Gradient bounds for solutions of elliptic and parabolic equations

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Abstract

Let L be a second order elliptic operator on $\mathbb{R}^d$ with a constant diffusion matrix and a dissipative (in a weak sense) drift $b \in L^p_{loc}$ with some $p > d$. We assume that L possesses a Lyapunov function, but no local boundedness of b is assumed. It is known that then there exists a unique probability measure μ satisfying the equation $L^*\mu = 0$ and that the closure of L in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geq 0}$ with the resolvent $\{G_\lambda\}_{\lambda>0}$. We prove that, for any Lipschitzian function $f \in L^1(\mu)$ and all $t, \lambda > 0$, the functions $T_t f$ and $G_\lambda f$ are Lipschitzian and

$$|\nabla T_t f(x)| \leq T_t |\nabla f|(x) \quad \text{and} \quad |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} G_\lambda |\nabla f|(x).$$

An analogous result is proved in the parabolic case.

Suppose that for every $t \in [0,1]$, we are given a a strictly positive definite symmetric matrix $A(t) = (a^{ij}(t))$ and a measurable vector field $x \mapsto b(t,x) = (b^1(t,x), \ldots, b^n(t,x))$.

Let L_t be the elliptic operator on $\mathbb{R}^d$ given by

$$L_t u(x) = \sum_{i,j\leq d} a^{ij}(t,x)\partial_{x_i}\partial_{x_j} u(x) + \sum_{i\leq d} b^i(t,x)\partial_{x_i} u(x). \tag{1}$$

Suppose that A and b satisfy the following hypotheses:

(Ha) $\sup_{t\in[0,1]}\big(\|A(t)\| + \|A(t)^{-1}\|\big) < \infty$, $\sup_{t\in[0,1]} \|b(t,\,\cdot\,)\|_{L^p(U)} < \infty$ for every ball U in $\mathbb{R}^d$ with some $p > d$, $p \geq 2$.

(Hb) b is *dissipative* in the following sense: for every $t \in [0,1]$ and every $h \in \mathbb{R}^d$, there exists a measure zero set $N_{t,h} \subset \mathbb{R}^d$ such that

$$\big(b(t,x+h) - b(t,x), h\big) \leq 0 \quad \text{for all } x \in \mathbb{R}^d \setminus N_{t,h}.$$

(Hc) for every $t \in [0,1]$, there exists a *Lyapunov function* V_t for L_t, i.e., a nonnegative C^2-function V_t such that $V_t(x) \to +\infty$ and $L_t V_t(x) \to -\infty$ as $|x| \to \infty$.

We consider the parabolic equation

$$\frac{\partial u}{\partial t} = L_t u, \quad u(0,x) = f(x), \tag{2}$$

where f is a bounded Lipschitz function. A locally integrable function u on $[0,1]\times\mathbb{R}^d$ is called a solution if, for every $t\in(0,1]$, one has $u(t,\,\cdot\,)\in W^{1,2}_{loc}(\mathbb{R}^d)$, the functions $\partial_{x_i}\partial_{x_j}u$ and $b^i\partial_{x_i}u$ are integrable on the sets $[0,1]\times K$ for every cube K in $\mathbb{R}^d$, and for every $\varphi\in C_0^\infty(\mathbb{R}^d)$ and all $t\in[0,1]$ one has

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx + \int_0^t\int_{\mathbb{R}^d} L_s\varphi(x)\,u(s,x)\,dxds.$$

In the case where A and b are independent of t, so that we have a single operator L, Hypotheses (Ha) and (Hc) imply (see [6] and [8]) that there exists a unique probability measure μ on $\mathbb{R}^d$ such that μ has a strictly positive continuous weakly differentiable density ϱ, $|\nabla\varrho|\in L^p_{loc}(\mathbb{R}^d)$, and $L^*\mu=0$ in the following weak sense:

$$\int Lu\,d\mu = 0 \quad \text{for all } u\in C_0^\infty(\mathbb{R}^d).$$

The closure $\overline{L}$ of L with domain $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geq 0}$ for which μ is invariant. Let $D(\overline{L})$ denote the domain of $\overline{L}$ in $L^1(\mu)$ and let $\{G_\lambda\}_{\lambda>0}$ denote the corresponding resolvent, i.e., $G_\lambda=(\lambda-\overline{L})^{-1}$. The restrictions of T_t and G_λ to $L^2(\mu)$ are contractions on $L^2(\mu)$. In particular, if $v\in D(\overline{L})$ is such that $\lambda v-\overline{L}v=g\in L^2(\mu)$, then $v\in L^2(\mu)$. Moreover, it follows by [8, Theorem 2.8] that one has $v\in H^{2,2}_{loc}(\mathbb{R}^d)$ and $\overline{L}v=Lv$ a.e., so that one has a.e.

$$\lambda v - Lv = g. \tag{3}$$

In fact, due to our assumptions on the coefficients of L one has even $v\in W^{p,2}_{loc}(\mathbb{R}^d)$ (see [10]). It has been shown in [3] that for every function $f\in L^1(\mu)$ that is Lipschitzian with constant C and all $t,\lambda>0$, the continuous version of the function T_tf is Lipschitzian with constant C, and the continuous version of $G_\lambda f$ is Lipschitzian with constant $\lambda^{-1}C$. Here we establish pointwise estimates in both cases and prove their parabolic analogue. The main results of this work are the following two theorems.

Theorem 1. *Suppose that A and b are independent of t and satisfy* (Ha), (Hb) *and* (Hc). *Then, for any Lipschitzian function $f\in L^1(\mu)$ and all $t,\lambda>0$, T_tf and $G_\lambda f$ have Lipschitzian versions such that*

$$|\nabla T_tf(x)|\leq T_t|\nabla f|(x) \quad \textit{and} \quad |\nabla G_\lambda f(x)|\leq \frac{1}{\lambda}G_\lambda|\nabla f|(x) \tag{4}$$

for the corresponding continuous versions. In particular,

$$\sup_{x,t}|\nabla T_tf(x)|\leq \sup_x|\nabla f(x)|, \quad \sup_x|\nabla G_\lambda f(x)|\leq\frac{1}{\lambda}\sup_x|\nabla f(x)|. \tag{5}$$

Theorem 2. *Suppose that A and b satisfy* (Ha), (Hb) *and* (Hc). *Then, for any bounded Lipschitzian function f there is a solution u of equation* (2) *such that for all t one has*

$$\sup_x|\nabla u(t,x)|\leq\sup_x|\nabla f(x)|. \tag{6}$$

In the case where $A = I$ and $b = 0$, estimate (6) has been established in [12], [13] for solutions of boundary value problems in bounded domains. It should be noted that gradient estimates of the type

$$\sup_x |\nabla u(x,t)| \le C(t) \sup_x |f(x)|$$

for solutions of parabolic equations have been obtained by many authors, see, e.g., [1], [2], [11], [15], and the references therein. Such estimates do not require (Hb) and one has $C(t) \to +\infty$ as $t \to 0$ or $t \to +\infty$. In contrast to this type of estimates, our theorems mean a contraction property on Lipschitz functions rather than a smoothing property. It is likely that some results of the cited works, established for sufficiently regular b, can be extended to more general drifts satisfying just (Ha), but not (Hb).

A short proof of the following result can be found in [3].

Lemma 1. *Suppose that b is infinitely differentiable, Lipschitzian, and strongly dissipative, so for some $\alpha > 0$, one has*

$$\big(b(x+h) - b(x), h\big) \le -\alpha(h,h) \quad \text{for all } x, h \in \mathbb{R}^d.$$

Then, for any $\lambda > 0$ and any smooth bounded Lipschitzian function f, one has pointwise

$$|\nabla G_\lambda f| \le G_\lambda |\nabla f|.$$

In particular, $\sup_x |\nabla G_\lambda f(x)| \le \lambda^{-1} \sup_x |\nabla f(x)|$.

Proof of Theorem 1. The estimate with the suprema has been proven in [3], and the stronger pointwise estimate can be derived from that proof. For the reader's convenience, instead of recursions to the steps of the proof in [3] we reproduce the whole proof and explain why it yields a stronger conclusion. We recall that if a sequence of functions on $\mathbb{R}^d$ is uniformly Lipschitzian with constant L and bounded at a point, then it contains a subsequence that converges uniformly on every ball to a function that is Lipschitzian with the same constant. Therefore, approximating f in $L^1(\mu)$ by a sequence of bounded smooth functions f_j with

$$\sup_x |\nabla f_j(x)| \le \sup_x |\nabla f(x)|,$$

it suffices to prove (5) for smooth bounded f. Moreover, due to Euler's formula $T_t f = \lim_n \left(\frac{t}{n} G_{\frac{t}{n}}\right)^n f$, it suffices to establish the resolvent estimate. First we construct a suitable sequence of smooth strongly dissipative Lipschitzian vector fields b_k such that $b_k \to b$ in $L^p(U, \mathbb{R}^d)$ for every ball U as $k \to \infty$. Let $\sigma_j(x) = j^{-d}\sigma(x/j)$, where σ is a smooth compactly supported probability density. Let $\beta_j := b * \sigma_j$. Then β_j is smooth and dissipative and $\beta_j \to b$, $j \to \infty$, in $L^p(U, \mathbb{R}^d)$ for every ball U. For every $\alpha > 0$, the mapping $I - \alpha\beta_j$ is a homeomorphism of $\mathbb{R}^d$ and the inverse mapping $(I - \alpha\beta_j)^{-1}$ is Lipschitzian with constant α^{-1} (see [9]). Let us consider the Yosida approximations

$$F_\alpha(\beta_j) := \alpha^{-1}\big((I - \alpha\beta_j)^{-1} - I\big) = \beta_j \circ (I - \alpha\beta_j)^{-1}.$$

It is known (see [9, Ch. II]) that $|F_\alpha(\beta_j)(x)| \le |\beta_j(x)|$, the mappings $F_\alpha(\beta_j)$ converge locally uniformly to β_j as $\alpha \to 0$, and one has

$$\big(F_\alpha(\beta_j)(x) - F_\alpha(\beta_j)(y), x-y\big) \le 0.$$

Thus, the sequence $b_k := F_{\frac{1}{k}}(b * \sigma_k) - \frac{1}{k}I$, $k \in \mathbb{N}$, is the desired one. For every $k \in \mathbb{N}$, let L_k be the elliptic operator defined by (1) with the same constant matrix A and drift b_k in place of b. Let $\mu_k = \varrho_k\,dx$ be the corresponding invariant probability measure and let $G_\lambda^{(k)}$ denote the associated resolvent family on $L^1(\mu_k)$. Since b_k is smooth, Lipschitzian and strongly dissipative, we obtain that $v_k := G_\lambda^{(k)} f$ is smooth, bounded, Lipschitzian and

$$\sup_x |v_k(x)| \le \frac{1}{\lambda}\sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v_k(x)| \le \frac{1}{\lambda}\sup_x |\nabla f(x)|$$

by the lemma. Moreover, for every ball $U \subset \mathbb{R}^d$, the functions v_k are uniformly bounded in the Sobolev space $W^{2,2}(U)$, since the mappings $|b_k|$ are bounded in $L^p(U)$ uniformly in k and f is bounded. This follows from the fact that for any solution $w \in W^{2,2}(U)$ of the equation

$$\sum_{i,j\le d} a^{ij}\partial_{x_i}\partial_{x_i} w + \sum_{i\le d} b^i \partial_{x_i}\partial_{x_i} w - \lambda w = g$$

one has $\|w\|_{W^{2,2}(U)} \le C\|w\|_{L^2(U)}$, where C is a constant that depends on U, A, and the quantity $\kappa := \|g\|_{L^2(U)} + \||b|\|_{L^p(U)}$ in such a way that as a function of κ it is locally bounded. Thus, the sequence $\{v_k\}$ contains a subsequence, again denoted by $\{v_k\}$, that converges locally uniformly to a bounded Lipschitzian function $v \in W^{2,2}_{loc}(\mathbb{R}^d)$ such that

$$\sup_x |v(x)| \le \lambda^{-1}\sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v(x)| \le \lambda^{-1}\sup_x |\nabla f(x)|,$$

and, in addition, the restrictions of v_k to any ball U converge to $v|_U$ weakly in $W^{2,2}(U)$.

Let $\widehat{L}$ be the elliptic operator with the same second order part as L, but with drift is $\widehat{b} = 2A\nabla\varrho/\varrho - b$. Then by the integration by parts formula

$$\int \psi L\varphi\,d\mu = \int \varphi \widehat{L}\psi\,d\mu \quad \text{for all } \psi, \varphi \in C_0^\infty(\mathbb{R}^d).$$

In addition, for any $\lambda > 0$, the ranges of $\lambda - L$ and $\lambda - \widehat{L}$ on $C_0^\infty(\mathbb{R}^d)$ are dense in $L^1(\mu)$. The operator $\widehat{L}$ also generates a Markov semigroup on $L^1(\mu)$ with respect to which μ is invariant. The corresponding resolvent is denoted by $\widehat{G}_\lambda$. For the proofs we refer to [7, Proposition 2.9] or [14, Proposition 1.10(b)] (see also [8, Theorem 3.1]).

Now we show that $v = G_\lambda f$. Note that $\varrho_k \to \varrho$ uniformly on balls according to [6], [5]. Hence, given $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in a ball U, we have

$$\int [\lambda v - Lv - f]\varphi\varrho\,dx = \lim_{k\to\infty}\int [\lambda v_k - L_k v_k - f]\varphi\varrho_k\,dx = 0$$

by weak convergence of v_k to v in $W^{2,2}(U)$ combined with convergence of b_k to b in $L^p(U,\mathbb{R}^d)$. Therefore, by the integration by parts formula

$$\int v(\lambda\varphi - \widehat{L}\varphi)\, d\mu = \int f\varphi\, d\mu$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The function $G_\lambda f$ is bounded and satisfies the same relation, so it remains to recall that if a bounded function u satisfies the equality

$$\int u(\lambda\varphi - \widehat{L}\varphi)\, d\mu = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $u = 0$ a.e., since $(\lambda - \widehat{L})\big(C_0^\infty(\mathbb{R}^d)\big)$ is dense in $L^1(\mu)$.

Now we turn to the pointwise estimate $|\nabla G_\lambda f(x)| \le \lambda^{-1} G_\lambda |\nabla f|(x)$. Suppose first that $f \in C_0^\infty(\mathbb{R}^d)$. The desired estimate holds for every $G_\lambda^{(k)}$ in place of G_λ. It has been shown above that $v = G_\lambda f$ is a weak limit of $v_k = G_\lambda^{(k)} f$ in $W^{2,2}(U)$ for every ball U. In addition, the functions $G_\lambda^{(k)}|\nabla f|$ converge weakly in $W^{2,2}(U)$ to the function $G_\lambda|\nabla f|$, which is also clear by the above reasoning. Since the embedding of $W^{2,2}(U)$ into $W^{2,1}(U)$ is compact, we may assume, passing to a subsequence, that $\nabla G_\lambda^{(k)} f(x) \to \nabla G_\lambda f(x)$ and $G_\lambda^{(k)}|\nabla f|(x) \to G_\lambda|\nabla f|(x)$ almost everywhere on U. Hence we arrive at the desired estimate. If f is Lipschitzian and has bounded support, we can find uniformly Lipschitzian functions $f_n \in C_0^\infty(\mathbb{R}^d)$ vanishing outside some ball such that $f_n \to f$ uniformly and $\nabla f_n \to \nabla f$ a.e. Then, by the same reasons as above, one has $G_\lambda|\nabla f_n| \to G_\lambda|\nabla f|$ and $\nabla G_\lambda f_n \to \nabla G_\lambda f$ in $L^2(U)$. Passing to an almost everywhere convergent subsequence we obtain a pointwise inequality. Finally, in the case of a general Lipschitzian function $f \in L^1(\mu)$, we can find uniformly Lipschitzian functions ζ_n such that $0 \le \zeta_n \le 1$ and $\zeta_n(x) = 1$ if $|x| \le n$. Let $f_n = f\zeta_n$. By the previous step we have

$$|\nabla G_\lambda f_n(x)| \le \lambda^{-1} G_\lambda |\nabla f_n|(x).$$

The functions f_n are uniformly Lipschitzian. Hence, for every ball U, the sequence of functions $G_\lambda f_n|_U$ is bounded in the norm of $W^{2,2}(U)$. In addition, the functions $G_\lambda|\nabla f_n|$ on U converge to $G_\lambda|\nabla f|$ in $L^2(U)$, since $|\nabla f_n| \to |\nabla f|$ in $L^2(\mu)$ by the Lebesgue dominated convergence theorem. Therefore, the same reasoning as above completes the proof. □

Proof of Theorem 2. Suppose first that A is piece-wise constant, i.e., there exist finitely many intervals $[0,t_1)$, $[t_1,t_2)$,…, $[t_n,1]$ such that $A(t) = A_k$ whenever $t_{k-1} \le t < t_k$, where each A_k is a strictly positive symmetric matrix. In addition, let us assume that there exist vector fields b_k such that $b(t,x) = b_k(x)$ whenever $t_{k-1} \le t < t_k$. Then we obtain a solution u by successively applying the semigroups $T_t^{(k)}$ generated by the elliptic operators with the diffusion matrices A_k and drifts b_k, i.e.,

$$u(t,x) = T_{t-t_{k-1}} T_{t_{k-1}} \cdots T_{t_1} f(x) \quad \text{whenever } t \in [t_{k-1}, t_k).$$

The conclusion of Theorem 2 in this case follows by Theorem 1. Our next step is to approximate A and b by mappings of the above form in such a way that the corresponding sequence of solutions would converge to a solution of our equation. Let us observe that, for an arbitrary sequence of such solutions u_k corresponding to piece-wise constant in time coefficients, for every compactly supported function φ on $\mathbb{R}^d$, the functions

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x) u_k(t,x)\,dx \tag{7}$$

are uniformly Lipschitzian provided that the operator norms of the matrix functions A_k are uniformly bounded and that the $L^p(K)$-norms of the vector fields $b_k(t,\,\cdot\,)$ are uniformly bounded for every fixed cube K in $\mathbb{R}^d$. This is clear, because (2) can be written as

$$\int_{\mathbb{R}^d} \varphi(x)u(t,x)\,dx = \int_0^t \int_{\mathbb{R}^d} [L_s\varphi(x)\,u(s,x) + \varphi(x)b^i(s,x)\partial_{x_i}u(s,x)]\,dx\,ds,$$

where in the case $u = u_k$ we have

$$|u(s,x)| \le \sup|f(x)| \quad \text{and} \quad |\nabla_x u(s,x)| \le \sup|\nabla f(x)|.$$

One can choose a subsequence in $\{u_k\}$ that converges to some function u on $[0,1]\times\mathbb{R}^d$ in the following sense: for every cube K in $\mathbb{R}^d$, the functions the restrictions of the functions u_k to $[0,1]\times K$ converge weakly to u in the space $L^2([0,1], W^{2,2}(K))$, where each u_k is regarded as a mapping $t \mapsto u_k(t,\,\cdot\,)$ from $[0,1]$ to $W^{2,2}(K)$. Passing to another subsequence we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x)u_k(t,x)\,dx = \int_{\mathbb{R}^d} \varphi(x)u(t,x)\,dx$$

for all $t \in [0,1]$ and all smooth compactly supported φ. Indeed, for a given function φ this is possible due to the uniform Lipschitzness of the functions (7). Then our claim is true for a countable family of functions φ, which, on account of the uniform boundedness of u_k, yields the claim for all φ. Therefore, it remains to find approximations A_k and b_k such that, for every function $\psi \in C[0,1]$, the integrals

$$\int_0^1 \psi(s) \int_{\mathbb{R}^d} [L_s^{(k)}\varphi(x)\,u_k(s,x) + \varphi(x)b_k^i(s,x)\partial_{x_i}u_k(s,x)]\,dx\,ds$$

would converge to the corresponding integral with A, b, and u. Clearly, it suffices to obtain the desired convergence for suitable countable families of functions φ_i and ψ_j. Let us fix two sequences $\{\psi_j\} \subset C[0,1]$ and $\{\varphi_i\} \subset C_0^\infty(\mathbb{R}^d)$ with the following property: every compactly supported square-integrable function v on $[0,1]\times\mathbb{R}^d$ can be approximated in L^2 by a sequence of finite linear combinations of products $\psi_j\varphi_i$. Let us consider the functions

$$\alpha_{i,j,k}(t) := a^{ij}(t)\psi_k(t), \quad \beta_{i,j,k}(t) := \psi_k(t)\int_{\mathbb{R}^d} b^i(s,x)\varphi_j(x)\,dx,$$

$$\theta_{k,i}(t) = \int_{[-k,k]^d} b_i(t,x)^2\,dx.$$

Let $\mathcal{F}$ denote the obtained countable family of functions extended periodically from $[0,1)$ to $\mathbb{R}$ with period 1. It is well known that, for almost every $s \in [0,1)$, the Riemannian sums $R_n(\theta)(s) = 2^{-n}\sum_{k=1}^{2^n}\theta(s+k2^{-n})$ converge to the integral of θ over $[0,1]$ for each $\theta \in \mathcal{F}$. It follows that one can find points $t_{n,l}$, $l = 1,\dots,N_n$, $n \in \mathbb{N}$, such that

$$0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,N_n} = 1$$

and, for every $\theta \in \mathcal{F}$, letting $\theta_n(t) := \theta(t_{n,l})$ whenever $t_{n,l-1} \le t < t_{n,l}$, one has

$$\int_0^1 \theta_n(t)\,dt \to \int_0^1 \theta(t)\,dt.$$

To this end, we pick a common point s_0 of convergence of the Riemann sums $R_n(\theta)(s_0)$ to the respective integrals and let $t_{n,l} = s_0 + l2^{-n}$ (mod1). By using the points $t_{n,l}$, one obtains the desired piece-wise constant approximations of A and b. Namely, let $A_n(t) = A(t_{n,l})$ and $b_n(t,x) = b(t_{n,l},x)$ whenever $t_{n,l-1} \le t < t_{n,l}$. As explained above, passing to a subsequence, we may assume that the corresponding solutions u_n converge to a function u such that, for every cube $K = [-m,m]^d$ in $\mathbb{R}^d$ and every $t \in (0,1]$, one has

$$u(t,\,\cdot\,)|_K \in W^{2,2}(K), \qquad \int_0^1 \|u(t,\,\cdot\,)\|^2_{W^{2,2}(K)}\,dt < \infty,$$

and for any function $\zeta \in L^2([0,1]\times K)$ there holds the equalities

$$\lim_{n\to\infty}\int_0^1\int_K \zeta(t,x)u_n(t,x)\,dx\,dt = \int_0^1\int_K \zeta(t,x)u(t,x)\,dx\,dt,$$

$$\lim_{n\to\infty}\int_0^1\int_K \zeta(t,x)\partial_{x_i}\partial_{x_j}u_n(t,x)\,dx\,dt = \int_0^1\int_K \zeta(t,x)\partial_{x_i}\partial_{x_j}u(t,x)\,dx\,dt,$$

$$\lim_{n\to\infty}\int_0^1\int_K \zeta(t,x)\partial_{x_i}u_n(t,x)\,dx\,dt = \int_0^1\int_K \zeta(t,x)\partial_{x_i}u(t,x)\,dx\,dt,$$

$$\lim_{n\to\infty}\int_0^1\int_K b_i^n(t,x)^2\,dx\,dt = \int_0^1\int_K b_i(t,x)^2\,dx\,dt.$$

Note that for any cube $K \subset \mathbb{R}^d$, the restrictions of the functions b_n^i to $[0,1]\times K$ converge to the restriction of b^i in the norm of $L^2([0,1]\times K)$. This is clear from the last displayed equality, which gives convergence of L^2-norms, along with convergence of the Riemann sums $R_n(\beta_{i,j,k})(s_0)$ to the integral of $\beta_{i,j,k}$ over $[0,1]$, which yields weak convergence (we recall that if a sequence of vectors h_n in a Hilbert space H converges weakly to a vector h and the norms of h_n converge to the norm of h, then there is norm convergence). It follows that for any $\psi \in C[0,1]$ and any $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in $[-m,m]^d$, we have

$$\lim_{n\to\infty}\int_0^1 \psi(t)a_n^{ij}(t)\int_{\mathbb{R}^d}\partial_{x_i}\partial_{x_j}\varphi(x)u_n(t,x)\,dx\,dt = \int_0^1 \psi(t)a^{ij}(t)\int_{\mathbb{R}^d}\partial_{x_i}\partial_{x_j}\varphi(x)u(t,x)\,dx\,dt.$$

In addition,

$$\lim_{n\to\infty}\int_0^1 \psi(t)\int_{\mathbb{R}^d}\varphi(x)\partial_{x_i}u_n(t,x)b_n^i(t,x)\,dx\,dt = \int_0^1 \psi(t)\int_{\mathbb{R}^d}\varphi(x)\partial_{x_i}u(t,x)b^i(t,x)\,dx\,dt.$$

This follows by norm convergence of b_n^i to b^i and weak convergence of $\varphi\partial_{x_i}u_n$ to $\varphi\partial_{x_i}u$ in $L^2([0,1]\times[-m,m]^d)$. Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, one has

$$\int_{\mathbb{R}^d}\varphi(x)u(t,x)\,dx\,dt = \int_{\mathbb{R}^d}\varphi(x)f(x)\,dx + \int_0^t\int_{\mathbb{R}^d}\varphi(x)L_t u(t,x)\,dx\,dt$$

for almost all $t \in [0,1]$, since the integrals of both sides multiplied by any function $\psi \in C_0^\infty(0,1)$ coincide. Taking into account the continuity of both sides (the left-hand side is Lipschitzian as explained above), we conclude that the equality holds for all $t \in [0,1]$. $\square$

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