Gradient bounds for solutions of elliptic and parabolic equations

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Abstract

Let \( L \) be a second order elliptic operator on \( \mathbb{R}^d \) with a constant diffusion matrix and a dissipative (in a weak sense) drift \( b \in L^p_{\text{loc}} \) with some \( p > d \). We assume that \( L \) possesses a Lyapunov function, but no local boundedness of \( b \) is assumed. It is known that then there exists a unique probability measure \( \mu \) satisfying the equation \( L^* \mu = 0 \) and that the closure of \( L \) in \( L^1(\mu) \) generates a Markov semigroup \( \{T_t\}_{t \geq 0} \) with the resolvent \( \{G_\lambda\}_{\lambda > 0} \). We prove that, for any Lipschitzian function \( f \in L^1(\mu) \) and all \( t, \lambda > 0 \), the functions \( T_t f \) and \( G_\lambda f \) are Lipschitzian and

\[
|\nabla T_t f(x)| \leq T_t |\nabla f|(x) \quad \text{and} \quad |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} G_\lambda |\nabla f|(x).
\]

An analogous result is proved in the parabolic case.

Suppose that for every \( t \in [0, 1] \), we are given a a strictly positive definite symmetric matrix \( A(t) = (a^{ij}(t)) \) and a measurable vector field \( x \mapsto b(t, x) = (b^1(t, x), \ldots, b^n(t, x)) \).

Let \( L_t \) be the elliptic operator on \( \mathbb{R}^d \) given by

\[
L_t u(x) = \sum_{i,j \leq d} a^{ij}(t, x) \partial_{x_i} \partial_{x_j} u(x) + \sum_{i \leq d} b^i(t, x) \partial_{x_i} u(x). \tag{1}
\]

Suppose that \( A \) and \( b \) satisfy the following hypotheses:

(Ha) \( \sup_{t \in [0,1]} (\|A(t)\| + \|A(t)^{-1}\|) < \infty, \sup_{t \in [0,1]} \|b(t, \cdot)\|_{L^p(U)} < \infty \) for every ball \( U \) in \( \mathbb{R}^d \) with some \( p > d, p \geq 2 \).

(Hb) \( b \) is dissipative in the following sense: for every \( t \in [0, 1] \) and every \( h \in \mathbb{R}^d \), there exists a measure zero set \( N_{t,h} \subset \mathbb{R}^d \) such that

\[
(b(t, x + h) - b(t, x), h) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}^d \setminus N_{t,h}.
\]

(Hc) for every \( t \in [0, 1] \), there exists a Lyapunov function \( V_t \) for \( L_t \), i.e., a nonnegative \( C^2 \)-function \( V_t \) such that \( V_t(x) \to +\infty \) and \( L_t V_t(x) \to -\infty \) as \( |x| \to \infty \).

We consider the parabolic equation

\[
\frac{\partial u}{\partial t} = L_t u, \quad u(0, x) = f(x), \tag{2}
\]
where \( f \) is a bounded Lipschitz function. A locally integrable function \( u \) on \([0, 1] \times \mathbb{R}^d\) is called a solution if, for every \( t \in (0, 1] \), one has \( u(t, \cdot) \in W^{1,2}_{loc}(\mathbb{R}^d) \), the functions \( \partial_x \partial_{x_j} u \) and \( b \partial_x u \) are integrable on the sets \([0, 1] \times K\) for every cube \( K \) in \( \mathbb{R}^d \), and for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and all \( t \in [0, 1] \) one has

\[
\int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx + \int_0^t \int_{\mathbb{R}^d} L_x \varphi(x) u(s, x) \, dx \, ds.
\]

In the case where \( A \) and \( b \) are independent of \( t \), so that we have a single operator \( L \), Hypotheses (Ha) and (Hc) imply (see [6] and [8]) that there exists a unique probability measure \( \mu \) on \( \mathbb{R}^d \) such that \( \mu \) has a strictly positive continuous weakly differentiable density \( \varrho \), \( \nabla \varrho \in L^p_{loc}(\mathbb{R}^d) \), and \( L^* \mu = 0 \) in the following weak sense:

\[
\int L u \, d\mu = 0 \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^d).
\]

The closure \( \overline{L} \) of \( L \) with domain \( C_0^\infty(\mathbb{R}^d) \) in \( L^1(\mu) \) generates a Markov semigroup \( \{T_t\}_{t \geq 0} \) for which \( \mu \) is invariant. Let \( D(\overline{L}) \) denote the domain of \( \overline{L} \) in \( L^1(\mu) \) and let \( \{G_\lambda\}_{\lambda > 0} \) denote the corresponding resolvent, i.e., \( G_\lambda = (\lambda - \overline{L})^{-1} \).

The restrictions of \( T_t \) and \( G_\lambda \) to \( L^2(\mu) \) are contractions on \( L^2(\mu) \). In particular, if \( v \in D(\overline{L}) \) is such that \( \lambda v - \overline{L} v = g \in L^2(\mu) \), then \( v \in L^2(\mu) \). Moreover, it follows by [8, Theorem 2.8] that one has \( v \in H^2_{loc}(\mathbb{R}^d) \) and \( \overline{L} v = L v \) a.e., so that one has a.e.

\[
\lambda v - L v = g. \tag{3}
\]

In fact, due to our assumptions on the coefficients of \( L \) one has even \( v \in W^{p,2}_{loc}(\mathbb{R}^d) \) (see [10]). It has been shown in [3] that for every function \( f \in L^1(\mu) \) that is Lipschitzian with constant \( C \) and all \( t, \lambda > 0 \), the continuous version of the function \( T_t f \) is Lipschitzian with constant \( C \), and the continuous version of \( G_\lambda f \) is Lipschitzian with constant \( \lambda^{-1} C \). Here we establish pointwise estimates in both cases and prove their parabolic analogue. The main results of this work are the following two theorems.

**Theorem 1.** Suppose that \( A \) and \( b \) are independent of \( t \) and satisfy (Ha), (Hb) and (Hc). Then, for any Lipschitzian function \( f \in L^1(\mu) \) and all \( t, \lambda > 0 \), \( T_t f \) and \( G_\lambda f \) have Lipschitzian versions such that

\[
|\nabla T_t f(x)| \leq T_t |\nabla f|(x) \quad \text{and} \quad |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} G_\lambda |\nabla f|(x) \tag{4}
\]

for the corresponding continuous versions. In particular,

\[
\sup_{x,t} |\nabla T_t f(x)| \leq \sup_x |\nabla f(x)|, \quad \sup_x |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} \sup_x |\nabla f(x)|. \tag{5}
\]

**Theorem 2.** Suppose that \( A \) and \( b \) satisfy (Ha), (Hb) and (Hc). Then, for any bounded Lipschitzian function \( f \) there is a solution \( u \) of equation (2) such that for all \( t \) one has

\[
\sup_x |\nabla u(t, x)| \leq \sup_x |\nabla f(x)|. \tag{6}
\]
In the case where $A = I$ and $b = 0$, estimate (6) has been established in [12], [13] for solutions of boundary value problems in bounded domains. It should be noted that gradient estimates of the type
\[ \sup_x |\nabla u(x,t)| \leq C(t) \sup_x |f(x)| \]
for solutions of parabolic equations have been obtained by many authors, see, e.g., [1], [2], [11], [15], and the references therein. Such estimates do not require (Hb) and one has $C(t) \to +\infty$ as $t \to 0$ or $t \to +\infty$. In contrast to this type of estimates, our theorems mean a contraction property on Lipschitz functions rather than a smoothing property. It is likely that some results of the cited works, established for sufficiently regular $b$, can be extended to more general drifts satisfying just (Ha), but not (Hb).

A short proof of the following result can be found in [3].

**Lemma 1.** Suppose that $b$ is infinitely differentiable, Lipschitzian, and strongly dissipative, so for some $\alpha > 0$, one has
\[ (b(x+h) - b(x), h) \leq -\alpha(h,h) \quad \text{for all } x, h \in \mathbb{R}^d. \]
Then, for any $\lambda > 0$ and any smooth bounded Lipschitzian function $f$, one has pointwise
\[ |\nabla G_\lambda f| \leq G_\lambda |\nabla f|. \]
In particular, $\sup_x |\nabla G_\lambda f(x)| \leq \lambda^{-1} \sup_x |\nabla f(x)|$.

**Proof of Theorem 1.** The estimate with the suprema has been proven in [3], and the stronger pointwise estimate can be derived from that proof. For the reader’s convenience, instead of recursions to the steps of the proof in [3] we reproduce the whole proof and explain why it yields a stronger conclusion. We recall that if a sequence of functions on $\mathbb{R}^d$ is uniformly Lipschitzian with constant $L$ and bounded at a point, then it contains a subsequence that converges uniformly on every ball to a function that is Lipschitzian with the same constant. Therefore, approximating $f$ in $L^1(\mu)$ by a sequence of bounded smooth functions $f_j$ with
\[ \sup_x |\nabla f_j(x)| \leq \sup_x |\nabla f(x)|, \]
it suffices to prove (5) for smooth bounded $f$. Moreover, due to Euler’s formula
\[ T_t f = \lim_n \left( \frac{t}{n} G_{\frac{t}{n}} \right)^n f, \]
it suffices to establish the resolvent estimate. First we construct a suitable sequence of smooth strongly dissipative Lipschitzian vector fields $b_k$ such that $b_k \to b$ in $L^p(U, \mathbb{R}^d)$ for every ball $U$ as $k \to \infty$. Let $\sigma_j(x) = j^{-d} \sigma(x/j)$, where $\sigma$ is a smooth compactly supported probability density. Let $\beta_j := b * \sigma_j$. Then $\beta_j$ is smooth and dissipative and $\beta_j \to b$, $j \to \infty$, in $L^p(U, \mathbb{R}^d)$ for every ball $U$. For every $\alpha > 0$, the mapping $I - \alpha \beta_j$ is a homeomorphism of $\mathbb{R}^d$ and the inverse mapping $(I - \alpha \beta_j)^{-1}$ is Lipschitzian with constant $\alpha^{-1}$ (see [9]). Let us consider the Yosida approximations
\[ F_\alpha(\beta_j) := \alpha^{-1} ((I - \alpha \beta_j)^{-1} - I) = \beta_j \circ (I - \alpha \beta_j)^{-1}. \]
It is known (see [9, Ch. II]) that \( |F_{\alpha}(\beta_j)(x)| \leq |\beta_j(x)| \), the mappings \( F_{\alpha}(\beta_j) \) converge locally uniformly to \( \beta_j \) as \( \alpha \to 0 \), and one has
\[
(F_{\alpha}(\beta_j)(x) - F_{\alpha}(\beta_j)(y), x - y) \leq 0.
\]
Thus, the sequence \( b_k := F_{\frac{1}{k}}(b * \sigma_k) - \frac{1}{k}I \), \( k \in \mathbb{N} \), is the desired one. For every \( k \in \mathbb{N} \), let \( L_k \) be the elliptic operator defined by (1) with the same constant matrix \( A \) and drift \( b_k \) in place of \( b \). Let \( \mu_k = \varrho_k dx \) be the corresponding invariant probability measure and let \( G^k(\lambda) \) denote the associated resolvent family on \( L^1(\mu_k) \). Since \( b_k \) is smooth, Lipschitzian and strongly dissipative, we obtain that \( v_k := G^k(\lambda) f \) is smooth, bounded, Lipschitzian and
\[
\sup_x |v_k(x)| \leq \frac{1}{\lambda} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v_k(x)| \leq \frac{1}{\lambda} \sup_x |\nabla f(x)|
\]
by the lemma. Moreover, for every ball \( U \subset \mathbb{R}^d \), the functions \( v_k \) are uniformly bounded in the Sobolev space \( W^{2,2}(U) \), since the mappings \( |b_k| \) are bounded in \( L^p(U) \) uniformly in \( k \) and \( f \) is bounded. This follows from the fact that for any solution \( w \in W^{2,2}(U) \) of the equation
\[
\sum_{i,j \leq d} a_{ij} \partial_{x_i} \partial_{x_j} w + \sum_{i \leq d} b_i \partial_{x_i} w - \lambda w = g
\]
one has \( \|w\|_{W^{2,2}(U)} \leq C \|w\|_{L^2(U)} \), where \( C \) is a constant that depends on \( U, A \), and the quantity \( \kappa := \|g\|_{L^2(U)} + \|b\|_{L^p(U)} \) in such a way that as a function of \( \kappa \) it is locally bounded. Thus, the sequence \( \{v_k\} \) contains a subsequence, again denoted by \( \{v_k\} \), that converges locally uniformly to a bounded Lipschitzian function \( v \in W^{2,2}(U) \) such that
\[
\sup_x |v(x)| \leq \lambda^{-1} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v(x)| \leq \lambda^{-1} \sup_x |\nabla f(x)|,
\]
and, in addition, the restrictions of \( v_k \) to any ball \( U \) converge to \( v|_U \) weakly in \( W^{2,2}(U) \).

Let \( \widehat{L} \) be the elliptic operator with the same second order part as \( L \), but with drift is \( \mathbf{b} = 2A \nabla \varrho / \varrho - b \). Then by the integration by parts formula
\[
\int \psi L \varphi \, d\mu = \int \varphi \widehat{L} \psi \, d\mu \quad \text{for all} \ \psi, \varphi \in C_0^\infty(\mathbb{R}^d).
\]
In addition, for any \( \lambda > 0 \), the ranges of \( \lambda - L \) and \( \lambda - \widehat{L} \) on \( C_0^\infty(\mathbb{R}^d) \) are dense in \( L^1(\mu) \). The operator \( \widehat{L} \) also generates a Markov semigroup on \( L^1(\mu) \) with respect to which \( \mu \) is invariant. The corresponding resolvent is denoted by \( \widehat{G}_\lambda \). For the proofs we refer to [7, Proposition 2.9] or [14, Proposition 1.10(b)] (see also [8, Theorem 3.1]).

Now we show that \( v = G_\lambda f \). Note that \( \varrho_k \to \varrho \) uniformly on balls according to [6], [5]. Hence, given \( \varphi \in C_0^\infty(\mathbb{R}^d) \) with support in a ball \( U \), we have
\[
\int [\lambda v - Lv - f] \varphi \varrho \, dx = \lim_{k \to \infty} \int [\lambda v_k - L_k v_k - f] \varphi \varrho_k \, dx = 0
\]
by weak convergence of $v_k$ to $v$ in $W^{2,2}(U)$ combined with convergence of $b_k$ to $b$ in $L^p(U, \mathbb{R}^d)$. Therefore, by the integration by parts formula
\[ \int v(\lambda \varphi - \hat{L}\varphi) \, d\mu = \int f \varphi \, d\mu \]
for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The function $G_{\lambda} f$ is bounded and satisfies the same relation, so it remains to recall that if a bounded function $u$ satisfies the equality
\[ \int u(\lambda \varphi - \hat{L}\varphi) \, d\mu = 0 \]
for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $u = 0$ a.e., since $(\lambda - \hat{L})(C_0^\infty(\mathbb{R}^d))$ is dense in $L^1(\mu)$.

Now we turn to the pointwise estimate $|\nabla G_{\lambda} f(x)| \leq \lambda^{-1} G_{\lambda} |\nabla f|(x)$. Suppose first that $f \in C_0^\infty(\mathbb{R}^d)$. The desired estimate holds for every $G_{\lambda}^{(k)}$ in place of $G_{\lambda}$. It has been shown above that $v = G_{\lambda} f$ is a weak limit of $v_k = G_{\lambda}^{(k)} f$ in $W^{2,2}(U)$ for every ball $U$. In addition, the functions $G_{\lambda}^{(k)} |\nabla f|$ converge weakly in $W^{2,2}(U)$ to the function $G_{\lambda} |\nabla f|$, which is also clear by the above reasoning. Since the embedding of $W^{2,2}(U)$ into $W^{2,1}(U)$ is compact, we may assume, passing to a subsequence, that $\nabla G_{\lambda}^{(k)} f(x) \to \nabla G_{\lambda} f(x)$ and $G_{\lambda}^{(k)} |\nabla f|(x) \to G_{\lambda} |\nabla f|(x)$ almost everywhere on $U$. Hence we arrive at the desired estimate. If $f$ is Lipschitzian and has bounded support, we can find uniformly Lipschitzian functions $f_n \in C_0^\infty(\mathbb{R}^d)$ vanishing outside some ball such that $f_n \to f$ uniformly and $\nabla f_n \to \nabla f$ a.e. Then, by the same reasons as above, one has $G_{\lambda} |\nabla f_n| \to G_{\lambda} |\nabla f|$ and $G_{\lambda} \nabla f_n \to G_{\lambda} \nabla f$ in $L^2(U)$. Passing to an almost everywhere convergent subsequence we obtain a pointwise inequality. Finally, in the case of a general Lipschitzian function $f \in L^1(\mu)$, we can find uniformly Lipschitzian functions $\zeta_n$ such that $0 \leq \zeta_n \leq 1$ and $\zeta_n(x) = 1$ if $|x| \leq n$. Let $f_n = f \zeta_n$. By the previous step we have
\[ |\nabla G_{\lambda} f_n(x)| \leq \lambda^{-1} G_{\lambda} |\nabla f_n|(x). \]
The functions $f_n$ are uniformly Lipschitzian. Hence, for every ball $U$, the sequence of functions $G_{\lambda} f_n|_U$ is bounded in the norm of $W^{2,2}(U)$. In addition, the functions $G_{\lambda} |\nabla f_n|$ on $U$ converge to $G_{\lambda} |\nabla f|$ in $L^2(U)$, since $|\nabla f_n| \to |\nabla f|$ in $L^2(\mu)$ by the Lebesgue dominated convergence theorem. Therefore, the same reasoning as above completes the proof.

**Proof of Theorem 2.** Suppose first that $A$ is piece-wise constant, i.e., there exist finitely many intervals $[0, t_1)$, $[t_1, t_2)$, \ldots, $[t_n, 1]$ such that $A(t) = A_k$ whenever $t_{k-1} \leq t < t_k$, where each $A_k$ is a strictly positive symmetric matrix. In addition, let us assume that there exist vector fields $b_k$ such that $b(t, x) = b_k(x)$ whenever $t_{k-1} \leq t < t_k$. Then we obtain a solution $u$ by successively applying the semigroups $T_t^{(k)}$ generated by the elliptic operators with the diffusion matrices $A_k$ and drifts $b_k$, i.e.,
\[ u(t, x) = T_{t-t_{k-1}} T_{t_{k-1}} \cdots T_{t_1} f(x) \quad \text{whenever} \ t \in [t_{k-1}, t_k). \]
The conclusion of Theorem 2 in this case follows by Theorem 1. Our next step is to approximate \( A \) and \( b \) by mappings of the above form in such a way that the corresponding sequence of solutions would converge to a solution of our equation. Let us observe that, for an arbitrary sequence of such solutions \( u_k \) corresponding to piece-wise constant in time coefficients, for every compactly supported function \( \varphi \) on \( \mathbb{R}^d \), the functions

\[
t(t) \mapsto \int_{\mathbb{R}^d} \varphi(x) u_k(t, x) \, dx
\]

are uniformly Lipschitzian provided that the operator norms of the matrix functions \( A_k \) are uniformly bounded and that the \( L^p(K) \)-norms of the vector fields \( b_k(t, \cdot) \) are uniformly bounded for every fixed cube \( K \) in \( \mathbb{R}^d \). This is clear, because (2) can be written as

\[
\int_{\mathbb{R}^d} \varphi(x) u(t, x) \, dx = \int_0^t \int_{\mathbb{R}^d} [L_s \varphi(x) u(s, x) + \varphi(x) b'(s, x) \partial_x u(s, x)] \, dx \, ds,
\]

where in the case \( u = u_k \) we have

\[
|u(s, x)| \leq \sup |f(x)| \quad \text{and} \quad \nabla_x u(s, x) \leq \sup |\nabla f(x)|.
\]

One can choose a subsequence in \( \{u_k\} \) that converges to some function \( u \) on \([0, 1] \times \mathbb{R}^d\) in the following sense: for every cube \( K \) in \( \mathbb{R}^d \), the functions the restrictions of the functions \( u_k \) to \([0, 1] \times K\) converge weakly to \( u \) in the space \( L^2([0, 1], W^{2,2}(K)) \), where each \( u_k \) is regarded as a mapping \( t \mapsto u_k(t, \cdot) \) from \([0, 1]\) to \( W^{2,2}(K) \). Passing to another subsequence we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x) u_k(t, x) \, dx = \int_{\mathbb{R}^d} \varphi(x) u(t, x) \, dx
\]

for all \( t \in [0, 1] \) and all smooth compactly supported \( \varphi \). Indeed, for a given function \( \varphi \) this is possible due to the uniform Lipschitzness of the functions (7). Then our claim is true for a countable family of functions \( \varphi \), which, on account of the uniform boundedness of \( u_k \), yields the claim for all \( \varphi \). Therefore, it remains to find approximations \( A_k \) and \( b_k \) such that, for every function \( \psi \in C_{\infty}(\mathbb{R}^d) \), the integrals

\[
\int_0^1 \psi(s) \int_{\mathbb{R}^d} [L_s^{(k)} \varphi(x) u_k(s, x) + \varphi(x) b'_k(s, x) \partial_x u_k(s, x)] \, dx \, ds
\]

would converge to the corresponding integral with \( A, b \), and \( u \). Clearly, it suffices to obtain the desired convergence for suitable countable families of functions \( \varphi_i \) and \( \psi_j \). Let us fix two sequences \( \{\psi_j\} \subset C_{\infty}(\mathbb{R}^d) \) and \( \{\varphi_i\} \subset C_{\infty}(\mathbb{R}^d) \) with the following property: every compactly supported square-integrable function \( v \) on \([0, 1] \times \mathbb{R}^d \) can be approximated in \( L^2 \) by a sequence of finite linear combinations of products \( \psi_j \varphi_i \). Let us consider the functions

\[
\alpha_{i,j,k}(t) := a^{(i)}(t) \psi_k(t), \quad \beta_{i,j,k}(t) := \psi_k(t) \int_{\mathbb{R}^d} b'(s, x) \varphi_j(x) \, dx,
\]
\[ \theta_{k,i}(t) = \int_{[-k,k]^d} b_i(t, x)^2 \, dx. \]

Let \( \mathcal{F} \) denote the obtained countable family of functions extended periodically from \([0, 1]\) to \(\mathbb{R}\) with period 1. It is well known that, for almost every \( s \in [0, 1) \), the Riemannian sums \( R_n(\theta)(s) = 2^{-n} \sum_{k=1}^{2^n} \theta(s + k2^{-n}) \) converge to the integral of \( \theta \) over \([0, 1]\) for each \( \theta \in \mathcal{F} \). It follows that one can find points \( t_{n,l} \), \( l = 1, \ldots, N_n, n \in \mathbb{N} \), such that

\[ 0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,N_n} = 1 \]

and, for every \( \theta \in \mathcal{F} \), letting \( \theta_n(t) := \theta(t_{n,l}) \) whenever \( t_{n,l-1} \leq t < t_{n,l} \), one has

\[ \int_0^1 \theta_n(t) \, dt \to \int_0^1 \theta(t) \, dt. \]

To this end, we pick a common point \( s_0 \) of convergence of the Riemann sums \( R_n(\theta)(s_0) \) to the respective integrals and let \( t_{n,l} = s_0 + l2^{-n} \, (\text{mod} 1) \). By using the points \( t_{n,l} \), one obtains the desired piece-wise constant approximations of \( A \) and \( b \). Namely, let \( A_n(t) = A(t_{n,l}) \) and \( b_n(t, x) = b(t_{n,l}, x) \) whenever \( t_{n,l-1} \leq t < t_{n,l} \). As explained above, passing to a subsequence, we may assume that the corresponding solutions \( u_n \) converge to a function \( u \) such that, for every cube \( K = [-m, m]^d \) in \( \mathbb{R}^d \) and every \( t \in (0, 1] \), one has

\[ u(t, \cdot) |_{K} \in W^{2,2}(K), \quad \int_0^1 \| u(t, \cdot) \|_{W^{2,2}(K)}^2 \, dt < \infty, \]

and for any function \( \zeta \in L^2([0, 1] \times K) \) there holds the equalities

\[
\begin{align*}
&\lim_{n \to \infty} \int_0^1 \int_K \zeta(t, x) u_n(t, x) \, dx \, dt = \int_0^1 \int_K \zeta(t, x) u(t, x) \, dx \, dt, \\
&\lim_{n \to \infty} \int_0^1 \int_K \zeta(t, x) \partial_x \partial_x u_n(t, x) \, dx \, dt = \int_0^1 \int_K \zeta(t, x) \partial_x \partial_x u(t, x) \, dx \, dt, \\
&\lim_{n \to \infty} \int_0^1 \int_K \zeta(t, x) \partial_x u_n(t, x) \, dx \, dt = \int_0^1 \int_K \zeta(t, x) \partial_x u(t, x) \, dx \, dt, \\
&\lim_{n \to \infty} \int_0^1 \int_K b_n^\circ(t, x)^2 \, dx \, dt = \int_0^1 \int_K b(t, x)^2 \, dx \, dt.
\end{align*}
\]

Note that for any cube \( K \subset \mathbb{R}^d \), the restrictions of the functions \( b_n^\circ \) to \([0, 1] \times K\) converge to the restriction of \( b^\circ \) in the norm of \( L^2([0, 1] \times K) \). This is clear from the last displayed equality, which gives convergence of \( L^2 \)-norms, along with convergence of the Riemann sums \( R_n(\beta_{i,j,k})(s_0) \) to the integral of \( \beta_{i,j,k} \) over \([0, 1]\), which yields weak convergence (we recall that if a sequence of vectors \( h_n \) in a Hilbert space \( H \) converges weakly to a vector \( h \) and the norms of \( h_n \) converge to the norm of \( h \), then there is norm convergence). It follows that for
any $\psi \in C[0, 1]$ and any $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in $[-m, m]^d$, we have

$$
\lim_{n \to \infty} \int_0^1 \psi(t) a_n^{ij}(t) \int_{\mathbb{R}^d} \partial_x \varphi(x) u_n(t, x) \, dx \, dt = \int_0^1 \psi(t) a^{ij}(t) \int_{\mathbb{R}^d} \partial_x \varphi(x) u(t, x) \, dx \, dt.
$$

In addition,

$$
\lim_{n \to \infty} \int_0^1 \psi(t) \int_{\mathbb{R}^d} \varphi(x) \partial_x u_n(t, x) b_n^i(t, x) \, dx \, dt = \int_0^1 \psi(t) \int_{\mathbb{R}^d} \varphi(x) \partial_x u(t, x) b^i(t, x) \, dx \, dt.
$$

This follows by norm convergence of $b_n^i$ to $b^i$ and weak convergence of $\varphi \partial_x u_n$ to $\varphi \partial_x u$ in $L^2([0, 1] \times [-m, m]^d)$. Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, one has

$$
\int_{\mathbb{R}^d} \varphi(x) u(t, x) \, dx \, dt = \int_{\mathbb{R}^d} \varphi(x) f(x) \, dx + \int_0^t \int_{\mathbb{R}^d} \varphi(x) L_t u(t, x) \, dx \, dt
$$

for almost all $t \in [0, 1]$, since the integrals of both sides multiplied by any function $\psi \in C_0^\infty(0, 1)$ coincide. Taking into account the continuity of both sides (the left-hand side is Lipschitzian as explained above), we conclude that the equality holds for all $t \in [0, 1]$. \qed

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**References**


