ON MORREY'S ESTIMATE OF SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. We give a complete proof of Morrey's estimate for $W^{1,p}$-norms of solutions of second order elliptic equations on a domain in terms of their L_1-norms. In addition, we investigate dependence of the constants in this estimate on the coefficients of the equation.

The purpose of this work is to provide a complete proof of an important result of Morrey announced in his book [1] only with a brief hint and giving an estimate of the $W^{1,p}$-norm of a solution of a second order elliptic equation on a domain in terms of the coefficients of the equation and the L_1-norm of the solution. In addition, we investigate dependence of the constants in this estimate on the coefficients of the equation. The precise formulations are given in the theorem below and its corollary (see also the concluding remark). It states that if a function u in the Sobolev class $W^{1,q}(G)$ satisfies the elliptic equation

$$\partial_{x_i}(a^{ij}\partial_{x_j}u + b^i u) - c^i\partial_{x_i}u - du = f - \partial_{x_i}e^i$$

in the weak sense (in the form of the integral identity below), where the usual summation rule is used, the matrix-valued mapping $(a^{ij})_{i,j\le n}$ is continuous, uniformly bounded and uniformly nondegenerate, the vector-valued mappings (b^i), (e^i), (c^i), and scalar functions d and f are integrable in suitable degrees, then one has

$$\|u\|_{W^{1,q}(G)} \le C\bigl(\|e\|_{L_q(G)} + \|f\|_{L_p(G)} + \|u\|_{L_1(G)}\bigr).$$

This result has found interesting applications in the study of weak elliptic equations for measures undertaken in [2], [3], and therefore it is important to have a complete proof. The main feature of this estimate, as compared to many related estimates found in the extensive literature on the subject (see, e.g., [4], [5], [6], [7]), is that it estimates the $W^{1,q}(G)$-norm of u via the $L_1(G)$-norm of u on the same domain. It is much easier to estimate the $W^{1,q}(G)$-norm of u via the $L_q(G)$-norm or to estimate the $W^{1,q}(G')$-norm of u on a subdomain $G'\subset G$ with compact closure via the $L_1(G)$-norm on the larger domain G. In addition, we investigate dependence of the constants in these estimates on the coefficients of the equation.

We retain the notation of [1], according to which $W^{1,q}(G)$ is the Sobolev class of all functions on a domain G that belong to $L_q(G)$ together with their weak first order derivatives. Let $B(x,r)$ denote the open ball of radius r centered at x. Throughout we use the symbols $\|f\|^0_{q,r}$, $\|f\|^0_{q,D}$, $\|f\|^1_{q,r}$ and $\|f\|^1_{q,D}$ to denote the norms in $L_q\bigl(B(0,r)\bigr)$, $L_q(D)$, $W^{1,q}\bigl(B(0,r)\bigr)$ and $W^{1,q}(D)$, respectively, where D is a domain in R^n. The standard symbols like $C(D)$ and $C^1(D)$ are employed to denote the classes of continuous and continuously differentiable functions. The subindex 0 in the notation like $C_0(D)$ means compact support. Let $u_{x_i} := \partial_{x_i}u$ stand for the partial derivative (possibly, in the Sobolev sense) with respect

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to the variable x_i. Set

$$G(0,r) = B(0,r) \cap \{(x_1, x_2, \dots, x_n)\colon\ x_n \geq 0\}.$$

Let G be a bounded domain of class C^1 in R^n. Let a function $u \in W^{1,q}(G)$, where $q > 1$, satisfy the integral identity

$$\int\limits_G v_{x_i}(a^{ij}u_{x_j} + b^i u + e^i) + v(c^i u_{x_i} + du + f)\,dx = 0 \tag{1}$$

for all $v \in C_0^1(G)$, where the coefficients a^{ij} are such that, for positive constants m, M, one has $m|\xi|^2 \leq \sum\limits_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq M|\xi|^2$ for all $x \in \bar{G}, \xi \in R^n$. The assumptions on the functions b^i, c^i, e^i, d, and f will be specified below, but in any case they are supposed to be measurable and integrable enough in order the above identity could make sense.

Definition 1. *We say that the coefficients a, b, c, and d satisfy the H_q^1-condition on a domain Γ if $a^{ij} \in C(\Gamma)$, and b^i, c^i, and d are measurable and*

(a) *if $\frac{n}{n-1} < q < n$, then $b^i, c^i \in L_n(\Gamma)$, $d \in L_{\frac{n}{2}}(\Gamma)$,*

(b) *if $q = n \geq 2$, then $\int\limits_{\Gamma \cap B(x,r)} (|b|^n + |c|^n + |d|^{\frac{n}{2}})\,dx \leq L_1^n r^{n\mu_1}$ for some $\mu_1 > 0$ and all $B(x,r)$,*

(c) *if $q > n$, then $b^i \in L_q(\Gamma)$, $c^i \in L_n(\Gamma)$, $d \in L_p(\Gamma)$, $p = \frac{nq}{n+q}$.*

We set

$$Q_r(e)(y) = \sum_{i=1}^{n} \int\limits_{B(0,r)} K_{x_i}(x-y)e^i(x)\,dx,$$

$$P_r(f)(y) = \int\limits_{B(0,r)} K(x-y)f(x)\,dx,$$

where

$$K(y) = \begin{cases} \dfrac{-1}{n(n-2)\alpha(n)|y|^{n-2}}, & n > 2 \\ \dfrac{1}{2\pi}\ln|y|, & n = 2. \end{cases}$$

Lemma 1. *The functions $P_r(f)$ and $Q_r(e)$ have the following properties:*

(a) *If $n = 2, q = 2$, then*

$\|P_r(f)\|_{2,r}^0 \leq C(\mu)L(2r)^{\mu+1}(1 + \ln\frac{1}{2r})$

whenever $\int\limits_{B(0,r)\cap B(x,\rho)} |f(y)|\,dy \leq L\rho^\mu$ for some $\mu > 0$ and all $B(x,\rho)$, $0 < r < \frac{1}{2}$;

if $n = 2, q > 2$, then

$\|P_r(f)\|_{q,r}^0 \leq C(n,q)\|f\|_{q,r}^0 r^2(1 + \ln(\frac{1}{2r}))$ *for all $f \in L_q\big(B(0,r)\big)$, $0 < r < \frac{1}{2}$;*

if $n \geq 2$, $1 \leq q < n$, then

$\|P_r(f)\|_{\frac{nq}{n-q},r}^0 \leq C(A,n,q)r\|f\|_{q,r}^0$ *for all $f \in L_q\big(B(0,r)\big)$ and $0 < r < A < \frac{1}{2}$;*

If $n > 2, q \geq 1$, then

$\|P_r(f)\|^0_{q,r} \le C(n,q)r^2\|f\|^0_{q,r}$ *for all* $f \in L_q\big(B(0,r)\big)$*;*

(b) *If* $n = 2, q = 2,$ *then*

$\|\nabla P_r(f)\|^0_{2,r} \le C(\mu)L(2r)^\mu$

provided that $\displaystyle\int\limits_{B(0,r)\cap B(x,\rho)} |f(y)|\,dy \le L\rho^\mu$ *for some* $\mu > 0$ *and all* $B(x,\rho)$*;*

if $n \ge 2,\ q \ge 1,$ *then*

$\|\nabla P_r(f)\|^0_{q,r} \le C(n,q)r\|f\|^0_{q,r}$ *for all* $f \in L_q\big(B(0,r)\big)$*;*

if $n \ge 2,\ 1 < q < n,$ *then*

$\|\nabla P_r(f)\|^0_{\frac{nq}{n-q},r} \le C(n,q,A)\|f\|^0_{q,r}$ *for all* $f \in L_q\big(B(0,r)\big)$ *and* $0 < r < A.$

(c) *If* $n \ge 2,\ q \ge 1,$ *then*

$\|Q_r(e)\|^0_{q,r} \le C(n,q)r\|e\|^0_{q,r}$ *for all* $e \in L_q\big(B(0,r)\big)$*;*

if $n \ge 2,\ q > 1,$ *then*

$\|\nabla Q_r(e)\|^0_{q,r} \le C(n,q)\|e\|^0_{q,r}$ *for all* $e \in L_q\big(B(0,r)\big)$*.*

Proof. 1. The fourth inequality in (a), the second inequality in (b) and the first inequality in (c) can be deduced from the following assertion [4, p. 157]:

if $1 \le q \le \infty$ and $f(x) \in L_p(\Omega)$, $0 \le \delta = \frac{1}{p} - \frac{1}{q} < \lambda$, and

$$V_\lambda(f) = \int\limits_\Omega |x-y|^{n(1-\lambda)} f(y)\,dy,$$

then one has

$$\|V_\lambda(f)\|_{L_q(\Omega)} \le \Big(\frac{1-\delta}{\lambda-\delta}\Big)^{(1-\delta)} \alpha(n)^{(1-\lambda)}|\Omega|^{\lambda-\delta}\|f\|_{L_q(\Omega)}.$$

In order to deduce the desired inequalities from this result, we note that, if $n > 2$, then $|K(x-y)| \le C(n)|x-y|^{2-n}$; if $n \ge 2$, then $|K_{x_i}(x-y)| \le C(n)|x-y|^{1-n}$. We set in the above assertion $\Omega = B(0,r)$, $\lambda = \frac{2}{n}$, $\delta = 0$ in the fourth inequality from (a), $\lambda = \frac{1}{n}$, $\delta = 0$ in the second inequality from (b) and the first enequality from (c).

2. Let us prove the first inequality from (a). Let us continue the function f by zero outside of $B(0,r)$ and set $\varphi(\rho) = \int\limits_{B(x,\rho)} |f(y)|\,dy$. Then

$$\int\limits_{B(0,r)} \big|\ln|x-y|\big|\,|f(y)|\,dy = \int\limits_0^{2r} |\ln\rho|\varphi'(\rho)\,d\rho \le \ln(\frac{1}{2r})\varphi(2r) + \int\limits_0^{2r} \rho^{-1}\varphi(\rho)\,d\rho$$
$$\le L(2r)^\mu \ln(\frac{1}{2r}) + \mu^{-1}L(2r)^\mu.$$

Integrating with respect to x we obtain the required estimate.

Let us prove the second inequality from (a). Note that if $f \in L_q(B(0,r))$, $n = 2, q > 2$, then

$$\int\limits_{B(0,r)\cap B(x,\rho)} |f(y)|\,dy \le c(n)\|f\|^0_{q,r}\rho^{2-\frac{2}{q}} \text{ for all } B(x,\rho).$$

Repeating the proof of the first inequality from (a) with $L = c(n)\|f\|^0_{q,r}$ and $\mu = 2 - \frac{2}{q}$ we obtain

$$\int\limits_{B(0,r)} \bigl|\ln|x-y|\bigr|\,|f(y)|\,dy \le c(n)\|f\|^0_{q,r}(2r)^{2-\frac{2}{q}}\ln(\frac{1}{2r}) + (2-\frac{2}{q})^{-1}\|f\|^0_{q,r}(2r)^{2-\frac{2}{q}}.$$

Integrating with respect to x we obtain the required estimate.

3. Let us prove the first inequality from (b). One has

$$\Bigl(\int\limits_{B(0,r)} |x-y|^{-1}|f(y)|\,dy\Bigr)^2 \le \int\limits_{B(0,r)} |x-y|^{-2\sigma}|f(y)|\,dy \int\limits_{B(0,r)} |x-y|^{2(\sigma-1)}|f(y)|\,dy,$$

where $\sigma \in (0,\mu/2)$. In order to estimate the first multiplier, we continue f by zero outside $B(0,r)$ and define $\varphi(\rho)$ as above. Then

$$\int\limits_{B(0,r)} |x-y|^{-2\sigma}|f(y)|\,dy = \int\limits_0^{2r} \rho^{-2\sigma}\varphi'(\rho)\,d\rho$$

$$= (2r)^{-2\sigma}\varphi(2r) + 2\sigma\int\limits_0^{2r} \rho^{-2\sigma-1}\varphi(\rho)\,d\rho \le L(2r)^{(\mu-2\sigma)} + 2\sigma(\mu-2\sigma)^{-1}L(2r)^{\mu-2\sigma}.$$

Therefore,

$$\Bigl(\int\limits_{B(0,r)}\Bigl(\int\limits_{B(0,r)} |x-y|^{-1}|f(y)|\,dy\Bigr)^2 dx\Bigr)^{\frac{1}{2}}$$

$$\le \Bigl(\int\limits_{B(0,r)}\Bigl(\int\limits_{B(0,r)} |x-y|^{-2\sigma}|f(y)|\,dy\Bigr)\Bigl(\int\limits_{B(0,r)} |x-y|^{2(\sigma-1)}|f(y)|\,dy\Bigr)dx\Bigr)^{\frac{1}{2}}$$

$$\le C(\mu,\sigma)L^{\frac{1}{2}}(2r)^{(\frac{1}{2}\mu-\sigma)}\Bigl(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |x-y|^{2(\sigma-1)}|f(y)|\,dy\,dx\Bigr)^{\frac{1}{2}}$$

$$\le C(\mu,\sigma)L(2r)^{\mu-\sigma}\Bigl(\int\limits_{B(0,r)} |x-y|^{2(\sigma-1)}\,dx\Bigr)^{\frac{1}{2}} \le C(\mu,\sigma)L(2r)^{\mu}.$$

4. Let us prove the third enequality from (b). Let us extend the function f by zero outside of $B(0,r)$. Then $P_r(f)(x) = P_A(f)(x)$, $\|f\|^0_{q,r} = \|f\|^0_{q,A}$. Using the fact that has already been proved in Steps 1,2,3 the estimate $\|\nabla^2 P_A(f)\|^0_{q,A} \le C(n,q)\|f\|^0_{q,A}$ (see [4, p. 217]), and Sobolev's inequality for $q<n$, we have

$$\|\nabla P_r(f)\|^0_{\frac{nq}{n-q},r} \le \|\nabla P_A(f)\|^0_{\frac{nq}{n-q},A} \le C_1(B(0,A),q,n)\|\nabla P_A(f)\|^1_{q,A}$$
$$\le C(B(0,A),q,n)\|f\|^0_{q,A} = C(B(0,A),q,n)\|f\|^0_{q,r}.$$

We conclude that the third inequality from (b) is proved.

In order to prove the second inequality from (c) using again that if $f \in L_q, q > 1$, then $P_A(f) \in W^{2,q}\big(B(0,A)\big)$ and

$$\|\nabla^2 P_A(f)\|^0_{q,A} \leq C(n,q)\|f\|^0_{q,A},$$

which is proved in [4, p. 217]. Noting that $Q_r(e)(y) = \sum\limits_{i=1}^{n} -\partial_{y_i} P_r(e^i)(y)$, we obtain the second inequality in (c).

5. Applying Sobolev's inequality for $q < n$ and the second inequality from (b) we obtain the third inequality from (a). This completes the proof of Lemma 1.

Lemma 2. *Suppose the coefficients of integral identity* (1) *satisfy the H^1_q-condition on $B(0,A)$, $0 < A < \frac{1}{2}$ with $q \geq \frac{n}{n-1}$ where an equality holds only if q=n=2. There is $r \in (0,A)$ such that if $u \in W^{1,q}\big(B(0,r)\big)$ satisfies equation* (1) *on $B(0,r)$, $\operatorname{supp} u \in B(0,r)$, $P_r(f) \in W^{1,q}\big(B(0,r)\big)$, $e \in L_q\big(B(0,r)\big)$, $\operatorname{supp} f \in B(0,r)$, $\operatorname{supp} e \in B(0,r)$, $a^{ij}(0) = \delta^{ij}$, then*

$$\|u\|_{W^{1,q}(B(0,r))} \leq C(\|e\|_{L_q(B(0,r))} + \|P_r(f)\|_{W^{1,q}(B(0,r))}),$$

where $C = C\big(a,b,c,d,B(0,A),r,q,n\big)$.

Proof. 1. Let u satisfy integral identity (1). If u is fixed, then identity (1) with $G = B(0,r)$ holds for all $v \in C^1_0\big(B(0,r)\big)$. Since u, e, f have compact support in $B(0,r)$, identity (1) holds for all $v \in C^1\big(B(0,r)\big)$. Let us consider the convolution ψ of the functions $\varphi \in C^1_0$ and K given by $\psi(x) = \int\limits_{B(0,r)} K(x-y)\varphi(y)\,dy$. Clearly, ψ is a C^1-function, moreover $\psi_{x_i}(x) = \int\limits_{B(0,r)} K_{x_i}(x-y)\varphi(y)\,dy$. Substituting ψ in place of v in (1) we have

$$\int\limits_{B(0,r)}\int\limits_{B(0,r)} \Big(K_{x_i}(x-y)\varphi(y)[a^{ij}(x)u_{x_j}(x) + b^i(x)u(x) + e^i(x)]$$
$$+ K(x-y)\varphi(y)[c^i(x)u_{x_i}(x) + d(x)u(x) + f(x)]\Big)\,dy\,dx = 0.$$

According to Fubini's theorem we obtain

$$\int\limits_{B(0,r)} \varphi(y) \int\limits_{B(0,r)} \Big(K_{x_i}(x-y)[a^{ij}u_{x_j} + b^i u + e^i] + K(x-y)[c^i u_{x_i} + du + f]\Big)\,dx\,dy = 0.$$

Since this equality holds for all $\varphi \in C^1_0\big(B(0,r)\big)$, we have for almost every $y \in B(0,r)$

$$\int\limits_{B(0,r)} \Big(K_{x_i}(x-y)[a^{ij}u_{x_j} + b^i u + e^i] + K(x-y)[c^i u_{x_i} + du + f]\Big)\,dx = 0. \tag{2}$$

Since $u \in W_0^{1,q}\big(B(0,r)\big)$, according to [4, p. 158] we obtain

$$u(y)=\sum_{i=1}^{n}\frac{1}{n\alpha(n)}\int\limits_{B(0,r)}\frac{(y_i-x_i)u_{x_i}(x)}{|x-y|^n}\,dx.$$

By using the function K, one can represent u as

$$u(y)=\int\limits_{B(0,r)}-K_{x_i}(x-y)\delta^{ij}u_{x_i}(x)\,dx. \tag{3}$$

Summing (2) and (3) we obtain

$$u(y)=\int\limits_{B(0,r)}K_{x_i}(x-y)[(a^{ij}-\delta^{ij})u_{x_j}+b^iu+e^i]\,dy+\int\limits_{B(0,r)}K(x-y)[c^iu_{x_i}+du+f]\,dx. \tag{4}$$

According to the definition of the functions $Q_r(e)$ and $P_r(f)$, identity (4) can be represented in the following way:

$$u(x)=Q_r((a-\delta)\nabla u+bu+e)+P_r(c\nabla u+du+f). \tag{5}$$

2.A. Let $\frac{n}{n-1}<q<n$, then $p=\frac{nq}{n+q}>1$. Let us note that $p=\frac{nq}{n+q}<n$. By Lemma 1 and (5) we obtain

$$\|u\|^0_{q,r}\le\|Q_r((a-\delta)\nabla u+bu+e)\|^0_{q,r}+\|P_r(c\;\nabla u+du)\|^0_{\frac{np}{n-p},r}+\|P_r(f)\|^0_{q,r}$$

$$\le C(n,q,A)r\Big(\max_{x\in B(0,r)}|a(x)-\delta|\|\nabla u\|^0_{q,r}+\|bu\|^0_{q,r}+\|e\|^0_{q,r}+\|c\;\nabla u\|^0_{p,r}+\|du\|^0_{p,r}\Big)$$

$$+\|P_r(f)\|^0_{q,r} \tag{6}$$

and

$$\|\nabla u\|^0_{q,r}\le\|\nabla Q_r((a-\delta)\nabla u+bu+e)\|^0_{q,r}+\|\nabla P_r(c\;\nabla u+du)\|^0_{\frac{np}{n-p},r}+\|\nabla P_r(f)\|^0_{q,r}$$

$$\le C(n,q,A)\Big(\max_{x\in B(0,r)}|a(x)-\delta|\|\nabla u\|^0_{q,r}+\|bu\|^0_{q,r}+\|e\|^0_{q,r}+\|c\;\nabla u\|^0_{p,r}+\|du\|^0_{p,r}\Big)$$

$$+\|\nabla P_r(f)\|^0_{q,r}. \tag{7}$$

Let us estimate separately the right-hand sides in these inequalities.

(a) By virtue of Hölder's inequality and the imbedding theorem for Sobolev spaces, we obtain

$$\|bu\|^0_{q,r}=\Big(\int\limits_{B(0,r)}|b(x)|^q|u(x)|^q\,dx\Big)^{\frac{1}{q}}$$

$$\le\Big(\int\limits_{B(0,r)}|b(x)|^n\,dx\Big)^{\frac{1}{n}}\Big(\int\limits_{B(0,r)}|u(x)|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}}\le C(n,q)\|b\|^0_{n,r}\|u\|^1_{q,r}.$$

(b) According to Hölder's inequality with the indices $\frac{n+q}{q}$ and $\frac{n+q}{n}$ we have

$$\|c\ \nabla u\|^0_{p,r} = \Big(\int\limits_{B(0,r)} |c(x)|^{\frac{nq}{n+q}} |\nabla u(x)|^{\frac{nq}{n+q}}\,dx\Big)^{\frac{n+q}{nq}}$$
$$\le \Big(\int\limits_{B(0,r)} |c(x)|^n\,dx\Big)^{\frac{1}{n}} \Big(\int\limits_{B(0,r)} |\nabla u(x)|^q\,dx\Big)^{\frac{1}{q}} \le \|c\|^0_{n,r}\|u\|^1_{q,r}.$$

(c) According to Hölder's inequality with the indices $\frac{n+q}{n-q}$ and $\frac{n+q}{2q}$ and the imbedding theorem, we obtain

$$\|du\|^0_{p,r} = \Big(\int\limits_{B(0,r)} |d(x)|^{\frac{nq}{n+q}} |u(x)|^{\frac{nq}{n+q}}\,dx\Big)^{\frac{n+q}{nq}}$$
$$\le \Big(\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}\,dx\Big)^{\frac{2}{n}} \Big(\int\limits_{B(0,r)} |u(x)|^{\frac{nq}{n-q}}\,dx\Big)^{\frac{n-q}{nq}} \le C(n,q)\|d\|^0_{\frac{n}{2},r}\|u\|^1_{q,r}.$$

By using these estimates and summing (6) and (7), we obtain

$$\|u\|^1_{q,r} \le C(n,q,A)(r+1)\Big[\Big(\max_{x\in B(0,r)}|a(x)-\delta| + \|b\|^0_{n,r} + \|c\|^0_{n,r} + \|d\|^0_{\frac{n}{2},r}\Big)\|u\|^1_{q,r} + \|e\|^0_{q,r}\Big]$$
$$+ \|P_r(f)\|^1_{q,r},$$

$$\|u\|^1_{q,r}\Big(1 - C(n,q,A)(r+1)\big[\max_{x\in B(0,r)}|a(x)-\delta| + \|b\|^0_{n,r} + \|c\|^0_{n,r} + \|d\|^0_{\frac{n}{2},r}\big]\Big)$$
$$\le C(n,q,A)(r+1)(\|e\|^0_{q,r} + \|P_r(f)\|^1_{q,r}).$$

By using the continuity of a and the absolute continuity of the Lebesgue integral, we choose $0<r<A$ such that

$$C_1(r) = 1 - C(n,q,A)(r+1)\big[\max_{x\in B(0,r)}|a(x)-\delta| + \|b\|^0_{n,r} + \|c\|^0_{n,r} + \|d\|^0_{\frac{n}{2},r}\big] > 0.$$

Then

$$\|u\|^1_{q,r} \le \frac{C(n,q,A)(r+1)}{C_1(r)}(\|e\|^0_{q,r} + \|P_r(f)\|^1_{q,r}).$$

2.B. Let $q>n$. In this case inequalities (6) and (7) hold true. Let us obtain some additional estimates.

(a) By the imbedding theorem for $q>n$ (see [4, p. 154]) we have

$$\|bu\|^0_{q,r} \le C(n,q)r^{\frac{q-n}{q}}\|b\|^0_{q,r}\|u\|^1_{q,r}.$$

(b) In a similar way we get the estimate

$$\|du\|^0_{p,r} \le C(n,q)r^{\frac{q-n}{q}}\|d\|^0_{p,r}\|u\|^1_{q,r}.$$

(c) In the same way as in part 2.A. (b) we obtain the estimate

$$\|c\ \nabla u\|^0_{p,r} \le \|c\|^0_{n,r}\|u\|^1_{q,r}.$$

By using this estimates and summing (6) and (7), we find

$$\|u\|^1_{q,r} \le C(n,q,A)(r+1)\Big[\Big(\max_{x\in B(0,r)}|a(x)-\delta| + r^{\frac{q-n}{q}}\|b\|^0_{q,r} + \|c\|^0_{n,r}$$

$$+r^{\frac{q-n}{q}}\|d\|^0_{p,r}\Big)\|u\|^1_{q,r} + \|e\|^0_{q,r}\Big] + \|P_r(f)\|^1_{q,r},$$

$$\|u\|^1_{q,r}\Big(1 - C(n,q,A)(r+1)\Big[\max_{x\in B(0,r)}|a(x)-\delta| + \ r^{\frac{q-n}{q}}\|b\|^0_{q,r} + \|c\|^0_{n,r} + r^{\frac{q-n}{q}}\|d\|^0_{p,r}\Big]\Big)$$

$$\le C(n,q,A)(r+1)\big(\|e\|^0_{q,r} + \|P_r(f)\|^1_{q,r}\big).$$

In the same way as in part 2.A with the corresponding constant $C_1(r)$, we obtain the required estimate.

2.C. Let $q = n > 2$. In this case inequalities (6) and (7) hold. Let us obtain some additional estimates.

(a) In order to estimate $\|bu\|^0_{n,r}$ we consider $(\|bu\|^0_{n,r})^n$. We use the well-known inequality

$$|u(x)| \le C(n)\int\limits_{B(0,r)} |x-y|^{1-n}|\nabla u(y)|\,dy,$$

which follows from (3). Then

$$\int\limits_{B(0,r)} |b(x)|^n|u(x)|^n\,dx = \int\limits_{B(0,r)} |b(x)|^n|u(x)|^{n-1}|u(x)|\,dx$$

$$\le C(n)\int\limits_{B(0,r)} |b(x)|^n|u(x)|^{n-1}\int\limits_{B(0,r)} |x-y|^{1-n}|\nabla u(y)|\,dy\,dx$$

$$= C(n)\int\limits_{B(0,r)}\int\limits_{B(0,r)} |b(x)|^n|u(x)|^{n-1}|x-y|^{1-n}|\nabla u(y)|\,dy\,dx$$

$$= C(n)\int\limits_{B(0,r)}\int\limits_{B(0,r)} \Big(|b(x)|^{n-1}|u(x)|^{n-1}|x-y|^{(1-\sigma)(1-n)}\Big)\Big(|\nabla u(y)||b(x)||x-y|^{\sigma(1-n)}\Big)\,dy\,dx.$$

According to Hölder's inequality the right-hand side is estimated by

$$C(n)\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |b(x)|^n|u(x)|^n|x-y|^{-n(1-\sigma)}\,dy\,dx\Big)^{\frac{n-1}{n}}\times$$

$$\times\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |b(x)|^n|\nabla u(y)|^n|x-y|^{n\sigma(1-n)}\,dy\,dx\Big)^{\frac{1}{n}}.$$

Let us estimate every multiplier separately. For the first multiplier we have

$$\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |b(x)|^n|u(x)|^n|x-y|^{-n(1-\sigma)}\,dy\,dx\Big)^{\frac{n-1}{n}}$$
$$\le \Big(\int\limits_{B(0,r)} |b(x)|^n|u(x)|^n\Big(\int\limits_{B(0,r)} |x-y|^{-n(1-\sigma)}\,dy\Big)\,dx\Big)^{\frac{n-1}{n}}$$
$$\le C(\sigma,n)r^{(n-1)\sigma}\Big(\int\limits_{B(0,r)} |b(x)|^n|u(x)|^n\,dx\Big)^{\frac{n-1}{n}}.$$

In order to estimate the second multiplier, we first estimate the following integral. Let

$$\varphi(\rho)=\int\limits_{B(y,\rho)} |b(x)|^n\,dx$$

and let b be continued by zero outside $B(0,r)$. Then we obtain

$$\int\limits_{B(0,r)} |b(x)|^n|x-y|^{n\sigma(1-n)}\,dx=\int\limits_0^{2r}\rho^{n\sigma(1-n)}\varphi'(\rho)\,d\rho$$
$$=(2r)^{n\sigma(1-n)}\varphi(2r)+n(n-1)\sigma\int\limits_0^{2r}\rho^{n(1-n)\sigma-1}\varphi(\rho)\,d\rho$$
$$\le L_1^n(2r)^{n(\mu_1-(n-1)\sigma)}\Big(1+(n-1)\sigma(\mu_1-(n-1)\sigma)^{-1}\Big)\le C(n,\mu_1,\sigma)L_1^n(2r)^{n(\mu_1-(n-1)\sigma)},$$

where $0<\sigma<\mu_1/(n-1)$. Now the second multiplier is estimated in the following way:

$$\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |b(x)|^n|\nabla u(y)|^n|x-y|^{n\sigma(1-n)}\,dy\,dx\Big)^{\frac{1}{n}}\le C(n,\mu_1,\sigma)L_1(2r)^{\mu_1-(n-1)\sigma}\|u\|^1_{n,r}.$$

These estimates yield that

$$\|bu\|^0_{n,r}\le C(n,\mu_1)L_1(2r)^{\mu_1}\|u\|^1_{n,r}.$$

(b) The estimate $\|c\ \nabla u\|^0_{\frac{n}{2},r}\le\|c\|^0_{n,r}\|u\|^1_{n,r}$ is obtained in the same way as in 2.A.

(c) Let us estimate $\|du\|^0_{\frac{n}{2},r}$. By using the same inequality as in (a) we find

$$\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n}{2}}\,dx = \int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n-2}{2}}|u(x)|\,dx$$
$$\le C(n)\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n-2}{2}}\int\limits_{B(0,r)} |x-y|^{1-n}|\nabla u(y)|\,dy\,dx$$
$$= C(n)\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n-2}{2}}|x-y|^{1-n}|\nabla u(y)|\,dy\,dx$$
$$= C(n)\int\limits_{B(0,r)}\int\limits_{B(0,r)} \Big(|d(x)|^{\frac{n-2}{2}}|u(x)|^{\frac{n-2}{2}}|x-y|^{(1-n)(1-\sigma)}\Big)\Big(|d(x)|^{\frac{1}{2}}|x-y|^{(1-n)\sigma+\delta}\Big)$$
$$\Big(|\nabla u(y)||d(x)|^{\frac{1}{2}}|x-y|^{-\delta}\Big)\,dy\,dx$$
$$\le C(n)\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n}{2}}|x-y|^{+\frac{n(1-n)(1-\sigma)}{n-2}}\,dy\,dx\Big)^{\frac{n-2}{n}}$$
$$\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|x-y|^{n(1-n)\sigma+n\delta}\,dy\,dx\Big)^{\frac{1}{n}}\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|\nabla u(y)|^n|x-y|^{-n\delta}\,dy\,dx\Big)^{\frac{1}{n}}$$

Let us estimate every multiplier. The first multiplier is estimated as follows:

$$\Big(\int\int |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n}{2}}|x-y|^{\frac{n(1-n)(1-\sigma)}{n-2}}\,dy\,dx\Big)^{\frac{n-2}{n}}$$
$$\le C(\sigma,n)r^{(n-1)\sigma-1}\Big(\int\limits_{B(0,r)} |d(x)|^{\frac{n}{2}}|u(x)|^{\frac{n}{2}}\,dx\Big)^{\frac{n-2}{n}}.$$

Now let us estimate the second multiplier. We have

$$\Big(\int\int |x-y|^{n(1-n)\sigma+n\delta}|d(x)|^{\frac{n}{2}}\,dy\,dx\Big)^{\frac{1}{n}}$$
$$\le \Big(\int\Big(\int |x-y|^{n(1-n)\sigma+n\delta}\,dy\Big)|d(x)|^{\frac{n}{2}}\,dx\Big)^{\frac{1}{n}}$$
$$\le C(n,\mu_1,\sigma)L_1 r^{(1-n)\sigma+\delta+1}r^{\mu_1} = C(n,\mu_1,\sigma)L_1 r^{(1-n)\sigma+\delta+1+\mu_1}.$$

Now let us estimate the third multiplier. We have

$$\Big(\int\int |d(x)|^{\frac{n}{2}}|\nabla u(y)|^n|x-y|^{-n\delta}\,dy\,dx\Big)^{\frac{1}{n}}$$
$$\le \Big(\int |\nabla u(y)|^n\Big(\int |d(x)|^{\frac{n}{2}}|x-y|^{-n\delta}\,dx\Big)\,dy\Big)^{\frac{1}{n}}$$
$$\le \|u\|^1_{n,r}C(n,\mu_1,\delta)L_1 r^{\mu_1-\delta}.$$

Let $0<\delta<\mu_1$, $\frac{1}{n-1}<\sigma<\frac{\delta+1}{n-1}$. Applying the obtained estimates we find

$$\|du\|^0_{\frac{n}{2},r} \le C(n,\mu_1)L_1^2(r)^{2\mu_1}\|u\|^1_{n,r}.$$

Summing (6) and (7), we have

$$\|u\|^1_{q,r} \le C(n,q,A,\mu_1)(r+1)\Big[\Big(\max_{x\in B(0,r)} |a(x)-\delta| + 2L_1 r^{\mu_1} + L_1^2 r^{2\mu_1}\Big)\|u\|^1_{q,r} + \|e\|^0_{q,r}\Big] + \|P_r(f)\|^1_{q,r}.$$

In the same way as in part 2.A we obtain the required estimate with the corresponding constant $C_1(r)$ given by

$$C_1(r) = 1 - C(n,q,A,\mu_1)(r+1)\Big[\max_{x\in B(0,r)} |a(x)-\delta| + 2L_1 r^{\mu_1} + L_1^2 r^{2\mu_1}\Big].$$

2.D. Let $n=q=2$, $p=1$. By using representation (5), we obtain

$$\begin{aligned}\|u\|^1_{2,r} &\le \|Q_r((a-\delta)\nabla u + bu + e)\|^0_{2,r} + \|P_r(c\,\nabla u + du)\|^0_{2,r} + \|P_r(f)\|^0_{2,r} \\ &+\|\nabla Q_r((a-\delta)\nabla u + bu + e)\|^0_{2,r} + \|\nabla P_r(c\,\nabla u + du)\|^0_{2,r} + \|\nabla P_r(f)\|^0_{2,r}. \qquad (8)\end{aligned}$$

In order to apply Lemma 1, we have to show that the functions $c\nabla u$ and du satisfy the conditions required in that lemma. According to the hypotheses of the present lemma we have

$$\int\limits_{B(0,r)} |c(x)||\nabla u(x)|\,dx \le \Big(\int\limits_{B(0,r)} |c(x)|^2\,dx\Big)^{\frac12}\Big(\int\limits_{B(0,r)} |\nabla u(x)|^2\,dx\Big)^{\frac12} \le L_1 r^{\mu_1}\|u\|^1_{2,r}.$$

By the Cauchy inequality

$$\int\limits_{B(0,r)} |d(x)||u(x)|\,dx \le \Big(\int\limits_{B(0,r)} |d(x)|\,dx\Big)^{\frac12}\Big(\int\limits_{B(0,r)} |d(x)||u(x)|^2\,dx\Big)^{\frac12}.$$

Let us consider the second multiplier. Since $|u(x)| \le C \int\limits_{B(0,r)} |x-y|^{-1}|\nabla u(y)|\,dy$, one has

$$\begin{aligned}\int\limits_{B(0,r)} |d(x)||u(x)|^2\,dx &\le C \int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)||u(x)||\nabla u(y)||x-y|^{-1}\,dy\,dx \\ &\le C\Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)||u(x)|^2|x-y|^{2(1-\sigma)}\,dy\,dx\Big)^{\frac12} \\ &\qquad \Big(\int\limits_{B(0,r)}\int\limits_{B(0,r)} |d(x)||\nabla u(y)|^2|x-y|^{-2\sigma}\,dy\,dx\Big)^{\frac12},\end{aligned}$$

which is estimated by

$$C(\mu_1,\sigma)L_1 r^{\mu_1}\|u\|^1_{2,r}\Big(\int\limits_{B(0,r)} |d(x)||u(x)|^2\,dx\Big)^{\frac12}.$$

Thus we conclude that

$$\int\limits_{B(0,r)} |d(x)||u(x)|\,dx \le C(\mu_1)L_1^2 r^{2\mu_1}\|u\|^1_{2,r}.$$

The estimate of $\|bu\|^0_{2,r}$ follows by the estimate from 2.C(a) for $n=2$, i.e.,

$$\|bu\|^0_{2,r} \le C(\mu_1)L_1 r^{\mu_1}\|u\|^1_{2,r}.$$

By using Lemma 1 and applying the obtained estimates, we have

$$\begin{aligned}\|u\|^1_{2,r} \le Cr\Big(\max_{x\in B(0,r)} |a(x)-\delta|\|u\|^1_{2,r} + C(\mu 1, A)L_1 r^{\mu_1}\|u\|^1_{2,r} + \|e\|^0_{2,r}\Big)\\ + C(\mu_1, A)L_1(2r)^{\mu_1+1}\big(1-\ln(2r)\big)\big(1+c(\mu_1,A)\big)\|u\|^1_{2,r}\\ + C\Big(\max_{x\in B(0,r)} |a(x)-\delta|\|u\|^1_{2,r} + C(\mu_1, A)L_1 r^{\mu_1}\|u\|^1_{2,r} + \|e\|^0_{2,r}\Big)\\ + C(\mu_1, A)L_1(2r)^{\mu_1}\big(1+c(\mu_1,A)\big)\|u\|^1_{2,r} + \|P_r(f)\|^1_{2,r}.\end{aligned}$$

Hence

$$\begin{aligned}\|u\|^1_{2,r}\Big(1 - C(r+1)\Big[\max_{x\in B(0,r)}|a(x)-\delta| + C(\mu_1,A)L_1 r^{\mu_1}\Big]\\ - C(\mu_1,A)L_1(2r)^{\mu_1}\big(1+r-r\ln(2r)\big)\big(1+c(\mu_1,A)\big)\Big)\\ \le C(r+1)C(\mu_1,A)L_1(2r)^{\mu_1}\big(1+r-r\ln(2r)\big)\big(1+c(\mu_1,A)\big)\big(\|e\|^0_{2,r} + \|P_r(f)\|^1_{2,r}\big).\end{aligned}$$

Choosing $r>0$ so that

$$\begin{aligned}0 < 1 - C(r+1)\Big[\max_{x\in B(0,r)}|a(x)-\delta| + C(\mu_1,A)L_1 r^{\mu_1}\Big]\\ - C(\mu_1,A)L_1(2r)^{\mu_1}\big(1+r-r\ln(2r)\big)\big(1+c(\mu_1,A)\big),\end{aligned}$$

we obtain the required estimate. Lemma 2 is proved.

Lemma 3. *Suppose that in the hypotheses of Lemma 2 we replace $B(0,A)$ by $G(0,A)$ and $B(0,r)$ by $G(0,r)$. Let the functions u, f, e vanish on the spherical part of the surface of the hemisphere $G(0,r)$, but not necessarily on its bottom $\sigma_r = G(0,r)\cap\{x_n=0\}$ and let $P_r(f)$ be the restriction to $G(0,r)$ of the former $P_r(\bar f)$ where $\bar f$ is the extension to $B(0,r)$ by the "negative reflection with respect to x_n". Then the conclusion of Lemma 2 holds.*

Proof. Let us set

$$\begin{aligned}D(x,y) &= D(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)\\ &= K\Big(\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n+y_n)^2}\Big),\end{aligned}$$

where K is the function defined before Lemma 1. Let $\varphi\in C^1_0\big(G(0,r)\big)$. Then the function

$$\psi(x) = \int\limits_{G(0,r)} \big(K(x-y)-D(x,y)\big)\varphi(y)\,dy$$

belongs to $C^1\big(\bar G(0,r)\big)$ and $\psi(x_1,x_2,\dots,x_n)=0$ if $x_n=0$. We extend ψ by zero outside of $G(0,r)$. Let us fix u satisfying (1). Then integral identity (1) holds for all $v\in C^1_0\big(G(0,r)\big)$. Let $w\in C^1\big(\bar G(0,r)\big)$ and let w vanish on the spherical part of the boundary of the hemisphere $G(0,r)$ (but not necessarily on its bottom σ_r). Then we can replace v by w in (1). In order to prove this fact we extend w by zero outside of $G(0,r)$ and consider the functions

$$w_\epsilon(x_1,x_2,\dots,x_n) = w(x_1,x_2,\dots,x_n-\epsilon).$$

Since w_ϵ belongs to the space $\mathrm{Lip}_0\big(G(0,r)\big)$ of Lipschitzian functions with compact support in $G(0,r)$, we can replace v in (1) by w_ϵ. Letting $\epsilon \to 0$ we obtain that the identity holds for w. Then we can replace v in (1) by ψ. Letting $v = \psi$ in the integral identity and using Fubini's theorem we find

$$\int\limits_{G(0,r)} \int\limits_{G(0,r)} \Big(\big(K_{x_i}(x-y) - D_{x_i}(x,y)\big)\varphi(y)[a^{ij}(x)u_{x_j}(x) + b^i(x)u(x) + e^i(x)]$$
$$+ \big(K(x-y) - D(x,y)\big)\varphi(y)[c^i(x)u_{x_i}(x) + d(x)u(x) + f(x)]\Big)\, dy\, dx = 0.$$

Hence

$$\int\limits_{G(0,r)} \varphi(y) \int\limits_{G(0,r)} \Big(\big(K_{x_i}(x-y) - D_{x_i}(x,y)\big)[a^{ij}u_{x_j} + b^i u + e^i]$$
$$+ \big(K(x-y) - D(x,y)\big)[c^i u_{x_i} + du + f]\Big)\, dx\, dy = 0.$$

Since this holds for all $\varphi \in C_0^1\big(G(0,r)\big)$, we have for almost every $y \in G(0,r)$

$$\int\limits_{G(0,r)} \Big(\big(K_{x_i}(x-y) - D_{x_i}(x,y)\big)[a^{ij}u_{x_j} + b^i u + e^i]$$
$$+\big(K(x-y) - D(x,y)\big)[c^i u_{x_i} + du + f]\Big)\, dx = 0. \tag{9}$$

Therefore,

$$\int\limits_{G(0,r)} K_{x_i}(x-y)[a^{ij}u_{x_j} + b^i u + e^i] + K(x-y)[c^i u_{x_i} + du + f]\, dx$$
$$- \int\limits_{G(0,r)} \Big(D_{x_i}(x,y)[a^{ij}u_{x_j} + b^i u + e^i] + D(x,y)[c^i u_{x_i} + du + f]\Big)\, dx = 0.$$

Let us consider the second integral and make the change of variables $z_i = x_i$, $i = 1, 2, \ldots, n$, $z_n = -x_n$. Let

$$G^-(0,r) = B(0,r) \cap \{x_n \le 0\}, \quad \vec{z}_{n-1} = (z_1, z_2, \ldots, z_{n-1}).$$

Then we obtain the integral

$$- \int\limits_{G^-(0,r)} \Big(D_{z_i}(\vec{z}_{n-1}, -z_n, y)[a^{ij}u_{z_j} + b^i u + e^i] + D(\vec{z}_{n-1}, -z_n, y)[c^i u_{z_i} + du + f]\Big)\, dz.$$

Note that $D(\vec{z}_{n-1}, -z_n, y) = K(z-y)$ and $D_{z_i}(\vec{z}_{n-1}, -z_n, y) = K_{z_i}(z-y)$ if $1 \le i < n$, $D_{z_n}(\vec{z}_{n-1}, -z_n, y) = -K_{z_i}(z-y)$ if $i = n$.

Let us extend $[a^{ij}u_{x_j} + b^i u + e^i]$ and $[c^i u_{x_i} + du + f]$ to $G^-(0,r)$ in the following way:

(a) $[a^{ij}u_{x_j} + b^i u + e^i](x_1, x_2, \ldots, x_n) = [a^{ij}u_{x_j} + b^i u + e^i](x_1, x_2, \ldots, -x_n)$ if $x_n < 0$ and $1 \le i < n$,

$[a^{ij}u_{x_j} + b^i u + e^i](x_1, x_2, \ldots, x_n) = -[a^{ij}u_{x_j} + b^i u + e^i](x_1, x_2, \ldots, -x_n)$ if $x_n < 0$ and $i = n$,

(b) $[c^i u_{x_i} + du + f](x_1, x_2, \ldots, x_n) = -[c^i u_{x_i} + du + f](x_1, x_2, \ldots, -x_n)$ if $x_n < 0$.

Then we obtain

$$\int\limits_{B(0,r)} \Big(K_{x_i}(x-y)[a^{ij}u_{x_j}+b^iu+e^i]+K(x-y)[c^iu_{x_i}+du+f]\Big)\,dx=0. \qquad (10)$$

Let us extend the function u to $G^-(0,r)$ as follows: $u(x_1,x_2,\dots,x_n)=u(x_1,x_2,\dots,-x_n)$ if $x_n<0$. The functions a^{ij},b^i,c^i,d are extended so that conditions (a) and (b) be fulfilled (it is obvious that the extension of a^{ij} is continuous at zero because $a^{ij}(0)=\delta^{ij}$). Then formula (3) holds for the new function u. Summing (10) and (3) we obtain the representation of u in form (4) from Lemma 2. The rest of the proof coincides with Step 2 of Lemma 2. Lemma 3 is proved.

Theorem. *Let G be a C^1-domain and let the coefficients of integral identity* (1) *satisfy the H^1_q-condition on $\Gamma\supset G$ with $q\ge\frac{n}{n-1}$, where an equality holds only if $q=n=2$. Suppose also that $e\in L_q(G)$, $f\in L_p(G)$, where $p=\frac{qn}{n+q}\ge 1$, and that the function f satisfies the following condition: $\int\limits_{\Gamma\cap B(x,r)}|f(y)|\,dy\le Lr^\mu$ for some $\mu>0$ and all $B(x,r)$ if $p=1$. Let $u\in W^{1,q}(G)$ satisfy* (1) *on G. Then*

$$\|u\|^1_{q,G}\le C\big(\|e\|^0_{q,G}+\|f\|^0_{p,G}+\|u\|^0_{1,G}\big).$$

If $q=n=2$, so $p=1$, the term $\|f\|^0_{p,G}$ must be replaced by L. The constant C depends only on n,m,M,q,G, and the functions a,b,c,d.

Proof. 1. At every point $x_0\in\bar G$ there are neighborhoods $U(x_0)$ and $W(x_0)$, $U(x_0)\subset W(x_0)\subset\Gamma$ and a one-to-one mapping ψ_{x_0} such that $W(x_0)$ and $U(x_0)$ are mapped onto $B(0,A)$ and $B(0,r)$, $0<r<A$, $A<\frac12$, respectively, if x_0 is an inner point, and the indicated neighborhoods are mapped onto $G(0,A)$ and $G(0,r)$ in the case of a boundary point. The number r will be chosen in Step 3. Moreover, the mapping ψ_{x_0} has the following properties:

(a) ψ_{x_0} and $\psi^{-1}_{x_0}$ belong to C^1,

(b) $\tilde a^{ij}\big(\psi_{x_0}(x_0)\big)=a^{km}(x_0)\dfrac{\partial\psi_{x_0,i}}{\partial x_m}\dfrac{\partial\psi_{x_0,j}}{\partial x_k}=\delta^{ij}$,

(c) the Jacobian of ψ_{x_0} equals some constant $J(x_0)$ and $C_1<|J(x_0)|<C_2$ for all $x_0\in\bar G$, where the constants C_1,C_2 depend only on m,M.

This follows by the assumption that $m|\xi|^2\le\sum\limits_{i,j=1}^n a^{ij}(x)\xi_i\xi_j\le M|\xi|^2$ for all $x\in\bar G$, $\vec\xi\in R^n$ and $G\in C^1$.

2. Since the system of neighborhoods $\{U(x_0)\}_{x_0\in\bar G}$ is a cover of $\bar G$ and $\bar G$ is compact, we can choose a finite subcover $\{U(x_k)\}_{1\le k\le K}$. We may choose a partition of unity $\zeta_1,\zeta_2,\dots,\zeta_K$, where each ζ_k belongs to C^1 and has support in $U(x_k)$. Let $w_k=\zeta_k u$, where u satisfies integral identity (1). Let

$$\tilde e^i=\zeta_k e^i-a^{ij}\zeta_{k,x_j}u;\quad \tilde f=\zeta_k f-c^i\zeta_{k,x_i}u+\zeta_{k,x_i}(a^{ij}u_{x_j}+b^iu+e^i). \qquad (12)$$

Then w_k also satisfies (1) with the coefficients e,f replaced by $\tilde e$ and $\tilde f$. Note that the supports of $w_k,\tilde e,\tilde f$ belong to $U(x_k)$.

3. Suppose that the theorem is false. Then there are sequences $\{u_m\}_{m=1}^\infty\in W^{1,q}(G)$, $\{e_m\}_{m=1}^\infty\in L_q(G)$, $\{f_m\}_{m=1}^\infty\in L_p(G)$ (or $\{L_m\}_{m=1}^\infty$ if $p=1$) such that

1) $\|u_m\|^1_{q,G} = 1$ for all $m \in N$,

2) $\{u_m\}$ converges weakly in $W^{1,q}(G)$,

3) $\{u_m\} \to 0$ in $L_1(G)$,

4) $\{e_m\} \to 0$ in $L_q(G)$,

5) $\{f_m\} \to 0$ in $L_p(G)$ if $p > 1$ and $L_m \to 0$ if $p = 1$.

Then $u_m \to 0$ in $L_q(G)$, which follows from 2) and 3) and the embedding theorem.

We have $u_m = \sum\limits_{k=1}^{K} \zeta_k u_m = \sum\limits_{k=1}^{K} w_{m,k}$. Then $\|u_m\|^1_{q,G} \le \sum\limits_{k=1}^{K} C_k \|w_{m,k}\|^1_{q,U(x_k)}$. The function $w_{m,k}$ satisfies the integral identity

$$\int\limits_{U(x_k)} \Big(v_{x_i}(a^{ij} w_{m,k,x_j} + b^i w_{m,k} + \tilde{e}^i_m) + v(c^i w_{m,k,x_i} + d w_{m,k} + \tilde{f}_m) \Big)\, dx = 0,$$

where $\tilde{e}_m$ and $\tilde{f}_m$ are defined as above in (12). By using the mapping ψ_{x_k} from Step 1 we obtain

$$\int\limits_{B(0,r)} \Big(v_{x_i}(a^{ij} w_{m,k,x_j} + b^i w_{m,k} + \tilde{e}^i_m) + v(c^i w_{m,k,x_i} + d w_{m,k} + \tilde{f}_m) \Big)\, dx = 0, \ \ a^{ij}(0) = \delta^{ij}. \quad (13)$$

In the boundary case the same holds for $G(0,r)$ in place of $B(0,r)$. Choosing $0 < r < A$ so small that Lemma 2 and Lemma 3 apply, we obtain

$$\|w_{m,k}\|^1_{q,r} \le C_1 (\|\tilde{e}_m\|^0_{q,r} + \|P_r(\tilde{f}_m)\|^1_{q,r}). \quad (14)$$

4. Let $p > 1$. Let us estimate $\|\tilde{e}_m\|^0_{q,r}$. We have

$$\|\tilde{e}_m\|^0_{q,r} \le \|\zeta_k e_m\|^0_{q,r} + \|a \nabla\zeta_k u_m\|^0_{q,r} \le C(\zeta_k, \nabla\zeta_k, M)(\|e_m\|^0_{q,r} + \|u_m\|^0_{q,r}).$$

Hence $\|\tilde{e}_m\|^0_{q,r} \to 0$ if $m \to \infty$. Let us estimate $\|P_r(\tilde{f}_m)\|^1_{q,r}$. By using Lemma 1 and the estimates from the proof of Lemma 2 and Lemma 3, we have

$$\begin{aligned} \|P_r(\tilde{f}_m)\|^1_{q,r} \le C(\zeta_k, A, r, q, n)\big(&\|f_m\|^0_{p,r} + \|c\|^0_{n,r}\|u_m\|^0_{q,r} + \|b\|^0_{n,r}\|u_m\|^0_{q,r} + \|e_m\|^0_{q,r}\big) \\ &+ \|P_r(\zeta_{k,x_i} a^{ij} u_{m,x_j})\|^1_{q,r}. \end{aligned} \quad (15)$$

Let us consider $\|P_r(\zeta_{k,x_i} a^{ij} u_{m,x_j})\|^1_{q,r}$. Note that $t_m = \zeta_{k,x_i} a^{ij} u_{m,x_j}$ converges weakly to zero in L_q, i.e., for all $\varphi \in L_p$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int\limits_{D_r} \varphi(x) t_m(x)\, dx \to 0 \quad \text{as } m \to \infty, \quad (16)$$

where D_r stands for $B(0,r)$ or $G(0,r)$ depending on the case (without or with boundary points) we consider. If $\psi \in L_p$, then $P_r(\psi) \in L_p$, $\nabla P_r(\psi) \in L_p$, which follows by Lemma 1. Replacing φ in (16) by $P_r(\psi)$ (or by $\nabla P_r(\psi)$ in the boundary case) and using Fubini's theorem we obtain

$$\int\limits_{D_r} \Big(\int\limits_{D_r} K(x-y)\psi(y)\, dy \Big) t_m(x)\, dx = \int\limits_{D_r} \psi(y) \Big(\int\limits_{D_r} K(x-y) t_m(x)\, dx \Big)\, dy$$

and similarly for $\nabla P_r(\psi)$. Then we conclude that the sequences $P_r(t_m)$ and $\nabla P_r(t_m)$ converge weakly to zero in L_q. Taking a subsequence we conclude that the sequence $P_r(t_m)$ converges weakly to zero in $W^{1,q}$. Then it strongly converges to zero in L_q. Since

$\nabla P_r(t_m)$ converges weakly to zero in L_q and the sequence $\nabla^2 P_r(t_m)$ is uniformly bounded in L_q (see Lemma 1), we obtain taking a subsequence that $\nabla P_r(t_m)$ converges weakly to zero in $W^{1,q}$. Then it strongly converges to zero in L_q. Thus we have that $P_r(t_m)$ converges to zero in $W^{1,q}$. Consequently, according to inequality (15), we obtain

$$\|P_r(\tilde{f}_m)\|^1_{q,r} \to 0 \quad \text{as } m \to \infty.$$

Then estimate (14) yields that $\|w_{m,k}\|^1_{q,r} \to 0$ as $m \to \infty$. This holds for each $k \le K$, hence $\|u_m\|^1_{q,G} \to 0$ as $m \to \infty$. This contradicts the fact that $\|u_m\|^1_{q,G} = 1$. In the case $n = q = 2$ the proof is similar. The theorem is proved.

Given a locally integrable function u, we set

$$\|u\|_{M^{\mu_1}(\Gamma)} = \inf\Big\{L\colon \int\limits_{\Gamma\cap B(x,r)} |u(x)|\,dx \le L^n r^{n\mu_1} \quad \text{for all } B(x,r)\Big\}.$$

Definition 2. *We say that a function u belongs to $M^{\mu_1}(\Gamma)$ if u is integrable on Γ and $\|u\|_{M^{\mu_1}(\Gamma)} < \infty$.*

Definition 3. *Let $\delta > 0$. We say that the coefficients a, b, c, and d satisfy the $H^1_{q,\delta}$-condition on a domain Γ if $a^{ij} \in C^{0,\delta}(\Gamma)$, the functions b^i, c^i, and d are measurable and*

(a) *if $\frac{n}{n-1} < q < n$, then $b^i, c^i \in L_{n+\delta}(\Gamma)$, $d \in L_{\frac{n+\delta}{2}}(\Gamma)$;*

(b) *if $q = n \ge 2$, then $|b|^n, |c|^n, |d|^{\frac{n}{2}} \in M^{\mu_1}(\Gamma)$;*

(c) *if $q > n$, then $b^i \in L_q(\Gamma)$, $c^i \in L_{n+\delta}(\Gamma)$, $d \in L_p(\Gamma)$, $p = \frac{nq}{n+q}$.*

Corollary. *Let us replace H^1_q in the hypotheses of the theorem by $H^1_{q,\delta}$ with some $\delta > 0$. Suppose that $\|a\|_{C^{0,\delta}}, \|b\|, \|c\|, \|d\| \le C_0$ and $c_1 \le m \le M \le c_2$ for some positive constants c_1, c_2, C_1, where the norms $\|b\|, \|c\|, \|d\|$ are defined as follows:*

(a) *if $\frac{n}{n-1} < q < n$, then $\|b\| = \|b\|_{L_{n+\delta}(\Gamma)}, \|c\| = \|c\|_{L_{n+\delta}(\Gamma)}$, $\|d\| = \|d\|_{L_{\frac{n+\delta}{2}}(\Gamma)}$;*

(b) *if $q = n \ge 2$, then $\|b\| = \||b|^n\|_{M^{\mu_1}(\Gamma)}, \|c\| = \||c|^n\|_{M^{\mu_1}(\Gamma)}$, $\|d\| = \||d|^{\frac{n}{2}}\|_{M^{\mu_1}(\Gamma)}$;*

(c) *if $q > n$, then $\|b\| = \|b\|_{L_q(\Gamma)}$, $\|c\| = \|c\|_{L_{n+\delta}(\Gamma)}$, $\|d\| = \|d\|_{L_p(\Gamma)}$, $p = \frac{nq}{n+q}$.*

Then there exists a number $C = C(n, q, G, C_0, c_1, c_2)$ such that the conclusion of the theorem holds.

Proof. 1. Suppose that the statement is false. Then there are sequences $\{u_m\}_{m=1}^\infty \in W^{1,q}(G)$, $\{e_m\}_{m=1}^\infty \in L_q(G)$, $\{f_m\}_{m=1}^\infty \in L_p(G)$ (or $\{L_m\}_{m=1}^\infty$ if $p = 1$) such that

(1) $\|u_m\|^1_{q,G} = 1$ for all $m \in N$,

(2) $\{u_m\}$ converges weakly in $W^{1,q}(G)$,

(3) $\{u_m\} \to 0$ in $L_1(G)$,

(4) $\{e_m\} \to 0$ in $L_q(G)$,

(5) $\{f_m\} \to 0$ in $L_p(G)$ if $p > 1$ and $L_m \to 0$ if $p = 1$.

Now $u_m \to 0$ in $L_q(G)$, which follows from 2) and 3) and the embedding theorem. There are sequences $\{a_m\}_{m=1}^\infty, \{b_m\}_{m=1}^\infty$, $\{c_m\}_{m=1}^\infty, \{d_m\}_{m=1}^\infty$ such that one has

$$\|a_m\|, \|b_m\|, \|c_m\|, \|d_m\| \leq C_0$$

and

$$\int\limits_G v_{x_i}(a_m^{ij} u_{m,x_j} + b_m^i u_m + e_m^i) + v(c_m^i u_{m,x_i} + d_m u_m + f_m)\, dx = 0$$

for all $v \in C_0^1(G)$. Then repeating part 3 of the proof of the theorem and using the above notation we obtain

$$\int\limits_{B(0,r)} \Big(v_{x_i}(a_m^{ij} w_{m,k,x_j} + b_m^i w_{m,k} + \tilde{e}_m^i) + v(c_m^i w_{m,k,x_i} + d_m w_{m,k} + \tilde{f}_m)\Big)\, dx = 0,\ a_m^{ij}(0) = \delta^{ij}.$$

2. Let us prove that there exists $r \in (0, A)$ so small that estimate (14) holds with a constant C_1 independent of m.

1) Since $\|a_m(x)\|_{C^{0,\delta}(B(0,A))} \leq C(A, C_0, c_1, c_2)$, the sequence $\{a_m(x)\}$ is equicontinuous at the origin.

2) If $\frac{n}{n-1} < q < n$, then by Hölder's inequality we have

$$\|b\|_{n,r} \leq c(n) r^{\frac{n+\delta}{\delta}} \|b\|_{n+\delta,r} \leq c(n) r^{\frac{n+\delta}{\delta}} C_0,$$

$$\|c\|_{n,r} \leq c(n) r^{\frac{n+\delta}{\delta}} \|c\|_{n+\delta,r} \leq c(n) r^{\frac{n+\delta}{\delta}} C_0,$$

$$\|d\|_{\frac{n}{2}} \leq c(n) r^{\frac{2(n+\delta)}{\delta}} \|d\|_{\frac{n+\delta}{2},r} \leq c(n) r^{\frac{2(n+\delta)}{\delta}} C_0.$$

According to part 2.A of the proof of Lemma 2 and 1), 2) we have

$$C_1(r) = 1 - C(n,q,A)(r+1)\Big[\max_{x \in B(0,r)} |a(x) - \delta| + \|b\|_{n,r}^0 + \|c\|_{n,r}^0 + \|d\|_{\frac{n}{2},r}^0\Big]$$

$$\geq 1 - C(n,q,A)(r+1)\Big[C(A, C_0, c_1, c_2) r + c(n, C_0) r^{\frac{n+\delta}{\delta}} + c(n, C_0) r^{\frac{n+\delta}{\delta}} + c(n, C_0) r^{\frac{2(n+\delta)}{\delta}}\Big].$$

There exists $r \in (0, A)$ so small that $C_1(r) > \frac{1}{2}$ and

$$\|w_{m,k}\|_{q,r}^1 \leq 2C(n,q,A)(r+1)\big(\|\tilde{e}_m\|_{q,r}^0 + \|P_r(\tilde{f}_m)\|_{q,r}^1\big).$$

We obtain the required estimate.

3) If $q > n$, then by Hölder's inequality we have

$$\|c\|_{n,r} \leq c(n) r^{\frac{n+\delta}{\delta}} \|c\|_{n+\delta,r} \leq c(n) r^{\frac{n+\delta}{\delta}} C_0.$$

According to part 2.B of the proof of Lemma 2 and 1), 3) we obtain the required estimate as above.

4) If $q = n$ then, we replace L_1 by C_0 in parts 2.C and 2.D of the proof of Lemma 2. Repeating the proof of the Lemma 2 and applying 1) we obtain the required estimate. So we have

$$\|w_{m,k}\|_{q,r}^1 \leq C_1(\|\tilde{e}_m\|_{q,r}^0 + \|P_r(\tilde{f}_m)\|_{q,r}^1),$$

where C_1 depends only on $n, q, r, A, C_0, c_1, c_2$.

3. The end of the proof essentially repeats the coresponding part of the proof of the theorem. Let us additionally note that one has the following estimates:

$$\|\tilde{e}_m\|_{q,r}^0 \leq \|\zeta_k e_m\|_{q,r}^0 + \|a \nabla \zeta_k u_m\|_{q,r}^0 \leq C(\zeta_k, \nabla \zeta_k, c_2, c_1, C_0)(\|e_m\|_{q,r}^0 + \|u_m\|_{q,r}^0),$$

$$\|P_r(\tilde{f}_m)\|^1_{q,r} \le C(n,q,r,A)\|\tilde{f}_m\|^0_{p,r} \le$$
$$C(n,q,r,A,\zeta_k,\nabla\zeta_k)\big(\|f_m\|^0_{p,r} + \|c_m\|^0_{n,r}\|u_m\|^0_{q,r} + \|b\|^0_{m,r}\|u_m\|^0_{q,r} + \|e_m\|^0_{q,r}\big)$$
$$+\|P_r(\zeta_{k,x_i}a^{ij}_m u_{m,x_j})\|^1_{q,r},$$

where $\|c_m\|, \|b_m\| \le C_0$. Since $\{u_{m,x_j}\}$ converges weakly to zero in L_q and $\|\zeta_{k,x_i}a^{ij}_m\|_{C^{0,\delta}} \le C(\nabla\zeta_k, C_0, c_1, c_2)$ we obtain that the sequence $t_m = \zeta_{k,x_i}a^{ij}_m u_{m,x_j}$ converges weakly to zero in L_q. The corollary is proved.

Remark. Let us consider a collection $F(\Gamma)$ of elliptic equations on a domain $G \subset \Gamma$

$$\partial_{x_i}(a^{ij}\partial_{x_j}u + b^i u) - c^i\partial_{x_i}u - du = f - \partial_{x_i}e^i,$$

where the set $\{a\}$ is compact in $C(\Gamma)$ and there are positive numbers c_1, c_2 such that $c_1|\xi|^2 \le \sum\limits_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le c_2|\xi|^2$ for all $x \in \bar{G}, \xi \in R^n$ and $a \in \{a\}$. Suppose the families $\{b\}, \{c\}, \{d\}$ satisfy the following conditions:

(a) if $\frac{n}{n-1} < q < n$, then $\{b\}, \{c\} \subset L_n(\Gamma)$, $\{d\} \subset L_{\frac{n}{2}}(\Gamma)$, and the families of functions $\{|b|^n\}, \{|c|^n\}, \{|d|^{\frac{n}{2}}\}$ have compact closure in the weak topology of $L_1(\Gamma)$;

(b) if $q = n \ge 2$, then the families $\{|b|^n\}, \{|c|^n\}, \{|d|^{\frac{n}{2}}\}$ are bounded in $M^{\mu_1}(\Gamma)$,

(c) if $q > n$, then the family $\{b\}$ is bounded in $L_q(\Gamma)$, the family $\{c\}$ is bounded in $L_n(\Gamma)$, the family $\{d\}$ is bounded in $L_p(\Gamma)$, where $p = \frac{nq}{n+q}$, and the family $\{|c|^n\}$ has compact closure in the weak topology of $L_1(\Gamma)$.

Let G be a C^1-domain and let $q \ge \frac{n}{n-1}$, where the equality holds only if $q = n = 2$. Suppose also that $e \subset L_q(G)$, $f \subset L_p(G)$, where $p = \frac{qn}{n+q} \ge 1$, and that the functions from f satisfy the following condition:

$$\int\limits_{\Gamma\cap B(x,r)} |f(y)|\,dy \le L(f)r^{\mu(f)} \quad \text{for some } \mu(f) > 0 \text{ and all } B(x,r) \text{ if } p = 1.$$

Let $u \in W^{1,q}(G)$ satisfy an equation from $F(\Gamma)$ in the weak sense. Then one has

$$\|u\|^1_{q,G} \le C\big(\|e\|^0_{q,G} + \|f\|^0_{p,G} + \|u\|^0_{1,G}\big).$$

If $q = n = 2$, so $p = 1$, then the term $\|f\|^0_{p,G}$ must be replaced by $L(f)$. The number C depends only on $n, q, G, \Gamma, c_0, c_1$, and the colection $F(\Gamma)$.

Proof. Recall that a set $Q \subset L_1(\Gamma)$ has compact closure in the weak topology of $L_1(\Gamma)$ if and only if Q is uniformly integrable and that Q is uniformly integrable if and only if Q has uniformly absolutely continuous integrals. Applying this fact and repeating the proof of the corollary we obtain the required result.

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