Keller-Osserman Conditions and Regular Evans Functions for Semilinear PDE

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Abstract

let f be a function from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$, continuous, increasing and derivable at zero. Let $X = \mathbb{R}^d (d \geq 2)$. For every open set $U \subset X$ we set: $\mathcal{H}_f(U) = \{u \in \mathcal{C}(U) : \Delta u = f(u)$ in the distributional sense$\}$. We define regular Evans function associated with f, U and Δ by the existence of element of $\mathcal{H}_f(U)$ tending to infinity at the regular boundary of U. We then introduce the KO (Keller-Osserman) property for f, by the existence of a natural number $d \geq 2$ such that every ball $B \subset X$ admits a regular Evans function. We then give nice and explicit characterisations for the validity of the KO property and examine the relationship between the KO condition, the Harnack principle and the Brelot convergence property. We prove that in the nonlinear case, and in contrast to the linear case, we do not have the equivalence between Harnack and Brelot . We continue the investigation of regular Evans functions in the case of uniformly elliptic or uniformly parabolic operators and where we replace the function f by a function ψ from $X \times \mathbb{R}$ to $\mathbb{R}$, which in contrast to many other authors, is not supposed to be convex or locally Lipschitzian.

Introduction

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$, continuous, increasing and derivable at zero. Let $X = \mathbb{R}^d (d \geq 2)$. For every open set $U \subset X$ we set:

$$\mathcal{H}_f(U) = \{u \in \mathcal{C}(U) : \Delta u = f(u)$$

We recall (see [BBM] or [B_1]) that $\mathcal{H}_f$ is the sheaf of harmonic functions for Bauer space. In the first section we define, for a relatively compact open subset U, regular Evans functions (associated with f, U and Δ) as an element of $\mathcal{H}_f^+(U)$ tending to infinity at the regular boundary of U.

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These smooth functions exploding at the regular part of the boundary were investigated by many authors, for a sampling of the literature see [K], [O], [D_1], [D_2], [DK], [B_1] and the references contained therein. In contrast to many other authors we do not suppose that f is convex or locally Lipschitzian.

In a first result we prove that every ball B admits a regular Evans function if and only if we have the following Harnack type inequality (see [B_1]):

For every domain U in X and every compact subset C in U, there exists $c \geq 0$ such that $u(x) \leq c$ for every $x \in C$ and $u \in \mathcal{H}_f^+(U)$.

In the next section we define the Keller-Osserman property for the function f, denoted by KO, by the existence of a natural number $d \geq 2$ such that every ball $B \subset \mathbb{R}^d$ admits a regular Evans function associated with f and Δ. We then prove that the condition $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$ is necessary and not sufficient for the KO property. We propose a somewhat modified version of the Keller-Osserman integrability condition (KOI) and the Keller limit ($K\ell$) condition and then prove that these conditions are equivalent to the KO condition introduced in this paper. The KOI and $K\ell$ conditions on f are nice explicit characterizations for the existence of regular Evans functions associated with f, B and Δ on $\mathbb{R}^d (d \geq 2)$ for every ball B. As a consequence we prove that $f_\alpha(t) = t(\log(1+|t|))^\alpha$ satisfies the KO condition if and only if $\alpha > 2$.

In the third section we examine the relationship between the KO condition, the Harnack principle and the Brelot convergence property. If the nonlinear harmonic Bauer space $(\mathbb{R}^d, \mathcal{H}_f)$ satisfies the generalized Harnack principle introduced in [B_1], we then show that the KO property is equivalent to $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$ and this yields with the previous sections the existence of nonlinear harmonic Bauer spaces obtained by semilinear perturbation of the Laplace equation, where the (generalized) Harnack inequality is not fulfilled and whereas in contrast to the linear potential theory (see e.g..[M] or [LW]), the convergence property of Brelot is valid. Furthermore we prove that for every $d \geq 2, a > e(\log e = 1), \beta \in]0,1]$ and $f_{\beta,a}(t) = t[\log(a+|t|)]^\beta, (\mathbb{R}^d, \mathcal{H}_{f_{\beta,a}})$ is a harmonic Bauer space satisfying the convergence property of Brelot but the (generalized) Harnack inequality is not valid. We then remark that since $f_{\beta,a}$ is negative definite, by e.g. [Fi] $(\Delta, f_{\beta,a})$ is a superprocess and this particular type of Branching processes may have nice and interesting properties.

In the following paragraph, we consider a second order elliptic differential operator L on $X = \mathbb{R}^d (d \geq 2)$ in the form:

$$Lu = \sum_{i,j=1}^{d} \ a_{ij} \ \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu.$$

If the coefficients a_{ij}, b_i, c are locally bounded, $c \leq 0$ and f satisfies the KO condition, we then prove that for every ball B in X, there exists a function $v \in \mathcal{C}^2(B)$ such that $Lv \leq f(v)$ on B and $\lim v = +\infty$ at the boundary of B.

If L is uniformly elliplic and if the coefficients of L are uniformly Hölder continuous and bounded on X, then the previous function v is ${}^{L}H_f$-hyperharmonic, where for every $U \subset X$

$$ {}^{L}H_f(U) = \{u \in \mathcal{C}(U) : L(u + \int {}^{L}G^V(\cdot, y) f(u(y))\lambda(dy)) = 0 \text{ for every } V \subset \overline{V} \subset U\}. $$

${}^{L}G^V$ is the Green function (see [RMH])associated with V and L. We then consider φ an elliptic admissible function (see Definition 4.3) and for $\psi(x, y) = y\varphi(x, y)$, we show that the existence of f satisfying the KO condition and $c \geq 0$ with $f(y) \leq \psi(x, y) = y\varphi(x, y)$ for every $x \in X$ and $y \in [c, +\infty[$ yields the existence on every ball B of a regular Evans function for L and ψ, i.e. there exists $u \in^{L} \mathcal{H}_\psi^+(B)$ such that $\lim u = +\infty$ at the boundary of $B,{}^{L}\mathcal{H}_\psi$ is defined in the same way as ${}^{L}\mathcal{H}_f$.

In the fifth paragraph we consider a second order parabolic differential operator L on $X = \mathbb{R}^d \times \mathbb{R}(d \geq 1)$ in the following form:

$$ Lu(x, t) = \sum_{i,j=1}^{d} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) + \sum_{i=1}^{d} b_i(x, t) \frac{\partial u}{\partial x_i}(x, t) + (cu)(x, t) - \frac{\partial u}{\partial t}(x, t). $$

If f is a function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the KO condition and if the coefficients of L are locally bounded, then on every bounded cylinder V in X we prove the existence of a function $v \in \mathcal{C}_x^2(V) \cap \mathcal{C}_t^1(V)$ such that $Lv \leq f(v)$ on V and $\lim v = +\infty$ on the heat boundary of V. As in the previous section we show that if in addition the coefficients of L are uniformly Hölder continuous then v is hyperharmonic in $(X,{}^{L}\mathcal{H}_f)$.
For a parabolic admissible function φ (see Definition 5.3) such that there exist f satisfying the KO condition and $c \geq 0$ with $f(y) \leq \psi(x, y) = y\varphi(x, y)$ for every $x \in X$ and $y \in [c, +\infty[$, we prove that every cylinder V and more generally every L-resolutive set U admits a regular Evans function associated with ψ and L i.e. there exists $u \in {}^{L}\mathcal{H}_\psi^+(U)$ such that $\lim\limits_{y \to z} u(y) = +\infty$ for every $z \in \partial U$ regular for the heat equation.

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1 Regular Evans functions

Let $X = \mathbb{R}^d (d \geq 2)$ and a function f from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$, continuous, increasing and derivable at zero. For every open set U in X we set: $\mathcal{H}_f(U) = \{u \in \mathcal{C}(U) : \Delta u = f(u) \text{ in the distributional sense}\}$. By [BBM] or [B$_1$] $\mathcal{H}_f$

is a (nonlinear) sheaf of continuous functions and $(X, \mathcal{H}_f)$ is a harmonic Bauer space having the same regular sets as the classical harmonic structure given by the Laplace equation. Further for every open set U and u locally bounded lower semicontinuous,it is easy to check that u is hyperharmonic in $(X, \mathcal{H}_f)$ (for the definition, see [BBM] or [B_1]) if and only if $\Delta u \leq f(u)$ in the distributional sense (DS) and Analog hypoharmonic functions are given by $\Delta u \geq f(u)$. For every relatively compact open set we have a minimum (comparison) principle as in the linear case.

Let B be a ball with center x_0 and radius R and G^B be the Green function for the Laplacian i.e. $\Delta G^B(\cdot, y) = -\epsilon_y$ in the distributional sense ,then $G^B(x, y) = G^B(\sigma(y), \sigma(x))$ for every rotation $\sigma \in S0(n)$.

For every $c > 0$, let $u \in \mathcal{C}^+(B)$ such that $c = u + \int G^B(\cdot, y) f(u(y)) dy$. Hence $\Delta u = f(u)$ in DS, $\lim\limits_{y \to z} u(x) = c$ for every $z \in \partial B$ and $u(x) = u(\sigma(x))$ for every $\sigma \in SO(n)$ with $\sigma(x_0) = x_0$. Let $v(t) = u(x)$ for $t = \|x - x_0\|$; we have $v \in \mathcal{C}^2([0, R])$ and $v'(0) = 0$. Furthermore:

$$\Delta u = v''(t) + \frac{d-1}{t} v'(t) = f(u(t)) \quad \text{for every} \quad t \in [0, R].$$

As in [K] we hence obtain $v' \geq 0$ and $v'' \geq 0$ on $[0, R]$.

Let now U be an open subset of X.

Definition 1.1. *We shall say that $u \in \mathcal{C}^+(U)$ is a regular Evans function associated with f, U and Δ if $u \in \mathcal{H}_f^+(U)$ and $\lim\limits_{x \to z} u(x) = +\infty$ for every regular point z at the boundary of U.*

Remark 1.2. *1. If $(X, \mathcal{H}_f)$ is linear (i.e. f is linear), then there is no regular Evans function on every relatively compact open set. This notion is strongly related to the nonlinear nature of f as we shall explain in §2.*

2. If U is a ball with center x_0 having a regular Evans function associated with f and Δ, then it admits a radial regular Evans function u(henceforth $u(x) = g(||x - x_0||)$ for every $x \in U$).

3. If U is a starlike domain and $f(t) = sgn(t)|t|^\alpha, \alpha > 1$, then by[$D_1$] or [$B_1$], there exists a unique regular Evans function associated with f , U and Δ. Dynkin calls this function minimal positive solution of a certain problem .

By [B_1] there exists a unique in the same way defined regular Evans function associated with f,U and the heat equation on $\mathbb{R}^d \times R$.

4. In the linear potential theory (see e.g. [CC] [BH]), notion of Evans function related to the irregular points of the boundary of U is introduced. Such functions does not exists if U is regular.

5. At the beginning of this century, G. Bouligand introduced for an open set a notion of barrier function which characterizes the regular points of the boundary.

In [B_1] we introduce a Harnack type inequality as follows: The Harnack inequality (or principle) is satisfied in $(X, \mathcal{H}_f)$ if for every domain U of X and every compact subset C in U, there exists two constants $c_1 \geq 0$ and $c_2 \geq 0$ such that

$$u(x) \leq c_1 u(y) + c_2 \quad \text{for every} \quad x, y \in C \quad \text{and} \quad u \in \mathcal{H}_f^+(U).$$

We have the following:

Proposition 1.3. *$(X, \mathcal{H}_f)$ satisfies the Harnack inequality with $c_1 = 0$ if and only if every ball B admits a regular Evans function associated with f, B and Δ.*

Proof. Let B be a ball and $u_n \in \mathcal{H}_f^+(B)$ with $u_n = n$ on ∂B. Then (u_n) is increasing and it easy to check that $u := \sup_n u_n \in \mathcal{H}_f^+(B)$, if and only if the Harnack inequality is valid on $(X, \mathcal{H}_f)$ with $c_1 = 0$. Further we have $\liminf_{x \to z} u(x) \geq \liminf_{x \to z} u_n(x) = n$ for every $n \in \mathbb{N}$ and hence $\lim_{x \to z} u(x) = +\infty$ for every $z \in \partial B$. $\square$

Remark 1.4. *If f is odd,the same proof as in the previous proposition gives the following inequality :*

For every domain U of X and every compact subset C in U, there exists a constant $c > 0$ such that

$$u(x) \leq c \quad \textit{for every} \quad x \in C \quad \textit{and} \quad u \in \mathcal{H}_f^+(U).$$

More generally the inequality is still valid, if every ball admits regular Evans functions associated respectively with f and $\widetilde{f}$,where $\widetilde{f}(x) = -f(-x)$.

2 Keller-Osserman properties

The investigation of smooth functions exploding at (the regular part of) the boundary (as E.B. Dynkin says in $[D_2]$) was a research subject of many mathematicians of this century, among others we refer to those of J. B. Keller [K] and R. Osserman [O].

Let $f : \mathbb{R} \to R$ with the same properties as in the previous section.

Definition 2.1. *We shall say that f satisfies the Keller-Osserman property, denoted by KO, if there exists a natural number $d \geq 2$ such that every ball $B \subset \mathbb{R}^d$ admits a regular Evans function associated with f, B and Δ.*
We shall denote by $K = \{f : \mathbb{R} \to R \text{ satisfying } KO\}$.

Remark 2.2. *1. The definition 2.1 is not the original Keller-Osserman condition given by an integral which we shall use later.*

2. It is easy to see by the minimum principle that if f, g are in K and $M > 0$, then $f + g$ and Mf are in K.

3. The KO property is still valid if we replace in the definition 2.1 for every ball by the existence of a sequence (R_n) decreasing to zero such that every ball with radius (R_n) .

4. As we will see ,the KO property does not depend on d, if it is valid for a suitable $d \geq 2$, then it is valid for every $d \geq 2$, but the corresponding regular Evans functions depend on d.

We have the following necessary condition for the validity of the KO property.

Proposition 2.3. *If $f \in K$, then $\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty$.*

Proof. Let $d \geq 2, R > 0$ and $B(x_0, R)$ be the ball with center $x_0 \in \mathbb{R}^d$ and radius R. Let $(\alpha_n)_n \subset \mathbb{R}$ increasing to infinity and $u = u_n$ is the unique $u \in \mathcal{H}_f^+(B)$ with $u = \alpha_n$ on ∂B, u is then radial and we set $u(t) = u(x)$ for $t = \|x - x_0\|$. We easily verify that $u \in \mathcal{C}^2([0, R]), u'(0) = 0$ and $u''(t) + \frac{d-1}{t}u'(t) = f(u(t))$ and therefore $(t^{d-1}u'(t))' = t^{d-1}f(u(t))$, by integration we obtain

$$u(t) = \int_0^t \frac{1}{r^{d-1}} \left[\int_0^r s^{d-1}f(u(s))ds\right] dr + u(0).$$

Since u and f are increasing, for t=R we get: $\alpha_n \leq f(\alpha_n)\frac{R^2}{2d} + u_n(0)$, where $u_n(0) = u(0)$.
The hypothesis yields $\lim_n u_n(0) < +\infty$ and then $1 \leq \lim_{n\to\infty} \inf \frac{f(\alpha_n)}{\alpha_n}\frac{R^2}{2d}$, R is positive and arbitrary we hence obtain $\lim_{n\to\infty} \inf \frac{f(\alpha_n)}{\alpha_n} = +\infty$. Since (α_n) is arbitrary we get the statement. ☐

The previous necessary condition is not sufficient as the following example shows: Let $f(t) = t\log(1 + |t|))$. An easy calculation shows that f satisfies the required conditions for this section and $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$.

Proposition 2.4. *The KO property is not satisfied by f.*

Proof. Let $R > 0$ such that $\frac{R^2}{2d} < 1$, $B = B(0, R)$ and $u_n =: u \in \mathcal{H}_f^+(B)$, with $\lim u = n$ at the boundary of B. Put $v = \log(1 + u)$, by an easy calculation we have:

$$\Delta v = \frac{\Delta u}{1+u} - \left(\frac{u'}{1+u}\right)^2 \leq \frac{f(u)}{1+u} = \frac{u\log(1+u)}{1+u} \leq v.$$

Therefore $(r^{d-1}v')' \leq r^{d-1}v$. Since $v'(0) = 0$ we hence obtain $t^{d-1}v'(t) \leq v(t)\frac{t^d}{d}$,$v'(t) \leq \frac{t}{d}v(t)$ and $\int_0^R v'(t)d \leq \int_0^R \frac{t}{d}v(t)dt$.Since v is isotone we get $v(R) - v(0) \leq v(R)\frac{R^2}{2d}$ and

$$\left(1 - \frac{R^2}{2d}\right)\log(1 + n) \leq \log(1 + u_n(0)).$$

$\frac{R^2}{2d} < 1$ yields that $\lim_{n\to\infty} u_n(0) = \infty$. This implies that $B(0, R)$ cannot have a regular Evans functions for every $R > 0$ with $\frac{R^2}{2d} < 1$ and hence the KO property is not valid for f. ☐

Definition 2.5. *We will say that f satisfies the Keller-Osserman integrability condition, denoted by KOI, if there exists $B > 0$ such that*

$$\int_B^{+\infty} \left[\int_0^x f(t)dt\right]^{-1/2} dx < +\infty.$$

By an elementary calculus we have :

Lemma 2.6. *If f satisfies KOI, then $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$.*

Definition 2.7. *We shall say that f lies the Keller limit condition, denoted by $K\ell$, if*

$$\lim_{A\to+\infty} \int_A^{+\infty} \left[\int_A^x f(t)dt\right]^{-1/2} dx \quad = \quad 0.$$

The previous limit was used by Keller [K] for the investigation of regular Evans functions associated with f, Balls and Δ.

Lemma 2.8. *$KOI \Rightarrow K\ell$.*

Proof. Let $B > 0$ such that the integral in 2.5 is finite. Let $A > B$. Put $g(x) = \int_A^{x+A} f(t)dt = \int_0^x f(u+A)du$. Since f is increasing we get $g(x) \geq \int_0^x f(u)du$ and then

$$I_A := \int_A^{+\infty} [g(x)]^{-1/2} dx \leq \int_A^{+\infty} \left[\int_0^x f(u)du\right]^{-1/2} dx.$$

KOI implies then $\lim\limits_{A\to+\infty} I_A = 0$. Furthermore we have

$$I_A = \int_{2A}^{+\infty} \left[\int_A^x f(u)du\right]^{-1/2} dx.$$

Since

$$J_A := \int_A^{+\infty} \left[\int_A^x f(u)du\right]^{-1/2} dx = \int_A^{2A} \left[\int_A^x f(t)dt\right]^{-1/2} dx + I_A.$$

it is then enough to prove that

$$\ell_A = \int_A^{2A} \left[\int_A^x f(t)dt\right]^{-1/2} dx$$

tends to zero as A tends to infinity. Indeed we have $f(t) \geq f(A)$ for $t > A$ and then

$$\left[\int_A^x f(t)dt\right]^{-1/2} \leq (f(A)^{-1/2}(x-A)^{-1/2}$$

and hence

$$\ell_A \leq f(A)^{-1/2} \int_A^{2A} (x-A)^{-1/2} dx = 2\left(\frac{A}{f(A)}\right)^{-1/2}$$

by Lemma 1, we have $\lim\limits_{x\to\infty} \frac{f(x)}{x} = +\infty$ and hence $\lim\limits_{A\to+\infty} \ell_A = 0$. □

Theorem 2.9. *Let $f : \mathbb{R} \to R$, then the following properties are equivalent:*
(1) KO
(2) KOI
(3) $K\ell$

Proof. By the previous Lemma 2, we have (2) ⇒ (3) . (3) ⇒ (2) is easy to check. The rest of the proof is inspired from [K ,Theorem 1] . Let $d \geq 2$, $R > 0$, $B(0,R) \subset \mathbb{R}^d$ the ball with radius R and center 0 and $u = u_n \in \mathcal{H}_f^+(B(0,R))$ with $\lim\limits_{\|x\|\to R} u(x) = n$. We shall prove that $\lim\limits_{n\to\infty} u_n(0) < +\infty$ if and only if f fulfills $K\ell$. Put $I_n = \int_{u_n(0)}^{n} \left[2\int_{u_n(0)}^{x} f(z)dz\right]^{-1/2} dx$, as in the proof of Theorem 1 of Keller [K] we then have:

$$(*) \quad I_n \leq R \leq (\sqrt{d})I_n$$

.
If f satisfies $K\ell$, then necessary $\lim\limits_{n\to\infty} u_n(0) < +\infty$ and therefore we have the $K0$ property. Conversely let $\ell = \lim\limits_{n\to\infty} u_n(0) < +\infty$, then from the inequality $(*)$ and since $R > 0$, we get $\lim\limits_{n\to\infty} \sup I_n < +\infty$. The inequality $\int_{u_n(0)}^{x} f(z)dz \leq \int_0^x f(z)dz$ yields $\int_{u_n(0)}^{+\infty} \left[\int_0^x f(z)dz\right]^{-1/2} dx \leq I_n$ and since $u_n(0)$ increases to ℓ , we get $\int_{\ell}^{+\infty} \left[\int_0^x f(z)dz\right]^{-1/2} dx < +\infty$.The $K0I$ condition is equivalent to the $K\ell$ condition we hence obtain the statement (1) ⇔ (3). □

Remark 2.10. *(1) By proposition 2.3, the function $f(t) = t(\log(1+|t|)$ does not satisfy the KO condition, then without calculation of the integral we have for every $B > 0$: $\int_B^{+\infty} \left[\int_0^x t\log(1+|t|)dt\right]^{-1/2} dx = +\infty$ and hence if $f_{\alpha,a}(t) = t(\log(a+|t|))^{\alpha}$ for $a \geq 1$ and $\alpha > 0$ we have:for every $\alpha \in]0,1], a \geq 1$ and every $B > 0 : \int_B^{+\infty} \left[\int_0^x f_{\alpha,a}(t)dt\right]^{-1/2} dx = +\infty$.*

(2) By an easy calculation of the integral we have: for every $a \geq 1$ $f_{\alpha,a} \in K$ if and only if $\alpha > 2$.

3 KO property, Harnack inequality and Brelot convergence property

Let $f : \mathbb{R} \to R$ with the same properties as the previous sections i.e. f continuous, increasing, derivable at zero and $f(0) = 0$. Let $d \geq 2$ and $(\mathbb{R}^d, \mathcal{H}_f)$ be the

(nonlinear) harmonic Bauer space given by f as in the first section. In [B_1] we introduced a convergence type property of Brelot as follows:
For every domain U in X and every monotone sequence $(u_n) \subset \mathcal{H}_f(U)$ such that there exists $x_0 \in U$ with $\lim\limits_n |u_n(x_0)| < +\infty$, we have $\lim\limits_n u_n \in \mathcal{H}_f(U)$.
In the linear case we have a Harnack inequality (see Section 2) with $c_2 = 0$ and it is well known (see e.g.[M] or[LW])that the Harnack inequality is equivalent to the convergence property of Brelot, however in the nonlinear case this equivalence fails to be true as we shall show.

Theorem 3.1. *Assume that the Harnack inequality is valid on $(\mathbb{R}^d, \mathcal{H}_f)$, then f lies the Keller-Osserman property if and only if $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$.*

Proof. By Proposition 2.3 we have only to prove the sufficient condition. Let f such that $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$. Let $R > 0$ and $B(0,R)$ be the ball with center 0 and radius R. Let $(u_n)_n \subset \mathcal{H}_f^+(B(0,R))$ with $\lim\limits_{\|x\|\to R} u_n(x) = n$. As in the proof of Proposition 2.2 we have

$$u_n(r) = \int_0^r \frac{1}{t^{d-1}} \left(\int_0^t s^{d-1} f\Big(u_n(s)\Big) ds \right) dt + u_n(0).$$

Therefore $u_n(r) \geq f(u_n(0)) \times \frac{r^2}{2d}$ for $r \in]0, R[$. By the Harnack inequality, for every $r \in]0, R[$ there exists $c_1 \geq 0, c_2 \geq 0$ with $u_n(r) \leq c_1 u_n(0) + c_2$ for every $n \in \mathbb{N}$. It follows then $f(u_n(0)) \times \frac{r^2}{2d} \leq c_1 u_n(0) + c_2$ and

$$\frac{f(u_n(0))}{u_n(0)} \leq c_1 \times \frac{2d}{r^2} + \frac{c_2}{u_n(0)} \leq c_1 \times \frac{2d}{r^2} + \frac{c_2}{u_1(0)}$$

and $\lim\limits_{n\to\infty} \sup \frac{f(u_n(0))}{u_n(0)} < +\infty$, this does imply that necessarily $\lim\limits_n u_n(0) < +\infty$ and hence, since $R > 0$ is arbitrary, we have $f \in K$. $\square$

Corollary 3.2. *Let $f_{\alpha,a}(t) = t(\log(a + |t|))^\alpha$ for $\alpha \in]0, \infty[$ and $a \geq 1$. then the Harnack inequality is not fulfilled in the nonlinear harmonic space defined by $f_{\alpha,a}$ for every $\alpha \in]0, 2]$ and $a \geq 1$.*

In the following we will show that, in contrast to the linear potential theory, the Brelot convergence property is satisfied in harmonic spaces where the Harnack inequality (principle) is not valid. Let $a \geq e(\log e = 1)$, $f_\alpha(t) = t(\log(a + |t|))^\alpha$ with $\alpha \in]0, 1]$ and $\mathcal{H}_\alpha$ be the nonlinear sheaf corresponding to the nonlinear perturbation (see [BBM]or [B_1]) of the Laplacian by f_α in $\mathbb{R}^d (d \geq 2)$ i.e $\mathcal{H}_\alpha = \mathcal{H}_{f_\alpha}$

Theorem 3.3. *$(\mathbb{R}^d, \mathcal{H}_\alpha)$ has the Brelot convergence property.*

Proof. Since f_α is odd and increasing, then for every open set U in $\mathbb{R}^d$, $u \in \mathcal{H}_\alpha(U)$ and λ real positive we have $(-u) \in \mathcal{H}_\alpha(U)$ and $(u + \lambda) \in \mathcal{H}^*_\alpha(U)$. By [$B_1$] Proposition 3.1, it is then enough to prove the convergence property of Brelot (as in the linear case) in the form of positive increasing sequences: Let U be a domain in $X = \mathbb{R}^d, x_0 \in U$ and $(h_n)_n \subset \mathcal{H}_\alpha^+(U)$ increasing with $\sup\{ h_n(x_0); n \in \mathbb{N} \} < +\infty$. Let $v_n := [\log(a+h_n)]^\alpha$, then :

$$\nabla v_n = \alpha[\log(a+h_n)]^{\alpha-1}\frac{\nabla h_n}{a+h_n}$$

and

$$\begin{aligned}\Delta v_n &= \alpha[\log(a+h_n)]^{\alpha-1}\frac{\nabla h_n}{a+h_n} + \alpha(\alpha-1)[\log(a+h_n)]^{\alpha-2}\left(\frac{\nabla h_n}{a+h_n}\right)^2 \\ &\quad -\alpha[\log(a+h_n)]^{\alpha-1}\left(\frac{\nabla h_n}{a+h_n}\right)^2.\end{aligned}$$

Therefore

$$\Delta v_n \leq \alpha[\log(a+h_n)]^{\alpha-1} \times \frac{h_n[\log(1+h_n)]^\alpha}{a+h_n}.$$

Since $\alpha \in]0,1]$ and $a \geq e$ we then obtain $\Delta v_n \leq v_n$ and hence v_n is superharmonic on U. Let $R > 0$ such that $B(x_0, R) \subset U$. For $r < R$, let $M_{x_0}^r(v_n)$ be the mean value of v_n on the sphere with center x_0 and radius r, therefore we have

$$0 \leq M_{x_0}^r(v_n) \leq v_n(x_0) \quad \text{and hence} \quad \sup M_{x_0}^r(v_n) < +\infty$$

for every $r < R$. On the other hand since $[\log(a+t)]^\alpha$ is concave for $\alpha \in]0,1]$ and $M_{x_0}^r$ is a probability measure, by the Jensen 's inequality we have :

$$M_{x_0}^r(v_n) = M_{x_0}^r[\log(a+h_n)]^\alpha \geq [\log(a + M_{x_0}^r h_n)]^\alpha$$

which yields $\sup M_{x_0}^r h_n < +\infty$ for every $r < R$. We denote by $H_{B(x_0,r)}h_n$ the Poisson integral of h_n on $B(x_0, r)$. Since $\Delta h_n = f_\alpha(h_n) \geq 0$, h_n is subharmonic and hence $h_n(x) \leq H_{B(x_0,r)}h_n(x)$ for every $x \in B(x_0, r)$. Since $H_{B(x_0,r)}h_n(x_0) = M_{x_0}^r h_n$, the classical Harnack inequality yields $(H_{B(x_0,r)}h_n)_n$ locally bounded on $B(x_0, r)$. Since U is a domain, we can easily prove that (h_n) is locally uniformly bounded on U, by the convergence property of Bauer in the space $(X, \mathcal{H}_\alpha)$ (see [BBM]or [B$_1$]) we get $\sup_n h_n \in \mathcal{H}_\alpha(U)$. □

Remark 3.4. *(1) Let $(u_n) \in \mathcal{H}_\alpha^+(U)$ where U is a domain and (u_n) is increasing. Assume that $\{u_n(x), x \in U\}_{n\in\mathbb{N}}$ are running in a competition to attain the infinity, then the low governed by the $\mathcal{H}_\alpha$ implies the following interpretation: If for one $x \in U$, the competitor $(u_n(x))_n$ is not able to attain the infinity (i.e. $\sup_n u_n(x) < +\infty$) then not all others. However, the non validity of the Keller-Osserman property together with the non validity of the Harnack inequality give rise to the following interpretation: there is domain U in X and a sequence $(u_n) \subset \mathcal{H}_\alpha^+(U)$ tending (running) to the infinity at every $x \in U$ but the distance between two* *different competitors $u_n(x)$ and $u_n(x')$ for $x \neq x'$ is not (in contrast to the linear case) controllable and may be infinite.*

(2) The previous theorem is also valid for $f(t) = t\log(1+|t|)$ with the same proof.

(3) Let $\alpha \in]0,1[$ and $a \geq e$ and if $\alpha = 1$, we take $a \geq 1$. Put $f_{\alpha,a}(t) = t[\log(a+|t|)]^\alpha$. By[$B_2$] there is a nonlinear semigroup Q_t on

$B_b(\mathbb{R}^{\ d})$ satisfying

$$\Delta Q_t - \frac{\partial Q_t}{\partial t} = f_{\alpha,a}(Q_t).$$

It is easy to check that $f_{\alpha,a}$ is a negative definite function and by [Fi] there is a superprocess $(\Delta, f_{\alpha,a})$. The study of such particular branching processes type may have nice an interesting properties.

4 The Keller-Osserman condition for elliptic partial differential operators of second order

Let $f : \mathbb{R} \to R$ satisfying the KO condition (see Section 2). Let $X = \mathbb{R}^d (d \geq 2)$ and L be a second order differential operator with the following form:

$$Lu = \sum_{i,j=1}^{d} a_{ij}\ \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i\ \frac{\partial u}{\partial x_i} - cu.$$

Theorem 4.1. *Assume that the coefficients a_{ij}, b_i, c are bounded and $c \geq 0$. Then for every ball $B = B(x_0, R)$ with center at x_0 and radius $R > 0$, there exists a positive $v \in \mathcal{C}^2(B)$ such that $Lv \leq f(v)$ on B and $\lim\limits_{x\to z} v(x) = +\infty$ for every z in the boundary of B.*

Proof. Let $s \in \mathcal{D}^+(B)$ be an infinitely derivable positive function with compact support in B and $s \neq 0$.Let G^B be the Green function for the Laplacian on B and $p = \int_B G^B(\cdot, t)s(t)dt$, then p is again infinitely derivable on B and the derivatives of p at any order are bounded on B. Furthermore $p > 0$ on B and $\lim\limits_{x\to z} p(x) = 0$ for $z \in \partial B$. Let $g = \frac{R}{1+p}$, then $g \in \mathcal{C}^\infty_+(B), \lim\limits_{x\to z} g(x) = R$ for $z \in \partial B$, $0 < g < R$ on B and all derivatives of g are bounded on B. Let φ be a function in $\mathcal{C}^2_+([0,R[)$ with φ', φ'' positive. Put $v(x) = \varphi(g(x))$ for every $x \in B$, then v is in $\mathcal{C}^2(B)$ and we have:

$$\frac{\partial v}{\partial x_i} = \varphi'(g)\frac{\partial g}{\partial x_i}, \frac{\partial^2 v}{\partial x_j \partial x_i} = \varphi''(g) \times \frac{\partial g}{\partial x_j}\frac{\partial g}{\partial x_i} + \varphi'(g)\frac{\partial^2 g}{\partial x_j \partial x_i}$$

hence

$$Lv = \varphi''(g)\sum_{i,j=1}^{d} a_{ij}\frac{\partial g}{\partial x_j}\frac{\partial g}{\partial x_i} + \varphi'(g)\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 g}{\partial x_i \partial x_j}$$
$$+\varphi'(g)\sum_{i=1}^{d} b_i\frac{\partial g}{\partial x_i} - cv,$$

the assumptions on the coefficients of L and the choice of φ yield: $Lv \leq M(\varphi''(g) + \frac{d-1}{g}\varphi'(g))$ with is a strictly positive constant M depending on the coefficients of L, the derivatives of g and R. By Remark 2.1 (2) $\frac{f}{M}$ satisfies the KO condition, hence there exists $\psi \in \mathcal{C}^2([0,R[)$ such that $\psi''(t) + \frac{d-1}{t}\psi'(t) = \frac{1}{M}f(\psi(t))$ and $\lim\limits_{t\to R}\psi(t) = +\infty$ we set $\varphi = \psi$ and thus $v = \psi(g)$ gives the statement. □

Remark 4.2. *The previous theorem is valid for every continuous g, such that there exists $M > 0$ and $f \in K$ with $f = g$ on $[M, +\infty[$. Indeed, putting $u = v+M$ where v is given by the previous theorem, we have $Lu \leq f(v) \leq f(v+M) = g(v+M)$ and $v+M$ satisfies the same conditions as v.*

In what follows, we shall assume that L is uniformly elliptic with uniformly Hölder continuous and bounded coefficients on $\mathbb{R}^d$. Then it is well known see [RMH] that L has the same regular sets as the Laplace operator, in particular balls are regular. Furthermore every relatively compact (because of the special case $d = 2$) open set in X admits a Green function G^U. Let $M_U(y) = \int G^U(y,z)\lambda(dz), \lambda$ Lebesgue measure on X. Then $M = (M_U)_U$ is a positive section of continuous and real potentials (see [BHH]).

We recall that for a positive section of continuous and real potential in a linear harmonic Bauer space, a local Kato-class K_M^{loc} related to M in the same way as [AS] in the classical case of the Laplacian was introduced in [BBM] or [BM] as the set of $f \in B(\mathbb{R}^d)$ such that $f \bullet M$ is again a positive section of real and continuous potential,$\bullet$ is the specific order in the cone of the superharmonic functions.

Let φ from $X \times \mathbb{R}$ to $\overline{\mathbb{R}}$ Borel measurable, from [BBM] or [BM] we recall the following notions :

a) φ is locally Kato-bounded,if for every $c \in \mathbb{R}_+^*$,there exists $g_c \in K_M^{loc}$ such that $|\varphi(x,y)| \leq g_c(x)$ for every $x \in \mathbb{R}^d$ and $y \in [-c,c]$.

b) φ is Kato-bounded,if there exists $g \in K_M^{loc}$ such that $|\varphi(x,y)| \leq g(x)$ for every x$\in \mathbb{R}^d$ and $y \in \mathbb{R}$.

c)φ is locally Kato-Lipschitzian if for every $c \in \mathbb{R}_+^*$,there exists $g_c \in K_M^{loc}$ such that :

$|\varphi(x,y) - \varphi(x,y')| \leq g_c(x)|y-y'|$ for every $x \in \mathbb{R}^d$ and $y \in [-c,c]$.

Kato-Lipschitzian φ are defined in the same way as Kato-bounded.

Remark 4.3. *It is easy to see that $g \in K_M^{loc}$ does not imply g locally bounded and hence locally Kato-bounded (resp.Lipschitzian) does not need be locally bounded (resp.Lipschitzian).*

Definition 4.4. *Let φ be a function from $X\times\mathbb{R}$ to $\overline{\mathbb{R}}$ and $\varphi^-(y,z)=\sup(-\varphi(y,z),0)$. φ is called elliptic admissible if it satisfies the following properties:*

(1) φ is locally Kato-bounded

(2) φ^- is Kato-bounded (e.g. $\varphi\geq 0$),

(3) for every $z\in X$, $\varphi(z,\cdot)$ is continuous and $\psi(z,y)=y\varphi(z,y)$ is increasing in y for fixed $z\in\mathbb{R}^d$.

Let $\mathcal{H}$ (resp.$\mathcal{H}^*$) be the harmonic (resp. hyperharmonic) sheaf associated with L i.e.

$\mathcal{H}(U)=\{u\in\mathcal{C}^2(U):Lu=0\}$. We then set

$\mathcal{H}_\psi(U)=\{u\in\mathcal{C}(U):u+\int G^V(\cdot,y)\psi(y,u(y))\lambda(dy)\in\mathcal{H}(V)$ for every $V\subset\overline{V}\subset U\}$ and

$\mathcal{H}^*_\psi(U)=\{u$ locally bounded and lower semicontinuous such that $u+\int G^V(\cdot,y)\psi(y,u(y))\lambda(dy)\in\mathcal{H}^*(V)$ for every $V\subset\overline{V}\subset U\}$, and similarly for $\mathcal{H}_{\psi^*}(U)$.

We recall the following results from [BBM] or[B_1] which is,by a simple proof, valid in more general settings like the parabolic case which will be investigated in the next section.

Theorem 4.5. *Let φ be an elliptic admissible function and $\psi(x,y)=y\varphi(x,y)$. Then $(X,\mathcal{H}_\psi)$ is a (nonlinear) Bauer space. The $\mathcal{H}_\psi$-regular and resolutive sets are given by the Kato bound of φ^- i.e. there exist $0<R\leq+\infty$ such that if U is regular and if diameter of U is smaller than R, then U is $\mathcal{H}_\psi$-regular . In particular if $\varphi\geq 0$ we have $R=+\infty$. For the regular sets, we have minimum (comparison) principle. Moreover if we denote by ${}^\psi H_U u$ the solution of the Dirichlet problem in $(X,\mathcal{H}_\psi)$, associated with U and $u\in\mathcal{C}(\partial U)$. The $\mathcal{H}_\psi$-hyperharmonic functions on U(i.e. ${}^\psi H_V u\leq u$ for $V\subset\overline{V}\subset U$ in a basis of regular sets) are given by $\mathcal{H}^*_\psi(U)$ and analog for the hypoharmonic functions.*

The following theorem is the key to the existence of regular Evans functions associated with f,balls and the operator L

Theorem 4.6. *Let $f\in K$ and v be a function given by Theorem 4.1 on a Ball B.Then v is $\mathcal{H}_f$-hyperharmonic on B (i.e. $v\in\mathcal{H}^*_f(B)$).*

Proof. Let U be a ball in X with $U\subset\overline{U}\subset B$, it is then enough to prove that ${}^fH_U v\leq v$. since $\inf\limits_{x\in\overline{U}} v(x)>0$, there exists by the theorem of Stone-Weierstrass $g_n\in\mathcal{C}^\infty(B)$ with $0\leq g_n(x)\leq v(x)+\frac{1}{n}$ for every $x\in\overline{U}$. Let $\beta=\sup\limits_{\overline{U}} v$.Since $t\to\frac{f(t)}{t}$ is continuous on $[0,\beta+1]$, there exists again by Stone-Weierstrass $(p_n)\subset\mathcal{C}^\infty(B)$ such that :

$\frac{f(t)}{t}<p_n(t)$ and $p_n(t)$ converges uniformly to $\frac{f(t)}{t}$ on $[0,\beta+1]$.

We assume first that for all i,j in {1,2,...,d}, $a_{ij}\in\mathcal{C}^1(V)$ with $B\subset\bar{B}\subset V$.

By [GT Theorem 15.12] there exists for every $n \in \mathbb{N}$, $h_n \in \mathcal{C}^{2,\alpha}(\overline{U})$ (α is the Hölder exponent of the coefficients of L) such that:

$$Lh_n = h_n|p_n(h_n)| \quad \text{and} \quad h_n = g_n \text{ on } \partial U.$$

We shall prove the following inequality : (1) $h_n \leq v + \frac{1}{n}$ for every $n \in \mathbb{N}$.
Since $g_n \leq v + \frac{1}{n}$, if we assume that the inequality (1) does not hold on U, then there exists $x_0 \in U$ such that $\inf\limits_U \left(v + \frac{1}{n} - h_n\right) = \left(v + \frac{1}{n} - h_n\right)(x_0) < 0$. On the other hand we have

$$\begin{aligned} 0 &\leq L(v + \frac{1}{n} - h_n)(x_0) \leq (f(v + \frac{1}{n}) - h_n|p_n(h_n)|)(x_0) \\ &\leq f(h_n(x_0)) - h_n(x_0)|p_n(h_n(x_0))| \\ &\leq h_n(x_0)\left[\frac{f(h_n(x_0))}{h_n(x_0)} - p_n(h_n(x_0))\right] \end{aligned}$$

the maximum principle yields $0 \leq \sup\limits_U h_n = \sup\limits_{\partial U} g_n \leq \beta + 1$. It follows $0 \leq h(x_0)\left[\frac{f(h_n(x_0))}{h_n(x_0)} - p_n(h_n(x_0))\right] < 0$ which is absurd and hence $h_n \leq v + \frac{1}{n}$ for every $n \in \mathbb{N}$.
By e.g. [RMH], L admits a Green Function ${}^L G^U$ on U and since $Lh_n = h_n p_n(h_n)$, we then obtain

$$H_B g_n = h_n + \int {}^L G_t^U (h_n p_n(h_n))(t)\lambda(dt).$$

where $H_U g$ is the solution of the Dirichlet problem for L ,U and g. By [H], the family $\{\int {}^L G_t^U (h_n p_n(h_n))(t)\lambda(dt), n \in \mathbb{N}.\}$ is equicontinuous on U, then so $\{h_n, n \in \mathbb{N}.\}$. Therefore there exists a subsequence which we denote again by $\{h_n\}$ converging locally uniformly to u. By the convergence theorem of Lebesgue and since $\{p_n(t)\}$ converges locally uniformly to $\frac{f(t)}{t}$ on $[0, \beta + 1]$ we obtain

$$H_U v = u + \int {}^L G_t^U f(u(t))\lambda(dt)$$

and hence by [BM] or [BBM] $u = {}^f H_U v$ and therefore ${}^f H_U v \leq v$.
let us now consider $a_{ij} \in \mathcal{C}^\alpha(V)$ .By an appropriate choice of an approximation of the unity, it is easy to find a sequence $(a_{ij}^n)_n \subset \mathcal{C}^1(V)$ converging uniformly to a_{ij} on V and such that $|a_{ij}^n| \leq |a_{ij}|$ for every $n \in \mathbb{N}$, $i, j \in \{1, ..., d\}$ and $L_n = \sum\limits_{i,j=1}^{d} a_{ij}^n \frac{\partial^2}{\partial x_i \partial x_j} + \sum\limits_{i=1}^{d} b_i \frac{\partial}{\partial x_i} - c$ is uniformly elliptic on V.The choice of a_{ij}^n and the proof of theorem 4.1 give that the function v constructed there satisfies even $L_n v \leq f(v)$ for every $n \in \mathbb{N}$. Let ${}^n H_U g$ be the solution of the Dirichlet problem associated with L_n , U and g .Let $(u_n) \subset \mathcal{C}(\bar{U})$ with

$$(*) \qquad {}^n H_U v = u_n + \int {}^{Ln} G_t^U f(u_n(t))\lambda(dt)$$

where ${}^{Ln}G^U$ is the Green Function for L_n on U . It follows that $u_n \leq v$ for every, $n \in \mathbb{N}$. . By [HS], we have ${}^{Ln}G_t^U \longrightarrow {}^{L}G_t^U$ for $n \longrightarrow \infty$ and there exists $\alpha > 0$ such that $0 \leq \frac{{}^{Ln}G^U}{{}^{L}G^U} \leq \alpha$, again by[HS Corollary5.3] ${}^{n}H_U v$ converges to $H_U v$ uniformly on $\bar{U}$. By the convergence theorem of Lebesgue we have $\int [{}^{Ln}G_t^U - {}^{L}G_t^U] f(u_n(t))\lambda(dt)$ converges to zero.Since (u_n) is bounded, it is easy to see (e.g. by[Me] or [H]) that the family $\{\int {}^{L}G_t^U f(u_n(t))\lambda(dt)\ n \in \mathbb{N}.\}$is equicontinuous on $\bar{U}$ and so (u_n). Let (v_n) be a subsequence of (u_n) converging uniformly to u on $\bar{U}$. the equality (*) yields :

$H_U v = u + \int {}^{L}G_t^U f(u(t))\lambda(dt)$ and hence (u_n) is uniformly convergent to u on $\bar{U}$ and then $u = {}^{f}H_U v \leq v$. Let

$g = v + \int {}^{L}G_t^U f(v(t))\lambda(dt)$ $\bar{U} \subset B$, we have for a ball $A \subset \bar{A} \subset U$:

$H_A g = H_A v + H_A(\int {}^{L}G_t^U f(v(t))\lambda(dt))$

$= {}^{f}H_A v + \int {}^{L}G_t^A f({}^{f}H_A v\,(t))\lambda(dt) + \int H_A {}^{L}G_t^U f(v(t))\lambda(dt))$

since ${}^{f}H_A v \leq v$ we get

$H_A g \leq v + \int {}^{L}G_t^A f(v\,(t))\lambda(dt) + \int H_A {}^{L}G_t^U f(v(t))\lambda(dt))$

$\leq v + \int {}^{L}G_t^U f(v\,(t))\lambda(dt) = g$, g is then L-hyperharmonic on U for every $U \subset \bar{U} \subset B$, by the same proof of the sheaf property for $\mathcal{H}_f^*$ as in [BBM]), we obtain the required statement $v \in \mathcal{H}_f^*(B)$. □

Theorem 4.7. *Assume that φ is elliptic admissible and there exists $f \in K$ and $M \geq o$ such that $f(y) \leq y\varphi(x,y)$ for every $x \in X$ and $y \in [M, +\infty[$. Then every ball B in X admits a regular Evans function associated with ψ, B and L i.e. there exists $u \in \mathcal{H}_\psi^+(B)$ with $\lim u = +\infty$ at the boundary of B.*

Proof. Let B be a ball and v be a function given by Theorem 4.1 on B

let $\widetilde{v} = v + M$, we then have $L\widetilde{v} = Lv + LM = Lv - cM \leq f(v) \leq f(v+M) = f(\widetilde{v})$. $\widetilde{v}$ satisfies the same conditions as v in Theorem 4.1. By 4.5 $\widetilde{v}$ is in $\mathcal{H}_f^*(B)$ and hence in $\mathcal{H}_\psi^*(B)$ where $\psi(x,y) = y\varphi(x,y)$. By the comparison principle we get ${}^{\psi}H_B(n) \leq \widetilde{v}$ and ${}^{\psi}H_B(n)$ is increasing, by the Bauer convergence property in $(X, \mathcal{H}_\psi)$, $\sup\{{}^{\psi}H_B(n), n \in \mathbb{N}\ \}$is then a regular Evans function associated with B, ψ and L. ${}^{\psi}H_B(n)$ is the solutions of the Dirichlet problem associated with the constant n and B in $(X, \mathcal{H}_\psi)$. □

5 The Keller-Osserman property for parabolic differential operators of second order

Let $X = \mathbb{R}^d \times \mathbb{R} (d \geq 1)$. For every $r > 0$ and $a < b$ in $\mathbb{R}$, we denote by $V(x_0, r, a, b)$ the corresponding cylinder i.e. $V(x_0, r, a, b) = \{(x,t) \in X : \|x - x_0\| < r$ and $a < t < b\}$ and $\mathcal{V}$ the set of such cylinders. We will denote by $\partial_h V$ the heat boundary of V defined by $\{(x,t) \in \partial V : a \le t < b\}$. Let L be a second order differential operator on X with the form

$$Lu(x,t) = \sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^{d} b_i(x,t)\frac{\partial u}{\partial x_i}(x,t) + cu - \frac{\partial u}{\partial t}$$

with locally bounded coefficients and $c \le 0$, as in the elliptic case we have the following:

Theorem 5.1. *Let $f \in K$, then for every $V \in \mathcal{V}$ there exists a function $u \in \mathcal{C}_x^2 \cap \mathcal{C}_t^1$ on V such that $Lu \le f(u)$ and $\lim_{y \to z} u(y) = +\infty$ for every $z \in \partial_h V$. $\mathcal{C}_x^2$ means two times continuously differentiable relative to the space variable x and $\mathcal{C}_t^1$ one time continuously differentiable relative to the time variable t.*

Proof. Let $V \in \mathcal{V}$. By [F, theorem 7 p127], there exists a strict positive function p on V such that all derivatives $\frac{\partial P}{\partial x_i}, \frac{\partial^2 P}{\partial x_i \partial x_j}, \frac{\partial P}{\partial t}$ $i,j \in \{1,\dots,d\}$ are uniformly Hölder continuous on V (Hence bounded) and $\lim_{y\to z} P(y) = 0$ for every $z \in \partial_h V$. Putting $g = \frac{1}{p+1}$, we get $\lim_{y\to z} g(y) = 1$ for $z \in \partial_h V, 0 < g < 1$ on V and all second derivatives at the space variables and the first derivative at the time t are bounded on V. further it is easy to see from remark1.2 that for every $c > 0$ there exists $v = v_c \in \mathcal{C}^2[0,1[$ with $t \to 1 \lim v_c(t) = \infty$ and such that $v''(t) + \frac{2}{t}v'(t) = cf(v(t))$. As in the elliptic case we have $Lv(g) \le M[v''(g) + \frac{2}{g}v'(g)]$ with $M > 0$ a constant depending on the coefficients of L, V and g. for $c = \frac{1}{M}$, $v = v_{1/M}$ and $u = v \circ g$, we then obtain the desired statement $Lu \le f(u)$. ☐

Remark 5.2. *In the Theorem 5.1, the hypothesis $c \ge 0$ can be easily replaced by c locally bounded. Indeed let $k > 0$ such that $c - k \le 0$ and $M = e^{k(b-a)}$ (V is given by $r > 0$ and $a, b \in \mathbb{R}$). Let $L_1 u = Lu - ku$, then L_1 satisfies the assumptions of the previous theorem on V, hence there exists $v \in \mathcal{C}_x^2 \cap \mathcal{C}_t^1$ such that $\lim_{y\to z} v(y) = +\infty$ for every $z \in \partial_h V$ and $L_1 v \le \frac{1}{M} f(v)$. Put $\tilde{v}(x,t) = e^{k(t-a)} v(x,t)$ fort $\in]a,b[$, we then obtain $L\tilde{v} = e^{k(t-a)}[L_1 v] \le e^{k(t-a)}\frac{1}{M} f(v) \le f(v) \le f(\tilde{v})$ (sincef is increasing).*

In what follows, we shall assume that the coefficients of L are uniformly Hölder continuous and bounded on $\mathbb{R}^d \times R$ and L is uniformly elliptic. We consider ${}^L\mathcal{H}$ the scheaf of solutions of L i.e. for every open set U in X ,${}^L\mathcal{H}(U) = \{u \in \mathcal{C}_x^2 \cap \mathcal{C}_t^1 : Lu = 0\}$. Then by e.g.[Gu] $(X, {}^L\mathcal{H})$ is a harmonic Bauer space in the sense of [CC], for every $V \in \mathcal{V}$ and $f \in \mathcal{C}(V)$ there exists a unique $u \in {}^L\mathcal{H}(V)$ with $\lim_{y\to z} u(y) = f(z)$ for every $z \in \partial_h V$. By [F] every open set U in X admits a Green function G^U. As in the elliptic case we introduce here the same Kato notions related to the section of continuous and reel potential defined by $M_U = \int G^U(\cdot, y)\lambda(dy), \lambda$ Lebesgue measure on X. Let $\varphi : X \times \mathbb{R} \to \overline{R}$ and $\psi : X \times \mathbb{R} \to \overline{R}$ with $\psi(x,y) = y\varphi(x,y)$.

Definition 5.3. *We shall say that φ is parabolic admissible if φ satisfies one of the following conditions:*

(1) φ is elliptic admissible (definition 4.3).

(2) φ is locally Kato-Lipschitzian, $\varphi(\cdot,0) \in K_M^{loc}$ and φ^- is Kato-bounded.

Again by [BBM] we have an analog of Theorem 4.4 given by the following:

Theorem 5.4. *let φ be a parabolic admissible function then $(X, \mathcal{H}_\psi)$ (defined in the same way as in the elliptic case) is a (nonlinear) Bauer space. The $\mathcal{H}_\psi$-regular sets are the same as those for L. There is a minimum (comparison) principle for regular sets and the hyperharmonic and hypoharmonic functions are given as in the Theorem 4.5.*

Proposition 5.5. *Let $f \in K, V \in \mathcal{V}$ and $v \in \mathcal{C}_x^2(V) \cap \mathcal{C}_t^1(V)$ the function determined in Theorem 5.1, then v is hyperharmonic on V for the harmonic structure given by $(X, \mathcal{H}_f)$ (i.e. $v \in \mathcal{H}_f^*(V)$)*

Proof. As in the elliptic case we consider $U \subset \overline{U} \subset V$ regular and $g_n \in \mathcal{C}^\infty(\overline{U})$ with $0 \leq g_n(x) \leq v(x) + \frac{1}{n}$ on $\overline{U}$. Let $\beta = \sup\limits_{\overline{U}} v$, $\frac{f(t)}{t}$ is continuous on $[0, \beta+1]$, there exists $(p_n) \subset \mathcal{P}$ (the set of polynomials on $\mathbb{R}$) such $\frac{f(t)}{t} < p_n(t)$ and $p_n(t)$ tends to $\frac{f(t)}{t}$ as n tends to infinity for every $t \in [0, \beta+1]$. Let G^U be the Green function associated with U and L, then there exists $(h_n) \subset \mathcal{C}^+(U)$ such that: $h_n + \int G_t^U h_n(t)|p_n(h_n(t)|)\lambda(dt) =^L H_U g_n$.${}^L H_U g$ is the solution of the Dirichlet problem associated with L ,U and g .Since h_n is bounded on V, the previous equality yields $h_n \in \mathcal{C}^1$ and hence $h_n|p_n(h_n)|$ is Hölder continuous on $\overline{U}$ and we have $Lh_n = h_n(t)|p(h_n(t))|$ .The rest of the proof is the same as in Proposition 4.5 . □

Let φ from $X \times \mathbb{R}$ to $\mathbb{R}$ be parabolic admissible.

Theorem 5.6. *Assume that there exists $f \in K$ and $M \geq 0$ such that $f(y) \leq y\varphi(x,y)$ for every $x \in X$ and $y \in [M, +\infty[$, then every $V \in \mathcal{V}$ admits a regular Evans Function u i.e. $u \in \mathcal{H}_\psi^+(V)$ with $\lim\limits_{y \to z} u(y) = +\infty$ for every $z \in \partial_h V$.*

Proof. The same as in 4.6. □

For the notion of resolutiv set the reader is referred to [CC]

Corollary 5.7. *For every L - resolutive set U in X there exists $u \in \mathcal{H}_\psi^+(U)$ such that $\lim\limits_{x \to z} u(x) = +\infty$ for every for the heat equation regular point z of the boundary of U.*

Proof. Since U is resolutive, we consider the solution ${}^L H_U(n)$ in the sense of Perron-Wiener-Brelot in $(X, {}^L\mathcal{H})$ and we consider ${}^f H_U(n) \in \mathcal{H}_f(U)$ given by

$$^L H_U(n) =^f H_U(n) + \int G^U(\cdot, z)^f H_U(n)(z) d\lambda(z).$$

${}^fH_U(n)$ is increasing and since ${}^fH_U(n) \leq^f H_V(n)$ for every $V \subset \overline{V} \subset U$ with $V \in \mathcal{V}$, we then have $0 \leq \sup^f H_U(n) < +\infty$ and then $0 \leq \sup_n {}^\psi H_U(n) < +\infty$. For every regular point z in ∂U, we have $\lim_{x\to z} {}^\psi H_U(n)(x) \geq n$ for every $n \in \mathbb{N}$, thus $u = \sup_n {}^\psi H_U(n)$ satisfies the desired statement.

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