

A REMARK ON SETS IN INFINITE DIMENSIONAL SPACES WITH FULL OR ZERO CAPACITY

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Abstract

We give a simple proof that for classical Dirichlet forms on infinite dimensional linear state spaces the intrinsic closure of a set of full measure has full capacity. Furthermore, we show that the $C_{1,q}$ -capacity of a set, enlarged by adding the linear span of a basis in the generalized Cameron-Martin space remains zero if it was zero for slightly bigger capacities a priori.

1. INTRODUCTION, FRAMEWORK AND A RESULT ON SETS WITH FULL CAPACITY

In infinite dimensional analysis the question whether a given set has zero or full capacity (in the sense that its complement has zero capacity) is much less studied than in finite dimensions. This question is of importance, since roughly speaking capacity zero sets are not hit by the underlying process whereas a set of full capacity carries the process for all times. The first aim of this paper is to give a simple analytic proof for the fact that the intrinsic closure of a set of full measure has full capacity (cf. Theorem 1.4 below). This fact is essentially known to experts. We refer e.g. to [7] where this result was proved for a class of Dirichlet forms with non-flat underlying state space. But there is no reference for this result for general classical Dirichlet forms of gradient type on linear state spaces. In this case there is quite an easy proof which we present below. The second aim of this paper is to prove a result one would expect, but appears to be new. Namely, we prove that the $C_{1,q}$ -capacity of a set, enlarged by adding all finite linear combinations of a basis in the generalized Cameron-Martin space, remains zero if it was zero for (slightly bigger) $C_{r,p}$ -capacities, $r > 1$, $p > q$, a priori (cf. Theorem 3.3 below). Let us first describe our framework, in which we strictly follow [2].

Let E be a separable Banach space over \mathbb{R} . Let E' denote its dual and $\mathcal{B}(E)$ its Borel σ -algebra. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space such that $H \subset E$ continuously and densely.

Identifying H with its dual H' by Riesz's isomorphism, we have

$$E' \subset H \subset E \quad (1.1)$$

where both embeddings are continuous and dense. In particular, it follows for the dualization $\langle \cdot, \cdot \rangle_E : E' \times E \rightarrow \mathbb{R}$ that

$$\langle l, h \rangle_E = \langle l, h \rangle_H \text{ for all } l \in E', h \in H. \quad (1.2)$$

Furthermore

$$H = \{z \in E \mid \sup\{\langle l, z \rangle_E \mid l \in E' \text{ with } \|l\|_H \leq 1\} < \infty\}. \quad (1.3)$$

The norm in E and H we denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ respectively.

Let

$$\mathcal{FC}_b^\infty := \{g(l_1, \dots, l_N) \mid N \in \mathbb{N}, g \in C_b^\infty(\mathbb{R}^N), l_1, \dots, l_N \in E'\}, \quad (1.4)$$

where $C_b^\infty(\mathbb{R}^N)$ denotes the set of all infinitely differentiable bounded functions with all derivatives bounded. For $u \in \mathcal{FC}_b^\infty(\mathbb{R}^N)$ and $z \in E$ we define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), h \rangle_H = \frac{\partial u}{\partial h}(z) := \frac{d}{dt} u(z + th)|_{t=0}. \quad (1.5)$$

Let μ be a probability measure on $(E, \mathcal{B}(E))$ and denote the corresponding real L^p -spaces by $L^p(E, \mu)$, $p \in [1, \infty]$, and define

$$\mathcal{E}_\mu(u, v) := \int_E \langle \nabla u(z), \nabla v(z) \rangle_H \mu(dz); \quad u, v \in \mathcal{FC}_b^\infty. \quad (1.6)$$

For a set \mathcal{D} of $\mathcal{B}(E)$ -measurable functions on E we denote the corresponding μ -classes by $\widetilde{\mathcal{D}}^\mu$. Throughout this paper we assume that the following hypothesis is fulfilled

- (H1) If $u \in \mathcal{FC}_b^\infty$ such that $u = 0$ μ -a.e., then $\nabla u = 0$ μ -a.e. and the (thus on $L^2(E, \mu)$ well-defined) positive definite symmetric bilinear form $(\mathcal{E}_\mu, \widetilde{\mathcal{FC}_b^\infty}^\mu)$ is closable on $L^2(E, \mu)$.

Under condition (H1) the Hilbert space H is sometimes called generalized Cameron-Martin space of μ . We refer e.g. to [5] for the definition of closability and denote the closure of $(\mathcal{E}_\mu, \widetilde{\mathcal{FC}_b^\infty}^\mu)$ on $L^2(E, \mu)$ by $(\mathcal{E}_\mu, H_0^{1,2}(E, \mu))$. Then $(\mathcal{E}_\mu, H_0^{1,2}(E, \mu))$ is a symmetric Dirichlet form (see e.g. [5]).

Remark 1.1 (i) For sufficient conditions for (H1) we refer to [2]. We note that those conditions are also necessary, if one requires all partial derivatives to be closable separately (see [2] for details).

(ii) Closability of the form $(\mathcal{E}_\mu, \widetilde{\mathcal{FC}_b^\infty}^\mu)$ on $L^2(E, \mu)$ is equivalent to the closability of the operator

$$\nabla : \widetilde{\mathcal{FC}_b^\infty}^\mu \subset L^2(E, \mu) \longrightarrow L^2(E \rightarrow H, \mu).$$

We denote its closure (whose domain is, of course, $H_0^{1,2}(E, \mu)$) again by ∇ .

If this operator is closable, then also for $p \geq 2$

$$\nabla : \widetilde{\mathcal{FC}_b^\infty}^\mu \subset L^p(E, \mu) \longrightarrow L^p(E \rightarrow H, \mu).$$

is closable. Indeed, if $u_n \rightarrow 0$ in $L^p(E, \mu)$ as $n \rightarrow \infty$ and $(\nabla u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(E \rightarrow H, \mu)$, then the same holds in $L^2(E, \mu)$ and $L^2(E \rightarrow H, \mu)$ respectively. By assumption it follows that $\nabla u_n \rightarrow 0$ in $L^2(E \rightarrow H, \mu)$ as $n \rightarrow \infty$, hence in μ -measure, so by Fatou's Lemma

$$\int \|\nabla u_n\|_H^p d\mu \leq \liminf_{m \rightarrow \infty} \int \|\nabla u_n - \nabla u_m\|_H^p d\mu.$$

But the right hand side can be made arbitrarily small.

(iii) Assuming that for $p \geq 1$

$$\nabla : \widetilde{\mathcal{F}C_b^\infty} \subset L^p(E, \mu) \longrightarrow L^p(E \rightarrow H, \mu) \text{ is closable,} \quad (1.7)$$

we can prove all what follows for $p \geq 1$ instead of $p = 2$ with entirely similar proofs. For simplicity we restrict, however, to the case $p = 2$. The definition of capacities, however, we give for all $p \geq 1$ below.

(iv) We refer to [2] and [1] for examples for μ satisfying (H1). These examples include the white noise measure on E , i.e. the centered Gaussian measure on $(E, \mathcal{B}(E))$ with Cameron-Martin space H . But many other Gaussian measures and moreover Gibbs measures from statistical mechanics are included.

If for $p \in [1, \infty)$ condition (1.7) holds, we denote the closure by $(\nabla, H_0^{1,p}(E, \mu))$. For notational convenience we then set as usual for $p \geq 1$

$$\|u\|_{1,p} := \left(\int (\|\nabla u\|_H^p + |u|^p) d\mu \right)^{1/p}, \quad u \in H_0^{1,p}(E, \mu). \quad (1.8)$$

Now we recall the definition of capacity and intrinsic metric.

Definition 1.1. (i) For $U \subset E$, U open, and $p \in [1, \infty)$, we set

$$C_{1,p}(U) := \inf\{\|u\|_{1,p}^p \mid u \in H_0^{1,p}(E, \mu), u \geq 1 \text{ } \mu\text{-a.e. on } U\}$$

and for arbitrary $A \subset E$

$$C_{1,p}(A) := \inf\{C_{1,p}(U) \mid A \subset U\}.$$

$C_{1,p}(A)$ is called capacity of A .

(ii) A function $f : A \mapsto \mathbb{R}$, $A \subset E$, is called $C_{1,p}$ -quasicontinuous if there exist closed sets $A_n \subset A$, $n \in \mathbb{N}$, such that $f \upharpoonright_{A_n}$ is continuous for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} C_{1,p}(E \setminus A_n) = 0$.

Definition 1.2. For $x, y \in E$ set

$$\rho(x, y) := \sup\{f(x) - f(y) \mid f \in \mathcal{F}C_b^\infty \text{ with } \|\nabla f\|_H \leq 1\}.$$

ρ is called intrinsic metric of $(\mathcal{E}_\mu, H_0^{1,2}(E, \mu))$.

The following is well-known. The proof is easy and included for the reader's convenience.

Lemma 1.3. Let $x, y \in E$. Then

$$\rho(x, y) = \|x - y\|_H,$$

where we set $\|z\|_H := +\infty$ if $z \in E \setminus H$.

Proof. Let $f \in \mathcal{F}C_b^\infty$ with $\|\nabla f\|_H \leq 1$ and assume $x - y \in H$. Then

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \int_{E'} \langle Df(y + t(x - y)), x - y \rangle_E dt \\ &= \int_0^1 \langle \nabla f(y + t(x - y)), x - y \rangle_H dt \\ &\leq \|x - y\|_H. \end{aligned}$$

Here Df denotes the Fréchet derivative of f . So,

$$\rho(x, y) \leq \|x - y\|_H \text{ for all } x, y \in E.$$

Conversely, let $\chi_n \in C_b^\infty(\mathbb{R})$ satisfying $|\chi_n| \leq n + 1$, $|\chi_n'| \leq 1$, $\chi_n(s) = s$ for all $s \in [-n, n]$. The for $l \in E'$ such that $\|l\|_H \leq 1$ we have $\|\nabla \chi_n(l)\|_H \leq 1$ for all $n \in \mathbb{N}$, hence for $x, y \in E$

$$\begin{aligned} \int_{E'} \langle l, x - y \rangle_E &\leq \sup_n (\chi_n(\int_{E'} \langle l, x \rangle_E) - \chi_n(\int_{E'} \langle l, y \rangle_E)) \\ &\leq \rho(x, y). \end{aligned}$$

So, by (1.3)

$$\|x - y\|_H \leq \rho(x, y).$$

□

For $A \subset E$ as usual we set

$$\rho_A(x) := \inf\{\rho(x, y) \mid y \in A\}, \quad x \in E.$$

Now we can formulate the main result of this section which we shall prove in the next section.

Theorem 1.4. *Assume hypothesis (H1) holds. Let $A \in \mathcal{B}(E)$ such that $\mu(A) = 1$. Then $C_{1,2}(\rho_A > 0) = 0$, i.e., the ρ -closure of A has full $C_{1,2}$ -capacity.*

2. PROOF OF THEOREM 1.4

Throughout this section hypothesis (H1) is assumed to hold. Before we can prove Theorem 1.4, we need the following lemma.

Lemma 2.1. *Let $K \subset E$ be $\|\cdot\|_E$ -compact and $c \in (0, \infty)$. Then ρ_K is $\mathcal{B}(E)$ -measurable and*

$$\rho_K \wedge c \in H_0^{1,2}(E, \mu) \text{ and } \|\nabla(\rho_K \wedge c)\|_H \leq 1.$$

Furthermore, $\rho_K \wedge c$ is $C_{1,2}$ -quasicontinuous.

Proof. Let $\{e_i \mid i \in \mathbb{N}\} \subset E'$ be an orthonormal basis of H separating the points of E , and for $n \in \mathbb{N}$ define $P_n : E \mapsto E_n := \text{span}\{e_1, \dots, e_n\}$ by

$$P_n z := \sum_{i=1}^n \int_{E'} \langle e_i, z \rangle_E e_i, \quad z \in E.$$

Fix $y \in E$. By a simple approximation argument on E_N we see that

$$u_n(x) := \|P_n x - P_n y\|_H \wedge c, \quad x \in E,$$

is a function in $H_0^{1,2}(E, \mu)$ with $\|\nabla u_n\|_H \leq 1$. Clearly

$$u_n(x) \uparrow v_y(x) := \|x - y\|_H \wedge c \text{ for all } x \in E. \tag{2.1}$$

Hence by [5, Chap. I, Lemma 2.12] for $y \in E$, $v_y \in H_0^{1,2}(E, \mu)$ with $\|\nabla v_y\|_H \leq 1$ and $u_n \rightarrow v_y$ weakly in $H_0^{1,2}(E, \mu)$, hence the Cesaro mean of a subsequence converges strongly in $H_0^{1,2}(E, \mu)$. A standard argument of Egorov type for capacities (cf. [5, Chap.III, Sect. 3]) implies that selecting a subsequence if necessary, this Cesaro-mean converges $C_{1,2}$ -quasiuniformly, i.e., uniformly on closed sets whose compliments have arbitrarily small $C_{1,2}$ -capacity. Hence by (2.1) v_y is $C_{1,2}$ quasicontinuous.

Claim. Let $\dim E < \infty$. Then the assertions of the lemma hold even without assuming K to be compact.

Since $\dim E < \infty$, we have $H = E$ and $\|\cdot\|_H$ and $\|\cdot\|_E$ are equivalent norms. Let $\{y_n \mid n \in \mathbb{N}\}$ be a countable $\|\cdot\|_H$ -dense subset of K and defining

$$v_N := \inf\{v_{y_1}, \dots, v_{y_N}\}, \quad N \in \mathbb{N},$$

we have

$$\rho_K \wedge c = \inf_N v_N \quad \text{on } E.$$

Furthermore, (cf. e.g. [5, Chap. IV, Sect 4]) $v_N \in H_0^1(E, \mu)$ with

$$\|\nabla v_N\|_H \leq \sup\{\|\nabla v_{y_1}\|_H, \dots, \|\nabla v_{y_N}\|_H\},$$

hence

$$\|\nabla v_N\|_H \leq 1 \quad \text{for all } N \in \mathbb{N}.$$

Therefore, the claim follows by the same arguments as above.

Now we go back to the general case. First we show that for all $x \in E$

$$\rho_{P_n K}(P_n x) \uparrow \rho_K(x) \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

(So, in particular, ρ_K is $\mathcal{B}(E)$ -measurable.)

Let $x \in E$. Obviously, $\rho_{P_n K}(P_n x)$ is increasing with n and

$$\sup_n \rho_{P_n K}(P_n x) \leq \rho_K(x)$$

(cf. (2.1)). To prove the dual inequality we may assume that $\sup_n \rho_{P_n K}(P_n x) < \infty$. Let $a \in (0, \infty)$ such that

$$\sup_n \rho_{P_n K}(P_n x) < a.$$

Then there exist $k_n \in K$ such that

$$\|P_n x - P_n k_n\|_H < a \quad \text{for all } n \in \mathbb{N}. \quad (2.3)$$

Since balls in H are weakly compact and K is compact in E , we can find a subsequence such that $k_{n_j} \xrightarrow{j \rightarrow \infty} k \in K$ w.r.t. $\|\cdot\|_E$ and $P_{n_j} x - P_{n_j} k_{n_j} \xrightarrow{j \rightarrow \infty} h \in H$ weakly in H . Hence for all $i \in \mathbb{N}$ by (1.2)

$$\begin{aligned} {}_{E'}\langle e_i, h \rangle_E &= \lim_{j \rightarrow \infty} {}_{E'}\langle e_i, P_{n_j} x - P_{n_j} k_{n_j} \rangle_E \\ &= \lim_{j \rightarrow \infty} {}_{E'}\langle e_i, x - k_{n_j} \rangle_E \\ &= {}_{E'}\langle e_i, x - k \rangle_E. \end{aligned}$$

Since $\{e_i \mid i \in \mathbb{N}\}$ separates the points of E , it follows that $x - k = h$ and by (2.3) that

$$\|x - k\|_H \leq \liminf_{j \rightarrow \infty} \|P_{n_j} x - P_{n_j} k_{n_j}\|_H \leq a. \quad (2.4)$$

In particular, $\rho_K(x) < \infty$.

Now suppose that for some $\varepsilon \in (0, \rho_K(x))$

$$\sup_n \rho_{P_n K}(P_n x) < \rho_K(x) - \varepsilon.$$

Then applying the above with $a := \rho_K(x) - \varepsilon$, we get a contradiction from (2.4). So, (2.2) is proved. Now the assertions follow from the claim and the same arguments used for its proof. \square

Proof of Theorem 1.4. By inner regularity there exist compact sets $K_n \subset A$, $n \in \mathbb{N}$, such that $\mu(K_n) \uparrow \mu(A) = 1$. Let

$$u_n := \rho_{K_n} \wedge 1, \quad n \in \mathbb{N}.$$

Then

$$u := \inf_n u_n = \lim_{n \rightarrow \infty} u_n \geq \rho_A \wedge 1 \quad \text{on } E. \quad (2.5)$$

Furthermore, $u = 0$ on $\bigcup_{n \geq 1} K_n$, hence $u = 0$ μ -a.e.

By Lemma 2.1 and the same arguments as in its proof we obtain that u is $C_{1,2}$ -quasi-continuous, hence by [5, Chap. III, Proposition 3.9] $C_{1,2}(\{u > 0\}) = 0$. But by (2.5), $\{\rho_A > 0\} \subset \{u > 0\}$ and the assertion is proved. \square

3. A RESULT ON SETS WITH ZERO CAPACITY

Let $\{h_1, h_2, \dots\}$ be an ONB of H and let E_n denote the linear span of $\{h_1, \dots, h_n\}$ and set $K := \bigcup_n E_n$. A sufficient condition on A for $\mu(A + K) = 0$ is given in [4]. Now we look for a condition which implies $C_{p,1}(A + K) = 0$. But we have to work under an additional quasi-invariance hypothesis.

For $k \in H$ define

$$\tau_k(z) := z - k, \quad z \in E. \quad (3.1)$$

(H2) For all $k \in H$, μ is k -quasi-invariant, i.e., $\mu \circ \tau_{sk}^{-1} \cong \mu$ for all $s \in \mathbb{R}$, and we assume the Radon-Nikodym derivatives

$$a_{sk}^\mu := \frac{d(\mu \circ \tau_{sk}^{-1})}{d\mu}, \quad s \in \mathbb{R},$$

to have the following properties:

(H2a) $a_{sk}^\mu \in \bigcap_{q \geq 1} L^q(E; \mu)$, for all $s \in \mathbb{R}$, and for all $q \in [1, \infty)$ the function $s \mapsto \|a_{sk}^\mu\|_q$ is locally bounded on \mathbb{R} .

(H2b) For all compact $C \subset \mathbb{R}$

$$\int_C \frac{1}{a_{sk}^\mu(z)} ds < \infty \quad \text{for } \mu\text{-a.e. } z \in E.$$

Here ds denotes Lebesgue measure on \mathbb{R} .

Choosing appropriate versions by [1, Prop. 2.4] we may always assume that $a_{sk}^\mu(z)$ is jointly measurable in s and z and that (H2b) holds for all $z \in E$ (rather than only μ -a.e. $z \in E$).

For examples of measures μ satisfying condition (H2) we refer to [6, Section 3]. As shown in [2] hypothesis (H2) implies (H1).

We need Sobolev spaces with differentiability index higher than 1. Analogue to the gradient operator ∇ , we define the iterated gradient ∇^2 on $\mathcal{F}C_b^\infty$ by

$$\langle \nabla^2 u(z), h_1 \otimes h_2 \rangle_{H \otimes H} := \frac{d^2}{dt ds} u(z + th_1 + s_2 h_2)|_{t=0}. \quad (3.2)$$

Assume that

$$(H3) \quad \nabla^2 : \widetilde{\mathcal{F}C_b^\infty}^\mu \subset L^p(E, \mu) \longrightarrow L^p(E \rightarrow H \otimes H, \mu) \text{ is closable for } p \geq 1. \quad (3.3)$$

We define

$$H_0^{2,p} := \left\{ u \in L^p \mid \|u\|_{2,p} := \left(\int (\|\nabla u\|_{H \otimes H}^p + \|\nabla u\|_H^p + |u|^p) d\mu \right)^{1/p} < \infty \right\} \quad (3.4)$$

and the fractional Sobolev spaces $H_0^{r,p}$ ($1 < p < \infty$, $1 < r < 2$) are defined by real interpolation as follows.

Definition 3.1. For $1 < p < \infty$, $1 < r < 2$ we define

$$H_r^p = (H_0^{1,p}, H_0^{2,p})_{r-1,p} \quad (3.5)$$

where (\cdot, \cdot) denotes the real interpolation space, see e.g. [3, 9].

The norm in $H_0^{r,p}$ is given by the discrete K-method:

$$\|f\|_{r,p} = \left(\sum_{n=1}^{\infty} |2^{n(r-1)} K(2^{-n}, f)|^p \right)^{1/p} < \infty. \quad (3.6)$$

where

$$K(\epsilon, f) = \inf \{ \|f_1\|_{1,p} + \epsilon \|f_2\|_{2,p}, f_1 + f_2 = f, f_1 \in H_0^{1,p}, f_2 \in H_0^{2,p} \} \quad (3.7)$$

It follows by a standard interpolation argument that $H_0^{r,p}$ is uniformly convex (see e.g. [4]) and we know from the denseness of $\mathcal{F}C_b^\infty$ in $H_0^{2,p}$ and [3, Th. 3.4.2] that $\mathcal{F}C_b^\infty$ is dense in $H_0^{r,p}$. A combination of these two facts implies that every $u \in H_0^{r,p}$ has a $C_{r,p}$ -quasicontinuous redefinition which we denote by \tilde{u} .

For $1 < p < \infty$, $1 \leq r \leq 2$ we define, for a $[0, \infty]$ -valued lower semi-continuous functions h on E

$$C_{r,p}(h) := \inf \{ \|u\|_{r,p}^p; u \in H_0^{r,p}, u \geq h, a.e. \},$$

and for an arbitrary $[-\infty, \infty]$ -valued function f on E

$$C_{r,p}(f) := \inf \{ C_{r,p}(h); h \text{ is l.s.c. and } h(x) \geq |f(x)| \forall x \}.$$

This definition is an extension of the previous one for sets in the sense that for any $B \subset E$,

$$C_{r,p}(B) = C_{r,p}(1_B).$$

The following result is parallel to Shigekawa [8] which is stated for Bessel capacities. We omit the proof which is the same as in [8]. Note that (3.8) is implicit in [8].

Theorem 3.2. Fix $q \in (1, \infty)$, $r \in (1, 2]$, $\beta \in (0, 1]$ and $0 \leq \gamma < \beta/q$. Then there exists a constant $C = C(q, r, \beta, \gamma)$ such that If $\xi : [0, T]^n \times E \mapsto \mathbb{R}$, $(t, z) \mapsto \xi_t(z)$ is measurable and if

$$A(\xi) := \sup_{s \neq t, s, t \in [0, T]^n} \frac{\|\xi_t - \xi_s\|_{r,q}^q}{|t - s|^{n+\beta}} < \infty,$$

then $\{\xi_t\}$ has a version $\{\tilde{\xi}_t\}$ such that $\tilde{\xi}_t$ is $C_{r,q}$ -quasi-continuous for every $t \in [0, T]^n$ and

$$C_{r,q} \left(\sup_{s \neq t} \frac{|\tilde{\xi}_t - \tilde{\xi}_s|}{|t - s|^\gamma} \right) \leq CA(\xi). \quad (3.8)$$

Now we state the main result of this section.

Theorem 3.3. *Suppose (H2) and (H3) hold and let $A \subset E$, $p > 1$. If for any n there exists a pair $(r_n, p_n) \in (1, 2] \times (1, p)$ with $r_n > np_n^{-1} + 1$ such that*

$$C_{r_n, p_n}(A) = 0, \quad \forall n,$$

then $C_{1,p}(A + K) = 0$.

Since capacities are continuous from below Theorem 3.3 immediately follows from Proposition 3.5 below. First we need a lemma.

Fix $n \in \mathbb{N}$ and define for $t \in [-M, M]^n$ and $f : E \mapsto \mathbb{R}$ a function $f(t) := f(\cdot + \sum_{i=1}^n t_i h_i)$.

Lemma 3.4. *Fix $n \in \mathbb{N}$ and let $p > q > 1$ and $r \in (1, 2]$. Then there exists a constant $C := C(p, q, r, T)$ such that for all $f \in H_0^{r,p}$*

$$\|f(t) - f(s)\|_{1,q} \leq C \|f\|_{r,p} |t - s|^{r-1} \quad (3.9)$$

for all $s, t \in [0, T]^n$.

Proof. By the same argument as in [6], we can prove that for any $T > 0$, $p > 1$, $q \in (0, p)$ there exists a constant $C_1 = C_1(p, q, T)$ such that for all $f \in H_0^{2,p}$ and $(s, t) \in [-T, T]^n \times [-T, T]^n$

$$\|f(\cdot + \sum_{i=1}^n t_i h_i) - f(\cdot + \sum_{i=1}^n s_i h_i)\|_{1,q} \leq C_1 |t - s| \|f\|_{2,p}. \quad (3.10)$$

Now let $f \in H_0^{r,p} \setminus \{0\}$. By (3.6), there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset H_0^{2,p}$ such that

$$\sum_{n=1}^{\infty} 2^{(r-1)pn} \left(2^{-n} \|f_n\|_{2,p} + \|f - f_n\|_{1,p} \right)^p \leq 2^p \|f\|_{r,p}^p. \quad (3.11)$$

Choose the unique $n \in \mathbb{N}$ such that

$$2^{-n}T < |t - s| \leq 2^{-n+1}T.$$

Then by (3.10), (3.11) for some constant $C_2 = C_2(p, q, T)$

$$\begin{aligned} \|f(t) - f(s)\|_{1,q} &\leq \|f(t) - f_n(t)\|_{1,q} + \|f_n(t) - f_n(s)\|_{1,q} + \|f_n(s) - f(s)\|_{1,q} \\ &\leq 4C_2 \|f\|_{r,p} 2^{-(r-1)n} + \|f_n\|_{2,p} 2^{-n+1} T C_1 \\ &\leq 4C_2 \|f\|_{r,p} 2^{-(r-1)n} (1 + T C_1) \\ &\leq \frac{4C_2(1 + T C_1)}{T^{r-1}} |t - s|^{r-1} \|f\|_{r,p} \end{aligned} \quad (3.12)$$

and the assertion is proved. \square

Proposition 3.5. *Let $T > 0$, $n \in \mathbb{N}$, $p > 1$ and $r \in (1, 2]$ such that $r > np^{-1} + 1$. Let $q \in (1, p)$ such that $r > nq^{-1} + 1$. Then there exists a constant $C = C(n, p, q, r, T) > 0$ such that for any $A \subset W$ we have*

$$C_{1,q}(A + M(T; h_1, \dots, h_n)) \leq C \cdot C_{r,p}^{\frac{q}{p}}(A) \quad (3.13)$$

where

$$M(T; h_1, \dots, h_n) := \left\{ \sum_{i=1}^n s_i h_i, |s_i| \leq T \right\}.$$

Proof. Set $sh := \sum_{i=1}^n s_i h_i$. By changing signs we only need to prove

$$C_{1,q}\left(\bigcup_{s \in [0, T]^n} (A + sh) \right) \leq C \cdot C_{r,p}^{\frac{q}{p}}(A). \quad (3.14)$$

Let I denote the set of all rational points in $[0, T]^n$. If $O \subset W$ is an open set then so is $\bigcup_{s \in [0, T]^n} (O + sh)$ and we have

$$\bigcup_{s \in [0, T]^n} (O + sh) = \bigcup_{s \in I} (O + sh)$$

Let e_O denote the (r, p) -equilibrium potential of O . Since $e_O \geq 1_O$, we have

$$\begin{aligned} C_{1,q}\left(\bigcup_{t \in [0, T]^n} (O + th) \right) &= C_{1,q}\left(\sup_{t \in I} 1_O(\cdot + th) \right) \\ &\leq C_{1,q}\left(\sup_{t \in I} e_O(\cdot + th) \right) \end{aligned}$$

We set $f(t) := e_O(\cdot + th)$. Applying Lemma 3.4 gives

$$\|f(t) - f(s)\|_{1,q} \leq C \|e_O\|_{r,p} |t - s|^{r-1}. \quad (3.15)$$

For $\gamma \in [0, r - 1 - nq^{-1}]$, by Theorem 3.2 there exists a $C_{1,q}$ -quasicontinuous modification $\{\xi_t(\cdot), t \in [0, T]^n\}$ of $\{e_O(\cdot + th), t \in [0, T]^n\}$ such that

$$C_{1,q} \left[\sup_{s, t \in [0, T]^n, s \neq t} \left(\frac{|\xi_t - \xi_s|}{|t - s|^\gamma} \right) \right] \leq C \|e_O\|_{r,p}^q$$

for some constant $C = C(n, p, q, r, T)$ which may be different from that in (3.15). In particular, taking $\gamma = 0$ we obtain

$$C_{1,q}\left(\sup_{s \neq t} |\xi_t - \xi_s| \right) \leq C \cdot \|e_O\|_{r,p}^q.$$

Hence

$$\begin{aligned} C_{1,q}\left(\sup_{t \in I} e_O(\cdot + th) \right) &= C_{1,q}\left(\sup_{t \in I} |\xi_t| \right) \\ &\leq C_{1,q}\left(\sup_{t \in [0, T]} |\xi_t - \xi_0| + |\xi_0| \right) \\ &\leq (C + 1) \cdot \|e_O\|_{r,p}^q \\ &= (C + 1) \cdot C_{r,p}^{\frac{q}{p}}(O). \end{aligned}$$

Thus (3.13) is proved for open sets. For general A , we have for $O \supset A$, O open

$$\begin{aligned} C_{1,q}\left(\bigcup_{s \in [0,T]^n} (A + sh)\right) &\leq C_{1,q}\left(\bigcup_{s \in [0,T]^n} (O + sh)\right) \\ &\leq (C + 1) \cdot C_{r,p}^{\frac{q}{p}}(O). \end{aligned} \quad (3.16)$$

Consequently,

$$\begin{aligned} C_{1,q}\left(\bigcup_{s \in [0,T]^n} (A + sh)\right) &\leq (C + 1) \cdot \inf\{C_{r,p}^{\frac{q}{p}}(O), O \supset A\} \\ &= (C + 1) \cdot C_{r,p}^{\frac{q}{p}}(A), \end{aligned} \quad (3.17)$$

as desired. \square

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