CONSTRUCTION OF ELLIPTIC DIFFUSIONS WITH REFLECTING BOUNDARY CONDITION AND AN APPLICATION TO CONTINUOUS N-PARTICLE SYSTEMS WITH SINGULAR INTERACTIONS

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ABSTRACT. We give a Dirichlet form approach for the construction and analysis of elliptic diffusions in $\overline{\Omega} \subset \mathbb{R}^n$ with reflecting boundary condition. The problem is formulated in an L^2-setting w.r.t. a reference measure μ on $\overline{\Omega}$ having an integrable, dx-a.e. positive, density ϱ w.r.t. the Lebesgue measure. The symmetric Dirichlet forms $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ we consider are the closure of the symmetric bilinear forms

$$\mathcal{E}^{\varrho,a}(f,g) = \sum_{i,j=1}^{n} \int_{\Omega} \partial_i f\, a_{ij}\, \partial_j g\, d\mu, \quad f,g \in \mathcal{D},$$
$$\mathcal{D} = \{f \in C(\overline{\Omega}) \,|\, f \in W^{1,1}_{\text{loc}}(\Omega), \mathcal{E}^{\varrho,a}(f,f) < \infty\},$$

in $L^2(\overline{\Omega},\mu)$, where a is a symmetric, elliptic, $n \times n$-matrix-valued measurable function on $\overline{\Omega}$. Assuming that Ω is an open, relatively compact set with boundary $\partial\Omega$ of Lebesgue measure zero and that ϱ is satisfying the Hamza condition, we can show that $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is a local, quasi-regular Dirichlet form. Hence, it has an associated self-adjoint generator $(L^{\varrho,a}, D(L^{\varrho,a}))$ and diffusion process $\mathbf{M}^{\varrho,a}$ (i.e, an associated strong Markov process with continuous sample paths). Furthermore, since $1 \in D(\mathcal{E}^{\varrho,a})$ (due to the Neumann boundary condition) and $\mathcal{E}^{\varrho,a}(1,1) = 0$, we obtain a conservative process $\mathbf{M}^{\varrho,a}$ (i.e., $\mathbf{M}^{\varrho,a}$ has infinite life time). Additionally assuming that $\sqrt{\varrho} \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ or that ϱ is bounded, Ω is convex and $\{\varrho = 0\}$ has codimension ≥ 2 we can show that the set $\{\varrho = 0\}$ has $\mathcal{E}^{\varrho,a}$-capacity zero. Therefore, in this case we even can construct an associated conservative diffusion process in $\{\varrho > 0\}$. This is essential for our application to continuous N-particle systems with singular interactions. Note that for the construction of the self-adjoint generator $(L^{\varrho,a}, D(L^{\varrho,a}))$ and the Markov process $\mathbf{M}^{\varrho,a}$ we do not need to assume any differentiability condition on ϱ and a. The following explicit representation of the generator we obtain for $\sqrt{\varrho} \in W^{1,2}(\Omega)$ and $a \in W^{1,\infty}(\Omega)$:

$$L^{\varrho,a} = \sum_{i,j=1}^{n} \partial_i(a_{ij}\, \partial_j) + \partial_i(\log \varrho)\, a_{ij}\, \partial_j.$$

Note that the drift term can be very singular, because we allow ϱ to be zero on a set of Lebesgue measure zero. Our assumptions in this paper even allow a drift which is not integrable w.r.t. the Lebesgue measure.

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1. Introduction

The elliptic diffusions we construct in this paper are associated with symmetric Dirichlet forms $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ which are the closure of the symmetric bilinear forms

$$\mathcal{E}^{\varrho,a}(f,g) = \sum_{i,j=1}^{n} \int_{\Omega} \partial_i f \, a_{ij} \, \partial_j g \, d\mu, \quad f,g \in \mathcal{D}, \tag{1.1}$$
$$\mathcal{D} = \{ f \in C(\overline{\Omega}) \mid f \in W^{1,1}_{\mathrm{loc}}(\Omega), \mathcal{E}^{\varrho,a}(f,f) < \infty \},$$

in $L^2(\overline{\Omega}, \mu)$. The measure μ on $\overline{\Omega} \subset \mathbb{R}^n$ we assume to have an integrable, dx-a.e. positive density ϱ w.r.t. the Lebesgue measure. Furthermore, we assume a to be a symmetric, elliptic, $n \times n$-matrix-valued measurable function on $\overline{\Omega}$. The set Ω we assume to be an open, relatively compact set with boundary $\partial\Omega$ of Lebesgue measure zero.

In the special case where a is the identity matrix and ϱ is a constant, the associated diffusion process is called reflected Brownian motion in $\overline{\Omega}$. For Ω with Lipschitz boundary, it has been constructed and studied in by Bass and Hsu [BH90], [BH91]. See also [WZ90] for another approach.

In the case where

$$\sigma = \sqrt{a} \quad \text{and} \quad b = \Big(\sum_{i=1}^{n} \partial_i (\log \varrho) \, a_{ij} \Big)_{1 \le j \le n} \tag{1.2}$$

are Lipschitz on $\overline{\Omega}$ and Ω is smooth, the process associated with $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ has been obtained as a solution to the corresponding stochastic differential equation by Lions and Sznitman [LS84].

Pardaux and Williams [PW94] investigated two methods for approximating the diffusion process associated with $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$. One is a conventional penalty approximation by diffusions defined on all of $\mathbb{R}^n$. The other one uses diffusions confined to $\overline{\Omega}$ by singular drifts that tend to infinity at the boundary of Ω. Comparing our assumptions with those in [PW94] we only assume a stronger ellipticity of a. However, in [PW94] the authors additionally to our conditions assume that a and ϱ are locally Lipschitz. Furthermore, there the authors assume that $\varrho > 0$. We can allow $\varrho = 0$ in Ω in a set of Lebesgue measure zero. This is essential for our application to continuous N-particle systems with singular interactions, see Theorem 5.4 and Remark 5.5 below. In the case where ϱ is bounded above and below by positive constants the diffusions we construct coincide with the ones obtained in [PW94], see [PW94, Rem. 3.10].

Our approach is rather as in [AKR03], where Albeverio, Kondratiev and Röckner used Dirichlet form techniques, see [Fuk80], [MR92], to construct the diffusion corresponding to $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ in the case $\Omega = \mathbb{R}^n$ and a the identity matrix. Our assumptions on ϱ for constructing the diffusion process corresponding to $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ are still more general as in [AKR03] and rather as used by Fukushima in [Fuk85] where the author also considered the case $\Omega = \mathbb{R}^n$. In the case of a compact $\overline{\Omega}$ as we consider, however, one has to deal with other difficulties caused by the boundary, see Remarks 2.7 and 2.18.

Furthermore, we got to know that Trutnau [Tru03] at the same time as we developed a Dirichlet form approach for the construction and analysis of reflected diffusions. Among others, there the author considers Dirichlet forms with the same assumptions on matrix a and density ϱ as we do. However, the diffusions studied in [Tru03] are corresponding to Dirichlet forms obtained as the closure of $C^\infty(\overline{\Omega})$. For our application to continuous N-particle systems with singular interactions it is essential to have sufficiently many functions in $D(\mathcal{E}^{\varrho,a})$, see e.g. the proof of Theorem 4.5 and the proofs of Corollary 4.7

and Proposition 5.3, respectively, given below. Hence we need to choose the Dirichlet form given by the closure of the larger space $\mathcal{D} \supset C^\infty(\overline{\Omega})$. After constructing the associated diffusion process $\mathbf{M}^{\varrho,a}$ by Dirichlet form techniques, in [Tru03] a Skorokhod decomposition of $\mathbf{M}^{\varrho,a}$ is given. This, in particular, describes $\mathbf{M}^{\varrho,a}$ as a process with reflecting boundary condition. Again, in the case where ϱ is bounded above and below by positive constants the diffusions we construct coincide with the ones obtained in [Tru03].

There are further articles on reflected diffusions, see e.g. [Che93], [Fra97] and [FT96], with results complementary to ours.

Our paper is organized as follows. In Section 2 we are analyzing the symmetric bilinear form (1.1). Assuming the Hamza condition (see Condition 2.2 below) in Proposition 2.6 we can show that $(\mathcal{E}^{\varrho,a},\mathcal{D})$ is closable. Hence its closure $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$ has an associated self-adjoint generator $(L^{\varrho,a},D(L^{\varrho,a}))$, see Remark 2.9. Furthermore, we can prove that $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$ is a conservative, local, quasi-regular Dirichlet form (see Remark 2.7 (iv), Propositions 2.12, 2.17 and 2.20). For having closability, enough functions in $D(\mathcal{E}^{\varrho,a})$ for our application to continuous N-particle systems, and quasi-regularity, simultaneously, a proper choice of Ω and $\mathcal{D}$ is crucial, see Remarks 2.7 and 2.18. The main result of Section 2 we present in Theorem 2.22. There we prove that $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$ has an associated conservative diffusion process $\mathbf{M}^{\varrho,a}$ taking values in $\overline{\Omega}$, i.e., an associated strong Markov process with continuous sample paths and infinite life time. Here quasi-regularity gives the existence of the process $\mathbf{M}^{\varrho,a}$. Locality, see Proposition 2.20, implies that $\mathbf{M}^{\varrho,a}$ has continuous sample paths. The fact that $\mathbf{M}^{\varrho,a}$ is conservative (i.e., has infinite life time) follows from $1 \in D(\mathcal{E}^{\varrho,a})$ and $\mathcal{E}^{\varrho,a}(1,1)=0$. Furthermore, in Theorem 2.22 we prove that $\mathbf{M}^{\varrho,a}$ is the unique diffusion process having μ as symmetrizing measure and solving the martingale problem for $(L^{\varrho,a},D(L^{\varrho,a}))$.

Since $\mathbf{M}^{\varrho,a}$ solves the martingale problem for $(L^{\varrho,a},D(L^{\varrho,a}))$, it can be considered as the solution of a stochastic differential equation. Our existence result in Theorem 2.22, however, is so general that we even have no explicit formula for its generator $(L^{\varrho,a},D(L^{\varrho,a}))$. Under the additional condition $\sqrt{\varrho} \in W^{1,2}(\Omega)$, $a \in W^{1,\infty}(\Omega)$ and Ω having Lipschitz boundary, in Theorem 3.2 we prove that

$$\mathcal{D}_{\mathrm{Neu}} := \left\{ f \in W^{2,\infty}(\Omega) \,|\, \partial_{a\nu} f(x) = 0 \text{ for all } x \in \partial\Omega \right\} \subset D(L^{\varrho,a}),$$

where ν denotes the outer normal w.r.t. $\partial\Omega$ and $a\nu$ is the linear transformation of ν under a. Furthermore, for all $f \in \mathcal{D}_{\mathrm{Neu}}$ we derive the representation

$$L^{\varrho,a} f = \sum_{i,j=1}^{n} \partial_i (a_{ij}\,\partial_j) f + \partial_i\,(\ln \varrho)\,a_{ij}\,\partial_j f. \tag{1.3}$$

Note that elements from $\mathcal{D}_{\mathrm{Neu}}$ have Neumann boundary condition. We assume that Ω has Lipschitz boundary so that the representation given in (1.3) holds for a larger class of functions from $D(L^{\varrho,a})$. For functions with compact support in Ω we get the representation as in (1.3) without assuming Ω having Lipschitz boundary, see Remark 3.4. Now, using Itô's formula, we find that the process $\mathbf{M}^{\varrho,a}$ solves the stochastic differential equation

$$\begin{aligned} &d\mathbf{X}_t = b(\mathbf{X}_t)\,dt + \sqrt{2a}(\mathbf{X}_t)\,d\mathbf{B}_t \quad \text{inside } \Omega,\\ &\text{with reflecting boundary condition,} \end{aligned} \tag{1.4}$$

for $\mathcal{E}^{\varrho,a}$-quasi all initial conditions in $\mathbf{X}_0 \in \overline{\Omega}$. Here being a solution is understood in the sense of the associated martingale problem and $(\mathbf{B}_t)_{t\geq 0}$ is a vector valued Brownian motion. The functions b is as in (1.2).

In Section 4 we analyze $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ from the potential theoretical point of view. Assuming that $\sqrt{\varrho} \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ or that ϱ is bounded, Ω is convex and $\{\varrho = 0\}$ has codimension two, in Theorem 4.5 we can prove that the set $\{\varrho = 0\}$ has $\mathcal{E}^{\varrho,a}$-capacity zero. Thus, we can restrict the Dirichlet form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ to $\{\varrho > 0\}$ as a conservative, local, quasi-regular Dirichlet form, see Corollary 4.7. This gives us an associated conservative diffusion process in $\{\varrho > 0\}$, see Corollary 4.8 .

Finally, as an application we construct a solution to the N-particle stochastic dynamic in $\Lambda \subset \mathbb{R}^d$. This dynamic takes values in the space of N-point configurations in Λ:

$$\Gamma_\Lambda^{(N)} := \{\gamma \subset \Lambda \,|\, \#(\gamma) = N\}$$

and solves weakly the following N-system of stochastic differential equations:

$$dx(t) = -\sum_{\substack{y(t)\in\, \mathbf{X}(t) \\ y(t)\,\neq\, x(t)}} \nabla\phi(x(t) - y(t))\, dt + \sqrt{2}\, dB^{x_0}(t) \quad \text{inside } \Gamma_\Lambda^{(N)},$$

$$\text{with reflecting boundary condition.} \tag{1.5}$$

Here $x(t) \in \mathbf{X}(t) \in \Gamma_\Lambda^{(N)}$ and $(B^{x_0})_{x_0 \in \gamma_0}$ are N independent Brownian motions starting in x_0. The existence of a weak solution to (1.5) for all initial conditions $\gamma_0 \in \Gamma_\Lambda^{(N)}$ except for a set of capacity zero we prove in Theorem 5.4. Our assumptions on the interaction potential allow quite singular interactions. In the case $d = 1$ we assume the interaction potential ϕ either to be strongly repulsive (SRP) and bounded below (BB), or repulsive (RP) and weakly differentiable (DL^2). In the case $d \geq 2$ we either have to assume the interaction potential ϕ to be repulsive (RP) and bounded below (BB), or just bounded, see below for a precise definition of (SRP), (RP), (BB) and (DL^2). In our construction we first consider the corresponding Dirichlet form $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ on $\Lambda^N \subset \mathbb{R}^n$, $n = N \cdot d$. The measure μ in this case is the canonical Gibbs measure corresponding to N interacting particles in Λ. Then due to (RP) or in the case of a bounded potential by capacity estimates provided in [Stu95] we have that the set of diagonals Dg in Λ^N has $\mathcal{E}_{\Lambda,N}$-capacity zero, see Remark 5.5. Hence, via the symmetry mapping

$$\text{sym}_\Lambda^{(N)} : \Lambda^N \setminus Dg \to \Gamma_\Lambda^{(N)}$$

$$\text{sym}_\Lambda^{(N)}(x_1, \ldots, x_N) = \{x_1, \ldots, x_N\},$$

we can construct a solution to (1.5).

The progress achieved in this paper may be summarized by the following list of main results:

- Construction of conservative diffusion processes with reflecting boundary condition under very mild assumptions on drift and diffusion part, see Theorem 2.22.
- Providing an explicit representation of the generator for functions with Neumann boundary condition, see Theorem 3.2. This representation enables us, via the martingale problem, to identify the processes we construct as weak solutions to the stochastic differential equation (1.4).
- Showing that the set, where the density ϱ of the symmetrizing measure μ is zero, has capacity zero, see Theorem 4.5. As a corollary we can construct the associated process on $\{\varrho > 0\}$, see Corollary 4.8.
- Construction of the N-particle, finite volume stochastic dynamic with reflecting boundary condition for singular interactions, see Theorem 5.4.

We consider this paper as basis for several other articles. For example, it provides the N-particle dynamic in a finite volume for singular interactions which is essential for proving an N/V-limit for infinite particle, infinite volume stochastic dynamics in continuous particle systems, see [GKR07]. Furthermore, there is a forthcoming article [FG06] in which we proved strong Feller properties for the semigroups corresponding to the diffusions constructed here. In [Gro07] it is planed to determine the spectral gap of the corresponding generators.

It might be possible to construct an N-particle dynamic for singular interactions by first regularizing the potential, using existing theory on stochastic differential equations to construct the corresponding approximating process and then attempt to take a weak limit. But then it is still open whether the weak limit solves the associated martingale problem. This property is important for the considerations in [GKR07] and follows directly from the Dirichlet form approach. Furthermore for the considerations in [GKR07] a Lyons-Zheng decomposition (see [LZ88], [LZ94]) of the N-particle dynamic into a forward and backward martingale is needed. The existence of such a decomposition is only guarantied for processes associated to Dirichlet forms.

The reflecting boundary conditions are needed to obtain a process with infinite life time. Note that Dirichlet form techniques allow so general boundaries (more general than Lipschitz) that even the notion of a reflection might not be well defined. Hence for such boundaries we have a reflection at the boundary in a generalized sense.

2. Dirichlet forms

We start with the symmetric bilinear form

$$\mathcal{E}^{\varrho,a}(f,g) = \sum_{i,j=1}^{n} \int_{\Omega} \partial_i f \, a_{ij} \, \partial_j g \; d\mu$$

on $L^2(\overline{\Omega},\mu)$. In the entire paper we assume a to be a symmetric, $n \times n$-matrix valued measurable function, which is uniformly globally strictly elliptic on Ω, i.e., there exists $\kappa > 0$ such that

$$\kappa^{-1} \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \kappa \sum_{i=1}^{n} \xi_i^2 \text{ for all } \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \ \mu\text{-a.e. } x \in \Omega.$$

We assume Ω to be an open, relatively compact set with boundary $\partial\Omega$ of Lebesgue measure zero. The measure μ we assume to have an integrable, dx-a.e. positive density w.r.t. the Lebesgue measure, i.e., $\mu = \varrho dx$, where $\varrho > 0$ dx-a.e. on $\overline{\Omega}$ and $\varrho \in L^1(\overline{\Omega}, dx)$. As domain of $\mathcal{E}^{\varrho,a}$ we consider

$$\mathcal{D} = \{ f \in C(\overline{\Omega}) \,|\, f \in W^{1,1}_{\text{loc}}(\Omega), \ \mathcal{E}^{\varrho,a}(f,f) < \infty \}.$$

Here $W^{1,1}_{\text{loc}}(\Omega)$ denotes the Sobolev space of weakly differentiable, locally integrable functions on Ω.

2.1. **Closability of the bilinear form $(\mathcal{E}^{\varrho,a},\mathcal{D})$.** We get started by recalling some basic facts on bilinear forms. For a detailed study see for example [MR92] or [Fuk80].

Definition 2.1. A bilinear form $(\mathcal{E}, D)$ on $L^2(\overline{\Omega},\mu)$ is said to be

(i) closed if the space D is dense in $L^2(\overline{\Omega},\mu)$ and complete with respect to the inner product

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + (f,g)_{L^2(\overline{\Omega},\mu)},$$

where

$$(f,g)_{L^2(\overline{\Omega},\mu)} = \int_{\overline{\Omega}} f(x)g(x)\mu(dx).$$

(ii) closable if the following condition is satisfied:

$$\text{If } f_k \in D,\ \mathcal{E}(f_k - f_l, f_k - f_l) \to 0 \text{ as } k,l \to \infty$$
$$\text{and } (f_k, f_k)_{L^2(\overline{\Omega},\mu)} \to 0 \text{ as } k \to \infty,\ \text{then } \mathcal{E}(f_k, f_k) \to 0 \text{ as } k \to \infty.$$

In our considerations the natural question arises, under which conditions the form $(\mathcal{E}^{\varrho,a}, \mathcal{D})$ is closable. A discussion of this problem can be found for example in [Fuk85]. To prove closability of such a form, we have to put some additional restrictions on the density ϱ. We define

$$R_\varrho(\Omega) := \left\{ x \in \Omega \,\middle|\, \int_{\{y\in\Omega \,|\, |x-y|\le\varepsilon\}} \varrho^{-1}(y)dy < \infty \text{ for some } \varepsilon > 0 \right\}.$$

Condition 2.2. *$\varrho = 0$ dx-a.e. on $\Omega \setminus R_\varrho(\Omega)$.*

Remark 2.3. $R_\varrho(\Omega)$ is open and $\varrho > 0$ dx-a.e. on $R_\varrho(\Omega)$. Obviously, $R_\varrho(\Omega)$ is the largest open set in Ω, such that $\varrho^{-1} \in L^1_{\text{loc}}(R_\varrho(\Omega), dx)$, see e.g. [MR92, Chap. 2]. Condition 2.2 is called the Hamza Condition.

Remark 2.4. Note that due to the assumption that $\varrho > 0$ dx-a.e. on $\overline{\Omega}$ and Condition 2.2, we have that $\Omega \setminus R_\varrho(\Omega)$ is of Lebesgue measure zero.

The next lemma will give us an estimate which is essential for proving closability of $(\mathcal{E}^{\varrho,a}, \mathcal{D})$. For a proof see [MR92, Chap. II, Lem. 2.2].

Lemma 2.5. Let Condition 2.2 be satisfied, $\varphi \in C_0^\infty(R_\varrho(\Omega))$ and $f \in L^2(\overline{\Omega}, \mu)$, then there exists $C_1(\varphi) < \infty$ such that

$$\left| \int_{R_\varrho(\Omega)} f\varphi\ dx \right| \le C_1(\varphi) \cdot \|f\|_{L^2(\overline{\Omega},\mu)}.$$

Proposition 2.6. *Consider the measure $\mu = \varrho dx$ with density function ϱ and suppose Condition 2.2 is satisfied. Then the symmetric bilinear form*

$$\mathcal{E}^{\varrho,a}(f,g) = \sum_{i,j=1}^n \int_\Omega \partial_i f\, a_{ij}\, \partial_j g\ d\mu$$

with domain

$$\mathcal{D} = \{ f \in C(\overline{\Omega}) \,|\, f \in W^{1,1}_{loc}(\Omega),\ \mathcal{E}^{\varrho,a}(f,f) < \infty \},$$

is closable on $L^2(\overline{\Omega}, \mu)$. The closure we denote by $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$.

Proof. Because of the ellipticity of a, we can restrict ourselves to the case, where a equals the identity on $\mathbb{R}^n$. In the entire paper we write $\mathcal{E}^{\varrho,a} = \mathcal{E}^\varrho$, if a equals the identity matrix. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}$ w.r.t $\mathcal{E}^\varrho$, i.e., $\mathcal{E}^\varrho(f_k - f_l, f_k - f_l) \to 0$ as $k,l \to \infty$. Suppose furthermore that $f_k \to 0$ in $L^2(\overline{\Omega}, \mu)$, i.e. $(f_k, f_k)_{L^2(\overline{\Omega},\mu)} \to 0$ as $n \to \infty$. We have to check whether $\mathcal{E}^\varrho(f_k, f_k) \to 0$ as $k \to \infty$, see Definition 2.1 (ii).

We know that for fixed $i \in \{1, \dots, n\}$, $(\partial_i f_k)_{k\in\mathbb{N}}$ converges to some h_i in $L^2(\overline{\Omega}, \mu)$, since $(\partial_i f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\overline{\Omega}, \mu)$ and $(L^2(\overline{\Omega}, \mu), \|\cdot\|_{L^2(\overline{\Omega},\mu)})$ is complete. Now we use Lemma 2.5 to obtain for $\varphi \in C_0^\infty(R_\varrho(\Omega))$:

$$\left|\int_{R_\varrho(\Omega)} h_i\varphi\,dx - \int_{R_\varrho(\Omega)} \partial_i f_k\,\varphi\,dx\right| \le \int_{R_\varrho(\Omega)} |\partial_i f_k - h_i|\,|\varphi|\;dx$$
$$\le C_1(\varphi)\,\|\partial_i f_k - h_i\|_{L^2(\overline{\Omega},\mu)} \to 0 \text{ as } k\to\infty.$$

Thus $\int_{R_\varrho(\Omega)} h_i\varphi\,dx = \lim_{k\to\infty}\int_{R_\varrho(\Omega)} \partial_i f_k\,\varphi\,dx$. This, together with an integration by parts, Hölder's inequality and the fact that $(f_k,f_k)_{L^2(\overline{\Omega},\mu)} \to 0$ as $k\to\infty$ implies:

$$\int_{R_\varrho(\Omega)} h_i\varphi\,dx = \lim_{k\to\infty}\int_{R_\varrho(\Omega)} \partial_i f_k\,\varphi\,dx = -\lim_{k\to\infty}\int_{R_\varrho(\Omega)} f_k\,\partial_i\varphi\,dx = 0.$$

Hence h_i is the zero element in the space $L^2(R_\varrho(\varrho),\mu)$. Thus h_i is the zero element in the space $L^2(\overline{\Omega},\mu)$, since $\partial\Omega$ has Lebesgue measure zero and $\varrho = 0$ on $\Omega\setminus R^\varrho(\Omega)$ by Condition 2.2. Thus we have proven $\mathcal{E}^\varrho(f_k,f_k)\to 0$ as $k\to\infty$. □

Remark 2.7.

(i) Note that the proof of Proposition 2.6 is based on the fact that $\partial\Omega$ has Lebesgue measure zero.

(ii) From the proof of Proposition 2.6 we can easily conclude that $\mathcal{E}^{\varrho,a}$ with the larger domain

$$\widetilde{D(\mathcal{E}^{\varrho,a})} := \{f\in L^2(\overline{\Omega},\mu)\,|\,f\in W^{1,1}_{\mathrm{loc}}(R_\varrho(\Omega)),\ \mathcal{E}^{\varrho,a}(f,f)<\infty\}$$

is closed. In general, however, it is not clear whether $\mathcal{D} = C(\overline{\Omega})\cap\widetilde{D(\mathcal{E}^{\varrho,a})}$ is dense in $\widetilde{D(\mathcal{E}^{\varrho,a})}$ w.r.t. $\sqrt{\mathcal{E}_1^{\varrho,a}}$. This property is needed to show quasi-regularity (see Section 2.3), which is essential for our construction of the associated Markov process in Theorem 2.22.

(iii) On the other hand, for our application to continuous N-particle systems sufficiently many functions in the domain of $\mathcal{E}^{\varrho,a}$ are needed. E.g., choosing $\mathcal{D} = C^1(\overline{\Omega})$, it is not clear whether the corresponding closure would have sufficiently many functions for proving that the set $\{\varrho = 0\}$ has capacity zero, see the proof of Theorem 4.5. This theorem in fact is essential for our application to continuous N-particle systems, see Remark 5.5.

(iv) Since $1\in\mathcal{D}$ and $\mathcal{E}^{\varrho,a}(1,1)=0$, the bilinear form $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$ is conservative. In the case where $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$ has an associated diffusion process $\mathbf{M}^{\varrho,a}$, see Theorem 2.22, this implies that $\mathbf{M}^{\varrho,a}$ has infinite life time.

Notation 2.8. Recall that $\Omega\setminus R_\varrho(\Omega)$ has Lebesgue measure zero. Thus, after the considerations above we set $\nabla f := (\partial_1 f,\dots,\partial_n f) := (h_1,\dots,h_n)$ for all $f\in D(\mathcal{E}^{\varrho,a})$.

Remark 2.9. By Friedrichs representation theorem (see e.g. [AST03, Theo. 4]) we have the existence of the self-adjoint generator $(L^{\varrho,a},D(L^{\varrho,a}))$ corresponding to $(\mathcal{E}^{\varrho,a},D(\mathcal{E}^{\varrho,a}))$, i.e., $D(L^{\varrho,a})\subset D(\mathcal{E}^{\varrho,a})$ and

$$\mathcal{E}^{\varrho,a}(f,g) = -\int_\Omega L^{\varrho,a}f\,g\,d\mu \quad \text{for all} \quad f\in D(L^{\varrho,a}),\ g\in D(\mathcal{E}^{\varrho,a}).$$

Of course, $(L^{\varrho,a},D(L^{\varrho,a}))$ generates a strongly continuous contraction semi-group

$$(T_t^{\varrho,a})_{t\ge 0} := (\exp(t\,L^{\varrho,a}))_{t\ge 0}\,,$$

see e.g. [Fuk85], [MR92].

2.2. **Markov Property of** $(\mathcal{E}^\varrho, \mathcal{D}(\mathcal{E}^\varrho))$**.**

Definition 2.10. A symmetric closed bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\overline{\Omega}, \mu)$ is called Markovian if one has that:

$$f \in \mathcal{D}(\mathcal{E}) \text{ implies } f^+ \wedge 1 \in \mathcal{D}(\mathcal{E}) \text{ and } \mathcal{E}(f^+ \wedge 1, f^+ \wedge 1) \leq \mathcal{E}(f, f),$$

where $f^+ := \max\{0, f\}$ and $f \wedge 1 := \min\{1, f\}$.

Remark 2.11. One can easily show, that for each $\varepsilon > 0$ there exists a real function $\varphi_\varepsilon(t),\ t \in \mathbb{R}$, such that

$$\varphi_\varepsilon(t) = t \text{ for all } t \in [0,1],\ -\varepsilon \leq \varphi_\varepsilon(t) \leq 1 + \varepsilon \text{ for all } t \in \mathbb{R},$$
$$\text{and } 0 \leq \varphi_\varepsilon(s) - \varphi_\varepsilon(t) \leq s - t,\ \text{whenever } t < s.$$

Then it is enough to check:

$$f \in D(\mathcal{E}) \text{ implies } \varphi_\varepsilon(f) \in D(\mathcal{E}) \text{ and } \mathcal{E}(\varphi_\varepsilon(f), \varphi_\varepsilon(f)) \leq \mathcal{E}(f, f),$$

to obtain that $(\mathcal{E}, D(\mathcal{E}))$ is Markovian. See e.g. [MR92, Chap. 1, Sec. 4].

Proposition 2.12. *Suppose that Condition 2.2 is satisfied. Then* $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ *is Markovian. A Markovian form is also called a Dirichlet form.*

Before we can prove the above proposition we need the following result from the theory of Sobolev spaces. For a proof we refer to [GT77, Lem. 7.5].

Lemma 2.13. Let $f \in C^1(\mathbb{R}),\ f' \in L^\infty(\mathbb{R})$ and $u \in W^{1,1}_{\text{loc}}(\Omega)$. Then $f(u) \in W^{1,1}_{\text{loc}}(\Omega)$ and

$$\partial_i(f(u)) = f'(u)\, \partial_i u.$$

Proof of Proposition 2.12. As before, by the ellipticity of a it is enough to consider the case where a equals the identity matrix. Let φ_ε be as in Remark 2.11 and let us take $f \in D(\mathcal{E}^\varrho)$. At first we consider $\varphi_\varepsilon(f)$ as a function in $L^2(\overline{\Omega}, \mu)$. Then we take $(f_k)_{k\in\mathbb{N}} \subset \mathcal{D}$ such that $f_k \to f$ in $(D(\mathcal{E}^\varrho), \sqrt{\mathcal{E}^\varrho_1})$ and additionally $f_k \to f$ μ-a.e. as $k \to \infty$. Obviously, $\varphi_\varepsilon(f_k) \in C(\overline{\Omega})$ for all $k \in \mathbb{N}$. Since $\varphi_\varepsilon \in C^1(\mathbb{R}),\ \varphi'_\varepsilon \in L^\infty(\mathbb{R})$ and $f_k \in W^{1,1}_{\text{loc}}(\Omega)$, we have $\varphi_\varepsilon(f_k) \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i(\varphi_\varepsilon(f_k)) = \varphi'_\varepsilon(f_k)\partial_i f_k$ for all $k \in \mathbb{N}$, by Lemma 2.13. Furthermore, we have

$$\|\partial_i(\varphi_\varepsilon(f_k))\|_{L^2(\overline{\Omega},\mu)} = \|\varphi'_\varepsilon(f_k)\partial_i f_k\|_{L^2(\overline{\Omega},\mu)} \leq \|\partial_i f_k\|_{L^2(\overline{\Omega},\mu)} < \infty,$$

by the properties of φ_ε, and therefore $\varphi_\varepsilon(f_k) \in D(\mathcal{E}^\varrho)$ for all $k \in \mathbb{N}$. Clearly, $\varphi_\varepsilon(f_k) \to \varphi_\varepsilon(f)$ in $L^2(\overline{\Omega}, \mu)$ as $k \to \infty$, since $\|\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f)\|_{L^2(\overline{\Omega},\mu)} \leq \|f_k - f\|_{L^2(\overline{\Omega},\mu)}$, again by the properties of φ_ε. Next we show that $(\varphi_\varepsilon(f_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $(D(\mathcal{E}^\varrho), \sqrt{\mathcal{E}^\varrho_1})$. Therefore we consider

$$\begin{aligned}&\mathcal{E}^\varrho_1(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l), \varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l))\\ &\quad = \mathcal{E}^\varrho(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l), \varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l)) + \big(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l), \varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l)\big)_{L^2(\overline{\Omega},\mu)}\end{aligned}$$

Since $\varphi_\varepsilon(f_k) \to \varphi_\varepsilon(f)$ in $L^2(\overline{\Omega}, \mu)$ we have

$$\big(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l), \varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l)\big)_{L^2(\overline{\Omega},\mu)} \to 0 \text{ as } k, l \to \infty.$$

Thus, it remains to consider

$$\sum_{i=1}^n \int_\Omega \left(\partial_i(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l))\right)^2\ d\mu$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{\Omega} \big(\varphi_\varepsilon'(f_k)\partial_i f_k - \varphi_\varepsilon'(f_l)\partial_i f_l\big)^2 \; d\mu \text{ (by applying Lemma 2.13)} \\
&= \sum_{i=1}^{n} \int_{\Omega} \big(\varphi_\varepsilon'(f_k)\left(\partial_i f_k - \partial_i f_l\right) + \big(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f_l)\big)\left(\partial_i f_l - \partial_i f + \partial_i f\right)\big)^2 \; d\mu \\
&\le 3\big(\|\varphi_\varepsilon'(f_k)\|_{\sup}^2 \|\nabla f_k - \nabla f_l\|^2_{L^2(\overline{\Omega},\mu)} + \|(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f_l))\nabla f\|^2_{L^2(\overline{\Omega},\mu)} \\
&\qquad + \|(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f_l))\|^2_{sup} \|\nabla f_l - \nabla f\|^2_{L^2(\overline{\Omega},\mu)}\big) \\
&\qquad \le 3\big(\|\nabla f_k - \nabla f_l\|^2_{L^2(\overline{\Omega},\mu)} + 4\|\nabla f_l - \nabla f\|^2_{L^2(\overline{\Omega},\mu)} \\
&\qquad + 2\|\big(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f)\big)\nabla f\|^2_{L^2(\overline{\Omega},\mu)} + 2\|\big(\varphi_\varepsilon'(f) - \varphi_\varepsilon'(f_l)\big)\nabla f\|^2_{L^2(\overline{\Omega},\mu)}\big).
\end{aligned}
$$

Since $|(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f))\nabla f|$ and $|(\varphi_\varepsilon'(f) - \varphi_\varepsilon'(f_l))\nabla f|$ are bounded by $g := 2|\nabla f| \in L^2(\overline{\Omega},\mu)$ and $\varphi_\varepsilon'(f_k) \to \varphi_\varepsilon'(f)$ μ-a.e. as $k \to \infty$, we have by using Lebesgue's dominated convergence theorem that

$$
\big\|\big(\varphi_\varepsilon'(f_k) - \varphi_\varepsilon'(f)\big)\nabla f\big\|_{L^2(\overline{\Omega},\mu)} + \big\|\big(\varphi_\varepsilon'(f) - \varphi_\varepsilon'(f_l)\big)\nabla f\big\|_{L^2(\overline{\Omega},\mu)} \to 0 \text{ as } k,l \to \infty. \quad (2.1)
$$

Thus, (2.1) together with $f_k \to f$ in $(D(\mathcal{E}^\varrho), \sqrt{\mathcal{E}_1^\varrho})$ implies that

$$
\mathcal{E}_1^\varrho(\varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l), \varphi_\varepsilon(f_k) - \varphi_\varepsilon(f_l)) \to 0 \text{ as } k,l \to \infty.
$$

Hence, $(\varphi_\varepsilon(f_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $D(\mathcal{E}^\varrho)$ w.r.t. $\sqrt{\mathcal{E}_1^\varrho}$. Thus convergent in $D(\mathcal{E}^\varrho)$ and

$$
\varphi_\varepsilon(f) = \lim_{k\to\infty} \varphi_\varepsilon(f_k) \in D(\mathcal{E}^\varrho).
$$

Furthermore,

$$
\begin{aligned}
\mathcal{E}^\varrho(\varphi_\varepsilon(f), \varphi_\varepsilon(f)) &= \lim_{k\to\infty} \mathcal{E}^\varrho(\varphi_\varepsilon(f_k), \varphi_\varepsilon(f_k)) = \lim_{k\to\infty} \sum_{i=1}^{n} \int_{\Omega} \left(\partial_i \varphi_\varepsilon(f_k)\right)^2 \; d\mu \\
&= \lim_{k\to\infty} \sum_{i=1}^{n} \int_{\Omega} \underbrace{|\varphi_\varepsilon'(f_k)|^2}_{\le 1} \left(\partial_i f_k\right)^2 \; d\mu \le \lim_{k\to\infty} \sum_{i=1}^{n} \int_{\Omega} \left(\partial_i f_k\right)^2 \; d\mu = \mathcal{E}^\varrho(f,f).
\end{aligned}
$$

Thus $(\mathcal{E}^\varrho, D(\mathcal{E}^\varrho))$ is Markovian. □

2.3. **Quasi-regularity of the Dirichlet form** $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$**.** To get started with quasi-regularity, we have to introduce some notions from analytic potential theory of Dirichlet forms. A detailed discussion of the theory needed in this section can be found in [MR92, Chap. III]. In this section $(\mathcal{E}, D(\mathcal{E}))$ denotes a Dirichlet form on $L^2(\overline{\Omega},\mu)$.

Definition 2.14.

(i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of $\overline{\Omega}$ is called an $\mathcal{E}$-nest if $\bigcup_{k\ge1} D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ w.r.t $\sqrt{\mathcal{E}_1}$, where

$$
D(\mathcal{E})_{F_k} := \big\{u \in D(\mathcal{E}) \,\big|\, u = 0 \ \mu\text{-a.e. on } \overline{\Omega} \setminus F_k\big\}.
$$

(ii) A subset $N \subset \Omega$ is called $\mathcal{E}$-exceptional if $N \subset \bigcap_{k\ge1}\big(\overline{\Omega} \setminus F_k\big)$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$. We say that a property of points in Ω holds $\mathcal{E}$-quasi-everywhere (abbreviated $\mathcal{E}$-q.e.), if the property holds outside some $\mathcal{E}$-exceptional set.

Next we introduce the notion of quasi-continuity.

Definition 2.15. An $\mathcal{E}$-q.e. defined function f on $\overline{\Omega}$ is called $\mathcal{E}$-quasi continuous if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f \in C(\{F_k\}) := \big\{f : A \to \mathbb{R} \,\big|\, \bigcup_{k\geq 1} F_k \subset A \subset \overline{\Omega},\ f|_{F_k} \text{ is continuous for every } k \in \mathbb{N}\big\}$.

Now we can define quasi-regularity:

Definition 2.16. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\overline{\Omega}, \mu)$ is called quasi-regular if there exists:

(i) an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets;
(ii) an $\sqrt{\mathcal{E}_1}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions;
(iii) a sequence of functions $u_l \in D(\mathcal{E}),\ l \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous μ-versions $\tilde{u}_l,\ l \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset \overline{\Omega}$ such that $\{\tilde{u}_l \,|\, l \in \mathbb{N}\}$ separates the points of $\overline{\Omega} \setminus N$.

Now we can state the main result of this section.

Proposition 2.17. *Suppose that Condition 2.2 is satisfied. Then $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is quasi-regular.*

Proof. Let us check, whether (i)-(iii) in Definition 2.16 are fulfilled. Obviously, $(F_k)_{k\in\mathbb{N}}$, $F_k = \overline{\Omega}$, $k \in \mathbb{N}$, is an $\mathcal{E}^{\varrho,a}$-nest consisting of compact sets. Since $D(\mathcal{E}^{\varrho,a})$ is the completion of $\mathcal{D}$ w.r.t. $\sqrt{\mathcal{E}_1^{\varrho,a}}$, we have $\mathcal{D} \subset C(\overline{\Omega})$ is dense in $D(\mathcal{E}^{\varrho,a})$ w.r.t. $\sqrt{\mathcal{E}_1^{\varrho,a}}$ and property (ii) is shown.

It remains to find a sequence of functions $\{u_l \in \mathcal{D},\ l \in \mathbb{N}\}$ which is separating points in $\overline{\Omega}$. Clearly, the countable set of polynomials with rational coefficients is a subset of $\mathcal{D}$ and, of course, separating points on $\overline{\Omega}$. □

Remark 2.18. In the proof of Proposition 2.17 we see that it is very useful to have compact $\overline{\Omega}$. In this case we can choose as $\mathcal{E}^{\varrho,a}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets simply $F_k = \overline{\Omega}$ for all $k \in \mathbb{N}$. Moreover, one can show that when replacing $\overline{\Omega}$ by an open subset of $\mathbb{R}^n$ the corresponding Dirichlet form, even in the case $\varrho = 1$ and a the identity matrix, is not quasi-regular, see [Fuk80, Exa. 1.2.3]. Furthermore, from the proof of Proposition 2.17 we can easily conclude that $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is even regular, see e.g. [Fuk80].

2.4. **Locality of the quasi-regular Dirichlet form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$.** An useful property of a Dirichlet form is its so-called locality.

Definition 2.19. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is said to be local if $\mathcal{E}(u, v) = 0$ for all $u, v \in D(\mathcal{E})$ with $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$ and $\operatorname{supp}(u), \operatorname{supp}(v)$ compact.

Proposition 2.20. *Suppose that Condition 2.2 is satisfied. Then $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is local.*

Proof. By [MR92, Chap. V, Exa. 1.12(ii)] it is sufficient to show that $\mathcal{D}$ is closed under multiplication and that for the weak gradient we have a product rule. Let $f, g \in \mathcal{D}$, then obviously $f \cdot g$ is continuous on $\overline{\Omega}$. Furthermore, since f, g are bounded with weak derivative in $L^1_{\text{loc}}(\Omega, dx)$, also $f \cdot g$ is weakly differentiable and $\nabla(f \cdot g)$ is in $L^1_{\text{loc}}(\Omega, dx)$. Furthermore, the product rule holds and

$$\nabla(f \cdot g) = \nabla f \cdot g + f \cdot \nabla g, \quad f, g \in \mathcal{D}.$$

Obviously, $\nabla(f \cdot g) \in L^2(\overline{\Omega}, \mu)$ and therefore $\mathcal{E}^{\varrho,a}(f \cdot g, f \cdot g) < \infty$, by ellipticity of a. □

Let us summarize the properties of the bilinear form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$.

Corollary 2.21. *Suppose that Condition 2.2 is satisfied. Then $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is a conservative, local, quasi-regular Dirichlet form.*

Proof. This follows directly from Propositions 2.6, 2.12, 2.17, 2.20 and Remark 2.7 (iv). □

With these properties we are given an associated Markov process.

Theorem 2.22. *Suppose that Condition 2.2 is satisfied. Then:*

(i) There exists a conservative diffusion process (i.e., a Markov process with continuous sample paths and infinite life time)

$$\mathbf{M}^{\varrho,a} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbf{P}_x^{\varrho,a})_{x\in\overline{\Omega}})$$

with state space $\overline{\Omega}$ which is properly associated with $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$, i.e., for all ($\mu$-versions of) $f \in L^2(\overline{\Omega}, \mu)$ and all $t > 0$ the function

$$x \mapsto \int_{\mathbf{\Omega}} f(\mathbf{X}_t)\, d\mathbf{P}_x^{\varrho,a}, \ \ x \in \overline{\Omega},$$

is a $\mathcal{E}^{\varrho,a}$-quasi-continuous version of $T^{\varrho,a} f$. $\mathbf{M}^{\varrho,a}$ is up to μ-equivalence unique. In particular, $\mathbf{M}^{\varrho,a}$ is μ-symmetric (i.e., $\int g\, T_t^{\varrho,a} f\, d\mu = \int f\, T_t^{\varrho,a} g\, d\mu$ for all $f, g : \overline{\Omega} \to [0,\infty)$ measurable) and has μ as an invariant measure.

(ii) The diffusion process $\mathbf{M}^{\varrho,a}$ is up to μ-equivalence the unique diffusion process having μ as symmetrizing measure and solving the martingale problem for $(L^{\varrho,a}, D(L^{\varrho,a}))$, i.e., for all $g \in D(L^{\varrho,a})$

$$g(\mathbf{X}_t) - g(\mathbf{X}_0) - \int_0^t L^{\varrho,a} g(\mathbf{X}_s)\, ds, \qquad t \geq 0,$$

is an $\mathbf{F}_t$-martingale under $\mathbf{P}_x^{\varrho,a}$ (hence starting in x) for $\mathcal{E}^{\varrho}$-quasi all $x \in \overline{\Omega}$.

In the above theorem $\mathbf{M}^{\varrho,a}$ is canonical, i.e., $\mathbf{\Omega} = C([0,\infty) \to \overline{\Omega})$, $\mathbf{X}_t(\omega) = \omega(t), \omega \in \mathbf{\Omega}$. The filtration $(\mathbf{F}_t)_{t\geq 0}$ is the natural "minimum completed admissible filtration", cf. [FOT94], Chap. A.2, or [MR92], Chap. IV, obtained from the σ-algebras $\sigma\{\omega(s)\,|\, 0 \leq s \leq t, \omega \in \mathbf{\Omega}\}$, $t \geq 0$. $\mathbf{F} := \mathbf{F}_\infty := \bigvee_{t\in[0,\infty)} \mathbf{F}_t$ is the smallest σ-algebra containing all $\mathbf{F}_t$ and $(\mathbf{\Theta}_t)_{t\geq 0}$ are the corresponding natural time shifts. For a detailed discussions of these objects we refer to [MR92].

Proof. (i): The proof follows directly from [MR92, Chap. V, Theo. 1.11], since we have already shown that $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is a conservative, local, quasi-regular Dirichlet form on $L^2(\overline{\Omega}, \mu)$.

(ii): Since $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is a quasi-regular Dirichlet form, the statement follows from [AR95, Theo. 3.4 (i)]. □

3. The generator of the Dirichlet form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$

In the previous section we have shown that $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is a Dirichlet form. Thus, the existence of the associated generator $(L^{\varrho,a}, D(L^{\varrho,a}))$ is already clear, see Remark 2.9. In this section we derive an explicit representation of $L^{\varrho,a}$ for certain subsets of its domain $D(L^{\varrho,a})$. This representation will be obtained by using the Gaussian integral formula (see e.g. [Alt02, A 6.8 (1)]). However, before doing so, we must impose some additional restrictions on the density function ϱ and matrix a.

Condition 3.1. *We assume that $\sqrt{\varrho} \in W^{1,2}(\Omega)$ and $a \in W^{1,\infty}(\Omega)$.*

Here $W^{1,2}(\Omega)$ is the Sobolev space of weakly differentiable, square-integrable functions and $W^{m,\infty}(\Omega)$, $m \in \mathbb{N}$, is the Sobolev space of m-times weakly differentiable, essentially bounded functions on Ω. By Sobolev's embedding theorem, see e.g. [Alt02, 8.13], we have $W^{m,\infty}(\Omega) \subset C^1(\overline{\Omega})$ for $m > 1$.

Theorem 3.2. *Let Ω have Lipschitz boundary and Conditions 2.2 and 3.1 be satisfied. Then*

$$\mathcal{D}_{Neu} := \left\{ f \in W^{2,\infty}(\Omega) \,|\, \partial_{a\nu} f(x) = 0 \text{ for all } x \in \partial\Omega \right\} \subset D(L^{\varrho,a})$$

and we have the representation

$$L^{\varrho,a} f = \sum_{i,j=1}^{n} \partial_i (a_{ij}\, \partial_j f) + \partial_i (\ln \varrho)\, a_{ij}\, \partial_j f \tag{3.1}$$

(here ν denotes the outer normal w.r.t. $\partial\Omega$ and $a\nu$ is the linear transformation of ν under a).

Remark 3.3. Now, using Itô's formula, from Theorems 2.22(ii) and 3.2 we can conclude that the process $\mathbf{M}^{\varrho,a}$ solves the stochastic differential equation

$$\begin{aligned} &d\mathbf{X}_t = b(\mathbf{X}_t)\, dt + \sqrt{2a}(\mathbf{X}_t)\, d\mathbf{B}_t, \\ &\text{with reflecting boundary condition,} \end{aligned}$$

inside Ω, for $\mathcal{E}^{\varrho,a}$-quasi all initial conditions $\mathbf{X}_0 \in \overline{\Omega}$. Here being a solution is understood in the sense of the associated martingale problem, $(\mathbf{B}_t)_{t\geq 0}$ is a vector valued Brownian motion and

$$b = \Big(\sum_{i=1}^{n} \partial_i (\log \varrho)\, a_{ij} \Big)_{1 \leq j \leq n}.$$

Proof of Theorem 3.2. In [MR92, Prop. 2.16] a characterization of the domain of the operator $L^{\varrho,a}$ is given. Namely,

$$D(L^{\varrho,a}) = \Big\{ u \in D(\mathcal{E}^{\varrho,a}) : v \mapsto \mathcal{E}^{\varrho,a}(u,v) \text{ is continuous w.r.t. } \sqrt{(\cdot,\cdot)_{L^2(\overline{\Omega},\mu)}} \text{ on } D(\mathcal{E}^{\varrho,a}) \Big\}.$$

Therefore we have to check that the linear operator

$$A_f : D(\mathcal{E}^{\varrho,a}) \to \mathbb{R}, \quad g \mapsto \int_\Omega \sum_{i,j=1}^{n} \partial_i f\, a_{ij}\, \partial_j g \; d\mu,$$

is continuous w.r.t. the norm of $L^2(\overline{\Omega}, \mu)$ for $f \in \mathcal{D}_{\text{Neu}}$. Since we can write $\nabla \varrho = \nabla(\sqrt{\varrho} \cdot \sqrt{\varrho})$ and ϱ satisfies Condition 3.1, we have by the product rule for Sobolev functions that $\varrho \in W^{1,1}(\Omega)$. Since $f \in W^{2,\infty}(\Omega)$ and $a \in W^{1,\infty}(\Omega)$, this implies that $u := \varrho\, g\, a_{ij}\, \partial_j f \in W^{1,1}(\Omega)$ for all $g \in \mathcal{D}$ and $i,j \in \{1,\ldots,n\}$. Thus, we can apply the Gaussian integral formula, see e.g. [Alt02, A 6.8 (1)], and obtain

$$\int_\Omega \partial_i u \; dx = \int_{\partial\Omega} u\, \nu_i \; d\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ and ν the outer normal w.r.t. $\partial\Omega$. Hence

$$\sum_{i,j=1}^{n} \int_\Omega \partial_i (\varrho\, g\; a_{ij} \partial_j f) \; dx = 0, \tag{3.2}$$

because

$$\sum_{i,j=1}^{n}\int_{\partial\Omega}(\varrho g a_{ij}\partial_j f)\nu_i \, d\mathcal{H}^{n-1} = \int_{\partial\Omega}\left(\sum_{i,j=1}^{n}\partial_j f a_{ij}\nu_i\right) g\varrho \, d\mathcal{H}^{n-1} = \int_{\partial\Omega}\partial_{a\nu} f g\varrho \, d\mathcal{H}^{n-1} = 0.$$

Applying the product rule for Sobolev functions to (3.2) and rearranging terms, we obtain

$$\sum_{i,j=1}^{n}\int_{\Omega}\partial_j f \, a_{ij}\,\partial_i g \, d\mu = -\sum_{i,j=1}^{n}\int_{\Omega}\big(\partial_i(a_{ij}\partial_j f) + \partial_i\varrho\,\varrho^{-1}\,a_{ij}\,\partial_j f\big)\, g \, d\mu. \tag{3.3}$$

Since $\mathcal{D}$ is dense in $\mathcal{D}(\mathcal{E}^{\varrho,a})$ w.r.t. $\sqrt{\mathcal{E}_1^{\varrho,a}}$ and $\partial_i\varrho\varrho^{-1} \in L^2(\overline{\Omega},\mu)$, we can extend (3.3) to all $g \in \mathcal{D}(\mathcal{E}^{\varrho,a})$. To show continuity let us estimate

$$\begin{aligned}
&\left|\sum_{i,j=1}^{n}\int_{\Omega}\partial_i f \, a_{ij}\,\partial_j g \, d\mu\right| \\
&\qquad\le \kappa\sum_{i,j=1}^{n}\left(\sqrt{\int_{\Omega}(\partial_i\partial_j f)^2 \, d\mu} + \sqrt{\int_{\Omega}(\partial_i\varrho\varrho^{-1}\partial_j f)^2 \, d\mu}\right)\sqrt{\int_{\Omega} g^2 \, d\mu} \\
&\qquad= \kappa\sum_{i,j=1}^{n}\left(\sqrt{\int_{\Omega}(\partial_i\partial_j f)^2 \, d\mu} + \sqrt{\int_{\Omega}\left(\frac{\partial_i\varrho}{\sqrt{\varrho}}\,\partial_j f\right)^2 \, dx}\right)\sqrt{\int_{\Omega} g^2 \, d\mu} \\
&\qquad\quad= \kappa\sum_{i,j=1}^{n}\left(\sqrt{\int_{\Omega}(\partial_i\partial_j f)^2 \, d\mu} + 2\sqrt{\int_{\Omega}(\partial_i\sqrt{\varrho}\,\partial_j f)^2 \, dx}\right)\sqrt{\int_{\Omega} g^2 \, d\mu},
\end{aligned} \tag{3.4}$$

where we used the ellipticity of a. Due to our assumptions on ϱ and f the integrals in (3.4) are finite. Hence for $f \in \mathcal{D}_{\mathrm{Neu}}$ the operator A_f is continuous and

$$\mathcal{E}^{\varrho,a}(f,g) = \sum_{i,j=1}^{n}\int_{\Omega}\partial_i f \, a_{ij}\,\partial_j g \;\varrho dx = \sum_{i,j=1}^{n}\int_{\Omega} -\big(\partial_i(a_{ij}\,\partial_j f) + \partial_i\varrho\,\varrho^{-1}\,a_{ij}\,\partial_j f\big)\, g \, d\mu$$

for all $f \in \mathcal{D}_{\mathrm{Neu}}$ and $g \in D(\mathcal{E}^{\varrho,a})$. Therefore, for all $f \in \mathcal{D}_{\mathrm{Neu}}$ the generator $L^{\varrho,a}$ is given by

$$L^{\varrho,a} f = \sum_{i,j=1}^{n}\partial_i(a_{ij}\,\partial_j f) + \partial_i\varrho\,\varrho^{-1}\,a_{ij}\,\partial_j f.$$

□

Remark 3.4.

(i) We stress that in $\mathcal{D}_{\mathrm{Neu}}$ only the normal derivative of the function f is forced to be zero at the boundary. The function f itself is allowed to take arbitrary values at the boundary. I.e., we have Neumann boundary conditions.

(ii) In the proof of Theorem 3.2 we need an L^2-bound of $\partial_i\varrho\varrho^{-1}$. Note that for $\varrho \in L^1(\overline{\Omega}, dx)$ (as we assume anyway) an equivalent condition to Condition 3.1 is $\nabla \ln\varrho \in L^2(\overline{\Omega},\mu)$.

(iii) Obviously, we get the representation of $L^{\varrho,a}$ as in (3.1) for $f \in C_0^\infty(\Omega)$ without assuming Ω having a Lipschitz boundary.

4. Some Potential theory of Dirichlet forms and its consequences

In this section we show that the set $\{\varrho = 0\} := \{x \in \overline{\Omega} \,|\, \varrho(x) = 0\}$ has capacity zero. As a consequence we can construct the associated process in $\{\varrho > 0\} := \{x \in \overline{\Omega} \,|\, \varrho(x) > 0\}$. This is very important for our construction of the N-particle stochastic dynamics with singular interactions, see Remark 5.5 below.

Definition 4.1. Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(\overline{\Omega}, \mu)$. The $\mathcal{E}$-capacity $\mathrm{cap}_{\mathcal{E}}(A)$ of an open set $A \subset \overline{\Omega}$ (here open has to be understood w.r.t. the trace topology on $\overline{\Omega}$) w.r.t. $(\mathcal{E}, D(\mathcal{E}))$ is defined by

$$\mathrm{cap}_{\mathcal{E}}(A) = \inf\{\mathcal{E}_1(f,f) \,|\, f \in \mathcal{D}(\mathcal{E}),\ f \geq 1 \ \mu\text{-a.e. on } A\},$$

and for an arbitrary set $A \subset \overline{\Omega}$ by

$$\mathrm{cap}_{\mathcal{E}}(A) = \inf\{\mathrm{cap}_{\mathcal{E}}(B) \,|\, B \text{ open },\ B \supset A\}.$$

For later use we state the following lemma proved in [Fuk80, Theo. 3.1.1].

Lemma 4.2. Let A_m, $m \in \mathbb{N}$, be an increasing sequence of subsets of $\overline{\Omega}$. Then

$$\mathrm{cap}_{\mathcal{E}}\left(\bigcup_{m \in \mathbb{N}} A_m\right) = \sup_{m \in \mathbb{N}} \mathrm{cap}_{\mathcal{E}}(A_m).$$

To make use of capacity estimates provided in [Stu95] we need the next definition and the following lemma.

Definition 4.3. In our situation a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ with $D(\mathcal{E}) \subset L^2(\overline{\Omega}, \mu)$ is called strongly regular, if the topology induced by the intrinsic metric

$$d(x,y) := \sup\Big\{u(x) - u(y) \,\Big|\, u \in D(\mathcal{E}) \cap C(\overline{\Omega}) \text{ with } \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \ \varrho \leq \varrho \text{ a.e. on } \overline{\Omega}\Big\}, \quad x, y \in \overline{\Omega},$$

coincides with the topology generated by the Euclidean metric.

Lemma 4.4. $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is strongly regular.

Proof. The intrinsic metric of the underlying Dirichlet form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is given by

$$d(x,y) := \sup\Big\{u(x) - u(y) \,\Big|\, u \in D(\mathcal{E}^{\varrho,a}) \cap C(\overline{\Omega}) \text{ with } \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \ \varrho \leq \varrho \text{ a.e. on } \overline{\Omega}\Big\} \text{ for } x, y \in \overline{\Omega}.$$

By assumption we have $\varrho > 0$ a.e. on $\overline{\Omega}$. Thus

$$d(x,y) = \sup\Big\{u(x) - u(y) \,\Big|\, u \in D(\mathcal{E}^{\varrho,a})) \cap C(\overline{\Omega}) \text{ with } \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \leq 1 \text{ a.e. on } \overline{\Omega}\Big\}$$

for $x, y \in \overline{\Omega}$. Since Ω is convex, it follows easily by the fundamental theorem of calculus that

$$d_{\mathrm{euc}}(x,y) = \sup\Big\{u(x) - u(y) \,\Big|\, u \in D(\mathcal{E}^{\varrho,a}) \cap C(\overline{\Omega}) \text{ with } |\nabla u|^2 \leq 1 \text{ a.e. on } \overline{\Omega}\Big\}$$

for $x, y \in \overline{\Omega}$, where d_{euc} is the metric induced by the Euclidean norm on $\mathbb{R}^n$. By the ellipticity of a we have

$$\kappa^{-1}|\nabla u|^2 \leq \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j u \leq \kappa |\nabla u|^2 \text{ for all } u \in D(\mathcal{E}^{\varrho,a}) \cap C(\overline{\Omega}).$$

Hence

$$\left\{ u \in D(\mathcal{E}^{\varrho,a}) \cap C(\overline{\Omega}) \,\middle|\, \sum_{i,j=1}^n a_{ij}\ \partial_i u \partial_j u \leq 1 \text{ a.e. on } \overline{\Omega} \right\}$$
$$\supset \left\{ u \in D(\mathcal{E}^{\varrho,a}) \cap C(\overline{\Omega}) \,\middle|\, \kappa\,|\nabla u|^2 \leq 1 \text{ a.e. on } \overline{\Omega} \right\}$$

and we obtain

$$d(x,y) \geq \kappa^{-\frac{1}{2}} d_{\mathrm{euc}}(x,y) \text{ for } x, y \in \overline{\Omega}.$$

An analogous argumentation as above yields $d(x,y) \leq \kappa^{\frac{1}{2}} d_{\mathrm{euc}}(x,y)$ for $x, y \in \overline{\Omega}$. So the intrinsic metric of the underlying Dirichlet form $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is equivalent to the Euclidean metric. Hence the topologies generated by them coincide, i.e. $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is strongly regular. □

Now we are ready to state and prove the main result of this section.

Theorem 4.5. *Suppose that the density function ϱ satisfies Condition 2.2. Furthermore, assume one of the following two conditions:*

(i) ϱ satisfies Condition 3.1 and is continuous on $\overline{\Omega}$.

(ii) Ω is convex and there exists $0 < C < \infty$ such that

$$\int_{B_r(\{\varrho=0\})} \varrho(x)dx \leq Cr^2 \quad \textit{as} \quad r \to 0.$$

Then

$$cap^{\varrho,a}(\{\varrho = 0\}) = 0,$$

where $cap^{\varrho,a} := cap_{\mathcal{E}^{\varrho,a}}$.

Proof. In situation (i) we know that $\psi := \sqrt{\varrho} > 0$ dx-a.e. and $\psi \in W^{1,2}(\Omega)$. For $\varepsilon > 0$ let $\psi_\varepsilon := (\psi \vee \varepsilon) \wedge 1$ and $f_\varepsilon = -\log(\psi_\varepsilon)$. Then f_ε is continuous on $\overline{\Omega}$, $\nabla f_\varepsilon = -\frac{\nabla \psi_\varepsilon}{\psi_\varepsilon}$ and

$$(\nabla f_\varepsilon, \nabla f_\varepsilon)_{\mathbb{R}^n} = (\nabla \psi_\varepsilon, \nabla \psi_\varepsilon)_{\mathbb{R}^n} \psi_\varepsilon^{-2} \leq (\nabla \psi_\varepsilon, \nabla \psi_\varepsilon)_{\mathbb{R}^n} \varrho^{-1} \in L^1(\overline{\Omega}, d\mu),$$

since $\psi_\varepsilon \in W^{1,2}(\Omega)$. Thus $f_\varepsilon \in \mathcal{D}(\mathcal{E}^{\varrho,a})$. We have

$$\begin{aligned}
\mathcal{E}_1^{\varrho,a}(f_\varepsilon, f_\varepsilon) &= \sum_{i,j=1}^n \int_\Omega \partial_i f_\varepsilon\, a_{ij}\, \partial_j f_\varepsilon\ \psi^2 dx + \int_{\overline{\Omega}} {f_\varepsilon}^2\, d\mu \\
&= \sum_{i,j=1}^n \int_\Omega \partial_i \psi_\varepsilon\, a_{ij}\, \partial_j \psi_\varepsilon\ \frac{\psi^2}{{\psi_\varepsilon}^2}\ dx + \int_{\overline{\Omega}} \log(\psi_\varepsilon)^2\, d\mu \\
&\leq \kappa \sum_{i=1}^n \int_\Omega (\partial_i \psi_\varepsilon)^2\ dx + \int_{\overline{\Omega}} \log(\psi_\varepsilon)^2\, d\mu < \infty \quad (4.1)
\end{aligned}$$

(in view of our assumptions on ψ). Let $\lambda > 0$. Then

$$\mathrm{cap}^{\varrho,a}(\{f_\varepsilon \;>\; \lambda\}) \;\leq\; \mathcal{E}_1^{\varrho,a}(\frac{1}{\lambda}f_\varepsilon, \frac{1}{\lambda}f_\varepsilon) \;\leq\; \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi_\varepsilon)^2 \; dx + \int_{\overline{\Omega}} \log(\psi_\varepsilon)^2 \, d\mu\right),$$

by Definition 4.1 and (4.1). Next we set $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$, and consider

$$A_m := \big\{ f_{\frac{1}{m}} > \lambda \big\}.$$

We observe that the sequence of sets A_m is increasing in $m \in \mathbb{N}$. Thus, for $A := \bigcup_{m=1}^\infty A_m$ Lemma 4.2 yields

$$\mathrm{cap}^{\varrho,a}(A) = \sup_{m\in\mathbb{N}} \mathrm{cap}^{\varrho,a}(A_m) \leq \sup_{m\in\mathbb{N}} \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi_{\frac{1}{m}})^2 \; dx + \int_{\overline{\Omega}} \log(\psi_{\frac{1}{m}})^2 \, d\mu\right).$$

Since in the above integrals we are dealing with functions pointwisely monotone increasing in $m \in \mathbb{N}$, the supreme coincides with

$$\begin{aligned} &\lim_{m\to\infty} \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi_{\frac{1}{m}})^2 \; dx + \int_{\overline{\Omega}} \log(\psi_{\frac{1}{m}})^2 \, d\mu\right) \\ &= \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi)^2 \; dx + \int_{\overline{\Omega}} \log(\psi)^2 \psi^2 \, dx\right) < \infty, \end{aligned}$$

due to our assumptions on ψ. Observe that $A = \big\{ \log(\psi) > \lambda \big\}$. Therefore,

$$\mathrm{cap}^{\varrho,a}\left(\big\{ \log(\psi) > \lambda\big\}\right) \leq \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi)^2 \; dx + \int_{\overline{\Omega}} \log(\psi)^2 \psi^2 \, dx\right).$$

Thus,

$$\begin{aligned} \mathrm{cap}^{\varrho,a}(\{\varrho = 0\}) &\leq \mathrm{cap}^{\varrho,a}\left(\big\{ \log(\varrho) > 2\lambda\big\}\right) = \mathrm{cap}^{\varrho,a}\left(\big\{ \log(\psi) > \lambda\big\}\right) \\ &\leq \frac{\kappa}{\lambda^2}\left(\sum_{i=1}^n \int_\Omega (\partial_i \psi)^2 \; dx + \int_{\overline{\Omega}} \log(\psi)^2 \psi^2 \, dx\right) \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

Situation (ii): We have due to Lemma 4.4 that $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ is strongly regular. By assumption the compact set $\{\varrho = 0\}$ is of μ-measure zero. Thus we can apply [Stu95, Theo. 3] and the statement follows. □

Remark 4.6. Some of the ideas for the proof in situation (i) of Theorem 4.5 we got from the proof of [Fuk85, Theo. 2].

Let $i : \{\varrho > 0\} \to \{\varrho > 0\}$ be the identity map. Since $\{\varrho = 0\} = \overline{\Omega} \setminus \{\varrho > 0\}$ has Lebesgue measure zero, we can consider the isometry $i^* : L^2(\{\varrho > 0\}, \mu) \to L^2(\overline{\Omega}, \mu)$ by defining $i^*(f)$ to be the μ-class represented by $\tilde{f} \circ i$ on $\{\varrho > 0\}$ for any measurable μ-version $\tilde{f}$ of $f \in L^2(\{\varrho > 0\}, \mu)$. Obviously, i^* is surjective and due to [MR92, Chap. VI, Exe. 1.1]

$$\begin{aligned} &\widehat{\mathcal{E}^{\varrho,a}}(f,g) := \mathcal{E}^{\varrho,a}(i^*(f), i^*(g)), \;\; f,g \in D(\widehat{\mathcal{E}^{\varrho,a}}), \\ &D(\widehat{\mathcal{E}^{\varrho,a}}) := \Big\{ f \in L^2(\{\varrho > 0\}, \mu) | f \in {i^*}^{-1}(D(\mathcal{E}^{\varrho,a}))\Big\} \end{aligned}$$

is a Dirichlet form on $L^2(\{\varrho > 0\}, \mu)$. $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$ is called the image Dirichlet form of $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$ under i.

Corollary 4.7. *Under the assumptions as in Theorem 4.5 $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$ is a conservative, local, quasi-regular Dirichlet form on $L^2(\{\varrho > 0\}, \mu)$.*

Proof. By Theorem 4.5 we know that $\mathrm{cap}^{\varrho,a}(\{\varrho = 0\}) = 0$ and this is equivalent to the fact that $\{\varrho = 0\}$ is an $\mathcal{E}^{\varrho,a}$-exceptional set, i.e.,

$$\{\varrho = 0\} \subset \bigcap_{k\geq 1} F_k^c \text{ for some } \mathcal{E}^{\varrho,a}\text{-nest } (F_k)_{k\geq 1},$$

see [AST03, Prop. 14 (3)]. Thus, $(F_k)_{k\in\mathbb{N}}$ is a sequence of compact sets in $\{\varrho > 0\}$. Note that functions from $D(\widehat{\mathcal{E}^{\varrho,a}})$ are the restrictions to $\{\varrho > 0\}$ of functions from $D(\mathcal{E}^{\varrho,a})$. Since $\bigcup_{k\geq 1} D(\mathcal{E}^{\varrho,a})_{F_k}$ is a dense set in $D(\mathcal{E}^{\varrho,a})$, $\bigcup_{k\geq 1} D(\widehat{\mathcal{E}^{\varrho,a}})_{F_k}$ is a dense set in $D(\widehat{\mathcal{E}^{\varrho,a}})$. Hence $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$ has a compact $\widehat{\mathcal{E}^{\varrho,a}}$-nest. Furthermore, the functions from $\mathcal{D}$ restricted to $\{\varrho > 0\} \subset \overline{\Omega}$, again are continuous functions. Now, since $D(\mathcal{E}^{\varrho,a})$ is the completion of $\mathcal{D}$ w.r.t. $\sqrt{\mathcal{E}_1^{\varrho,a}}$ in $L^2(\overline{\Omega}, \mu)$ and $\{\varrho = 0\}$ is of Lebesgue measure zero, the continuous functions in $D(\widehat{\mathcal{E}^{\varrho,a}})$ are dense in $D(\widehat{\mathcal{E}^{\varrho,a}})$ w.r.t. $\sqrt{\widehat{\mathcal{E}_1^{\varrho,a}}}$. Clearly, the countable set of polynomials with rational coefficients is separating points on $\{\varrho > 0\}$. Locality and conservativity are clear by locality and conservativity of $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$, see Definition 2.19 and Remark 2.7(iv). □

The generator of $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$ we denote by $(\widehat{L^{\varrho,a}}, D(\widehat{L^{\varrho,a}}))$. The strongly continuous contraction semigroup generated by $(\widehat{L^{\varrho,a}}, D(\widehat{L^{\varrho,a}}))$ we denote by $(\widehat{T^{\varrho,a}}_t)_{t\geq 0}$. Then we have the following corollary.

Corollary 4.8. *Under the assumptions as in Theorem 4.5 there exists a conservative diffusion process* $\widehat{\mathbf{M}^{\varrho,a}}$ *in* $\{\varrho > 0\}$ *associated with the Dirichlet form* $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$. *Furthermore, all statements of Theorem 2.22 hold true if* $(\mathcal{E}^{\varrho,a}, D(\mathcal{E}^{\varrho,a}))$, $(L^{\varrho,a}, D(L^{\varrho,a}))$, $(T_t^{\varrho,a})_{t\geq 0}$ *and* $\mathbf{M}^{\varrho,a}$ *are replaced by* $(\widehat{\mathcal{E}^{\varrho,a}}, D(\widehat{\mathcal{E}^{\varrho,a}}))$, $(\widehat{L^{\varrho,a}}, D(\widehat{L^{\varrho,a}}))$, $(\widehat{T^{\varrho,a}}_t)_{t\geq 0}$ *and* $\widehat{\mathbf{M}^{\varrho,a}}$, *respectively.*

5. An application to continuous N-particle systems with singular interactions

Let $d \in \mathbb{N}$. A pair potential (without hard core) is a Borel measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Let us fix our assumptions on the potential ϕ.

(RP): (*Repulsion*) There exists a continuous decreasing function $\Phi : (0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \Phi(t) = \infty$ and $R_1 > 0$ such that

$$\phi(x) \geq \Phi(|x|) \quad \text{for} \quad |x| \leq R_1.$$

Furthermore the potential is bounded from above on

$$\{x \in \mathbb{R}^d \,|\, \kappa \leq |x|\} \text{ for all } \kappa > 0.$$

(SRP): (*Strong repulsion*) There exists $R_1 > 0$ such that

$$\phi(x) \geq -\ln(|x|) \quad \text{for} \quad 0 < |x| \leq R_1.$$

Furthermore the potential is bounded from above on

$$\{x \in \mathbb{R}^d \,|\, \kappa \leq |x|\} \text{ for all } \kappa > 0.$$

(BB): (*Bounded below*) There exists $0 \leq B < \infty$ such that

$$\phi(x) \geq -B \text{ for all } x \in \mathbb{R}^d.$$

(DL²): (*Differentiability and L^2*) The function $\exp(-\phi)$ is weakly differentiable on $\mathbb{R}^d$, ϕ is continuous on $\mathbb{R}^d \setminus \{0\}$ and weakly differentiable on $\mathbb{R}^d$. The gradient $\nabla\phi$, considered as a dx-a.e. defined function on $\mathbb{R}^d$, satisfies

$$\nabla\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \exp(-\phi)dx).$$

Remark 5.1. Note that for many typical potentials in Statistical Physics, we have $\phi \in C^\infty(\mathbb{R}^d \setminus \{0\})$. For such "outside the origin regular" potentials, condition (DL^2) nevertheless does not exclude a singularity at the point $0 \in \mathbb{R}^d$.

Let $N, d \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$, such that $\overline{\Omega} := \Lambda^N \subset \mathbb{R}^{N\cdot d}$, is the closure of an open, relatively compact set, having boundary $\partial(\Lambda^N)$ of Lebesgue measure zero. On Λ^N we consider the density function

$$\varrho_{\Lambda,N}(x) = \frac{1}{Z_{\Lambda,N}} \exp\Big(-\sum_{1\le i<j\le N} \phi(x_i - x_j)\Big), \quad x = (x_1, \ldots, x_N) \in \Lambda^N,$$

where

$$Z_{\Lambda,N} := \int_{\Lambda^N} \exp\Big(-\sum_{1\le i<j\le N} \phi(x_i - x_j)\Big)\, dx^{\otimes N}.$$

Proposition 5.2. *If $d = 1$ we suppose either condition (A) or (B):*

(A) ϕ satisfies conditions (SRP), (BB) and Λ is convex.

(B) ϕ satisfies conditions (RP) and (DL²).

If $d \ge 2$ we suppose either condition (C) or (D):

(C) ϕ to satisfies conditions (RP), (BB) and Λ is convex.

(D) ϕ is bounded and Λ is convex.

Then in all situations $(\mathcal{E}_{\Lambda,N}, \mathcal{D})$ is closable. Its closure $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ is a conservative, local, quasi-regular Dirichlet form on $L^2(\Lambda^N, \mu_{\Lambda,N})$. Its generator we denote by $(L_{\Lambda,N}, D(L_{\Lambda,N}))$.

Proof. In each situation it is easy to check that Condition 2.2 holds. Thus the statement follows from Corollary 2.21. □

As before, Proposition 5.2 now implies the existence of a corresponding conservative diffusion process in Λ^N. Λ^N, however, has no direct interpretation as continuous N-particle system. This leads us to the configuration space over $\mathbb{R}^d$, which is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, \#(\gamma_\Lambda) < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d\},$$

where $\#$ denotes the number of elements of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where ε_x is the Dirac measure at x, $\sum_{x\in\emptyset} \varepsilon_x :=$ zero measure, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Hence, via this identification, Γ can be equipped with the vague topology. We set $\Gamma_\Lambda := \{\gamma \in \Gamma \,|\, \gamma \subset \Lambda\}$. The space of N-point configurations in Λ is defined by

$$\Gamma_\Lambda^{(N)} := \{\gamma \subset \Lambda \,|\, \#(\gamma) = N\} \subset \Gamma_\Lambda \subset \Gamma.$$

To define more structure on $\Gamma_\Lambda^{(N)}$ we may use the following natural mapping

$$\mathrm{sym}^{(N)} : \widetilde{\Lambda^N} \to \Gamma_\Lambda^{(N)}$$

$$\mathrm{sym}^{(N)}(x_1, \ldots, x_N) := \{x_1, \ldots, x_N\},$$

where

$$\widetilde{\Lambda^N} := \{(x_1, \ldots, x_N) \in \Lambda^N \,|\, x_k \neq x_j \text{ if } k \neq j\}.$$

We assume these mappings to generate the topology and corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda^{(N)})$ on $\Gamma_\Lambda^{(N)}$. Obviously, this σ-algebra coincides with the Borel σ-algebra inherited from Γ equipped with its vague topology. Of course, the product measure $dx^{\otimes N}$ can be considered on $\widetilde{\Lambda^N}$. Let $dx^{(N)} := dx^{\otimes N} \circ \left(\mathrm{sym}^{(N)}\right)^{-1}$ denote the corresponding measure on $\Gamma_\Lambda^{(N)}$.

In order to construct the N-particle stochastic dynamics, we are interested in the image Dirichlet form $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ of $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ under $\mathrm{sym}^{(N)}$. Consider the measure $\mu_\Lambda^{(N)} := \mu_{\Lambda,N} \circ (\mathrm{sym}^{(N)})^{-1}$. $\mu_\Lambda^{(N)}$ is the canonical N-particle Gibbs measure in Λ with empty boundary conditions on $(\Gamma_\Lambda^{(N)}, \mathcal{B}(\Gamma_\Lambda^{(N)}))$. Define an isometry $(\mathrm{sym}^{(N)})^* : L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)}) \to L^2(\Lambda^N, \mu_{\Lambda,N})$ by setting $(\mathrm{sym}^{(N)})^* F$ to be the $\mu_{\Lambda,N}$-class represented by $\tilde{F} \circ \mathrm{sym}_\Lambda^{(N)}$ on $\widetilde{\Lambda^N}$ for any $\mu_\Lambda^{(N)}$-version $\tilde{F}$ of $F \in L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})$ (note that the set of diagonals $Dg := \Lambda^N \setminus \widetilde{\Lambda^N}$ has $\mu_{\Lambda,N}$-measure zero).

Note that the subspace

$$L^2_{\mathrm{sym}}(\Lambda^N, \mu_{\Lambda,N}) := (\,\mathrm{sym}^{(N)})^*(L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})) \subset L^2(\Lambda^N, \mu_{\Lambda,N})$$

is the closed subspace of symmetric functions from $L^2(\Lambda^N, \mu_{\Lambda,N})$. Using this mapping one can define a bilinear form $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ as the image bilinear form of $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ under $\mathrm{sym}^{(N)}$:

$$\mathcal{E}_\Lambda^{(N)}(F,G) := \mathcal{E}_{\Lambda,N}((\,\mathrm{sym}^{(N)})^* F, (\mathrm{sym}^{(N)})^* G), \qquad F, G \in D(\mathcal{E}_\Lambda^{(N)}),$$

where

$$D(\mathcal{E}_\Lambda^{(N)}) := ((\mathrm{sym}^{(N)})^*)^{-1}(D(\mathcal{E}_{\Lambda,N}) \cap L^2_{\mathrm{sym}}(\mu_{\Lambda,N})).$$

Proposition 5.3. *If $d = 1$ we suppose either condition (A) or (B):*

(A) ϕ satisfies conditions (SRP), (BB) and Λ is convex.

(B) ϕ satisfies conditions (RP) and (DL^2).

If $d \geq 2$ we suppose either condition (C) or (D):

(C) ϕ to satisfies conditions (RP), (BB) and Λ is convex.

(D) ϕ is bounded and Λ is convex.

Then in all situations the bilinear form $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ is a conservative, local, quasi-regular Dirichlet form on $L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})$. Its generator is given by

$$L_\Lambda^{(N)} = ((\mathrm{sym}^{(N)})^*)^{-1} \circ L_{\Lambda,N} \circ (\mathrm{sym}^{(N)})^*,$$
$$D(L_\Lambda^{(N)}) = ((\mathrm{sym}^{(N)})^*)^{-1}(D(L_{\Lambda,N}) \cap L^2_{\mathrm{sym}}(\mu_{\Lambda,N})).$$

Of course, $(L_\Lambda^{(N)}, D(L_\Lambda^{(N)}))$ generates a strongly continuous contraction semi-group

$$T_\Lambda^{(N)}(t) := \exp(t L_\Lambda^{(N)}), \quad t \geq 0.$$

Proof. In situations (A) and (C) we have $Dg = \{\varrho_{\Lambda,N} = 0\}$ due to conditions (SRP) and (RP), respectively. Since Dg has codimension $\geq d$ condition (ii) of Theorem 4.5 is fulfilled and $\mathrm{cap}_{\mathcal{E}_{\Lambda,N}}(Dg) = 0$. In situation (B) we also have $Dg = \{\varrho_{\Lambda,N} = 0\}$ due to condition (RP). Thus $\mathrm{cap}_{\mathcal{E}_{\Lambda,N}}(Dg) = 0$, because condition (i) in Theorem 4.5 is satisfied. In situation (D) we have $\{\varrho_{\Lambda,N} = 0\} = \emptyset$, since ϕ is bounded. $Dg = \Lambda^N \setminus \widetilde{\Lambda^N}$ is of codimension $\geq d$. $(\mathcal{E}^{\varrho_{\Lambda,N}}, D(\mathcal{E}^{\varrho_{\Lambda,N}}))$ is strongly regular (as shown in the proof of Lemma 4.4). Thus we have due to [Stu95, Theo. 3] that $\Lambda^N \setminus \widetilde{\Lambda^N}$ is of capacity zero with respect to $(\mathcal{E}^{\varrho_{\Lambda,N}}, D(\mathcal{E}^{\varrho_{\Lambda,N}}))$. Therefore, as in the proof of Corollary 4.7, we obtain that $(\mathcal{E}_\Lambda^{(N)}, \mathcal{D}(\mathcal{E}_\Lambda^{(N)}))$ is a conservative, local, quasi-regular Dirichlet form on $L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})$. Just take $\mathrm{sym}^{(N)}$ and $(\mathrm{sym}^{(N)})^*$ instead of the mappings i and i^*, respectively, and use that the isometry

$$(\mathrm{sym}^{(N)})^* : L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)}) \to L^2_{\mathrm{sym}}(\Lambda^N, \mu_{\Lambda,N})$$

is surjective. □

Theorem 5.4. *If $d = 1$ we suppose either condition (A) or (B):*

(A) ϕ satisfies conditions (SRP), (BB) and Λ is convex.

(B) ϕ satisfies conditions (RP) and (DL^2).

If $d \geq 2$ we suppose either condition (C) or (D):

(C) ϕ to satisfies conditions (RP), (BB) and Λ is convex.

(D) ϕ is bounded and Λ is convex.

Then in all situations:

(i) There exists a conservative diffusion process (i.e., a strong Markov process with continuous sample paths and infinite life time)

$$\mathbf{M}_\Lambda^{(N)} = \big(\mathbf{\Omega}_\Lambda^{(N)}, \mathbf{F}_\Lambda^{(N)}, (\mathbf{F}_\Lambda^{(N)}(t))_{t\geq 0}, (\mathbf{\Theta}_\Lambda^{(N)}(t))_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\Lambda^{(N)}(x))_{x\in\Gamma_\Lambda^{(N)}}\big)$$

in $\Gamma_\Lambda^{(N)}$ which is properly associated with $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$, i.e., for all ($\mu_\Lambda^{(N)}$-versions of) $F \in L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})$ and all $t > 0$ the function

$$x \mapsto \int_{\mathbf{\Omega}_\Lambda^{(N)}} F(\mathbf{X}(t))\, d\mathbf{P}_\Lambda^{(N)}(x), \qquad x \in \Gamma_\Lambda^{(N)},$$

is an $\mathcal{E}_\Lambda^{(N)}$-quasi-continuous version of $T_\Lambda^{(N)}(t)F$. $\mathbf{M}_\Lambda^{(N)}$ is up to $\mu_\Lambda^{(N)}$-equivalence unique. In particular, $\mathbf{M}_\Lambda^{(N)}$ is $\mu_\Lambda^{(N)}$-symmetric and has $\mu_\Lambda^{(N)}$ as an invariant measure.

(ii) The diffusion process $\mathbf{M}_\Lambda^{(N)}$ is up to $\mu_\Lambda^{(N)}$-equivalence the unique diffusion process having $\mu_\Lambda^{(N)}$ as symmetrizing measure and solving the martingale problem for $(L_\Lambda^{(N)}, D(L_\Lambda^{(N)}))$, i.e., for all $G \in D(L_\Lambda^{(N)})$

$$G(\mathbf{X}(t)) - G(\mathbf{X}(0)) - \int_0^t L_\Lambda^{(N)} G(\mathbf{X}(s))\, ds, \qquad t \geq 0,$$

is an $\mathbf{F}_\Lambda^{(N)}(t)$-martingale under $\mathbf{P}_\Lambda^{(N)}(x)$ (hence starting in x) for $\mathcal{E}_\Lambda^{(N)}$-quasi all $x \in \Gamma_\Lambda^{(N)}$.

In the above theorem $\mathbf{M}_\Lambda^{(N)}$ is canonical, i.e., $\mathbf{\Omega}_\Lambda^{(N)} = C([0,\infty) \to \Gamma_\Lambda^{(N)})$, $\mathbf{X}(t)(\omega) = \omega(t), \omega \in \mathbf{\Omega}_\Lambda^{(N)}$. The filtration $(\mathbf{F}_\Lambda^{(N)}(t))_{t\geq 0}$ is the natural "minimum completed admissible filtration", cf. [FOT94, Chap. A.2], or [MR92, Chap. IV], obtained from the σ-algebras $\sigma\{\omega(s) \,|\, 0 \leq s \leq t, \omega \in \mathbf{\Omega}_\Lambda^{(N)}\}$, $t \geq 0$. $\mathbf{F}_\Lambda^{(N)} := \mathbf{F}_\Lambda^{(N)}(\infty) := \bigvee_{t\in[0,\infty)} \mathbf{F}_\Lambda^{(N)}(t)$ is the smallest σ-algebra containing all $\mathbf{F}_\Lambda^{(N)}(t)$ and $(\mathbf{\Theta}_\Lambda^{(N)}(t))_{t\geq 0}$ are the corresponding natural time shifts. For a detailed discussions of these objects we refer to [MR92].

Proof. Due to Proposition 5.3 the proof is just the same as the proof of Theorem 2.22. □

Remark 5.5. Notice that the fact $\mathrm{cap}_{\mathcal{E}_{\Lambda,N}}(Dg) = 0$, i.e., the $\mathcal{E}_{\Lambda,N}$-capacity of the set of diagonals in Λ^N is zero, is essential for proving Proposition 5.3, which in turn yields Theorem 5.4. In situations (A), (B) and (C) it is due to the fact that the interaction potential ϕ is repulsive, see condition (RP) or (SRP). Thus, we get from condition (RP) or (SRP) that the N-particle stochastic dynamics $\mathbf{M}_\Lambda^{(N)}$ in Λ stays in the configuration space $\Gamma_\Lambda^{(N)}$ of single configurations, i.e., we never have more then one particle in one position. In other words, the repulsive interaction potential ϕ prevents particles from interpenetrating each another. In situation (D) we obtain $\text{cap}_{\mathcal{E}_{\Lambda,N}}(Dg) = 0$ by the fact that the diagonals have codimension ≥ 2. This, of course, is only true for $d \geq 2$.

Another way to construct the process $\mathbf{M}_\Lambda^{(N)}$ is to use first Corollary 4.8 to construct the corresponding process in $\{\varrho_{\Lambda,N} > 0\} = \Lambda^N \setminus Dg = \widetilde{\Lambda^N}$ and then the mapping $\text{sym}^{(N)} : \widetilde{\Lambda^N} \to \Gamma_\Lambda^{(N)}$ to construct $\mathbf{M}_\Lambda^{(N)}$ as the image process under $\text{sym}^{(N)}$. This approach also needs $\text{cap}_{\mathcal{E}_{\Lambda,N}}(Dg) = 0$, which in this situation yields that the process in Λ^N does not hit the diagonals $Dg \subset \Lambda^N$. This is of essential importance, since otherwise it would not be possible to apply the mapping $\text{sym}^{(N)}$, which is only defined outside the diagonals.

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