligned}$$

In the remaining case $p = n/(n-1)$ we apply (2.3) and find

$$\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{p,-1} \leq C(n,n',r')\|u\partial_{x_k}(a^{ik}\partial_{x_i}\zeta_j)\|_{L^{r'}(\mathbb{R}^n)}.$$

We choose r' sufficiently close to 1 and estimate the right-hand side by Hölder's inequality through $\text{const}\|u\|_{L^p(B_j)}$, which is possible, since $q = n + \gamma > n$ and the quantities $\|\Theta_A\|_{L^q(B_j)}$ are uniformly bounded.

Taking into account that $\zeta_j|_{K_j} = 1$, we obtain from (2.5) and the foregoing estimates

$$\|\nabla u\|_{L^p(K_j)}^p \leq C_2 [\|\zeta_j L_A u\|_{p,-1}^p + \|u \operatorname{Tr} A\|_{L^p(B_j)}^p]. \quad (2.6)$$

Let us observe that there is a constant N_0 such that for every $\nu \in W^{p,-1}(\mathbb{R}^n)$ one has

$$\sum_{j=1}^{\infty} \|\zeta_j \nu\|_{p,-1}^p \leq N_0 \|\nu\|_{p,-1}^p. \quad (2.7)$$

Indeed, one has $\nu = g_0 - \sum_{i=1}^n \partial_{x_i} g_i$, where $g_i \in L^p(\mathbb{R}^n)$. To this end, it suffices to write $\nu = g_0 - \Delta g_0$, $g_0 \in W^{p,1}(\mathbb{R}^n)$, and set $g_i := \partial_{x_i} g_0$. Then

$$N_1 \sum_{i=0}^n \|g_i\|_{L^p(\mathbb{R}^n)} \leq \|\nu\|_{p,-1} \leq N_2 \sum_{i=0}^n \|g_i\|_{L^p(\mathbb{R}^n)} \quad (2.8)$$

with some constants N_1 and N_2 independent of ν , because $I - \Delta$ is an isomorphism between $W^{p,1}(\mathbb{R}^n)$ and $W^{p,-1}(\mathbb{R}^n)$. Since $\|\zeta_j g_0\|_{p,-1} \leq \|\zeta_j g_0\|_{L^p(\mathbb{R}^n)}$ and

$$\|\zeta_j \partial_{x_i} g_i\|_{p,-1} = \|\partial_{x_i}(\zeta_j g_i) - g_i \partial_{x_i} \zeta_j\|_{p,-1} \leq (1+M)\|g_i\|_{L^p(B_j)},$$

by (2.8) we find that

$$\begin{aligned} \sum_{j=1}^{\infty} \|\zeta_j \nu\|_{p,-1}^p &\leq (n+1)^p \sum_{j=1}^{\infty} \left[\|\zeta_j g_0\|_{p,-1}^p + \sum_{i=1}^n \|\zeta_j \partial_{x_i} g_i\|_{p,-1}^p \right] \\ &\leq (1+M)^p (n+1)^p \sum_{j=1}^{\infty} \sum_{i=0}^n \|g_i\|_{L^p(B_j)}^p \leq (1+M)^p (n+1)^p \sum_{i=0}^n \|g_i\|_{L^p(\mathbb{R}^n)}^p \\ &\leq (1+M)^p (n+1)^p N_1^{-p} \|\nu\|_{p,-1}^p. \end{aligned}$$

Therefore, (2.6) yields

$$\|\nabla u\|_{L^p(\mathbb{R}^n)}^p \leq NC_2 \|u \operatorname{Tr} A\|_{L^p(\mathbb{R}^n)}^p + N_0 C_2 \|L_A u\|_{p,-1}^p.$$

Clearly, $u \in L^p(\mathbb{R}^n)$ by our assumptions that $u \operatorname{Tr} A \in L^p(\mathbb{R}^n)$ and $A \geq \varepsilon \cdot I$. So, the proof in the case of L_A is complete.

The case of \mathcal{L}_A is similar. We only note that

$$\mathcal{L}_A(\zeta_j u) = \zeta_j \mathcal{L}_A u + 2\partial_{x_i}(a^{ik} u \partial_{x_k} \zeta_j) - u \partial_{x_i}(a^{ik} \partial_{x_k} \zeta_j). \quad (2.9)$$

The $W^{p,-1}$ -norm of the right-hand side is estimated along the same lines as in the first case. \square

Part of the hypotheses of Theorem 2.1 is the integrability of order bigger than 1. Since we intend to apply this theorem to measures, it is desirable to ensure such an integrability for measures satisfying equations of the type $L^* \mu = \nu$. The next three theorems serve this purpose. Before proving them we recall some known facts. We need the following classical result (see, e.g., [13, Ch. 2, Ch. 5], [14, §3.1.1]). Suppose that A is uniformly continuous and A and A^{-1} are uniformly bounded. Then, for every $\lambda > 0$, the operator $L_A - \lambda$ is an isomorphism between $W^{p,2}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. In particular, for every $v \in L^p(\mathbb{R}^n)$, there exists $u \in W^{p,2}(\mathbb{R}^n)$ such that $L_A u - \lambda u = v$ and

$$\|u\|_{p,2} \leq M_p \|v\|_{L^p(\mathbb{R}^n)},$$

where M_p depends only on p, n , the bounds on A and A^{-1} , and the modulus of continuity of A . An analogous result holds for \mathcal{L}_A on $W^{p,1}(\mathbb{R}^n)$. For the reader's convenience, we include it with a proof involving our basic assumptions.

Proposition 2.2. *Let A be uniformly continuous on \mathbb{R}^n and satisfy (C1) for p and let A and A^{-1} be uniformly bounded. Then, for every sufficiently large $\lambda > 0$, the operator $\mathcal{L}_A - \lambda$ is an isomorphism between $W^{p,1}(\mathbb{R}^n)$ and $W^{p,-1}(\mathbb{R}^n)$. In particular, for every $v \in W^{p,-1}(\mathbb{R}^n)$, there exists $u \in W^{p,1}(\mathbb{R}^n)$ such that $\mathcal{L}_A u - \lambda u = v$ and*

$$\|u\|_{p,1} \leq N_p \|v\|_{p,-1}, \quad (2.10)$$

where N_p depends only on p , $\sup_x [\|A(x)\| + \|A(x)^{-1}\|]$, and the modulus of continuity of A .

Proof. The case of the space $W_0^{p,1}(B)$ on a ball B is considered in [15], where the following estimate is established:

$$\sqrt{\lambda} \|u\|_{L^p(\mathbb{R}^n)} + \|u\|_{p,1} \leq N_p \|\mathcal{L}_A u - \lambda u\|_{p,-1}. \quad (2.11)$$

Moreover, N_p depends only on p , B , $\sup_x [\|A(x)\| + \|A(x)^{-1}\|]$, and the modulus of continuity of A . The global result is deduced from this local one in a standard way by establishing first (2.11), possibly with another constant, for functions $u \in C_0^\infty(\mathbb{R}^n)$. Such an estimate is verified by considering sequences of functions ζ_j as in the proof above and

dealing with $\mathcal{L}_A(\zeta_j u)$. Then we use (2.9) and the same bounds on $\|u \partial_{x_i}(a^{ik} \partial_{x_k} \zeta_j)\|_{p,-1}$ as above to deduce from (2.11) the estimate

$$\sqrt{\lambda} \|\zeta_j u\|_{L^p(\mathbb{R}^n)} + \|\zeta_j u\|_{p,1} \leq N_p \|\zeta_j(\mathcal{L}_A u - \lambda u)\|_{p,-1} + C \|u\|_{L^p(B_j)}$$

with some constant C that depends on p , n , and A . Therefore,

$$\lambda^{p/2} \|u\|_{L^p(\mathbb{R}^n)}^p + \|u\|_{p,1}^p \leq N N_0 2^p N_p^p \|\mathcal{L}_A u - \lambda u\|_{p,-1}^p + N 2^p C^p \|u\|_{L^p(\mathbb{R}^n)}^p,$$

whence we obtain (2.10) by choosing $\lambda > 4C^2 N^{2/p}$. \square

It is worth noting that in Theorem 2.1, unlike the proposition, we do not assume that A is bounded. It should be also noted that the proposition only guarantees the uniqueness of a solution to the equation $\mathcal{L}_A F - F = G \in W^{p,-1}$ in the class $W^{p,1}$, not in larger classes of distributions to which our measure μ may belong a priori.

Theorem 2.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that $L_{A,b}^* \mu = \nu \in W^{p,-1}$ for some $p \in (1, \frac{n}{n-1})$, $|b| \in L^1(|\mu|)$. Suppose that A is uniformly continuous and $c_1 \cdot I \leq A \leq c_2 \cdot I$ for some $c_1, c_2 > 0$. Then μ has a density in $L^r(\mathbb{R}^n)$ for every $r \in [1, p]$.*

In the case of $\mathcal{L}_{A,b}$ the same is true under the additional assumption that $a^{ij} \in W_{loc}^{1,1}(\mathbb{R}^n)$ and $\partial_{x_j} a^{ij} \in L^1(|\mu|)$ for every i .

Proof. Let us consider the case of $L_{A,b}$. We know that μ has a density ϱ , hence

$$|b| \varrho \in L^1(\mathbb{R}^n) \subset W^{p,-1}(\mathbb{R}^n),$$

because $p < n/(n-1)$. For any test function φ we have

$$\begin{aligned} \left| \int (L_A \varphi - \varphi) d\mu \right| &= \left| - \int \varphi d\mu - \int (b, \nabla \varphi) d\mu + \nu(\varphi) \right| \\ &\leq C \sup_x [|\varphi(x)| + |\nabla \varphi(x)|] + C \|\varphi\|_{p',1}, \end{aligned} \quad (2.12)$$

where C is a number independent of φ . If we fix a ball $U \subset \mathbb{R}^n$, then for all $\varphi \in C_0^\infty(U)$ we have

$$\|\varphi\|_{p',1} \leq C(U) \sup_{x \in U} [|\varphi(x)| + |\nabla \varphi(x)|],$$

hence (2.12) yields

$$\left| \int (L_A \varphi - \varphi) d\mu \right| \leq C(1 + C(U)) \sup_{x \in U} [|\varphi(x)| + |\nabla \varphi(x)|]$$

for all $\varphi \in C_0^\infty(U)$. According to [4] one has $\varrho \in L_{loc}^r(\mathbb{R}^n)$ for any $r < n/(n-1)$. Hence estimate (2.12) extends to all $\varphi \in W_0^{p',2}(\mathbb{R}^n)$. Note that $p' > n$, hence we have the continuous embedding $W^{p',2}(\mathbb{R}^n) \subset C_b^1(\mathbb{R}^n)$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$. There exists a function $\varphi \in W^{p',2}(\mathbb{R}^n)$ such that $L_A \varphi - \varphi = \psi$ and

$$\sup_x [|\varphi(x)| + |\nabla \varphi(x)|] + \|\varphi\|_{p',1} \leq M \|\varphi\|_{p',2} \leq M_0 \|\psi\|_{L^{p'}(\mathbb{R}^n)},$$

where M and M_0 are independent of ψ . Let us take a sequence of functions $\zeta_k \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \zeta_k \leq 1$, $\zeta_k(x) = 0$ if $|x| \geq k+1$, $\zeta_k(x) = 1$ if $|x| \leq k$, and

$$\sup_k \sup_x [|\nabla \zeta_k(x)| + \|D^2 \zeta_k(x)\|] = M_1 < \infty.$$

We have

$$\begin{aligned} L_A(\zeta_k \varphi) - \zeta_k \varphi &= \zeta_k(L_A \varphi - \varphi) + 2(A \nabla \zeta_k, \nabla \varphi) + \varphi L_A \zeta_k \\ &= \zeta_k \psi + 2(A \nabla \zeta_k, \nabla \varphi) + \varphi L_A \zeta_k. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \zeta_k \psi \, d\mu &= \int (L_A(\zeta_k \varphi) - \zeta_k \varphi) \, d\mu - 2 \int (A \nabla \zeta_k, \nabla \varphi) \, d\mu - \int \varphi L_A \zeta_k \, d\mu \\ &\leq C \sup_x [|\zeta_k(x) \varphi(x)| + |\nabla(\zeta_k(x) \varphi(x))|] + C \|\zeta_k \varphi\|_{p',1} \\ &\quad + (2c_2 M_1 M_0 + M_0 M_1 n^2 c_2) \|\psi\|_{L^{p'}(\mathbb{R}^n)} |\mu|(\{k \leq |x| \leq k+1\}). \end{aligned}$$

The last term on the right converges to zero as $k \rightarrow \infty$. Therefore, we obtain

$$\int \psi(x) \varrho(x) \, dx \leq C \sup_x [|\varphi(x)| + M_1 |\varphi(x)| + |\nabla \varphi(x)|] + C \|\varphi\|_{p',1} \leq M_0 C (1 + M_1) \|\psi\|_{L^{p'}(\mathbb{R}^n)},$$

which yields $\varrho \in L^p(\mathbb{R}^n)$. Since $\varrho \in L^1(\mathbb{R}^n)$, one has $\varrho \in L^r(\mathbb{R}^n)$ for all $r \in [1, p]$. In the case of $\mathcal{L}_{A,b}$ it suffices to note that $\mathcal{L}_{A,b}^* \mu = L_{A,b_0}^* \mu$ with $b_0^i = b^i - \partial_{x_j} a^{ij} \in L^1(|\mu|)$. \square

The next result is a generalization of [16, Theorem 3.1] and a partial generalization of a result in [2]. We impose weaker assumptions than in [16], where A^{-1} is bounded and $|b| \in L^2(\mu)$ (in addition, the same local assumptions as below are imposed along with a condition a bit stronger than (2.13)); as compared to [2] (where A is uniformly Lipschitzian, A and A^{-1} are uniformly bounded, $|b| \in L^2(\mu)$), we weaken the assumptions on A , but add an extra local condition on b . That extra condition is not needed if we know in advance that μ has a locally bounded density in $W_{loc}^{2,1}(\mathbb{R}^n)$. It should be noted that unlike most other results in this paper, this theorem deals with probability measures and fails for signed measures.

Theorem 2.4. *Suppose that $\mu \in \mathcal{P}(\mathbb{R}^n)$ satisfies the equation $\mathcal{L}_{A,b}^* \mu = 0$, where A is continuous and nondegenerate, $a^{ij} \in W_{loc}^{p,1}(\mathbb{R}^n)$ with some $p > n$, $|b| \in L_{loc}^p(\mu)$. Suppose in addition that $|A^{-1/2}b| \in L^2(\mu)$ and that*

$$\liminf_{r \rightarrow \infty} \int_{r \leq |x| \leq 2r} \left[r^{-2} \|A(x)\| + r^{-1} \Theta_A(x) \right] \mu(dx) = 0. \quad (2.13)$$

Then μ has a density $\varrho \in W_{loc}^{2,1}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{\sqrt{A} \nabla \varrho}{\varrho} \right|^2 \, d\mu \leq \int_{\mathbb{R}^n} |A^{-1/2}b|^2 \, d\mu. \quad (2.14)$$

In particular, under the additional assumption $A \geq \varepsilon \cdot I$, $\varepsilon > 0$, one has $\sqrt{\varrho} \in W^{2,1}(\mathbb{R}^n)$, $\varrho \in L^{n/(n-2)}(\mathbb{R}^n)$ if $n > 2$ and $\varrho \in L^s(\mathbb{R}^n)$ for all $s \in [1, \infty)$ if $n = 2$.

Proof. The desired estimate is obtained by formally substituting $\varphi = \log \varrho$ in (1.3) and integrating by parts in the term with \mathcal{L}_A . In order to justify this straightforward procedure (see, e.g., [9], [16]), we need certain local regularity of μ , hence better local integrability of b . In order to avoid this extra assumption, a smoothing procedure (by convolutions or heat semigroups) was used in [2], [10]; however, smoothing requires certain global conditions on A . By the local theory, we know that μ has a continuous density $\varrho \in W_{loc}^{p,1}(\mathbb{R}^n)$. Hence (1.3) with $L = \mathcal{L}_{A,b}$ extends to $\varphi \in W_0^{2,1}(\mathbb{R}^n)$. Integrating by parts we obtain

$$\int_{\mathbb{R}^n} (A \nabla \varphi, \nabla \varrho) \, dx = \int_{\mathbb{R}^n} (b, \nabla \varphi) \varrho \, dx.$$

We fix a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 1$ if $|x| \leq 1$ and set $\zeta_j(x) := \zeta(x/j)$. Letting $\varrho_{k,\delta} := \min(k, \max(\varrho, \delta))$ for small $\delta > 0$ and large $k > 0$, $\Omega_{k,\delta} := \{\delta < \varrho < k\}$,

and taking $\varphi = \zeta_j^2 \log \varrho_{k,\delta}$ we arrive at the equality

$$\begin{aligned} \int_{\mathbb{R}^n} \left(A \nabla \varrho, \frac{\nabla \varrho}{\varrho} \right) \zeta_j^2 I_{\Omega_{k,\delta}} \varrho \, dx + 2 \int_{\mathbb{R}^n} (A \nabla \varrho, \nabla \zeta_j) \log \varrho_{k,\delta} \zeta_j \, dx \\ = \int_{\mathbb{R}^n} I_{\Omega_{k,\delta}} \left(b, \frac{\nabla \varrho}{\varrho} \right) \zeta_j^2 \varrho \, dx + 2 \int_{\mathbb{R}^n} (b, \nabla \zeta_j) \zeta_j \log \varrho_{k,\delta} \varrho \, dx. \end{aligned}$$

Integrating by parts in the second term on the left we find

$$\begin{aligned} S_{j,k,\delta} &:= \int_{\Omega_{k,\delta}} \left(A \nabla \varrho, \frac{\nabla \varrho}{\varrho} \right) \zeta_j^2 \varrho \, dx \\ &= 2 \int_{\Omega_{k,\delta}} \left(\frac{\nabla \varrho}{\varrho}, A \nabla \zeta_j \right) \zeta_j \varrho \, dx + 2 \int_{\mathbb{R}^n} \operatorname{div}(\zeta_j A \nabla \zeta_j) \log \varrho_{k,\delta} \varrho \, dx \\ &\quad + \int_{\Omega_{k,\delta}} \left(b, \frac{\nabla \varrho}{\varrho} \right) \zeta_j^2 \varrho \, dx + 2 \int_{\mathbb{R}^n} (b, \nabla \zeta_j) \zeta_j \log \varrho_{k,\delta} \varrho \, dx \\ &\leq \sqrt{S_{j,k,\delta}} \left(2 \|I_{\Omega_{k,\delta}} \sqrt{A} \nabla \zeta_j\|_{L^2(\mu, \mathbb{R}^n)} + \|A^{-1/2} b\|_{L^2(\mu, \mathbb{R}^n)} \right) + R_{j,k,\delta}, \end{aligned}$$

where

$$R_{j,k,\delta} := 2 \int_{\mathbb{R}^n} \operatorname{div}(\zeta_j A \nabla \zeta_j) \log \varrho_{k,\delta} \varrho \, dx + 2 \int_{\mathbb{R}^n} (b, \nabla \zeta_j) \zeta_j \log \varrho_{k,\delta} \varrho \, dx.$$

Keeping k and δ fixed, we observe that, given $\varepsilon > 0$, for all sufficiently large numbers j of the form $j = r_l$ with $r_l \rightarrow \infty$ chosen according to (2.13), the quantity $R_{j,k,\delta}$ can be made smaller than ε in absolute value. Indeed, it follows by the hypotheses and the estimates

$$\sup_x |\nabla \zeta_j(x)| \leq j^{-1} \sup_x |\nabla \zeta(x)|, \quad \sup_x |\partial_{x_i} \partial_{x_m} \zeta_j(x)| \leq j^{-2} \sup_x |\partial_{x_i} \partial_{x_m} \zeta(x)|$$

that for all $j = r_l$ the first term in the expression for $R_{j,k,\delta}$ can be estimated by

$$M_{k,\delta} r_l^{-2} \int_{\{r_l \leq |x| \leq 2r_l\}} \|A(x)\| \mu(dx) + M_{k,\delta} r_l^{-1} \int_{\{r_l \leq |x| \leq 2r_l\}} \Theta_A(x) \mu(dx),$$

where $M_{k,\delta} = 2M(\log k - \log \delta)$ with a constant M that depends on the maxima of the first and second derivatives of ζ . Similarly, by the Cauchy inequality and the estimate $|b(x)| \leq \|A^{1/2}(x)\| \|A^{-1/2}(x)b(x)\|$ the second term is majorized by

$$r_l^{-1} M_{k,\delta} \|A^{-1/2} b\|_{L^2(\mu, \mathbb{R}^n)} \left(\int_{r_l \leq |x| \leq 2r_l} \|A(x)\| \mu(dx) \right)^{1/2}.$$

Therefore, for all $l > l(k, \delta)$, one has $S_{r_l, k, \delta} \leq \|A^{-1/2} b\|_{L^2(\mu, \mathbb{R}^n)} + \varepsilon$. This yields that the integrals of $|\sqrt{A} \nabla \varrho / \varrho|^2$ over the sets $\Omega_{k,\delta}$ against μ are majorized by the same quantity. Letting $k \rightarrow \infty$ and $\delta \rightarrow 0$ we see that $|\sqrt{A} \nabla \varrho / \varrho| \in L^2(\mu)$ and obtain the desired bound. Note that in [16] similar estimates have been employed with $\nabla \varrho / \varrho$ and b in place of $\sqrt{A} \nabla \varrho / \varrho$ and $A^{-1/2} b$, which yields less precise bounds. \square

Remark 2.5. (i) Condition (2.13) is fulfilled if

$$|\nabla a^{ij}(x)| \leq C_0 + C_1 |x|.$$

If μ is known to have finite first moment, i.e., $|x| \in L^1(\mu)$, then a quadratic growth of $|\nabla a^{ij}|$ is allowed.

(ii) Condition (2.13) can be replaced by the assumption that for some $r > 0$ one has

$$\liminf_{R \rightarrow \infty} \int_{R \leq |x| \leq R+r} \left[\|A(x)\| + \Theta_A(x) \right] \mu(dx) = 0. \quad (2.15)$$

This condition is weaker on the part of Θ_A , but is stronger on the part of $\|A\|$; for uniformly bounded A , it is weaker. The only difference in the proof is that we take a sequence ζ_j such that $\zeta_j(x) = 1$ if $|x| \leq R_l$, $\zeta_j(x) = 0$ if $|x| \geq R_l + r$, and the first and second derivatives of the functions ζ_j are uniformly bounded in j .

(iii) Note also that if A is uniformly bounded and satisfies (C1), then (2.13) is ensured by the assumption that $\liminf_{r \rightarrow \infty} r^{n-1} \mu(\{|x| \geq r\}) = 0$, which is fulfilled, e.g., if $|x|^{n-1} \in L^1(\mu)$. The latter can be effectively verified in terms of A and b by the Lyapunov functions method (see [5], [6]).

Estimate (2.14) can be regarded as the estimate

$$\int \left\langle \frac{\nabla \varrho}{\varrho}, \frac{\nabla \varrho}{\varrho} \right\rangle d\mu \leq \int \langle b, b \rangle d\mu$$

with respect to the Riemannian geometry generated by A . See [10] for such estimates in the case of a Riemannian manifold M . It is shown in [10] under certain assumptions on M (e.g., that the Ricci curvature is bounded from below and that the Riemannian volumes of balls of any fixed positive radius are separated from zero) that if $|b| \in L^2(M, \lambda)$, where λ is the Riemannian volume on M , then one has

$$\int \left\langle \frac{\nabla \varrho}{\varrho}, \frac{\nabla \varrho}{\varrho} \right\rangle d\lambda \leq \int \langle b, b \rangle d\lambda.$$

In the situation of Theorem 2.4, we do not know whether the natural estimate (2.14) holds without any extra local assumptions on b and without (2.13). However, there is an important special case when (2.13) is not needed.

Theorem 2.6. *Let A be continuous and nondegenerate, $a^{ij} \in W_{loc}^{p,1}(\mathbb{R}^n)$, $|b| \in L_{loc}^p(\mathbb{R}^n)$ with some $p > n$. Suppose there exists a function $V \in W_{loc}^{2,2}(\mathbb{R}^n)$ such that*

$$V(x) \rightarrow +\infty \quad \text{and} \quad \mathcal{L}_{A,b}V(x) \rightarrow -\infty \quad \text{as } |x| \rightarrow +\infty.$$

Assume also that there are $c_1, c_2 > 0$ such that

$$\mathcal{L}_{A,b}V \leq c_1 - c_2|A^{-1/2}b|^2$$

outside some ball. Then there exists a measure $\mu \in \mathcal{P}(\mathbb{R}^n)$ such that one has $\mathcal{L}_{A,b}^*\mu = 0$ and $|\sqrt{A}\nabla\varrho|^2/\varrho \in L^1(\mathbb{R}^n)$.

If, in addition, there is a positive Borel function θ on $[0, +\infty)$ such that $\lim_{t \rightarrow \infty} \theta(t) = +\infty$ and

$$\mathcal{L}_{A,b}V \leq c_1 - c_2\theta(|A^{-1/2}b|)|A^{-1/2}b|^2$$

outside some ball, then $|A^{-1/2}b| \in L^2(\mu)$ and (2.14) holds.

Proof. We recall that the existence of μ has been shown in [5] by taking positive functions $f_j \in W^{p,1}(B_j)$, $B_j = B(0, j)$, such that

$$\operatorname{div}(A\nabla f_j - f_j b) = 0, \quad f_j|_{\partial B_j} = 1.$$

Then μ is obtained as a weak limit of $\|f_j\|_{L^1(B_j)}^{-1} f_j \cdot dx$. By multiplying the above equation by $\log f_j$ (which vanishes on ∂B_j) and integrating by parts we obtain

$$\int_{B_j} \left(A\nabla f_j, \frac{\nabla f_j}{f_j} \right) dx = \int_{B_j} (b, \nabla f_j) dx = \int_{B_j} (A^{-1/2}b, A^{1/2}\nabla f_j) dx.$$

The same is true for the normalized probability densities $\varrho_j := f_j \|f_j\|_{L^1(B_j)}^{-1}$, whence by the Cauchy inequality we find

$$\int_{B_j} \left(A \nabla \varrho_j, \frac{\nabla \varrho_j}{\varrho_j} \right) dx \leq \int_{B_j} |A^{-1/2} b|^2 \varrho_j dx.$$

The right-hand side is uniformly bounded in j due to the estimate $\mathcal{L}_{A,b} V \leq c_1 - c_2 |A^{-1/2} b|^2$ outside some ball, because it was shown in [5] that the integrals of $|\mathcal{L}_{A,b} V| \varrho_j$ over B_j are uniformly bounded. This yields the integrability of $|\sqrt{A} \nabla \varrho|^2 / \varrho$, since $\varrho_j \rightarrow \varrho$ locally uniformly (see [5]).

The existence of a function θ with the properties mentioned in the formulation yields the uniform boundedness of the integrals of $I_{B_j} \theta(|A^{-1/2} b|) |A^{-1/2} b|^2 \varrho_j$, which ensures convergence of the integrals of $I_{B_j} |A^{-1/2} b|^2 \varrho_j$ to the integral of $|A^{-1/2} b|^2 \varrho$. \square

Theorem 2.7. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that $\mathcal{L}_{A,b}^* \mu = \nu$. Suppose that*

- (a) *A and b satisfy (1.5) with some $\alpha > n$, $A \geq \varepsilon I$ with $\varepsilon > 0$,*
- (b) *$|b| \in L^\beta(|\mu|)$, $\beta > 1$.*
- (c) *A satisfies condition (C1) for β and is uniformly continuous.*

Assume also that the density ϱ of μ belongs to $L^{\beta_0}(\mathbb{R}^n)$ with some $\beta_0 > 1$, which is automatically the case if A is bounded.

(i) *Let $1 < \beta < n$ and let $\nu \in W^{\theta, -1}$ for all $\theta \in (1, \frac{n}{n-\beta+1})$. Then $\varrho \in W^{r, 1}(\mathbb{R}^n)$ for all $r \in (1, \frac{n}{n-\beta+1})$. Moreover, if μ is nonnegative, then the same is true for $r = \frac{n}{n-\beta+1}$.*

(ii) *Let $\beta = n$ and $\nu \in W^{\theta, -1}(\mathbb{R}^n)$ for all $\theta \in (1, n)$. Then $\varrho \in W^{r, 1}(\mathbb{R}^n)$ for all $r \in (1, n)$.*

(iii) *Let $n < \beta \leq \alpha$ and $\nu \in W^{\theta, -1}(\mathbb{R}^n)$ for all $\theta \in (1, \beta]$. Then $\varrho \in W^{r, 1}(\mathbb{R}^n)$ for any $r \in (1, \beta]$. In particular, $\varrho \in L^\infty(\mathbb{R}^n)$.*

The same is true in the case of $L_{A,b}$ provided that one has additionally $\partial_{x_j} a^{ij} \in L^\beta(|\mu|)$ for each i .

Proof. (i) We apply the reasoning from [16] with some simplifications due to Theorem 2.1. Suppose we know that $\varrho \in L^{\beta_k}(\mathbb{R}^n)$ for some $\beta_k \geq \beta_0$. Let

$$p_k = \frac{\beta \beta_k}{\beta + \beta_k - 1}. \quad (2.16)$$

By Hölder's inequality with exponent $t = \beta/p_k$ we obtain

$$|b| = |\varrho|^{1-1/\beta} |\varrho|^{1/\beta} |b| \in L^{p_k}(\mathbb{R}^n).$$

Hence $\partial_{x_i} (b^i \varrho) \in W^{p_k, -1}(\mathbb{R}^n)$. By the same reasoning $\partial_{x_i} (b^i \varrho) \in W^{s, -1}(\mathbb{R}^n)$ for every $s \in (1, p_k]$. Hence $\mathcal{L}_A \varrho \in W^{s, -1}(\mathbb{R}^n)$ for each $s \in (1, p_k]$, which yields $\varrho \in W^{s, 1}(\mathbb{R}^n)$ whenever $s \in (1, p_k]$. By the Sobolev embedding theorem $\varrho \in L^{\beta_{k+1}}(\mathbb{R}^n)$ with

$$\beta_{k+1} = \frac{n \beta \beta_k}{n(\beta + \beta_k - 1) - \beta \beta_k}.$$

Starting from β_0 and iterating we obtain the sequence $\{\beta_k\}$ that is increasing and, as one can easily verify, converges to $\frac{n}{n-\beta}$. By using (2.16) once again we conclude that $\varrho \in W^{r, 1}(\mathbb{R}^n)$ for all $r \in (1, n/(n-\beta+1))$. Other cases are similar. We only note that in the case $\beta > n$ we first obtain that $\varrho \in W^{p, 1}(\mathbb{R}^n)$ with $p \in (n, \beta)$, which yields that $\varrho \in L^\infty(\mathbb{R}^n)$. Now $|b| \in L^\beta(\mathbb{R}^n)$. The case of $L_{A,b}$ reduces to that of \mathcal{L}_{A,b_0} as above. \square

Corollary 2.8. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ satisfy the equation $\mathcal{L}_{A,b}^* \mu = 0$. Suppose that there is $\alpha > n$ such that $|b| \in L^\alpha(\mu)$, $A \in W_{loc}^{\alpha,1}$ is uniformly continuous and satisfies (C1) and (2.13), $A \geq \varepsilon I$, $\varepsilon > 0$. Then $\varrho \in W^{\alpha,1}(\mathbb{R}^n)$, in particular, $\varrho \in L^\infty(\mathbb{R}^n)$.*

In particular, the conclusion holds true if one has (1.7) and

$$A \geq \varepsilon I, \quad b^i \in L^\alpha(\mu), \quad \text{Tr } A \in L^1(\mu), \quad (1 + |x|)^{-1} \Theta_A \in L^1(\mu).$$

It is clear that ϱ may not belong to the class $W^{p,2}$ unless we require certain regularity of b . The following theorem extends [16, Theorem 4.7], where somewhat stronger assumptions on A were employed. Since essentially the same reasoning as in [16, Lemma 4.5, Theorem 4.7] applies along with our sharper regularity results, we only briefly comment on the proof.

Theorem 2.9. *Suppose that $\alpha \geq 2n$, $A \geq \varepsilon I$ with $\varepsilon > 0$, $a^{ij}, b^i \in W_{loc}^{\alpha,1}$, A is uniformly continuous and satisfies condition (C1) for α . Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ satisfy $\mathcal{L}_{A,b}^* \mu = \nu$, where $|b| \in L^\alpha(\mu)$ and $\nu \in L^r(\mathbb{R}^n)$ for all $r \in (1, \alpha]$. Assume also that $\text{div } b \in L^\alpha(\mu)$ and $|\nabla a^{ij}| \in L^\theta(\mu)$ with $\theta \geq \max(n^2, \alpha)$. Then $\varrho \in W^{r,2}(\mathbb{R}^n)$ for all $r \in (1, \alpha/2)$. In particular, $|\nabla \varrho| \in L^\infty(\mathbb{R}^n)$ if $\alpha > 2n$.*

If $a^{ij}, |\nabla a^{ij}| \in L^\infty(\mathbb{R}^n)$, then the conclusion is true for any solution $\mu \in \mathcal{M}(\mathbb{R}^n)$.

Proof. Suppose first that μ is a probability measure. We know that μ has a bounded density $\varrho \in W^{\alpha,1}(\mathbb{R}^n)$. In addition, it is known from the local theory that $\varrho \in W_{loc}^{\alpha,2}$. Therefore,

$$L_A \varrho = -\partial_{x_i} a^{ij} \partial_{x_j} \varrho + \varrho \text{div } b + (b, \nabla \varrho) + \nu.$$

Since ϱ is bounded, we obtain $\varrho \text{div } b \in L^s(\mathbb{R}^n)$ if $s \leq \alpha$. By using that $|\nabla \varrho|/\varrho \in L^2(\mu)$, which follows by Theorem 2.4 (note that $\Theta_A \in L^\theta(\mu)$ and $\|A(x)\| \leq c_1 + c_2|x|$ by the uniform continuity of A), we obtain from Hölder's inequality

$$|\partial_{x_i} a^{ij} \partial_{x_j} \varrho|^r |\partial_{x_i} a^{ij}|^r |\partial_{x_j} \varrho|^{r-1} \varrho^{1/2} |\partial_{x_j} \varrho| \varrho^{-1/2} \in L^1(\mu),$$

provided that $\frac{1}{2} + \frac{r}{\theta} + \frac{r-1}{\alpha} = 1$, i.e., $r = \theta \frac{\alpha+2}{\alpha+\theta}$. Note that $r \geq n$ and the equality is only possible if $\alpha = 2n$ and $\theta = n^2$.

In addition, $(b, \nabla \varrho) \in L^p(\mathbb{R}^n)$, where $1 < p \leq \alpha/4 + 1/2$. Indeed, we have

$$|(b, \nabla \varrho)|^p \leq |b|^p |\nabla \varrho|^{p-1} \varrho^{1/2} |\nabla \varrho| \varrho^{-1/2}.$$

The integral of the right-hand side is finite, since $|\nabla \varrho| \varrho^{-1/2} \in L^2(\mathbb{R}^n)$ and $|b|^{2p} |\nabla \varrho|^{2p-2} \varrho \in L^1(\mathbb{R}^n)$, which is verified by Hölder's inequality with exponent $t = \frac{\alpha}{2p}$, which requires that $|\nabla \varrho| \in L^s(\mathbb{R}^n)$ with $s = \alpha \frac{2p-2}{\alpha-2p} \leq \alpha$. A reasoning similar to that of the proof of Theorem 2.1 shows that $\varrho \in W^{p,2}(\mathbb{R}^n)$. Moreover, one has $\varrho \in W^{s,2}(\mathbb{R}^n)$ for any $s \in (1, p]$, which is proved by similar estimates. By using this estimate and the fact that $|\nabla \varrho| \in L^\alpha(\mathbb{R}^n)$, one easily deduces by iteration that $|\nabla \varrho| \in L^s(\mathbb{R}^n)$ for all $s \in (1, +\infty)$ (see [16, Theorem 4.7]). Now $\varrho \in W^{r,2}(\mathbb{R}^n)$ for any $r < \alpha/2$. If $\alpha > 2n$, this yields $|\varrho| \in L^\infty(\mathbb{R}^n)$, hence we finally obtain $\varrho \in W^{\alpha/2,2}(\mathbb{R}^n)$. In the case $a^{ij}, |\nabla a^{ij}| \in L^\infty(\mathbb{R}^n)$, some estimates above simplify and we do not need the integrability of $|\nabla \varrho|^2/\varrho$. \square

Remark 2.10. If μ is a signed measure satisfying equation (1.1), then one might ask whether $|\mu|$ is also a solution to the same equation. In general, this is not true even if $A = I$ and b is smooth, see [8]. Of course, this is not surprising for locally integrable solutions, e.g., the absolute value of a harmonic function may not be harmonic, but for globally integrable ones the question is more interesting. In particular, if μ is an invariant measure for a semigroup $(T_t)_{t \geq 0}$ whose generator extends $(L_{A,b}, C_0^\infty)$, then $|\mu|$ is also an invariant measure (see, e.g., [1]). In the situation where any probability measure

satisfying (1.1) possesses a positive continuous density, which is the case if $a^{ij} \in W_{loc}^{p,1}(\mathbb{R}^n)$, $|b| \in L_{loc}^p(\mathbb{R}^n)$ with $p > n$, non-uniqueness of solutions to (1.1) in $\mathcal{P}(\mathbb{R}^n)$ always yields signed solutions whose absolute values are not solutions. Under our typical assumptions on A and b , equation (1.1) has at most one solution in $\mathcal{P}(\mathbb{R}^n)$. However, we do not know whether in such a case the space of solutions in $\mathcal{M}(\mathbb{R}^n)$ is at most one dimensional.

3. POINTWISE ESTIMATES OF DENSITIES

Now we turn to pointwise bounds of solutions. The idea is simple: in order to show that $|\varrho(x)| \leq C\Psi(x)$ for some positive function Ψ , one has to consider the measure μ_0 with density ϱ/Ψ and verify that this measure satisfies an equation of the type considered in Theorem 2.7. This idea was employed in [16] in the case of exponential functions. Case (iii) of the example below gives the bound from [16] under slightly weaker assumptions.

Theorem 3.1. *Suppose that μ is a probability measure satisfying the equation $\mathcal{L}_{A,b}^*\mu = 0$, where A satisfies the hypotheses of Theorem 2.7(iii) with some $\beta > n$ and $|b| \in L^\beta(\mu)$. Let $\Phi \in W_{loc}^{1,1}(\mathbb{R}^n)$ be a positive function such that for some $\theta > n$ one has*

$$\Phi \in L^1(\mu), \quad |\nabla\Phi| \in L^\theta(\mu), \quad \partial_{x_i}a^{ij} \in L^n(\mu).$$

Then the density ϱ of μ satisfies the estimate $\varrho(x) \leq C\Phi(x)^{-1}$ with some constant C .

Proof. We may assume that $n < \theta \leq \beta$. Let us consider the bounded positive measure $\mu_0 = \Phi \cdot \mu$. It is easily verified that

$$\mathcal{L}_{A,b}^*\mu_0 = -(b, \nabla\Phi)\varrho - \partial_{x_i}a^{ij}\partial_{x_j}\Phi\varrho + 2\partial_{x_i}(a^{ij}\partial_{x_j}\Phi\varrho).$$

It suffices to show that the right-hand side belongs to $W^{s,-1}(\mathbb{R}^n)$ for all $s \in (1, \theta]$. Clearly, $a^{ij}\partial_{x_j}\Phi\varrho \in L^s(\mathbb{R}^n)$ whenever $s \leq \theta$ by the boundedness of a^{ij} and ϱ and the inclusion $|\nabla\Phi| \in L^\theta(\mu)$. In addition, $L^{n\theta/(n+\theta)}(\mathbb{R}^n) \subset W^{\theta,-1}(\mathbb{R}^n)$, so it remains to verify that $(b, \nabla\Phi)\varrho$ and $\partial_{x_i}a^{ij}\partial_{x_j}\Phi\varrho$ belong to $L^{n\theta/(n+\theta)}(\mathbb{R}^n)$, which by the boundedness of ϱ reduces to proving that $(b, \nabla\Phi)$ and $\partial_{x_i}a^{ij}\partial_{x_j}\Phi$ belong to $L^{n\theta/(n+\theta)}(\mu)$. The latter is true by Hölder's inequality, since $\partial_{x_i}a^{ij} \in L^n(\mu)$ and $|\nabla\Phi| \in L^\theta(\mu)$. The same reasoning shows that $(b, \nabla\Phi)\varrho$ and $\partial_{x_i}a^{ij}\partial_{x_j}\Phi\varrho$ belong to $W^{s,-1}(\mathbb{R}^n)$ for any s with $n/(n-1) < s \leq \theta$. Finally, these functions belong also to $W^{s,-1}(\mathbb{R}^n)$ if $1 < s \leq n/(n-1)$, because $(b, \nabla\Phi)$ and $\partial_{x_i}a^{ij}\partial_{x_j}\Phi$ belong to all $L^r(\mu)$ with $1 \leq r \leq n\theta/(n+\theta)$ and ϱ is bounded. Therefore, by Theorem 2.7 one has $\Phi\varrho \in W^{\theta,1}(\mathbb{R}^n)$. In particular, $\Phi\varrho \in L^\infty(\mathbb{R}^n)$. \square

Example 3.2. Suppose that $\mu \in \mathcal{P}(\mathbb{R}^n)$ satisfies the equation $\mathcal{L}_{A,b}^*\mu = 0$ and that A is uniformly nondegenerate, uniformly Lipschitzian and uniformly bounded, $|b| \in L^p(\mu)$ for some $p > n$.

(i) If $\Psi \in W_{loc}^{1,1}(\mathbb{R}^n)$ is a positive function such that

$$\Phi \in L^1(\mu), \quad |\nabla\Phi| \in L^\theta(\mu), \quad \theta > n,$$

then $\varrho(x) \leq C\Phi(x)^{-1}$.

(ii) Let $k > 1$ and suppose that $|x|^r \in L^1(\mu)$ for some $r > (k-1)n$. Then, letting $\Phi(x) = |x|^k$, we obtain $\varrho(x) \leq C|x|^{-k}$.

(iii) Suppose that

$$\exp(\alpha|x|^\beta) \in L^1(\mu), \quad |b(x)| \leq C_0 + C_1 \exp(\alpha_0|x|^\beta),$$

where $\alpha, \beta, C_0, C_1 > 0$, $\alpha_0 < \alpha/n$. Then, for any $\kappa < \beta/n$, there is $C > 0$ such that $\varrho(x) \leq C \exp(-\kappa|x|^\beta)$.

It is surprising that the above estimates, very rough at a first glance, are in fact sufficiently precise. It was proved in [16] that if $\mu \in \mathcal{P}(\mathbb{R}^n)$, $a^{ij} \in C_b^3(\mathbb{R}^n)$, A is uniformly nondegenerate, $b^i \in C^2(\mathbb{R}^n)$ and, for some $\beta > 1$ and $c > 0$ one has

$$|b(x)| + |Db(x)| + |D^2b(x)| \leq C(1 + |x|^{\beta-1}), \quad \limsup_{|x| \rightarrow \infty} |x|^{-\beta}(b(x), x) = -c,$$

then $\exp[-K_1(1 + |x|^\beta)] \leq \varrho(x) \leq \exp[-K_2(1 + |x|^\beta)]$ with some $K_1, K_2 > 0$. It would be interesting to study the question whether such an estimate holds under weaker assumptions on A and b not involving the second derivative of b . The two-sided estimate from [16] gives a sufficient condition implying that $|\nabla\varrho/\varrho| \in L^p(\mu)$ for the logarithmic gradient of μ and any $p \geq 2$. So far this is the only known general result in this direction. Even if $A = I$, it is not known whether the condition $|b| \in L^p(\mu)$ with $p > 2$ always implies that $|\nabla\varrho/\varrho| \in L^p(\mu)$ as in the case $p = 2$.

In a similar manner one obtains upper bounds on $|\nabla\varrho|$.

Proposition 3.3. *Suppose that in Theorem 3.1 we have additionally $\Phi \in W_{loc}^{1,2}$ and*

$$\varrho|\nabla\Phi| \in L^\infty(\mathbb{R}^n), \quad |b|, |\nabla\Phi|, \partial_{x_i}a^{ij}\partial_{x_j}\Phi, L_A\Phi, |A\nabla\Phi| \in L^r(\mu), \quad r > 2n.$$

Then $|\nabla\varrho(x)| \leq C\Phi(x)^{-1}$.

Proof. The measure μ_0 satisfies the equation with the right-hand side

$$-(b, \nabla\Phi)\varrho + \partial_{x_i}a^{ij}\partial_{x_j}\Phi\varrho + 2L_A\Phi\varrho + 2(A\nabla\Phi, \nabla\varrho).$$

We know that ϱ and $\nabla\varrho$ are bounded. Hence the right-hand side is in $L^s(\mathbb{R}^n)$ for any $s \in (1, r/2]$, which yields $\Phi\varrho \in W^{\theta,2}(\mathbb{R}^n)$, $1 < \theta < r/2$. Therefore, $\nabla(\Phi\varrho)$ is bounded, whence the claim follows. \square

Example 3.4. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ satisfy the equation $\mathcal{L}_{A,b}\mu = 0$ and let A be uniformly nondegenerate, uniformly Lipschitzian and uniformly bounded with $|b|, \operatorname{div} b \in L^p(\mu)$ for some $p > 2n$.

(i) Let $\Phi(x) = |x|^k$, $k \geq 1$, and let $|x|^m \in L^1(\mu)$, where $m > 2n(k-1)$. Then

$$|\nabla\varrho(x)| \leq C(1 + |x|)^{-k}.$$

(ii) Let $\Phi(x) = \exp(K|x|^\beta)$ and let $\exp(M|x|^\beta) \in L^1(\mu)$, where $M > 2nK$. Then

$$|\nabla\varrho(x)| \leq C \exp(-K|x|^\beta).$$

By using the method of Lyapunov functions, one can give effective conditions for the existence of polynomial or exponential moments for μ . For example, if $A(x) \leq \Lambda I$ and $(b(x), x) \leq -K < -\Lambda n$ outside some ball, then letting $V(x) = (x, x)^\gamma$ with $1 < \gamma < 1 + \frac{K-\Lambda n}{2}$, we obtain outside some ball

$$L_{A,b}V(x) \leq 2\gamma(x, x)^{\gamma-1}[\Lambda n + 2(\gamma - 1) + (b(x), x)] \leq -\kappa|x|^{2\gamma-2},$$

where $\kappa > 0$. Hence $|x|^{2\gamma-1} \in L^1(\mu)$. Stronger decay of $(b(x), x)$ yields exponential integrability (see [6], [16]). Certainly, the required integrability of the coefficients can be also deduced from such estimates provided we know certain bounds on the coefficients. There are cases where the L^p -integrability of A and b with respect to μ comes naturally without any known bounds on A and b . For example, if we have a diffusion with an invariant measure on an infinite dimensional space, say, \mathbb{R}^∞ , then, under broad assumptions, the projection of μ on \mathbb{R}^n satisfies equation (1.1) whose coefficients are obtained by taking projections and conditional expectations. Even if we had polynomial bounds on the coefficients of the infinite dimensional diffusion, no such bounds can be guaranteed after taking conditional expectations. However, the membership in L^p is preserved by

conditional expectation. This also shows that the study of integrability with respect to μ is worthwhile, since it can be much more adequate than with respect to Lebesgue measure.

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