ON THE RECONSTRUCTION OF THE DRIFT OF A DIFFUSION FROM TRANSITION PROBABILITIES WHICH ARE PARTIALLY OBSERVED IN SPACE

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ABSTRACT. The problem of reconstructing the drift of a diffusion in $\mathbb{R}^d$, $d \geq 2$, from the transition probability density observed outside a domain is considered. The solution of this problem also solves a new inverse problem for a class of parabolic partial differential equations. This work considerably extends [2] in terms of generality, both concerning assumptions on the drift coefficient, and allowing for non-constant diffusion coefficient. Sufficient conditions for solvability of this type of inverse problem for $d = 1$ are also given.

1. Introduction

Let $(x_t, \mathbb{P}^x)$ be the weak solution of the stochastic equation

$$x_t = x + \int_0^t c(x_s)\, ds + \int_0^t \sqrt{a(x_s)}\, dw_s \tag{1}$$

where c is a measurable vector field on $\mathbb{R}^d$, $d \geq 2$, a is symmetric positive-definite matrix, and $w.$ is a standard d-dimensional Brownian motion. Denoting with $p(x,t,y)$ the transition probability density of $(x_t, \mathbb{P}^x)$, our aim is to reconstruct c from the observation of $p(x,t,y)$ for $(x,y) \in \Lambda^c \times \Lambda^c$, $t \in [0,T]$, where Λ is a bounded domain in $\mathbb{R}^d$ and $T > 0$. We point out that this is an entirely different problem from the ones which are usually studied in filtering theory (or statistics of processes where the observations are partial with respect to time instead of space).

The problem we handle in the present paper was introduced and solved in [2] under the assumptions that $a(x)$ is constant, $c = \nabla\varphi$ with $\varphi \in C^2(\mathbb{R}^d)$, and under growth conditions on c such that a strong solution to (1) exists with infinite lifetime. The results were rediscussed in [5], where more detailed proofs can be found, and an approach to the case of diagonal diffusion through a random time change is sketched. We improve all these results in several directions. In particular, we impose only integrability assumptions on φ, we require only the existence of weak solutions instead of strong ones, and we allow Λ to be unbounded. Moreover, we consider much more general classes of SDEs with variable diffusion coefficients. We also obtain a partial solution of the problem in the case of the drift not being a gradient field and for one-dimensional diffusions. In their full generality, however, these latter problems were and remain unsolved (to the best of our knowledge).

The problem at hand admits a purely analytic interpretation. Namely, Kolmogorov's classical work implies that $p(x,t,\cdot)$ solves the parabolic partial differential equation

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(PDE)

$$\begin{cases} \dfrac{\partial p}{\partial t} = L^*_{c,a} p, \quad t > 0 \\ p(x,0,\cdot) = \delta_x(\cdot), \end{cases} \tag{2}$$

where $L^*_{c,a}$ is the formal adjoint of the operator

$$\begin{aligned} L_{c,a}u(x) &= \frac{1}{2}\operatorname{tr}\big(a(x)D^2u(x)\big) + \langle c(x), Du(x)\rangle \\ &= \frac{1}{2}a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + c^i(x)\frac{\partial}{\partial x_i}u(x) \end{aligned}$$

(here and throughout the paper we adopt the convention of summation over repeated indices). Therefore our problem can be reformulated as an inverse problem for PDEs: from the knowledge of the solution of (2) in Λ^c alone, determine the coefficient c of the first-order term (as a function on $\mathbb{R}^d$). To the best of our knowledge the problem has not been addressed in the literature on inverse problems for PDEs (see e.g. [8], [15]), and our solution can also be seen as a probabilistic solution to this analytic problem.

The paper is organized as follows: we collect basic assumptions, definitions, and some known results in section 2. In section 3 we derive a representation formula for the transition probabilities of diffusions whose generators satisfy certain conditions (this class, in particular, contains distorted Brownian motion – see e.g. [6]). We also prove some consequences of this representation formula that allow us to reconstruct from the transition densities of a diffusion a function of its drift coefficient (in section 4), and eventually its drift (in section 5). Section 6 deals with extensions such as unbounded Λ and one-dimensional diffusions.

2. Preliminaries

Unless otherwise stated, we shall work under the following standing assumptions:

(i) the operator $L_{c,a}$ is uniformly elliptic, i.e.

$$\langle a(x)\xi, \xi\rangle \geq \delta|\xi|^2 \quad \forall x,\, \xi \in \mathbb{R}^d,$$

for some constant $\delta > 0$;

(ii) equation (1) admits a unique weak solution in $\mathbb{R}^d$;

(iii) the transition probability measures of all considered diffusions admit a continuous density with respect to Lebesgue measure.

Remark 1. By a classical result of Zvonkin and Krylov [22], a weak solution of equation (1) exists and is unique if the coefficients are bounded and a is continuous. A detailed study of conditions implying that a (generalized) diffusion has continuous probability densities can be found in [10] (see also [18] and [9]).

We shall make use of some Banach spaces. We denote by $L^p := L^p(\mathbb{R}^d;\mathbb{R})$, for $p \in [1,+\infty[$, the space of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$|f|^p_{L^p} := \int_{\mathbb{R}^d} |f(x)|^p\, dx < \infty.$$

For $s \in \mathbb{R}$, let $H^{p,s} = (1-\Delta)^{-s/2}L^p$ be the usual space of Bessel potentials on $\mathbb{R}^d$. The norm in $H^{p,2}$ can be taken to be equivalent to the one of $W^{p,2}$, the usual Sobolev space of function with (generalized) derivatives up to order 2 in L^p. $H^{p,s}_{loc}$ is the space of functions f such that $f\zeta \in H^{p,s}$ for all $\zeta \in C^\infty_0(\mathbb{R}^d)$. The following Sobolev embedding theorem holds: $f \in H^{p,s}_{loc}(\mathbb{R}^d)$ implies that $f \in C^\alpha(\mathbb{R}^d)$, $\alpha := s - d/p$. In particular $f \in H^{d,2}_{loc}$ implies $f \in C^1(\mathbb{R}^d)$.

Let us briefly recall some results from pinned diffusions connected to representation of transition probability densities of diffusions (see [17] and [19] for more details). Let $(x_t, \mathbb{P}^x)$ be a diffusion process on $\mathbb{R}^d$, endowed with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Define a probability measure $\mathbb{P}_T^{x,y}$ on $\sigma(\mathcal{F}_t : t < T)$ by

$$\frac{d\mathbb{P}_T^{x,y}}{d\mathbb{P}^x}\bigg|_{\mathcal{F}_t} = \frac{p(x_t, T-t, y)}{p(x,T,y)} \quad \forall t < T.$$

The diffusion process $((x_t)_{t\leq T}, \mathbb{P}_T^{x,y})$ is called a pinned diffusion, or a diffusion conditioned on $x_0 = x$ and $x_T = y$.
The following Cameron-Martin formula for pinned diffusions was proved in [19].

Lemma 2. *Let $(x_t, \mathbb{P}^x)$ be a diffusion process with generator*

$$L_{b,a} f = \frac{1}{2}\operatorname{tr}(a(x)D^2 f(x)) + \langle b(x), Df(x)\rangle.$$

Let $c : \mathbb{R}^d \to \mathbb{R}$ be measurable and such that

$$\mathbb{E}^x\Big[\exp\Big(\frac{1}{2}\int_0^T |a^{-1/2}(x_s)c(x_s)|^2\, ds\Big)\Big] < \infty.$$

Let p_b and p_{b+c} be the transition density functions corresponding to the diffusions with generators $L_{b,a}$ and $L_{b+c,a}$, respectively. Suppose that

$$y \mapsto \mathbb{E}_T^{x,y}\Big[\exp\Big(\int_0^T \langle a^{-1/2}(x_t)c(x_t), dw_t\rangle - \frac{1}{2}\int_0^T |a^{-1/2}(x_t)c(x_t)|^2\, dt\Big)\Big]$$

is continuous. Then one has

$$\frac{p_{b+c}(x,T,y)}{p_b(x,T,y)} = \mathbb{E}_T^{x,y}\Big[\exp\Big(\int_0^T \langle a^{-1/2}(x_t)c(x_t), dw_t\rangle - \frac{1}{2}\int_0^T |a^{-1/2}(x_t)c(x_t)|^2\, dt\Big)\Big]. \tag{3}$$

3. Transition densities for a class of diffusions

From now on we assume that the hypotheses of Lemma 2 are satisfied and that there exists a function $\psi : \mathbb{R}^d \to \mathbb{R}$, $\psi \in H^{d,2}$, such that $\nabla\psi(x) = a^{-1}(x)c(x)$. This assumption, even though it looks quite restrictive, contains many important examples. Let us consider, for instance, the case of unit diffusion coefficient, i.e. Markov processes with generator $Lu = \frac{1}{2}\Delta u + \langle c, \nabla u\rangle$. Assume that c is regular and L admits an infinitesimally invariant measure $\nu(dx) = \rho(x)\, dx$, i.e. that

$$\int_{\mathbb{R}^d} Lf\, \nu(dx) = 0 \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

(see e.g. [4] for more details on these notions). It is well known that the diffusion is reversible if and only if L is symmetric in $L^2(\mathbb{R}^d, \nu)$. One can also prove that L is symmetric in $L^2(\mathbb{R}^d, \nu)$ if and only if $2c = \nabla(\log\rho)$. Therefore, our problem of drift reconstruction can be solved for a large class of reversible diffusions. However, the general case of c not being a gradient field is unfortunately not within the reach of our method, except for some special situations discussed in the last section.

In this section we shall specialize Lemma 2 to the class of diffusions just introduced, and deduce some important corollaries.

Proposition 3. *One has*

$$p_{b+c}(x,t,y) = p_b(x,t,y)\exp\big(\psi(y) - \psi(x)\big)\,\mathbb{E}_t^{x,y}\Big[\exp\Big(-\int_0^t V(x_s)\, ds\Big)\Big],$$

where $V(x) := L_{b,a}\psi(x) + \frac{1}{2}|a^{1/2}(x)\nabla\psi(x)|^2$.

Proof. By an application of Itô's lemma (as in [16]) we get

$$\begin{aligned}\psi(x_t)-\psi(x_0) &= \int_0^t \langle \nabla\psi(x_s), a^{1/2}(x_s)\,dw_s\rangle + \int_0^t L_{b,a}\psi(x_s)\,ds \\ &= \int_0^t \langle a^{-1/2}(x_s)c(x_s), dw_s\rangle + \int_0^t L_{b,a}\psi(x_s)\,ds.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^t \langle a^{-1/2}(x_s)c(x_s), dw_s\rangle - \frac{1}{2}\int_0^t |a^{-1/2}(x_s)c(x_s)|^2\,ds \\ &= \psi(x_t)-\psi(x_0) - \int_0^t L_{b,a}\psi(x_s)\,ds - \frac{1}{2}\int_0^t |a^{-1/2}(x_s)c(x_s)|^2\,ds \\ &= \psi(x_t)-\psi(x_0) - \int_0^t L_{b,a}\psi(x_s)\,ds - \frac{1}{2}\int_0^t |a^{1/2}(x_s)\nabla\psi(x_s)|^2\,ds \\ &= \psi(x_t)-\psi(x_0) - \int_0^t V(x_s)\,ds,\end{aligned}$$

By a simple rewriting of (3) we get the desired result. □

If $b=0$, $a=I$, i.e. x_t is Brownian motion, we recover a formula already obtained in [2], although under more regularity assumptions:

Corollary 4. *Assume that c satisfies the hypotheses of Lemma 2 with $b=0$ and $a=I$. Setting $V(x)=\frac{1}{2}(|\nabla\psi(x)|^2+\Delta\psi(x))$, one has*

$$p_c(x,t,y) = p_0(x,t,y)\exp\big(\psi(y)-\psi(x)\big)\,\mathbb{E}_t^{x,y}\Big[\exp\Big(-\int_0^t V(w_s)\,ds\Big)\Big],$$

where $p_0(x,t,y)$ is the transition probability density of Brownian bridge.

Given a domain $\Lambda\subset\mathbb{R}^d$ and x, $y\in\partial\Lambda$, we shall denote by $(\gamma_x^{xy})_{s\in[0,t]}$ the (straight) line joining x with y in time t, i.e. the function

$$\gamma_s^{xy} = x+(y-x)\frac{s}{t}, \quad 0\le s\le t.$$

The following proposition will be crucial for the solution of our problem. First, let us observe that the elementary relation

$$\int_0^t V(\gamma_s^{xy})\,ds = t\int_0^1 V(x+(y-x)s)\,ds$$

holds.

Proposition 5. *Assume that V is bounded from below and satisfies the following property:*

$$V\circ\gamma^{xy}\in L^1(\mathbb{R};\mathbb{R}),\ |\gamma_n-\gamma^{xy}|_{L^\infty(\mathbb{R};\mathbb{R}^d)}\to 0 \ \Rightarrow\ |V\circ\gamma_n - V\circ\gamma^{xy}|_{L^1(\mathbb{R};\mathbb{R})}\to 0,$$

where γ_n are continuous curves in $\mathbb{R}^d$ with endpoints x, y. Then $V\circ\gamma^{xy}\in L^1(\mathbb{R};\mathbb{R})$ for almost all x, y (with respect to Lebesgue measure) and one has

$$\lim_{t\to 0}\frac{p_{b+c}(x,t,y)}{p_b(x,t,y)} = e^{\psi(y)-\psi(x)} \tag{4}$$

and

$$\lim_{t\to 0}\frac{1}{t}\Big(\log\frac{p_{b+c}(x,t,y)}{p_b(x,t,y)} - (\psi(y)-\psi(x))\Big) = -\int_0^1 V(x+(y-x)s)\,ds. \tag{5}$$

Proof. It clearly follows from $\psi \in H^{d,2}$ that $V \in L^d$, hence $V_\Lambda := V 1_\Lambda \in L^1$. Therefore, by Fubini's theorem, $\int_{\gamma^{xy}} V_\Lambda(s)\,ds < \infty$ for almost all $x,\, y \in \mathbb{R}^d$, as desired. We claim that

$$\lim_{t\to 0} \frac{\mathbb{E}^{x,y}_t\Big[\exp\Big(-\int_0^t V(x_s)\,ds\Big)\Big]}{\exp\Big(-\int_0^t V(\gamma^{xy}_s)\,ds\Big)} = 1.$$

Since one has, as a consequence of $V \circ \gamma^{xy} \in L^1$ (in this proof L^1 stands for $L^1(\mathbb{R};\mathbb{R})$),

$$\lim_{t\to 0} -\int_0^t V(\gamma_s)\,ds = -\lim_{t\to 0} t \int_0^1 V(x + (y-x)s)\,ds = 0,$$

we just need to prove that

$$\lim_{t\to 0} \mathbb{E}^{x,y}_t\Big[\exp\Big(-\int_0^t V(x_s)\,ds\Big)\Big] = 1.$$

In fact, we observe that for any constant $\varepsilon > 0$, the continuity of the paths of $x_\cdot$ implies that

$$\lim_{t\to 0} \mathbb{P}^{x,y}_t\Big(|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} \geq \varepsilon\Big) = 0,$$

where L^∞ stands for $L^\infty([0,t];\mathbb{R}^d)$. Moreover,

$$\begin{aligned}
&\mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big] \\
=\;& \mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} < \varepsilon\Big] \\
&+ \mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} \geq \varepsilon\Big]
\end{aligned}$$

Using the elementary inequality $|e^x - 1| \leq 1 \vee e^x$, we can write

$$\begin{aligned}
\Big|\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big| &\leq 1 \vee \exp\Big(\int_0^t -V(x_s)\,ds\Big) \\
&\leq 1 \vee e^{t(-\inf V)},
\end{aligned}$$

hence

$$\begin{aligned}
&\lim_{t\to 0}\Big|\mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} \geq \varepsilon\Big]\Big| \\
\leq\;& \lim_{t\to 0} \mathbb{E}^{x,y}_t\Big[\Big|\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big|\;\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} \geq \varepsilon\Big] \\
\leq\;& \lim_{t\to 0}\Big(1 \vee e^{t(-\inf V)}\Big)\, \mathbb{P}^{x,y}_t\big(|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} \geq \varepsilon\big) = 0.
\end{aligned}$$

Similarly, using the elementary inequality $|e^x - 1| \leq e^{|x|} - 1$, we have

$$\begin{aligned}
&\lim_{t\to 0}\Big|\mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t -V(x_s)\,ds\Big) - 1\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} < \varepsilon\Big]\Big| \\
\leq\;& \lim_{t\to 0} \mathbb{E}^{x,y}_t\Big[\exp\Big(\int_0^t |-V(x_s) + V(\gamma_s)|\,ds\Big)\Big|\,|x_\cdot - \gamma^{xy}_\cdot|_{L^\infty} < \varepsilon\Big] e^{t|V\circ\gamma^{xy}|_{L^1}} - 1 \\
\leq\;& \lim_{t\to 0} e^{\delta_\varepsilon} e^{t|V\circ\gamma^{xy}|_{L^1}} - 1 = 0,
\end{aligned}$$

where we have used the following immediate consequence of the hypotheses: for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\int_0^t |V(\gamma_s^{xy}) - V(x_s)|\, ds = |V \circ \gamma^{xy} - V \circ x|_{L^1} < \delta_\varepsilon$$

whenever $|x_. - \gamma_.^{xy}|_{L^\infty} < \varepsilon$. This concludes the proof of the claim. Assertion (4) now follows immediately from the claim just proved and the previous proposition. Assertion (5) follows by

$$\frac{\mathbb{E}_t^{x,y}\Big[\exp\Big(\int_0^t -V(x_s)\, ds\Big)\Big]}{\exp\Big(\int_0^t -V(\gamma_s^{xy})\, ds\Big)} = 1 + o(t)$$

for $t \to 0$, hence

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\Big(\log \frac{p_{b+c}(x,t,y)}{p_b(x,t,y)} - \big(\psi(y) - \psi(x)\big)\Big) \\
= \ &\lim_{t\to 0} \frac{1}{t} \log \mathbb{E}_t^{x,y}\Big[\exp\Big(\int_0^t -V(x_s)\, ds\Big)\Big] \\
= \ &\lim_{t\to 0} \frac{1}{t} \log\Big(\exp\Big(\int_0^t -V(\gamma_s)\, ds\Big) + o(t)\Big) \\
= \ &\lim_{t\to 0} \frac{1}{t} \log \exp\Big(t \int_0^1 -V(x + (y-x)s)\, ds\Big) \\
= \ &\int_0^1 -V(x + (y-x)s)\, ds,
\end{aligned}$$

which finishes the proof. □

Remark 6. If $x_.$ is Brownian motion, Aizenman and Simon [1] proved that V is of Kato class if and only if

$$\lim_{t\to 0} \sup_x \mathbb{E}^x\Big[\int_0^t |V(x_s)|\, ds\Big] = 0,$$

without assuming that V is lower bounded. Using Kasminskii's lemma (see e.g. [18]) it is then immediate to deduce that if V is of Kato class, then one also has

$$\lim_{t\to 0} \sup_x \mathbb{E}^x\Big[\exp\Big(\int_0^t |V(x_s)|\, ds\Big)\Big] = 1. \tag{6}$$

Moreover, Chung and Zhao proved that (6) continues to hold for any Hunt process $x_.$, under the only assumption that V is of Kato class (see [11], Proposition 3.8). We also refer to [3] and [7] for related results on generalized Schrödinger operators and forms and associated Markov processes.

4. Reconstruction of V

In this section we assume that the hypotheses of Lemma 2 and Proposition 5 are satisfied.

We shall then show that the transition probabilities of $(x_t, \mathbb{P}^x)$ determine the X-ray transform of V_Λ (in the sense of, e.g., [14]), which in turn yields V_Λ by a Fourier transform argument. In particular, from equations (4) and (5) of Proposition 5 it immediately follows that

$$F(x,y) := \int_0^1 V(x + (y-x)s)\, ds, \quad x, y \in \partial\Lambda$$

is determined by the transition probabilities of $(x_t, \mathbb{P}^x)$. Moreover, one immediately recognizes that F is the X-ray transform of V_Λ. Let us fix some notation: we represent a line γ as a pair $\gamma = (\omega, z)$, where $\omega \in \mathbb{S}^{n-1}$ is a unit vector in the direction of γ and $z \in \gamma \cap \omega^{\perp}$. Then the line integral $\int_\gamma f(x)\,dx$ is denoted by

$$\hat{f}(\gamma) = \hat{f}(\omega, z) = P_\omega f(z).$$

As argued before, since we know that $V_\Lambda \in L^1$, Fubini's theorem implies that for each $\omega \in \mathbb{S}^{d-1}$, $P_\omega V_\Lambda(z)$ is defined for almost all $z \in \omega^\perp$. Moreover, we have, for $p \in \omega^\perp$,

$$\tilde{V}_\Lambda(p) = \int_{\omega^\perp} P_\omega V_\Lambda(z) e^{i\langle p,z\rangle}\,dz$$

(a result often called slice-projection theorem, see e.g. [14]). One uniquely recovers V_Λ by taking the inverse Fourier transform of $\tilde{V}_\Lambda(p)$. Summarizing, we have proved the following

Theorem 7. *The restriction of V to the domain Λ can be uniquely reconstructed from the transition probabilities of $(x_t, \mathbb{P}^x)$.*

Remark 8. If our only aim were to reconstruct the function V_Λ from its X-ray or Radon transform, even more generality could be allowed, up to the situation where V is a distribution. For results on inverting the Radon transform of a distribution, see e.g. [12], [20], [14].

5. Reconstruction of the drift

As in the previous section, we assume that the hypotheses of Lemma 2 and Proposition 5 are satisfied.

Set $u(x) = e^{\psi(x)-\psi(y)}$, where y is any (fixed) point on the boundary of Λ. Then u satisfies the following Dirichlet boundary value problem for a second-order elliptic operator:

$$\begin{cases} \frac{1}{2}a^{ij}u_{x_ix_j} + b^i u_{x_i} = V(x)u, & x \in \Lambda \\ u(x) = e^{\psi(x)-\psi(y)}, & x \in \partial\Lambda. \end{cases} \tag{7}$$

This is easily seen as a consequence of the definition of V and of the following simple calculations:

$$\begin{aligned} u_{x_i} &= u\psi_{x_i} \\ u_{x_ix_j} &= u_{x_j}\psi_{x_i} + u\psi_{x_ix_j} = u(\psi_{x_j}\psi_{x_i} + \psi_{x_ix_j}) \\ a^{ij}u_{x_ix_j} &= \big(a^{ij}\psi_{x_ix_j} + |a^{1/2}\nabla\psi|^2\big)u, \end{aligned}$$

where the last step is justified by

$$a^{ij}\psi_{x_j}\psi_{x_i} = \langle a\nabla\psi, \nabla\psi\rangle = \langle a^{1/2}\nabla\psi, a^{1/2}\nabla\psi\rangle = |a^{1/2}\nabla\psi|^2.$$

If (7) is uniquely solvable, then we are able to recover $\psi(x)$ for $x \in \Lambda$ uniquely. In fact we have:

Proposition 9. *Suppose a^{ij} are differentiable and $V \in L^\infty_+(\Lambda)$. Then there exists a unique solution $u \in H^{2,1}(\Lambda)$ of the Dirichlet problem (7).*

Proof. Since $\psi \in C^1(\Lambda)$, as follows by Sobolev embeddings, then $f(x) := e^{\psi(x)-\psi(y)} \in C^1(\Lambda) \subset H^{2,1}(\Lambda)$. Moreover, $L_{b,a}$ is strictly elliptic and $b = \nabla\psi \in C(\Lambda)$, hence b is bounded. We can now appeal to Theorem 8.9 of [13], which yields the existence and uniquess of a solution to (7), as claimed. ☐

Remark 10. Using more general results on elliptic PDEs, one can remove the unpleasant assumption of V being bounded, at the cost of added technicalities. In particular, using the existence and uniquess results of [21], one can replace $V \in L^\infty(\Lambda)$ by $g \in L^{d/2}$ in the hypotheses of Proposition 9, where

$$g := (a^{-1})_{ij}(\tilde{b}^i + \tilde{b}^j), \quad \tilde{b}^j := b^j - \frac{1}{2}a^{ij}_{x_i}.$$

The details (mostly calculations) are left to the reader. The assumption $V \geq 0$ is used to guarantee that the spectrum of the operator $L_{b,a} - V(x)$ (considered between appropriate spaces of integrable functions) does not contain zero. If we are willing to accept this level of generality, sacrificing a bit of concreteness, then we can dispense with the assumption of V being positive, and simply assume that zero is not an eigenvalue of $L_{b,a} - V(x)$. For further details we refer to [13] and [21], where a Fredholm alternative for this type of operators is established.

If a is the identity matrix, hence $c = \nabla\psi$, we can obtain stronger results. In particular, the Dirichlet problem (7) reduces to the Dirichlet problem for the time-independent Schrödinger operator with Hamiltonian $-\frac{1}{2}\Delta + V$:

$$\begin{cases} \frac{1}{2}\Delta u = V(x)u, & x \in \Lambda \\ u(x) = e^{\psi(x)-\psi(y)}, & x \in \partial\Lambda, \end{cases} \tag{8}$$

for which there exists a rich literature. We can apply, for instance, Theorem 4.7 of [11] (see also [1]). We shall denote by τ_Λ the first exit time of Brownian motion from the domain Λ.

Theorem 11. *Assume that V is of local Kato class, Λ is bounded and regular, and*

$$x \mapsto \mathbb{E}^x\Big[\exp\Big(-\int_0^{\tau_\Lambda} V(w_s)\,ds\Big)\Big] \tag{9}$$

is bounded. Then there exists a unique solution $u \in C(\overline{\Lambda})$ of the boundary value problem (8).

Proof. The conditions on V and Λ are needed in order to apply the above mentioned results of [11]. Moreover, since $\psi \in C^1$ as follows by Sobolev embeddings, and thus $f(x) := e^{\psi(x)-\psi(y)} \in C(\partial\Lambda)$, part (iv) of Theorem 4.7 in *ibid.* ensures that

$$u(x) = \mathbb{E}^x\Big[\exp\Big(-\int_0^{\tau_\Lambda} V(w_s)\,ds\Big)f(w_{\tau_\Lambda})\Big]$$

is the unique solution of $(-\frac{1}{2}\Delta + V)u = 0$ such that $u \in C(\overline{\Lambda})$ and $u(x) = e^{\psi(x)-\psi(y)}$ on $\partial\Lambda$. □

Remark 12. A simple sufficient condition guaranteeing that V is of Kato class is $V \in L^{p/2}$, $p > d/2$ (see e.g. [1], p. 233). Therefore, if $\psi \in H^{p,2}$ with $p > d$, V is of Kato class. On the other hand, we were unable to find simple sufficient conditions ensuring that (9) is bounded. Let us mention, however, that each of the following analytic conditions is sufficient:

(i) $\int_0^\infty T_t 1\,dt$ is bounded;
(ii) $-\frac{1}{2}\Delta + V \geq 0$ (in the sense of operators), or equivalently:
(iii) The spectrum of $-\frac{1}{2}\Delta + V$ is contained in $]0, +\infty[$,

where we have denoted by T_t the semigroup generated by $-\frac{1}{2}\Delta + V$. For more informations see [1] and [11], p. 126.

Once u is obtained, one recovers ψ immediately, and hence c. We have proved the following result

Theorem 13. *Assume that the boundary value problem (7) admits a unique solution. Then the transition probabilities $p_{b+c}(x,t,y)$ for x, $y \in \Lambda^c$, $t \in [0,T]$, $T > 0$, determine c uniquely.*

Proof. It is just a combination of the previous steps. In particular, one proceeds as follows:

(1) Obtain the X-ray transform of V_Λ from the transition probabilities $p_{b+c}(x,t,y)$;
(2) Invert the X-ray transform obtaining V_Λ;
(3) Solve the elliptic PDE (7) obtaining $\psi(x) = \log u(x) + \psi(y)$;
(4) Obtain $c = a\nabla\psi$.

□

6. Some extensions

6.1. $a^{-1}c$ **not a gradient field.** It is clear that our approach strongly relies on the assumption that the $a^{-1}c$ is a gradient field. When this is not the case, we can only give a rather involved sufficient condition to reduce the problem to a more tractable one. We shall assume for simplicity that $x.$ is a $L_{c,I}$ diffusion, without knowing a priori that c is a gradient field. We also assume that the technical assumptions introduced so far are in place when needed.

Proposition 14. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a C^2 diffeomorphism mapping Λ into a bounded domain, and such that*

$$\left[\nabla f(f^{-1}(x))\nabla f(f^{-1}(x))^*\right]^{-1} L_{c,I} f(f^{-1}(x)) = \nabla\psi(x) \tag{10}$$

for some $\psi : \mathbb{R}^d \to \mathbb{R}$. Then the transition probabilities of $(x_t, \mathbb{P}^x)$ uniquely determine the drift c.

Proof. Itô's formula for $f_i(x_t)$, the i-th component of f, gives

$$f_i(x_t) - f_i(x_0) = \int_0^t \langle \nabla f_i(x_s), dw_s\rangle + \int_0^t L_{c,I} f_i(x_s)\, ds,$$

that is

$$f(x_t) - f(x_0) = \int_0^t \nabla f(x_s)\, dw_s + \int_0^t L_{c,I} f(x_s)\, ds,$$

or equivalently, defining $y. = f(x.)$

$$y_t = y + \int_0^t L_{c,I} f(f^{-1}(y_s))\, ds + \int_0^t \nabla f(f^{-1}(y_s))\, dw_s. \tag{11}$$

The hypotheses imply that one can reconstruct the drift and the transition probabilities of the diffusion (11). But this is enough to recover the transition probabilities of $x.$ as well, as the following obvious identities show:

$$\mathbb{P}\big(x_t = y | x_0 = x\big) = \mathbb{P}\big(f^{-1}(y_t) = y | f^{-1}(y_0) = x\big) = \mathbb{P}\big(y_t = f(y) | y_0 = f(x)\big).$$

It is well known that the transition probabilities of an $L_{c,I}$-diffusion determine the drift coefficient c uniquely. □

6.2. **Unbounded** Λ**.** We assumed Λ to be bounded in order to obtain existence and uniqueness of solutions for the Dirichlet problems (7) and (8). However, in some cases this assumption can be relaxed. For instance, imposing enough boundedness of the coefficients in (7), one can obtain existence and uniqueness results without assuming that Λ is bounded.

Proposition 15. *Assume that* a^{ij} *are differentiable,* $e^{-|x|^2}\|a(x)\| \to 0$ *for* $|x| \to \infty$*, and* V *is non-negative (in the generalized sense). Then there exists a unique solution of (7).*

Proof. Let $v(x) = u(x)e^{\varphi(x)} = e^{\psi(x)+\varphi(x)}$, $\varphi(x) = -|x|^2$. Since $\psi \in H^{d,2}$, hence $\psi \in C^1$ by Sobolev embedding, ψ is bounded on Λ, and so is u. Moreover, it is immediate to show that $v \in H^{2,1}(\Lambda)$. Then one has

$$L_{b-2a\nabla\varphi,a}v = L_{b,a}v - \langle 2a\nabla\varphi, \nabla v\rangle,$$

and $\nabla v = (\nabla\psi + \nabla\varphi)v$,

$$\begin{aligned} L_{b,a}v &= \big(L_{b,a}(\psi+\varphi) + |a^{1/2}\nabla(\psi+\varphi)|^2\big)v \\ &= \big(L_{b,a}\psi + |a^{1/2}\nabla\psi|^2 + L_{b,a}\varphi + |a^{1/2}\nabla\varphi|^2 + 2\langle a^{1/2}\nabla\varphi, a^{1/2}\nabla\psi\rangle\big)v \\ &= \Big(\frac{1}{2}\operatorname{tr}(a\psi_{xx}) + \langle b + 2a\nabla\varphi, \nabla\psi\rangle + |a^{1/2}\nabla\psi|^2 + L_{b,a}\varphi + |a^{1/2}\nabla\varphi|^2\Big)v, \end{aligned}$$

therefore

$$\begin{aligned} L_{b-2a\nabla\varphi,a}v &= L_{b,a}v - \langle 2a\nabla\varphi, \nabla v\rangle \\ &= L_{b,a}v - \langle 2a\nabla\varphi, \nabla\psi + \nabla\varphi\rangle v \\ &= L_{b,a}v - \langle 2a\nabla\varphi, \nabla\psi\rangle v - 2|a^{1/2}\nabla\varphi|^2 v \\ &= \Big(\frac{1}{2}\operatorname{tr}(a\psi_{xx}) + \langle b, \nabla\psi\rangle + |a^{1/2}\nabla\psi|^2 + L_{b,a}\varphi - |a^{1/2}\nabla\varphi|^2\Big)v \\ &= \big(V(x) + d(x)\big)v, \end{aligned}$$

where $d := L_{b,a}\varphi - |a^{1/2}\nabla\varphi|^2$. That is, v solves the equation

$$L_{b-2a\nabla\varphi,a}v = \big(V(x) + d(x)\big)v, \tag{12}$$

with boundary condition $v(x) = f(x)e^{\varphi(x)}$ on $\partial\Lambda$. One can apply now Theorem 8.9 of [13] to determine existence and uniqueness of a solution of (12) in $H^{2,1}$. In fact, if $L_{b,a}$ is strictly elliptic, so is also $L_{b-2a\nabla\varphi,a}$, and the continuity and growth condition on a imply that $b - 2a\nabla\varphi$ is bounded. Finally, there is no loss of generality in assuming that $V \geq 0$ implies $V(x) + d(x) \geq 0$: if it were not so, we could make $|d|$ arbitrarily small by using as cut-off function $\varphi(x) = e^{-\kappa|x|^2}$, without altering the previous results. Then the unique solution to (7) is given by $u(x) = v(x)e^{-\varphi(x)}$. □

In the case of unit diffusion a stronger statement can be made, as follows by the results in Chapter 5 of [11].

Proposition 16. *Let* $d \geq 3$*,* $V \in L^1(\Lambda)$ *and Kato class. If* $u_f \not\equiv \infty$ *in* Λ*, then the solution of (8) is given by*

$$u_f(x) = \mathbb{E}^x\bigg[\exp\Big(-\int_0^{\tau_\Lambda} V(w_s)\,ds\Big)f(w_{\tau_\Lambda})\bigg].$$

Proof. As before, $\psi \in H^{d,2}$ implies $\psi \in C^1$, hence that ψ is bounded on $\partial\Lambda$, and also that $f(x) := e^{\psi(x)-\psi(y)} \in L^\infty_+(\partial\Lambda)$. Then by Theorem 5.18 and 5.19 of [11] we obtain that u_f solves $(-\frac{1}{2}\Delta + V)u = 0$ and $u_f \in C_b(\overline{\Lambda})$. Finally, u_f satisfies the boundary condition as an immediate consequence of its definition and continuity in the closure of Λ. □

6.3. **One-dimensional case.** Using the considerations of the previous subsections it is possible to give a solution, at least in some special cases, to the problem posed in [2] of reconstructing the drift of a one-dimensional diffusion with $a = 1$. In particular, let us assume that the transition probability density $p_1(x, t, y)$ of the diffusion

$$dx_1(t) = c_1(x_1(t))\, dt + dw_1(t)$$

is known for x, $y \in]0, 1[^c$, $t \in [0, T]$, with T a fixed positive constant. As before, our aim is to determine $c_1(x)$ for $x \in [0, 1]$.
Define an $\mathbb{R}^2$-valued diffusion as weak solution of the following SDE:

$$\begin{cases} dx_1(t) = c_1(x_1(t))\, dt + dw_1(t) \\ dx_2(t) = c_2(x_2(t))\, dt + dw_2(t), \end{cases} \tag{13}$$

where c_2 is a smooth, bounded function, and w_1, w_2 are standard independent Brownian motions. In more compact notation, we can write

$$dx(t) = c(x(t))\, dt + dw(t),$$

with $c(x_1, x_2) = (c_1(x_1), c_2(x_2))$, $w = (w_1, w_2)$. It is clear that one can recover the transition probabilities of x_1 from those of x, since x_1 and x_2 are independent.
Trying to reconstruct the vector field c (which is trivially a gradient field: $c = \nabla\psi$, $\psi(x_1, x_2) = \int_0^{x_1} c_1(s)\, ds + \int_0^{x_2} c_2(s)\, ds$), one is faced with two problems: $\Lambda =]0, 1[\times\mathbb{R}$ is unbounded, and V_Λ is not in L^1. One can try, however, to consider the problem of drift reconstruction for $y_\cdot = f(x_\cdot)$, with f a C^2 diffeomorphism.

Proposition 17. *Assume that there exist $c_2 : \mathbb{R} \to \mathbb{R}$ and a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (10) and such that the corresponding Dirichlet problem (7) admits a unique solution on $f(\Lambda)$. Then c_1 can be uniquely reconstructed from the transition probabilities of $x_1(\cdot)$.*

Proof. The transition probabilities of $x_1(\cdot)$ uniquely determine the transition probabilities of $x(\cdot)$ outside the (unbounded) domain $\Lambda =]0, 1[\times\mathbb{R}$. If f exists such that (10) holds, then the the problem of reconstructing the drift of $y(\cdot) := f(x(\cdot))$ is well posed, and it is equivalent (under the technical assumptions introduced in the previous sections) to the solvability of the Dirichlet problem (7) on the domain $f(\Lambda)$. Therefore, assuming the latter problem admits a unique solution, this yields the transition probabilities of $y(\cdot)$, hence those of $x(\cdot)$ because f is a bijection. As already observed, the transition probabilities of a diffusion with generator $L_{c,I}$ uniquely determine c. $\square$

It is clear that the last proposition is not constructive and simply gives sufficient conditions for the solvability of the problem of drift reconstruction in dimension one. As in the case of higher dimensional diffusions with $a^{-1}c$ not being a gradient field, these sufficient conditions seem difficult to check. However, since we essentially rely on the above described representation with the drift being a gradient field, this seems to be the best we can achieve by our present method.

References

1. M. Aizenman and B. Simon, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), no. 2, 209–273. MR **MR644024 (84a:**35062)
2. S. Albeverio, Ph. Blanchard, S. Kusuoka, and L. Streit, *An inverse problem for stochastic differential equations*, J. Statist. Phys. **57** (1989), no. 1-2, 347–356. MR **91d:**60131
3. S. Albeverio, Ph. Blanchard, and Z. M. Ma, *Feynman-Kac semigroups in terms of signed smooth measures*, Random partial differential equations (Oberwolfach, 1989), Internat. Ser. Numer. Math., vol. 102, Birkhäuser, Basel, 1991, pp. 1–31. MR **MR1185735 (93i:**60140)

4. S. Albeverio, V. Bogachev, and M. Röckner, *On uniqueness of invariant measures for finite- and infinite-dimensional diffusions*, Comm. Pure Appl. Math. **52** (1999), no. 3, 325–362. MR **MR1656067 (2000a:**60111)
5. S. Albeverio and S. Hartmann, *A spatial inverse problem for stochastic differential equations*, Preprint, 2003.
6. S. Albeverio, Yu. Kondratiev, and M. Röckner, *Strong Feller properties for distorted Brownian motion and applications to finite particle systems with singular interactions*, Finite and infinite dimensional analysis in honor of Leonard Gross (New Orleans, LA, 2001), Amer. Math. Soc., Providence, RI, 2003, pp. 15–35. MR **MR1966885 (2004c:**60221)
7. S. Albeverio and Z. M. Ma, *Perturbation of Dirichlet forms—lower semiboundedness, closability, and form cores*, J. Funct. Anal. **99** (1991), no. 2, 332–356. MR **MR1121617 (92i:**47039)
8. Yu. Ya. Belov, *Inverse problems for partial differential equations*, VSP, Utrecht, 2002. MR **2004j:**35293
9. V. I. Bogachev, G. Da Prato, and M. Röckner, *Existence of solutions to weak parabolic equations for measures*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 753–774. MR **MR2044056 (2004k:**60208)
10. V. I. Bogachev, N. V. Krylov, and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations **26** (2001), no. 11-12, 2037–2080. MR **MR1876411 (2002m:**60117)
11. K. L. Chung and Z. Zhao, *From Brownian motion to Schrödinger's equation*, Springer-Verlag, Berlin, 1995. MR **MR1329992 (96f:**60140)
12. I. M. Gel′fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized functions. Vol. 5 – integral geometry and representation theory*, Academic Press, New York, 1966. MR 55 #8786e
13. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR **2001k:**35004
14. S. Helgason, *The Radon transform*, second ed., Birkhäuser, Boston, MA, 1999. MR **MR1723736 (2000m:**44003)
15. V. Isakov, *Inverse problems for partial differential equations*, Springer-Verlag, New York, 1998. MR **99b:**35211
16. N. V. Krylov, *Itô's stochastic integral equations*, Theor. Probability Appl. **14** (1969), 330–336. MR MR0270462 (42 #5350)
17. T. J. Lyons and W. A. Zheng, *On conditional diffusion processes*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 3-4, 243–255. MR **MR1069520 (91m:**60148)
18. N. I. Portenko, *Generalized diffusion processes*, Naukova Dumka, Kiev, 1982, In Russian. English translation: American Mathematical Society, Providence, RI, 1990. MR **MR1104660 (92b:**60074)
19. Z. Qian and W. Zheng, *A representation formula for transition probability densities of diffusions and applications*, Stochastic Process. Appl. **111** (2004), no. 1, 57–76. MR 2 049 569
20. A. G. Ramm and A. I. Katsevich, *The Radon transform and local tomography*, CRC Press, Boca Raton, FL, 1996. MR **MR1384070 (97g:**44009)
21. N. S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa (3) **27** (1973), 265–308. MR 51 #6113
22. A. K. Zvonkin and N. V. Krylov, *Strong solutions of stochastic differential equations*, Proceedings of the School and Seminar on the Theory of Random Processes (Druskininkai, 1974), Part II (Russian), Inst. Fiz. i Mat. Akad. Nauk Litovsk. SSR, Vilnius, 1975, English translation: Selecta Math. Sovietica 1(1), 19–61, 1981, pp. 9–88. MR MR0426154 (54 #14100)

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