Euclidean Gibbs measures of quantum crystals: existence, uniqueness and a priori estimates

Sergio Albeverio[1], Yuri Kondratiev[2], Tatiana Pasurek[2], and Michael Röckner[2]

[1] Institut für Angewandte Mathematik, Universität Bonn, D-53155
[2] Fakultät für Mathematik, Universität Bielefeld, D-33615

Abstract. We give a review of recent results obtained by the authors on the existence, uniqueness and a priori estimates for Euclidean Gibbs measures corresponding to quantum anharmonic crystals. Especially we present a new method to prove existence and a priori estimates for Gibbs mesures on loop lattices, which is based on the alternative characterization of Gibbs measures in terms of their logarithmic derivatives through integration by parts formulas. This method allows us to get improvements of essentially all related existence results known so far in the literature. In particular, it applies to general (non necessary translation invariant) interactions of unbounded order and infinite range given by many-particle potentials of superquadratic growth. We also discuss different techniques for proving uniqueness of Euclidean Gibbs measures, including Dobrushin's criterion, correlation inequalities, exponential decay of correlations, as well as Poincaré and log-Sobolev inequalities for the corresponding Dirichlet operators on loop lattices. In the special case of ferromagnetic models, we present the strongest result of such a type saying that uniqueness occurs for sufficiently small values of the particle mass.

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1 Introduction

The aim of this paper is twofold: First, we give the reader an elementary introduction in the mathematical theory of *quantum lattice systems* (*QLS*, for short). In Statistical Physics they are commonly viewed as models for *quantum crystals* (see, e.g., [1], [30], [33]). Second, we present recent results on the existence, uniqueness and a priori estimates for the corresponding Euclidean Gibbs measures obtained by the authors in [2]–[13], and we demonstrate how methods of Stochastic Analysis can successfully be applied to this topic.

According to common knowledge, a mathematical description of equilibrium properties of quantum systems might be given in terms of their Gibbs states defined on proper algebras of observables (cf. [20]). Such an algebraic approach is especially applicable to spin models with finite dimensional physical Hilbert spaces associated with every single particle. But, unfortunately, in the realization of this general concept for the quantum lattice systems considered in our papers there occur important difficulties (see, e.g., the discussion in [3]). In order to overcome these difficulties we shall take the *Euclidean* (or *path space*) *approach,* which is conceptually analogous to the well-known Euclidean strategy in quantum field theory (see, e.g., [30], [33], [47]). This analogy was pointed out and first implemented to quantum lattice systems in [1]; for recent developments see the review articles [13], [3] and an extensive bibliography therein. Actually, the Euclidean approach remains so far the only method which allows to construct and study Gibbs states for infinite systems of quantum particles described by unbounded operators. Briefly speaking, it transforms the problem of giving a proper meaning to a *quantum Gibbs state* G_β into the problem of studying a certain *Euclidean Gibbs measure* μ on the *loop lattice* $\Omega := [C(S_\beta)]^{\mathbb{Z}^d}$ (cf. Sect. 3 below for rigorous definitions). Here $\beta := 1/T > 0$ is the inverse (absolute) temperature and $C(S_\beta)$ is the space of continuous functions (i.e., loops) on a circle $S_\beta \cong [0, \beta]$ of length β. For a more detailed discussion of the relations between quantum and Euclidean Gibbs states we refer to [3], [5] (for an explanation of main ideas see also Sect. 2 below).

As a consequence, various probabilistic techniques become available for investigating equilibrium properties of quantum infinite-particle systems (cf. the review on this account included in Sect. 5). But, as compared with classical lattice systems, the situation with Euclidean Gibbs measures is much more complicated, since now the spin (i.e., loop) spaces themselves are *infinite dimensional* and their topological features should be taken into account carefully. Also, as is typical for non-compact spin spaces, we have to restrict ourselves to the set $\mathcal{G}_t$ of *tempered* Gibbs measures μ, which we specify by some natural support conditions (cf. definitions (15) and (22) below).

Among possible applications of Stochastic Analysis in Quantum Statistical Physics, we especially present *a new method for proving existence and a priori estimates* for tempered Euclidean Gibbs measures, which is based on the alternative description of Gibbs measures in terms of integration by parts and was developed by the authors in [6]–[8], [13]. It allows us to obtain improvements and generalizations of essentially all corresponding existence results for Gibbs measures known so far in the literature. Moreover, this method seems to be quite universal for lattice models and gives additional structural inside.

The organization of this paper is as follows. Section 2 is devoted to general aspects of the theory of Euclidean Gibbs measures. Here we introduce the models of quantum lattice systems ("anharmonic crystals"), concentrating on the simplest case of the *QLS Model I* with *harmonic pair interaction* between *nearest neighbors* only. In Sect. 3 we recall details on the corresponding Gibbsian formalism for Euclidean Gibbs measures μ on the loop lattice Ω. In Subsect. 4.1 we formulate our main *Theorems 1–6* on the existence, uniqueness and a priori estimates for tempered Euclidean Gibbs measures $\mu \in \mathcal{G}_t$. In Subsect. 4.2 we discuss the above mentioned alternative description of $\mu \in \mathcal{G}_t$ in terms of their shift–Radon–Nikodym derivatives (cf. *Theorem 7*) and its infinitesimal form, i.e., in terms of their logarithmic derivatives via the integration by parts formulas (cf. *Theorem 8*). In Sect. 5 we outline some possible generalizations of our method to the *QLS Models II–IV* with *many-particle interaction* of possibly *infinite range*. In Sect. 6 we discuss fundamental problems and basic methods in the study of Euclidean Gibbs states (e.g., Dobrushin's existence and uniqueness criteria, correlation inequalities, exponential decay of correlations for $\mu \in \mathcal{G}_t$, Poincare and log-Sobolev inequalities for the corresponding Dirichlet operators $\mathbb{H}_\mu$), as well as compare our results with those previously obtained by other authors.

The results presented in the paper have been obtained within the DFG-Schwerpunktprogramm *"Interacting Stochastic Systems of High Complexity"*, Research Projects AL 214/17 *"Stochastic Differential Equations on Infinite Dimensional Manigfolds"* (duration 01.05.1997–30.04.1999) and RO 1195/5 *"Analysis of Gibbs Measures via Integration by Parts and Quasi-Invariance"* (duration 01.05.1999 – 30.04.2003). One basic idea our research is the systematical applications of the methods of *Infinite Dimensional Stochastic Analysis* to the study of the equilibrium properties of infinite particle systems in Statistical Mechnics, which entirely corresponds to the general aim of the whole Schwerpunktprogramm. More presicely, the research performed by the authors can be naturally placed to the following main topics of the Schwerpunktprogramm: *1. "Interacting Systems of Statistical Physics"* and *6. "Stochastic Analysis"*. Moreover, within the Schwerpunktprogramm there have been highly stimulating discussions (e.g., during our various meetings) with Professors A. Bovier, J.-D. Deuschel, H.-O. Georgii, F. Götze, and H. Spohn concerned with applications to Statistical Mechanics and Quantum Field Theory (cf. the corresponding contributions in this volume).

2 A simple model of quantum anharmonic crystal

In order to fix the main ideas and make the reader more familiar with the topic, we start with the following simplest model of a quantum crystal, which was extensively studied in the literature.

Particular QLS Model I: harmonic pair interaction. Let $\mathbb{Z}^d$ be the integer lattice in the Euclidean space $(\mathbb{R}^d, |\cdot|)$, $d \in \mathbb{N}$. We consider an infinite system of interacting quantum particles performing one-dimensional (i.e., polarized) oscillations with displacements $q_k \in \mathbb{R}$ around their equilibrium positions at points $k \in \mathbb{Z}^d$. Each particle individually is described by the quantum mechanical Hamiltonian

$$\mathbb{H}_k := -\frac{1}{2\mathfrak{m}} \frac{d^2}{dq_k^2} + \frac{a^2}{2} q_k^2 + V(q_k) \tag{1}$$

acting in the (physical) Hilbert state space $\mathcal{H}_k := L^2(\mathbb{R}, dq_k)$. Here $\mathfrak{m}$ $(= \mathfrak{m}_{ph}/\hbar^2) > 0$ is the (reduced) mass of the particles and $a^2 > 0$ is their rigidity w.r.t. the harmonic oscillations. Concerning the anharmonic self-interaction potential, we suppose that $V \in C^2(\mathbb{R})$, i.e., twice continuously differentiable, and, moreover, that it satisfies the following growth condition:

Assumption (V). *There exist some constants* $P > 2$ *and* $K_V, C_V > 0$ *such that for all* $k \in \mathbb{Z}^d$ *and* $q \in \mathbb{R}$

$$K_V^{-1}|q|^{P-l} - C_V \leq (\mathrm{sgn} q)^l \cdot V^{(l)}(q) \leq K_V|q|^{P-l} + C_V, \quad l = 0, 1, 2.$$

Next, we add the *harmonic nearest-neighbor interaction*

$$W(q_k, q_{k'}) := \frac{J}{2}(q_k - q_{k'})^2$$

with intensity $J > 0$, the sum being taken over all (unordered) pairs $\langle k, k' \rangle$ in $\mathbb{Z}^d$ such that $|k - k'| = 1$. The whole system is then described by a heuristic Hamiltonian of the form

$$\mathbb{H} := -\frac{1}{2\mathfrak{m}} \sum_{k \in \mathbb{Z}^d} \frac{d^2}{dq_k^2} + \frac{a^2}{2} \sum_{k \in \mathbb{Z}^d} q_k^2 + \sum_{k \in \mathbb{Z}^d} V_k(q_k) + \frac{J}{2} \sum_{\langle k, k' \rangle \subset \mathbb{Z}^d} (q_k - q_{k'})^2. \quad (2)$$

Actually, the infinite-volume Hamiltonian (2) cannot be defined directly as a mathematical object and is represented by the local (i.e., indexed by finite volumes $\Lambda \Subset \mathbb{Z}^d$) Hamiltonians

$$\mathbb{H}_\Lambda := \sum_{k \in \Lambda} \mathbb{H}_k + \frac{J}{2} \sum_{\langle k, k' \rangle \subset \Lambda} (q_k - q_{k'})^2 \quad (3)$$

(as self-adjoint and lower bounded Schrödinger operators) acting in the Hilbert spaces $\mathcal{H}_\Lambda := \otimes_{k \in \Lambda} \mathcal{H}_k$.

Lattice systems of the above type (as well as their generalizations discussed below) are commonly viewed in quantum statistical physics as mathematical models of a crystalline substance (for more physical background see, e.g., [3], [30], [33], [42]). The study of such systems is especially motivated by the reason, that they provide a mathematically rigorous and physically quite realistic description for the important phenomenon of phase transitions (i.e., non-uniqueness of Gibbs states). So, if the potential V has a double-well shape, in the large mass limit $\mathfrak{m} \to \infty$ the QLS (2) may undergo (ferroelectic) structural phase transitions connected with the appearance of macroscopic displacements of particles for low temperatures $T < T_{cr}(\mathfrak{m})$ (for the mathematical justification of this effect see, e.g., [3], [5], [35]).

Remark 1. **(i)** Typical potentials satisfying Assumption (**V**) are polynomials of even degree and with a positive leading coefficient, i.e.,

$$V(q) := P(q) := \sum_{1 \leq l \leq 2n} b_l q^l \ \text{ with } b_{2n} > 0 \ \text{ and } \ n \geq 2. \quad (4)$$

In this case one speaks about so-called *ferromagnetic* $P(\phi)$*–models*, which also can naturally be looked upon as lattice discretizations of quantum $P(\phi)$–fields (cf. [33], [47]). Let us also mention a special choice in (4), when

$$P(q) := \sum_{0 \leq l \leq n} b_{2l} q^{2l} \ \text{ with } \ b_{2l} \geq 0 \ \text{ for all } \ 2 \leq l \leq n. \quad (5)$$

Since $b_2 \in \mathbb{R}$ can be a large negative number, such polynomials may have arbitrary deep double wells. The last systems are technically more suitable for the study of critical behaviour and the influence of quantum effects, since then one can use not only the FKG correlation inequalities, which are standard for ferromagnetic pair interactions, but also more advanced (e.g., GKS, Lebowitz) inequalities relying on the additional symmetry properties of the one-partical potential (5) (cf. [3] as well as Sect. 6 below).

(ii) All subsequent constructions remain true if one takes for $W(q_k, q_{k'})$ the general ferromagnetic interaction $U(q_k - q_{k'})$ given by a nonnegative convex function $U \in C^2(\mathbb{R} \to \mathbb{R})$ satisfying

$$0 \leq \inf_{\mathbb{R}} U'' \leq \sup_{\mathbb{R}} U'' < \infty, \quad 0 \leq U(q) = U(-q), \ \forall q \in \mathbb{R}. \tag{6}$$

As was already mentioned in Sect. 1, we take the *Euclidean approach* based on a path space representation for the quantum Gibbs states corresponding to the quantum mechanical system (2). Here we only illustrate these deep relations between quantum states and measures on loop spaces following the initial paper [1]) (see also [3], [5], [6] for the extended and up-to-date presentation).

Let us fix some $\beta := 1/T > 0$ having the meaning of inverse (absolute) temperature. Due to Assumption $(\mathbf{V_i})$, for each $k \in \mathbb{Z}^d$ the one-particle Hamiltonian $\mathbb{H}_k$ is a self-adjoint operator with trace class semigroup $e^{-\tau \mathbb{H}_k}$, $\tau \geq 0$. On the algebra $\mathcal{A}_k := \mathcal{L}(\mathcal{H}_k)$ of all bounded linear operators in $\mathcal{H}_k$, we may then define the authomorphism group $\alpha_{\theta,k}$, $\theta \in \mathbb{R}$, and the quantum Gibbs state $G_{\beta,k}$ acting respectively by

$$\alpha_{\theta,k}(B) := e^{i\theta\mathbb{H}_k} B e^{-i\theta\mathbb{H}_k}, \quad G_{\beta,k}(B) := Tr(Be^{-\beta\mathbb{H}_k})Tr(e^{-\beta\mathbb{H}_k}), \quad B \in \mathcal{A}_k.$$

For any finite set of multiplication operators $(B_i)_{i=1}^n \in L^\infty(\mathbb{R}^d)$ we next construct the so-called temperature *Euclidean Green functions*

$$\begin{gathered} \Gamma_{\beta,k}^{B_1,\ldots,B_n}(\tau_1, ..., \tau_n) := Tr_{\mathcal{H}_k}\left(\prod_{i=1}^n e^{-(\tau_{i+1}-\tau_i)\mathbb{H}_k} B_i\right) / Tr(e^{-\beta\mathbb{H}_k}), \\ 0 \leq \tau_1 \leq ... \leq \tau_n \leq \tau_{n+1} := \tau_1 + \beta. \end{gathered} \tag{7}$$

These functions have analytic continuations to the complex domain $0 < \operatorname{Re} z_1 < ... < \operatorname{Re} z_n < \beta$ with the boundary values at $z_i = -i\tau_i$:

$$\Gamma_{\beta,k}^{B_1,\ldots,B_n}(-i\tau_1, ..., -i\tau_n) = G_{\beta,k}\left(\prod_{i=1}^n \alpha_{\tau_i,k}(B_i)\right). \tag{8}$$

Moreover, it should be noted that relation (8) *uniquely* determines the Gibbs state $G_{\beta,k}$. A further *crucial* observation is that the Green functions (7) may be represented (by the Feynman–Kac formula) as the moments

$$\Gamma_{\beta,k}^{B_1,\ldots,B_n}(\tau_1, ..., \tau_n) = E_{\mu_{\beta,k}}\left(\prod_{i=1}^n B_i(\omega_k(\tau_i))\right) \tag{9}$$

of a certain probability measure μ_k on the loop space

$$C_\beta := \{\omega_k \in C([0,\beta] \to \mathbb{R}) \mid \omega_k(0) = \omega_k(\beta)\}. \quad (10)$$

More precisely,

$$d\mu_k(\omega_k) = \frac{1}{Z} E_\beta^{(x,x)} \left\{ - \exp \int_0^\beta \left[\frac{a^2}{2} \omega_k^2(\tau) + V(\omega_k(\tau)) \right] d\tau \right\} dx, \quad (11)$$

where Z is a normalization constant and $E_\beta^{(x,x)}$ is the conditional expectation, given that $\omega_k(0) = \omega_k(\beta) = x$, w.r.t. a Brownian bridge process of length β. So, we get a *one-to-one correspondence* between the quantum Gibbs state $G_{\beta,k}$, Euclidean Green functions (7) and the path measure μ_k. Respectively, for all local Hamiltonians $\mathbb{H}_\Lambda$ in volumes $\Lambda \Subset \mathbb{Z}^d$, relations similar to (7)–(9) are valid for the associated Gibbs states $G_{\beta,\Lambda}$ on the algebra $\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda)$ and the measure μ_Λ on the loop space $[C_\beta]^\Lambda$. This gives a possible way to construct the limiting states when $\Lambda \nearrow \mathbb{Z}^d$, and hence motivates us to consider Gibbs measures μ on the "*temperature loop lattice*" $\Omega := [C_\beta]^{\mathbb{Z}^d}$. We stress that so far there is *no method at all within operator theory* which allows to prove convergence of local Gibbs states in our situation. What is important, is the fact that (analogously to the well-known Osterwalder-Schrader reconstruction theorem in Euclidean field theory, see e.g. [30], [33], [47]) from each such Gibbs measure μ it is possible to *reconstruct* the quantum Gibbs state G_β of the system (2). For the above reasons the measures μ will be called *Euclidean Gibbs states (in the temperature loop space representation)* for the quantum lattice system (2). We proceed with their rigorous definition in Sect. 3 below.

3 Definition of Euclidean Gibbs measures

Here we briefly describe the corresponding *Euclidean Gibbsian formalism* just for the concrete class of quantum lattice systems (2); for a detailed exposition and an extensive bibliography we refer the reader to [3], [6].

Let $S_\beta \cong [0,\beta]$ be a circle of length β considered as a compact Riemannian manifold with Lebesgue measure $d\tau$ as a volume element and distance $\rho(\tau,\tau') := \min(|\tau-\tau'|, \beta - |\tau-\tau'|)$, $\tau,\tau' \in S_\beta$. For each $k \in \mathbb{Z}^d$, we denote by

$$L_\beta^r := L^r(S_\beta \to \mathbb{R}, d\tau), \quad r \ge 1,$$
$$C_\beta^\alpha := C^\alpha(S_\beta \to \mathbb{R}), \quad \alpha \ge 0,$$

the standard Banach spaces of all integrable resp. (Hölder) continuous functions (i.e., loops) $\omega_k : S_\beta \to \mathbb{R}$. In particular, C_β with sup-norm $|\cdot|_{C_\beta}$ will be treated as the *single spin space*, whereas $H := L_\beta^2$ with the inner product $(\cdot,\cdot)_H := |\cdot|_H^2$ as the Hilbert space tangent to C_β.

As the *configuration space* for the infinite volume system we define the space of all loop sequences over $\mathbb{Z}^d$

$$\Omega := [C_\beta]^{\mathbb{Z}^d} = \left\{ \omega = (\omega_k)_{k\in\mathbb{Z}^d} \,\middle|\, \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d},\ \omega_k \in C_\beta \right\}. \quad (12)$$

We endow Ω with the *product topology* (i.e., the weakest topology such that all finite volume projections

$$\Omega \ni \omega \mapsto \mathbb{P}_\Lambda \omega := \omega_\Lambda := (\omega_k)_{k\in\Lambda} \in C_\beta^\Lambda =: \Omega_\Lambda, \quad \Lambda \Subset \mathbb{Z}^d,$$

are continuous) and with the corresponding *Borel σ-algebra* $\mathcal{B}(\Omega)$ (which also coincides with the σ-algebra generated by all cylinder sets

$$\{\omega \in \Omega \mid \omega_\Lambda \in \Delta_\Lambda\}, \quad \Delta_\Lambda \in \mathcal{B}(\Omega_\Lambda), \ \Lambda \Subset \mathbb{Z}^d \ ).$$

Let $\mathcal{M}(\Omega)$ denote the set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. Next, we define the subset of *(exponentially) tempered configurations*

$$\Omega_t := \left\{ \omega \in \Omega \;\middle|\; \forall \delta \in (0,1): \ ||\omega||_{-\delta} := \left[\sum\nolimits_{k\in\mathbb{Z}^d} e^{-\delta|k|} |\omega_k|^2_{L^2_\beta}\right]^{\frac{1}{2}} < \infty \right\}. \quad (13)$$

and respectively the subset of tempered measures supported by $\Omega_t \in \mathcal{B}(\Omega)$, i.e.,

$$\mathcal{M}_t := \{\mu \in \mathcal{M}(\Omega) \mid \mu(\Omega_t) = 1\}. \quad (14)$$

In the context below, Ω_t will be always considered as a locally convex Polish space with the topology induced by the system of seminorms $\left(||\omega||_{-\delta}, |\omega_k|_{C_\beta}\right)_{\delta>0, \ k\in\mathbb{Z}^d}$.

Heuristically (cf. the discussion in Sect. 2 above), the Euclidean Gibbs measures μ we are interested in have the following representation

$$d\mu(\omega) := Z^{-1} \exp\{-\mathcal{I}(\omega)\} \prod_{k\in\mathbb{Z}^d} d\gamma_\beta(\omega_k), \quad (15)$$

where Z is a normalization factor, γ_β is a centered Gaussian measure on $(C_\beta, \mathcal{B}(C_\beta))$ with correlation operator A_β^{-1}, and $A_\beta := -\mathfrak{m}\Delta_\beta + a^2\mathbf{1}$ is the shifted Laplace–Beltrami operator on the circle S_β. Respectively $\mathcal{I}$ is defined as the map

$$\Omega \ni \omega \longmapsto \mathcal{I}(\omega) := \int_{S_\beta} \left[\sum_{k\in\mathbb{Z}^d} V(\omega_k) + \sum_{<k,k'>\subset\mathbb{Z}^d} W(\omega_k, \omega_{k'}) \right] d\tau, \quad (16)$$

which can be viewed as a potential energy functional describing an interacting system of loops $\omega_k \in C_\beta$ indexed by $k \in \mathbb{Z}^d$. Of course it is impossible to use this presentation for μ literally, since the series in (16) do not converge in any sense. Relying on the *Dobrushin–Lanford–Ruelle (DLR) formalism* (cf. [23], [32]), a rigorous meaning can be given to the measures μ as random fields on $\mathbb{Z}^d$ with a prescribed family of *local specifications* $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ in the following way:

For every finite set $\Lambda \Subset \mathbb{Z}^d$, we define a probability kernel π_Λ on $(\Omega, \mathcal{B}(\Omega))$: for all $\Delta \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$

$$\pi_\Lambda(\Delta|\xi) := Z_\Lambda^{-1}(\xi) \int_{\Omega_\Lambda} \exp\{-\mathcal{I}_\Lambda(\omega|\xi)\} \mathbf{1}_\Delta(\omega_\Lambda, \xi_{\Lambda^c}) \prod_{k\in\Lambda} d\gamma_\beta(\omega_k) \quad (17)$$

(where $\mathbf{1}_\Delta$ denotes the indicator on Δ). Here $Z_\Lambda(\xi)$ is the normalization factor and

$$\mathcal{I}_\Lambda(\omega|\xi) := \int_{S_\beta} \left[\sum_{k\in\Lambda} V(\omega_k) + \sum_{<k,k'>\subset\Lambda} W(\omega_k, \omega_{k'}) + \sum_{k\in\Lambda,\ k'\in\Lambda^c} W(\omega_k, \xi_{k'}) \right] d\tau \tag{18}$$

is the interaction in the volume Λ, subject to the boundary condition $\xi_{\Lambda^c} := (\xi_{k'})_{k'\in\Lambda^c}$ in the complement $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$. Obviously, $\inf_{\omega\in\Omega} \mathcal{I}_\Lambda(\omega|\xi) > -\infty$ and the RHS in (18) makes sense for the potentials V, W we deal here with. Moreover,

$$\int_\Omega \exp\left\{ \lambda |\omega_k|_{C^\alpha_\beta} \right\} d\pi_\Lambda(d\omega|\xi) < \infty, \quad \forall\lambda > 0,\ \alpha \in \alpha \in [0, 1/2) \tag{19}$$

since the Gaussian measure γ_β has such exponential moments. An important point is the consistency property for $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$: for all $\Lambda \subset \Lambda' \Subset \mathbb{Z}^d$, $\xi \in \Omega$ and $\Delta \in \mathcal{B}(\Omega)$

$$(\pi_{\Lambda'}\pi_\Lambda)(\Delta|\xi) := \int_\Omega \pi_{\Lambda'}(d\omega|\xi)\pi_\Lambda(\Delta|\omega) = \pi_{\Lambda'}(\Delta|\xi). \tag{20}$$

Definition 1. *A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called Euclidean Gibbs measure for the specification $\Pi := \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ (corresponding to the quantum lattice system (2) at inverse temperature $\beta > 0$) if it satisfies the DLR equilibrium equations: for all $\Lambda \Subset \mathbb{Z}^d$ and $\Delta \in \mathcal{B}(\Omega)$*

$$\mu\pi_\Lambda(\Delta) := \int_\Omega \mu(d\omega)\pi_\Lambda(\Delta|\omega) = \mu(\Delta). \tag{21}$$

Fixing $\beta > 0$, let $\mathcal{G}$ denote the set of all such measures μ. We shall be mainly concerned with the subset $\mathcal{G}_t$ of *tempered* Gibbs measures supported by Ω_t, i.e.,

$$\mathcal{G}_t := \mathcal{G} \cap \mathcal{M}_t = \{\mu \in \mathcal{G} \mid \mu(\Omega_t) = 1\}. \tag{22}$$

Remark 2. **(i)** It is worthwhile to compare our results on quantum systems with the analogous classical ones. The large-mass limit $\mathfrak{m} \to \infty$ (or $\hbar \to 0$) of model (2) gives us an infinite system of interacting classical particles moving in the external field V. Such system is described by the potential energy functional

$$\mathbb{H}_{cl}(q) = \frac{a^2}{2} \sum_{k\in\mathbb{Z}^d} q_k^2 + \sum_{k\in\mathbb{Z}^d} V(q_k) + \sum_{<k,k'>\subset\mathbb{Z}^d} W(q_k, q_{k'}) \tag{23}$$

on the configuration space $\Omega_{cl} := \mathbb{R}^{\mathbb{Z}^d} \ni \{q_k\}_{k\in\mathbb{Z}^d} := q$ (cf. [3]). Again, the formal Hamiltonian (23) does not make sense itself and is represented by the local Hamiltonians

$$\mathbb{H}_{cl,\Lambda}(q|y) := \frac{a^2}{2}\sum_{k\in\Lambda} q_k^2 + \sum_{k\in\Lambda} V(q_k) + \sum_{<k,k'>\subset\Lambda} W(q_k, q_{k'}) + \sum_{k\in\Lambda,\ k'\in\Lambda^c} W(q_k, y_{k'}) \tag{24}$$

in the volumes $\Lambda \Subset \mathbb{Z}^d$ given the boundary conditions $y \in \Omega_{cl}$. The corresponding Gibbs states $\mu \in \mathcal{G}_{cl}$ at inverse temperature $\beta > 0$ are defined as probability measures on Ω_{cl} satisfying the DLR equations $\mu\pi_\Lambda = \mu$, $\Lambda \Subset \mathbb{Z}^d$, with the family of local specifications

$$\pi_\Lambda(\Delta|y) := Z_\Lambda^{-1}(y)\int_{\mathbb{R}^\Lambda} \exp\{-\beta\mathbb{H}_{cl,\Lambda}(q|y)\}\,\mathbf{1}_\Delta(q_\Lambda, y_{\Lambda^c})\prod_{k\in\Lambda} dq_k,$$
$$\Delta \in \mathcal{B}(\Omega_{cl}), \quad y \in \Omega_{cl}. \tag{25}$$

Starting from the pioneering papers [15], [21], [40], [45], such unbounded spin systems have been under intensive investigation in classical statistical mechanics (for recent developments see, e.g., [13], [17], [50]). However, it should be noted that the results obtained in those papers principally do not apply in the more complex situation of quantum systems with infinite dimensional spin spaces as considered here.

(ii) The necessity of restricting to configurations $\omega \in \Omega_t$ will appear, once we proceed to getting uniform moment estimates on Gibbs measures (cf. Theorem 2 below). On the other hand, our definition of temperedness (as well as its modification to the classical systems (23) with $|q_k|$ substituting $|\omega_k|_{L^2_\beta}$) is less restrictive (and simpler) than those usually used in the literature (for comparison, see e.g. [15], [21]). So, obviously, $\Omega_t \supseteq \Omega_{(s)t}$ resp. $\mathcal{G}_t \supseteq \mathcal{G}_{(s)t}$, where the subsets of all (*"slowly increasing"*) *tempered* configurations resp. measures are defined by

$$\Omega_{(s)t} := \left\{ \omega\in\Omega \;\middle|\; \exists p = p(\omega) > 0 :\ ||\omega||_{-p} := \left[\sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2p}|\omega_k|^2_{L^2_\beta}\right]^{\frac{1}{2}} < \infty \right\},$$
$$\mathcal{G}_{(s)t} := \{\mu\in\mathcal{G} \mid \exists p = p(\mu) > 0 :\ ||\omega||_{-p} < \infty \ \text{ for } \ \mu\text{-a.e. } \omega\in\Omega\}. \tag{26}$$

In turn, $\mathcal{G}_{(s)t}$ contains the subset $\mathcal{G}_{(ss)t}$ of the so-called *Ruelle type "superstable"* Gibbs measures named after the earlier papers [40], [46] on the classical case. In the context of translation invariant quantum systems with many particle interactions at most of quadratic growth, such measures were introduced in [44] by the following support condition

$$\Omega_{(ss)t} := \left\{ \omega\in\Omega \;\middle|\; \sup_{N\in\mathbb{N}} \left[(1+2N)^{-d}\sum_{|k|\le N} |\omega_k|^2_{L^2_\beta}\right] < \infty \right\},$$
$$\mathcal{G}_{(ss)t} := \left\{\mu\in\mathcal{G} \;\middle|\; \mu(\Omega_{(ss)t}) = 1\right\}. \tag{27}$$

4 Formulation of the main results

Now we are ready to present our main results for the Euclidean Gibbs measures concerning the following two directions:

– **existence, uniqueness, and a priori estimates for** $\mu \in \mathcal{G}_t$;
– **alternative description of** $\mu \in \mathcal{G}_t$ **in terms of their Radon–Nikodym derivatives and integration by parts formulas**.

For the sake of simplicity we confine ourselves to the concrete set up of the QLS Model I. We assume that all the conditions on the interaction potentials imposed in Sect. 2 are fulfiled without mentioning this again in the formulations of our statements. It is worth noting that, even for this mostly studied model, all the results presented here (as well as their trivial modifications for classical lattice systems like (23) performed in [13]) either are completely new or essentially improve previous ones obtained by other authors. How far they can be extended to more general interactions will be discussed in Sect. 4 below.

4.1 Existence, uniqueness and a priori estimates for Euclidean Gibbs measures

The following provides us with basic information for any further investigation of $\mu \in \mathcal{G}_t$:

Theorem 1. (Existence of Tempered Gibbs States, cf. [6]–[8]) *(i) For all values of the mass* $\mathfrak{m} > 0$ *and the inverse temperature* $\beta > 0$:

$$\mathcal{G}_t \neq \emptyset.$$

(ii) Moreover, (in all translation invariant systems treated below) there exists at least one translation invariant $\mu \in \mathcal{G}_{(ss)t} \subseteq \mathcal{G}_t$ *satisfying the (even stronger than (27)) support condition for all* $Q \geq 1$ *and* $\alpha \in [0, 1/2)$*:*

$$\sup_{N \in \mathbb{N}} \left\{ (1+2N)^{-d} \sum\nolimits_{|k| \leq N} |\omega_k|^Q_{C^\alpha_\beta} \right\} < \infty \quad \text{for } \mu\text{-a.e. } \omega \in \Omega. \tag{28}$$

Theorem 2. (A Priori Estimates on Tempered Gibbs States, cf. [6]–[8]) *Every* $\mu \in \mathcal{G}_t$ *is supported by the set of Hölder loops* $\bigcap_{\alpha \in [0,1/2)} [C^\alpha_\beta]^{\mathbb{Z}^d}$*. Actually, for all* $Q \geq 1$ *and* $\alpha \in [0, 1/2)$

$$\sup_{\mu \in \mathcal{G}_t} \sup_{k \in \mathbb{Z}^d} \int_\Omega |\omega_k|^Q_{C^\alpha_\beta} d\mu(\omega) < \infty. \tag{29}$$

Corollary 1. *The set* $\mathcal{G}_t$ *is compact w.r.t. the topology of weak convergence of measures on any of spaces* $[C^\alpha_\beta]^{\mathbb{Z}^d}$, $\alpha \in [0, 1/2)$*, equipped by the system of seminorms* $|\omega_k|_{C^\alpha_\beta}$, $k \in \mathbb{Z}^d$.

In particular, by Prokhorov's tightness criterion, the existence result for $\mu \in \mathcal{G}_t$ immediately follows from the moment estimate (30) below, which holds for the family $\pi_\Lambda(d\omega|\xi = 0)$, $\Lambda \Subset \mathbb{Z}^d$, uniformly in volume. Besides, the a priori estimates for the probability kernels $\pi_\Lambda(d\omega|\xi)$ of the local specification $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{Z}^d}$ subject to the fixed boundary condition $\xi \in \Omega_t$, stated in Theorems 3 and 4 below, are also of an independent interest and have various applications.

Theorem 3. (Moment Estimates Uniformly in Volume, cf. [8]) *Let us fix any boundary condition $\xi \in \Omega$. Then for all $\delta > 0$, $\alpha \in [0, 1/2)$ and $Q \geq 1$*

$$\sup_{k \in \mathbb{Z}^d} |\xi_k|_{C^\alpha_\beta} < \infty \Longrightarrow \sup_{\Lambda \Subset \mathbb{Z}^d} \sup_{k \in \mathbb{Z}^d} \int_\Omega |\omega_k|^Q_{C^\alpha_\beta} \pi_\Lambda(d\omega|\xi) =: C_{Q,\xi} < \infty, \tag{30}$$

$$\xi \in \Omega_t \Longrightarrow \sup_{\Lambda \Subset \mathbb{Z}^d} \sum_{k \in \Lambda} e^{-\delta|k|} \int_\Omega |\omega_k|^Q_{C^\alpha_\beta} \pi_\Lambda(d\omega|\xi) =: C'_{Q,\xi} < \infty. \tag{31}$$

Theorem 4. (Dobrushin Type Exponential Bound, cf. [8]) *For any given $\lambda, \sigma > 0$, there exist a constant $A > 0$ and a matrix $\mathbf{I} = (I_{k,j})_{k,j \in \mathbb{Z}^d}$ with entries $I_{k,j} \geq 0$ and with the bounded operator norms*

$$||\mathbf{I}||_\delta := \sup_{k \in \mathbb{Z}^d} \left\{ \sum_{k' \in \mathbb{Z}^d} I_{k,j} \exp \delta|k - j| \right\} < \infty, \quad \forall \delta > 0,$$

$$||\mathbf{I}||_0 = ||\mathbf{I}||_{l^1(\mathbb{Z}^d)} := \sup_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} I_{k,j} < \sigma, \tag{32}$$

such that for all $k \in \mathbb{Z}^d$ and $\xi \in \Omega_t$

$$\int_\Omega \exp \lambda |\omega_k|_{L^2_\beta} \, d\pi_{\{k\}}(d\omega|\xi) \leq \exp \lambda \left(A + \sum_{j \in \mathbb{Z}^d} I_{k,j} |\xi_j|_{L^2_\beta} \right). \tag{33}$$

Of course, we cannot provide here the full proofs of the theorems formulated above. But it should be mentioned in this respect that we propose a new method of proving existence and a priori estimates for Gibbs measures, which is based on their alternative characterization via integration by parts (cf. Theorem 8 in Subsect. 4.2). This method and hence the statements of Theorems 1–4 obtained by it easily extend to general many-particle interactions (cf. Sect. 5). In contrast, the uniqueness results quoted below essentially use the concrete structure of the one-particle and pair potentials $V(q_k)$ and $W(q_k, q_{k'}) := \frac{J}{2}(q_k - q_{k'})^2$.

Theorem 5. (Uniqueness of Tempered Gibbs States, cf. [11], [12]) *Suppose that the anharmonic self-interaction possesses a decomposition*

$$V = V_0 + U,$$

where $V_0 \in C^2(\mathbb{R})$ *is a strictly convex function with polynomially bounded derivatives (i.e., satisfying Assumption* (V_0)*) and* $U \in C_b(\mathbb{R})$ *is a bounded perturbation (describing the presence of possible wells). Denote*

$$b := \inf_{\mathbb{R}} V_0'' > 0, \quad \delta(U) := \sup_{\mathbb{R}} U - \inf_{\mathbb{R}} U < \infty.$$

Then, for all values of the mass $\mathfrak{m} > 0$, *the set* $\mathcal{G}_t$ *consists of exactly one point, provided the following relation between the parameters is satisfied:*

$$\frac{e^{\beta\,\delta(U)}}{2d + J^{-1}(a^2 + b^2)} < \frac{1}{2d}. \tag{34}$$

Theorem 6. (Uniqueness in $P(\phi)$-models by small mass, cf. [2]–[8]) *For the quantum lattice model (2) with the polynomial interaction of the form (5), there exists* $\mathfrak{m}_* > 0$ *such that, for all* $\mathfrak{m} \in (0, \mathfrak{m}_*)$ *and all temperatures* $\beta > 0$, *the set* $\mathcal{G}_t$ *consists of exactly one point.*

A general presentation of the "*state of the art*" in the literature concerning these problems will be given in Sect. 6.

4.2 Flow and integration by parts characterization of Euclidean Gibbs measures

Here we briefly discuss the main ingredients of our new approach for proving existence and uniform a priori estimates for Euclidean Gibbs measures (cf. Theorems 1–4). A basic idea of our method is to use an *alternative characterization of Gibbs measures via their Radon–Nikodym derivatives or via integration by parts* (instead of the usual one in terms of local specifications through the Dobrushin–Lanford–Ruelle equations (21)). Such alternative descriptions of Gibbs measures have long been known for a number of specific models in statistical mechanics and field theory (see, e.g., [28]–[31], [36], [45]). But for the quantum lattice systems under consideration, a complete characterization of the measures $\mu \in \mathcal{G}_t$ in terms of their Radon–Nikodym derivatives has first been proved in [10], Theorem 4.6. Assuming that the interaction potentials are differentiable, it was further shown in [6]–[8] that this description of Gibbs measures is equivalent to their characterization as differentiable measures satisfying integration by parts formulas.

So, we start with the *flow* description of $\mu \in \mathcal{G}_t$ in terms of their Radon–Nikodym derivatives w.r.t. shift transformations of the configuration space Ω. Let us consider $\mathcal{H} := l^2(\mathbb{Z}^d \to L^2_\beta)$ with the scalar product $< \omega, \omega >_{\mathcal{H}} = ||\omega||^2_{\mathcal{H}} := \sum_{k\in\mathbb{Z}^d} |\omega_k|^2_{L^2_\beta}$ as the tangent Hilbert space to Ω. We fix an orthonormal basis in $\mathcal{H}$ consisting of the vectors $h_i := \{\delta_{k-j}\varphi_n\}_{j\in\mathbb{Z}^d}$ indexed by $i = (k, n) \in \mathbb{Z}^{d+1}$, where $\{\varphi_n\}_{n\in\mathbb{Z}} \subset C^\infty_\beta$ is the complete orthonormal system of eigenvectors of the operator A_β in $H := L^2_\beta$, i.e., $A_\beta\varphi_n = \lambda_n\varphi_n$ with $\lambda_n = (2\pi n/\beta)^2\mathfrak{m} + a^2$.

Theorem 7. (Flow Description of $\mu \in \mathcal{G}_t$, cf. [6]–[10])**.** *Let $\mathcal{M}_t^a$ denote the set of all probability measures $\mu \in \mathcal{M}_t$ which are quasi-invariant w.r.t. the shifts $\omega \longmapsto \omega + \theta h_i$, $\theta \in \mathbb{R}$, $i = (k,n) \in \mathbb{Z}^{d+1}$, with Radon–Nikodym derivatives*

$$a_{\theta h_i}(\omega) := \exp\left\{-\theta(A_\beta\varphi_n, \omega_k)_H - \frac{\theta^2}{2}(A_\beta\varphi_n, \varphi_n)_H\right\} \quad (35)$$
$$\times \exp\int_{S_\beta}\left\{V(\omega_k) - V(\omega_k + \theta\varphi_n) + \sum_{j:\ |j-k|=1}[W(\omega_k, \omega_j) - W(\omega_k + \theta\varphi_n, \omega_j)]\right\}d\tau.$$

Then $\mathcal{G}_t = \mathcal{M}_t^a$.

However, in applications it is more convenient to use not the flow characterization itself, but its infinitesimal form which we shall describe now. To this end we define functions (which will turn out to be the *partial logarithmic derivatives* of measures $\mu \in \mathcal{M}_t^a$ along directions h_i) for $i = (k,n) \in \mathbb{Z}^{d+1}$ by

$$b_{h_i}(\omega) := \frac{\partial}{\partial\theta}a_{\theta h_i}(\omega)\Big|_{\theta=0} = -(A_\beta\varphi_n, \omega_k)_H - (F_k^{V,W}(\omega), \varphi_n)_H, \quad \omega \in \Omega_t. \quad (36)$$

Here $F_k^{V,W} : \Omega \to C_\beta$ is the nonlinear Nemytskii-type operator acting by

$$F_k(\omega) := V'(\omega_k) + \sum\nolimits_{j:\ |j-k|=1} \partial_q W_{\{k,j\}}(q, q')\big|_{q=\omega_k,\ q'=\omega_{k'}}. \quad (37)$$

For every $i = (k,n) \in \mathbb{Z}^{d+1}$, we denote by $C^1_{\mathrm{dec},i}(\Omega_t)$ the set of all functions $f : \Omega_t \to \mathbb{R}$ which are bounded and continuous together with their partial derivatives $\partial_{h_i} f$ in the direction h_i and, moreover, satisfy the extra decay condition

$$\sup_{\omega\in\Omega_t}\left|f(\omega)\left(1 + |\omega_k|_{L^1_\beta} + |F_k(\omega)|_{L^1_\beta}\right)\right| < \infty. \quad (38)$$

Of course, $fb_i \in L^\infty(\mu)$ for any $f \in C^1_{\mathrm{dec},i}(\Omega_t)$ and any $\mu \in \mathcal{M}_t$, even though we do not know a priori whether $b_{h_i} \in L^1(\mu)$. Since the interaction potentials are assumed to be differentiable, next we can show that the above flow characterization of $\mu \in \mathcal{G}_t$ is equivalent to their characterization as *differentiable measures* solving the *integration by parts* (for short, *IbP*) equations

$$\partial_{h_i}\mu(d\omega) = b_{h_i}(\omega)\mu(d\omega)$$

with the prescribed logarithmic derivatives b_{h_i}.

Theorem 8. ((IbP)-Description of $\mu \in \mathcal{G}_t$, *cf. [6]–[8]*)**.** *Let $\mathcal{M}_t^b$ denote the set of all probability measures $\mu \in \mathcal{M}_t$ which satisfy the (IbP)-formula*

$$\int_\Omega \partial_{h_i} f(\omega)d\mu(\omega) = -\int_\Omega f(\omega)b_{h_i}(\omega)d\mu(\omega) \quad (39)$$

for all test functions $f \in C^1_{\mathsf{dec},i}(\Omega_t)$ and all directions h_i, $i = \mathbb{Z}^{d+1}$. Then $\mathcal{G}_t = \mathcal{M}_t^b$.

Remark 3. **(i)** Let us stress that the b_{h_i} just depend on the given potentials V and W, and hence are the same for all $\mu \in \mathcal{G}_t$ associated with the heuristic Hamiltonian (2). The measures given by the probability kernels $\pi_\Lambda(d\omega|\xi)$ of the local specification Π satisfy Theorems 7, 8, but only in directions h_i, $i = (k, n)$ with $k \in \Lambda \Subset \mathbb{Z}^d$. Since $a_{\theta h_i}$ and b_{h_i} are continuous locally bounded functions on Ω_t, the latter implies that every accumulation point of the family $\left\{\pi_\Lambda(d\omega|\xi) \,\middle|\, \Lambda \Subset \mathbb{Z}^d, \xi \in \Omega_t\right\}$ is surely Gibbs.

(ii) In Stochastic Analysis, solutions μ to the (IbP)-formula (39) are also called *symmetrizing* measures. For further connections to reversible diffusion processes and Dirichlet operators in infinite dimensions we refer e.g. to [9], [10], [13], [19].

(iii) The key point of the proofs of Theorems 1–4 stated in Subsect. 4.1 is that (according to Theorem 6) $\mu \in \mathcal{G}_t$ resp. $\pi_\Lambda(d\omega|\xi)$ are viewed as the solutions of the infinite system (39) of first order PDE's. Due to the assumptions on the potentials $V$and W imposed above, the corresponding vector fields $b_{h_i}, i \in \mathbb{Z}^{d+1}$, possess certain *coercivity properties* w.r.t. the tangent space $\mathcal{H}$. This enables us to employ an analog of the *Lyapunov function method* well-known from finite dimensional PDE's to get uniform moment estimates (29–33). It should be noted that this approach has been first implemented in [13], however in the much simpler situation of the classical spin systems (23). Since the concrete technique used in those papers does not apply to loop spaces, in [6]–[8] we develop its proper (highly non-trivial) modification for the quantum case, which involves a *"single spin space analysis"* depending on the spectral properties of the elliptic operator A.

5 Possible generalizations of QLS Model I

Here we briefly discuss how to modify the previous setting in order to include many-particle interaction potentials.

Particular QLS Model II: pair interaction of superquadratic growth. Let us first consider the following generalization of the QLS (2) described by a heuristic Hamiltonian of the form

$$\mathbb{H} := -\frac{1}{2\mathfrak{m}} \sum_{k\in\mathbb{Z}^d} \frac{d^2}{dq_k^2} + \frac{a^2}{2} \sum_{k\in\mathbb{Z}^d} q_k^2 + \sum_{k\in\mathbb{Z}^d} V(q_k) + \sum_{\langle k,k'\rangle\subset\mathbb{Z}^d} W(q_k, q_{k'}). \tag{40}$$

The one-particle potential $V \in C^2(\mathbb{R} \to \mathbb{R})$ satisfies the same Assumption $(\mathbf{V_0})$ as in Sect. 2, i.e., has asymptotic behaviour at infinity as a polynomial of order $P > 2$. Concerning the pair potential, we suppose that $W \in C^2(\mathbb{R}^2 \to \mathbb{R})$ has respectively at most polynomial growth of any order $R < P$:

Assumption $(\mathbf{W_0})$. *There exist constants $R \in [2, P)$ and $K_W, C_W > 0$ such that for all $q, q' \in \mathbb{R}$*

$$|\partial_q^{(l)} W(q, q')| \le K_W \left(|q|^{R-l} + |q'|^{R-l}\right) + C_W, \quad l = 0, 1, 2.$$

Remark 4. A trivial example for pair potentials which satisfy $(\mathbf{W_0})$ are the polynomials $W(q,q') := \sum_{l=0}^{2r}(q-q')^l$ of even degree $2r < P$. In other words, our assumptions mean that the pair interaction is dominated by the single-particle one, which implies the so-called *lattice stabilization.* The case $P = R$ is also allowed, but it needs a more accurate analysis (cf. [7], [8]).

As compared with the initial QLS model (2), the only principal difference in dealing with its generalization (40) is that we should proper change the notion of temperedness. Now we define the subset of *tempered configurations* by

$$\Omega_t^R := \left\{ \omega \in \Omega \;\middle|\; \forall \delta \in (0,1): \; ||\omega||_{-\delta,R} := \left[\sum\nolimits_{k\in\mathbb{Z}^d} e^{-\delta|k|} |\omega_k|^2_{L^R_\beta} \right]^{\frac{1}{2}} < \infty \right\}. \tag{41}$$

Note that for $R = 2$ this is just the previous definition (14). Then all our main Theorems 1–6 presented in Sect. 4.1 *remain true*, as soon as in their formulation we substitute the single spin space L^2_β by L^R_β and respectively specify the subset $\mathcal{G}_t$ of *tempered Gibbs measures* as those supported by $\Omega_t := \Omega_t^R$. Let us stress that (even in the case of translation invariant interactions we now deal with) we cannot guarantee that (outside the uniqueness regime) any tempered Gibbs measure will be invariant w.r.t. lattice translations. So, the above set $\mathcal{G}_t^R$ is the largest one so that for any of its points μ we are technically able to get moment estimates like (29) *uniformly* w.r.t. the lattice parameter $k \in \mathbb{Z}^d$.

Particular QLS Model III: pair interaction of infinite range. A further generalization of the QLS Models (2) and (40) concerns the case of *not necessarily translation-invariant* pair interaction of possibly *infinite range.* Namely, let us consider a model described by a heuristic Hamiltonian of the form

$$\begin{aligned} \mathbb{H} := & -\frac{1}{2\mathfrak{m}} \sum_{k\in\mathbb{Z}^d} \frac{d^2}{dq_k^2} + \frac{a^2}{2} \sum_{k\in\mathbb{Z}^d} q_k^2 + \sum_{k\in\mathbb{Z}^d} V_k(q_k) \\ & + \sum_{\{k,k'\}\subset\mathbb{Z}^d} W_{\{k,k'\}}(q_k, q_{k'}). \end{aligned} \tag{42}$$

The one-particle potentials $V_k \in C^2(\mathbb{R})$ satisfy the same Assumption $(\mathbf{V_0})$ as before, but with $P > 2$ and $K_V, C_V > 0$ which are *uniform* for all $k \in \mathbb{Z}^d$. The two-particle interactions (taken over all unordered pairs $\{k,k'\} \subset \mathbb{Z}^d$, $k \neq k'$) are given by symmetric functions $W_{\{k,k'\}} \in C^2(\mathbb{R}^2)$ satisfying the following growth condition:

Assumption $(\mathbf{W_0^*})$: *There exist some constants* $R \in [2, P)$ *and* $J_{k,k'} \geq 0$ *such that for all* $\{k,k'\} \subset \mathbb{Z}^d$ *and* $q, q' \in \mathbb{R}$:

$$|\partial_q^{(l)} W_{\{k,k'\}}(q,q')| \leq J_{k,k'} \left(1 + |q|^{R-l} + |q'|^{R-l}\right), \quad l = 0,1,2.$$

For the matrix $\mathbf{J} := \{J_{k,j}\}$ we can allow different rates of decay (for instance, polynomial or exponential), when the distance $|k - j|$ between the points of the lattice gets large:

Assumption $(\mathbf{J_0})$. *For all* $p \geq 0$ *or resp. some* $\delta > 0$

$$\text{(i)} \ ||\mathbf{J}||_p := \sup_{k\in\mathbb{Z}^d} \left\{ \sum_{j\in\mathbb{Z}^d\setminus\{k\}} J_{k,j}(1+|k-j|)^p \right\} < \infty,$$
$$\text{(ii)} \quad ||\mathbf{J}||_\delta := \sup_{k\in\mathbb{Z}^d} \left\{ \sum_{j\in\mathbb{Z}^d\setminus\{k\}} J_{k,j} e^{\delta|k-j|} \right\} < \infty.$$

Obviously, (ii) is stronger than (i). Again, we first need to choose the proper notion of the temperedness, which essentially depends on the decay rate of the pair interaction. A new issue caused by the *infinite range* of the interaction is that one also has to check that the probability kernels $\pi_\Lambda(d\omega|\xi)$ are well defined for all boundary conditions $\xi \in \Omega_t$. So, in view of Assumption $(\mathbf{J_0})(\mathbf{i})$, we define the subset $\Omega^R_{(s)t} \subset \Omega^R_t$ of (*"slowly increasing"*) *tempered* configurations by

$$\Omega^R_{(s)t} := \left\{ \omega \in \Omega \;\middle|\; \exists p = p(\omega) > 0 : \ ||\omega||_{-p,R} := \left[\sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2p} |\omega_k|^2_{L^R_\beta} \right]^{\frac{1}{2}} < \infty \right\}. \tag{43}$$

Respectively, we introduce the subset of tempered Gibbs measures

$$\mathcal{G}^R_{(s)t} := \{ \mu \in \mathcal{G} \mid \exists p = p(\mu) > d : \ ||\omega||_{-p,R} < \infty \ \forall \omega \in \Omega \ (\operatorname{mod} \mu) \} . \tag{44}$$

Then our main theorems about existence and a priori estimates for the tempered Euclidean Gibbs measures *remain true*, provided in their formulation one substitutes the single spin spaces L^2_β by L^R_β and, respectively, Ω_t by $\Omega^R_{(s)t}$ and $\mathcal{G}_t$ by $\mathcal{G}^R_{(s)t}$. In the formulation of Theorems 3 and 4 describing the properties of the probability kernels $\pi_\Lambda(d\omega|\xi)$ one also needs obvious changes (e.g., by claiming that $||\mathbf{I}||_p < \infty$ for some $p \geq 0$ instead of $||\mathbf{I}||_\delta < \infty$ for all $\delta > 0$ as before), which are discussed in [5], [6]. On the other hand, if we want to deal with the larger subset $\mathcal{G}^R_t \supset \mathcal{G}^R_{(s)t}$ and completely keep the previous setup of the QLS Model II, we should correspondingly impose the stronger Assumption $(\mathbf{J_0})(\mathbf{ii})$ on the decay of matrix $\mathbf{J}$.

General QLS Model IV: many particle interactions of unbounded order and infinite range. Here we mean the systems described by a heuristic infinite dimensional Hamiltonian

$$\mathbb{H} = -\frac{1}{2\mathfrak{m}} \sum_{k\in\mathbb{Z}^d} \frac{d^2}{dq_k^2} + \frac{a^2}{2} \sum_{k\in\mathbb{Z}^d} q_k^2 + \sum_{k\in\mathbb{Z}^d} V_k(q_k)$$
$$+ \sum_{n=2}^{\infty} \sum_{\{k_1,\dots,k_n\}\subset\mathbb{Z}^d} W_{\{k_1,\dots,k_n\}}(q_{k_1},\dots,q_{k_n}), \tag{45}$$

where the n-particle interaction potentials (taken over all finite sets $\{k_1,...,k_n\} \subset \mathbb{Z}^d$ consisting of $n \geq 2$ different points) are given by twice continuously differentiable *symmetric* functions $W_{\{k_1,\dots,k_n\}} \in C^2(\mathbb{R}^{dn})$. Again, the statements of Theorems 1–4 (in the same formulation as that for the QLS Model III) still hold, if one uses e.g. the following modification of $(\mathbf{W}_0^*)$ and $(\mathbf{J}_0)$:

Assumption (W). There exist $R \geq 2$, $I \geq 0$ and symmetric matrices $\{J_{k_1,...,k_n}\}_{(k_1,...,k_n)\in\mathbb{Z}^{nd}}$ with positive entries, such that for all $n \in \mathbb{N}$, $\{k_1, ..., k_n\} \subset \mathbb{Z}^d$ and $q_1, ..., q_n \in \mathbb{R}^d$

$$|\nabla_{q_1}^{(l)} W_{\{k_1,...,k_M\}}(q_1, ..., q_n)|(1 + \sum_{m=1}^{n} |q_m|^l) \leq J_{k_1,...,k_n} \sum_{m=1}^{n} |q_m|^R + I, \quad l = 0, 1, 2.$$

Assumption (J). *For all $p \geq 0$ or (even stronger) for some $\delta > 0$*

(i) $||\mathbf{J}||_p := \sum_{n=2}^{\infty} n^R \sup_{k_1\in\mathbb{Z}^d} \left\{ \sum_{\{k_2,...,k_n\}\subset\mathbb{Z}^d} J_{k_1,...,k_n} \left(1 + \sum_{m=1}^{n} |k_1 - k_m|^p\right) \right\} < \infty,$

(ii) $||\mathbf{J}||_\delta := \sum_{n=2}^{\infty} n^R \sup_{k_1\in\mathbb{Z}^d} \left\{ \sum_{\{k_2,...,k_n\}\subset\mathbb{Z}^d} J_{k_1,...,k_n} \exp\left(\delta \sum_{m=1}^{n} |k_1 - k_m|\right) \right\} < \infty.$

It should be particularly emphasized that our technique based on the (IbP)-formula (39) obviously extends even to the above non-trivial interactions, which were not covered at all by any previous work (cf. the discussion in Sect. 6). All details on the general QLS Model IV may be found in [6], [8]. Of course, from the physical point of view, it is more realistic to consider systems of D-dimensional quantum oscillators on the lattice $\mathbb{Z}^d$ $(d, D \in \mathbb{N})$ with interaction potentials $V_k : \mathbb{R}^D \to \mathbb{R}$, $W_{\{k_1,...,k_M\}} : (\mathbb{R}^D)^M \to \mathbb{R}$. Also in this case our method works. In this respect, we refer the interested reader to [7] where such a *multidimensional* version of the particular QLS Model II was treated.

6 Comments on Theorems 1–6

In order to give the reader a wider insight into the subject, we present here a systematic account of the *fundamental problems, basic methods, known results and possible nearest goals* in the study of Euclidean Gibbs measures on loop lattices.

I. Existence problem. As is typical for systems with noncompact (in our case, infinite-dimensional) spin spaces, even the initial question of whether the set $\mathcal{G}_t$ is not empty (and hence the positive answer on it given by **Theorem 1** for the QLS Models I–IV) is far from trivial. A useful observation in this respect is that, under natural assumptions on the interaction, any accumulating point of the family π_Λ, $\Lambda \Subset \mathbb{Z}^d$, is certainly Gibbs. Depending on the specific class of quantum lattice models one deals with, the required convergence $\pi_{\Lambda^{(N)}} \to \mu$, $\Lambda^{(N)} \nearrow \mathbb{Z}^d$, and thus the existence of $\mu \in \mathcal{G}_t$, are proved by the following main methods listed below:

(i) General Dobrushin's criterion for existence of Gibbs distributions [23]**.** The validity of the sufficient conditions of the *Dobrushin existence theorem* for some classical unbounded spin systems (23) has been verified, e.g., in [15], [21], [48] (however, under assumptions on the interaction potentials more restrictive than $(\mathbf{V}_0)$ and $(\mathbf{W}_0)$). Contrary to the classical case, the same problem for quantum lattice systems was not covered at all by any previous work.

More precisely, in order to apply the Dobrushin criterion to quantum lattice systems, one should estimate in a proper way the expectations $E_{\pi_\Lambda(d\omega|\xi)}(F(\omega_k))$, $k \in \Lambda \Subset \mathbb{Z}^d$, of some *compact* function $F : C_\beta \to \mathbb{R}_+ \cup \{\infty\}$. For this reason, the Dobrushin type moment estimate in the spin space L^R_β, which we prove in **Theorem 3** as an extension to the quantum case of the corresponding result in [15], is yet not enough for the existence criterion. Naturally one could try to take for F the norm-function in C^α_β with $0 \le \alpha < 1/2$, except that so far no technical means were available to get such moment estimates in Hölder spaces starting from the DLR equations. A proper improvement of Theorem 4 will be the subject of a forthcoming paper of the authors.

(ii) Ruelle's technique of superstability estimates (cf. the original papers [40], [46] and resp. [44] for its extension to the quantum case)**.** This technique in particular requires that the interaction is *translation invariant* and the many-particle potentials have *at most quadratic growth* (i.e., $(\mathbf{W_0})$ holds with $R = 2$). As was shown in [44], for the subclass of boundary conditions $\xi \in \Omega_{(ss)t} \subset \Omega_{(s)t}$ (for instance, such that $\sup_{k\in\mathbb{Z}^d} |\xi_k|_{L^2_\beta} < \infty$) the family of probability kernels $\pi_\Lambda(d\omega|\xi)$, $\Lambda \Subset \mathbb{Z}^d$, $\Lambda \nearrow \mathbb{Z}^d$, is tight (in the sense of local weak convergence on Ω) and has at least one accumulation point μ from the subset of superstable Gibbs measures $\mathcal{G}_{(ss)t} \subset \mathcal{G}_t$ defined by (27). This technique also shows that any $\mu \in \mathcal{G}_{(ss)t}$ is a priori of sub-Gaussian growth.

(iii) Cluster expansions is one of the most powerful methods for the study of Gibbs fields, but it works only in a *perturbative regime*, i.e., when an effective parameter of the interaction is small. In particular, various versions of this technique imply both existence and also uniqueness (but in some weaker than the DLR sense) of the associated infinite volume Gibbs distributions (see, e.g., [41], [42] and references therein).

(iv) Method of correlations inequalities involves more detailed information about the structure of the interaction (for instance, whether they are ferromagnetic or convex, cf. Remark 1). Starting from a number of correlations inequalities (such as *FKG, GKS, Lebowitz, Brascamp-Lieb* etc.) commonly known for classical lattice systems, by a lattice approximation technique (similar to the one used in Euclidean field theory) one can extend them to the quantum case (cf., e.g., [3]).

(v) Method of reflection positivity (as a part of **(iv)**) applies to *translation invariant* systems with *nearest-neighbours pair interactions* (i.e., when $V_k := V$, $W_{\{k,k'\}} := W$, and $W_{\{k,k'\}} = 0$ if $|k-k'| > 1$). The proper modification of this technique for the QLS (2) gives the existence of so-called *periodic* Gibbs states at least under Assumptions $(\mathbf{V_0})$, $(\mathbf{W_0})$ imposed in Sect. 3. Moreover, the reflection positivity method can also be used to study phase transitions in such models with the double-well anharmonicity V. This has been implemented under certain conditions (in the dimension $d \ge 3$ and for large enough $\beta, \mathfrak{m} >> 1$), e.g., in [3], [24], [35].

(v) Method of stochastic dynamics (also referred to in quantum physics as "*stochastic quantization*"; see, e.g., [25], [13] and the related bibliography therein). In this method the Gibbs measures are directly treated as invariant (more precise, reversible) distributions for the so-called Glauber or Langevin stochastic dynamics. However, some additional technical assumptions are required on the interaction (among them *at most quadratic growth* of the pair potentials $W_{\{k,k'\}}(q,q')$) in order to ensure the solvability of the corresponding stochastic evolution equations in infinite dimensions (not to mention the extremely difficult ergodicity problem for them). This method has been first applied in [13] to prove existence of Euclidean Gibbs states for the particular QLS model (2).

II. A priori estimates for measures in $\mathcal{G}_t$. Theorem 2 above contributes to the fundamental problem of getting *uniform estimates* on correlation functionals of Gibbs measures in terms of parameters of the interaction. This problem was initially posed for classical lattice systems in [15], [21] and is closely related with the compactness of the set of tempered Gibbs states (cf. **Corollary** from Theorem 2); we refer also to [13] for a detailed discussion of the classical lattice case. There are very few results in the literature about a priori integrability properties of tempered Gibbs measures on loop or path spaces (see, for instance, [31], [43] for the case of Euclidean $P(\phi)_1$-fields and resp. [13] for the case of quantum anharmonic crystals). All of them are based on the method of stochastic dynamics just mentioned above. It is worth noting that the other methods listed under I (i)–(v) give also some estimates on limit points for π_Λ, $\Lambda \Subset \mathbb{Z}^d$, but not uniformly for all $\mu \in \mathcal{G}_t$. Besides, the finiteness of the moments of the measures $\mu \in \mathcal{G}_t$ is also useful for the study of Gibbs measures by means of the associated Dirichlet operators $\mathbb{H}_\mu$ in the spaces $L^p(\mu)$, $p \geq 1$, (this is known as the Holley–Stroock approach). In particular, by [9], [10] μ is an *extreme point* (or pure phase) in $\mathcal{G}_t$, if and only if the corresponding Markov semigroup $\exp(-t\mathbb{H}_\mu)$, $t \geq 0$, is *ergodic* in $L^2(\mu)$ (which extends the well known results in [36] related to the Ising model).

III. Uniqueness problem. The validity of the sufficient conditions of the *Dobrushin uniqueness criterion* [Do70] for the QLS's (42) with *convex pair interactions of at most quadratic growth* has been first verified in [11], [12]. In doing so, the coefficients of Dobrushin's matrix were estimated by means of log-Sobolev inequalities proved on the single loop spaces L^2_β and the uniqueness of $\mu \in \mathcal{G}_t$ was established for small values of the inverse temperature $\beta \in (0, \beta_0)$, but under conditions *independent of the particle mass* $\mathfrak{m} > 0$ (and hence in the quasiclassical regime also). The exact statement for the particular QLS Model I is contained in our main **Theorem 5** above. For a *special class* of *ferromagnetic models* with the polynomial self-interaction (5), these results have been essentially improved in the recent series of papers [2]–[4]. The latter papers propose a new technique which combines the classical ideas of [15], [40] based on the use of FKG and GKS correlation inequalities with the spectral analysis of one-site oscillators (1) specific for the quantum case. The strongest result of such type, obtained in [4] and quoted here as **Theorem 6**, establishes the uniqueness of $\mu \in \mathcal{G}_t$ in the small-mass domain $\mathfrak{m} \in (0, \mathfrak{m}_0)$ *uniformly at all values of* $\beta > 0$. This provides a mathematical justification for the well-known physical phenomenon that structural phase transition for a given mass $\mathfrak{m} > 0$ can be suppressed not only by thermal fluctuations (i.e., high temperatures $\beta^{-1} > \beta_{\rm cr}^{-1}$), but for the light particles (with $\mathfrak{m} < \mathfrak{m}_{\rm cr}$) also by the quantum fluctuations (i.e., tunneling in a double-well potential) simultaneously at all temperatures $\beta > 0$. On the other hand, for small masses $\mathfrak{m} << 1$, the convergence of cluster expansions (independently of the boundary condition) has even been proved uniformly for all values of the temperature including the ground state case $\beta = \infty$, see [42]. However, in the case of unbounded spin systems such convergence of cluster expansions does not yet imply the DLR uniqueness. The uniqueness of $\mu \in \mathcal{G}_t$ in the QLS (42) with *superquadratic growth* of the many-particle interaction is a completely open problem, which will be the subject of our forthcoming paper. Another important and long-standing analytical problem is to find sufficient conditions for the uniqueness of symmetrizing measures in infinite dimensional spaces satisfying (IbP)-formulas like (39) with the given logarithmic derivatives b_{h_i} (for particular results on this topic see [19]).

Although our results are mainly concerned with the first three problems described above, for completeness of the exposition we also mention the following important directions:

IV. Decay of correlations. First of all, the exponential decay of spin correlations for Gibbs measures is one of the standard applications of the *Dobrushin contraction technique* (cf., e.g., [23], [27]). In particular, it implies that for $\mu \in \mathcal{G}_t$ (which in this case is a priori unique)

$$Cov_\mu\left(f(\omega_k,\varphi)_{L^2_\beta},\ g(\omega_{k'},\varphi')_{L^2_\beta}\right) \leq \mathcal{K}\exp(-\varepsilon|k-k'|)||f'||_{L^\infty}||g'||_{L^\infty}||\varphi||_{L^2_\beta}||\varphi'||_{L^2_\beta}$$

with some $K, \varepsilon > 0$ uniformly for all $k, k' \in \mathbb{Z}^d$, $\varphi, \varphi' \in L^2_\beta$ and $f, g \in C^1_b(\mathbb{R})$ (cf. [12]). Another general analytical approach to the decay of correlations is based on the *spectral gap estimates* for the corresponding Dirichlet operator $\mathbb{H}_\mu$ (see [17], [38], [50] for its realization for the classical spin systems (23). For quantum systems, however, the problem of getting the spectral gap estimates for $\mathbb{H}_\mu$ (or equivalently, for all $\mathbb{H}_{\mu_{\pi\Lambda(d\omega|\xi)}}$ uniformly w.r.t. volume and boundary conditions) has not yet been studied in the literature (except the trivial case of strictly convex interaction potentials) and will be the subject of our forthcoming paper. On the other hand, for the quantum ferromagnets with polynomial self-interactions like (5), the exponential decay as $|k-k'| \to \infty$ of the two-point correlations $Cov_{\mu_{\Lambda(d\omega|\xi)}}(\omega_k(\tau), \omega_{k'}(\tau'))$, uniformly in $\tau, \tau' \in S_\beta$, $\Lambda \Subset \mathbb{Z}^d$ and $\xi \in \Omega_t$, has been used in [4] as a crucial step for proving uniqueness for $\mu \in \mathcal{G}_t$. Moreover, for such quantum systems one expects the *complete equivalence* between the Dobrushin-Shlosman mixing conditions, exponential decay of correlations and Poincaré and log-Sobolev inequalities for the corresponding Dirichlet operators $\mathbb{H}_\mu$ (this equivalence has been first shown in [49] for lattice systems with compact spins and respectively extended in [50] to classical ferromagnetic systems with unbounded spins). Poincaré and log-Sobolev inequalities are commonly accepted in the literature as important tools to describe the links between the relaxation of stochastic dynamics and its equilibrium (e.g., Gibbs) measures (cf., e.g., [50], [13]).

V. Phase transitions. There are basically *two general methods* for proving existence of phase transitions (i.e., non-uniqueness of $\mu \in \mathcal{G}_t$) for *low temperatures* β^{-1}, namely, the *reflection positivity* (for $d \geq 3$) and the *energy-entropy (Peierls-type) argument* (for $d \geq 2$). However, in practice their successful applications to quantum lattice systems have been limited so far to ferromagnetic $P(\phi)$-models (cf., e.g., [14], [24], [33], [35]). The first method (already mentioned in Item I.(v)) enables one to prove the positivity of *a long-range order parameter* $\lim_{|\Lambda|\to\infty} E_{\mu_{per,\Lambda}} \left[\sum_{k\in\Lambda} \omega_k(\tau)\right]^2 /|\Lambda|^2$ for large enough $\mathfrak{m} > \mathfrak{m}_0$ and $\beta > \beta_0(\mathfrak{m}_0)$ via the so-called infrared (Gaussian) bounds on two-point correlation functions $E_{\mu_{per,\Lambda}}\omega_k(\tau)\omega_{k'}(\tau)$ w.r.t. the local Gibbs measures $\mu_{\mathrm{per},\Lambda}$ with periodic boundary conditions (cf. [24]). The second method has originally been discovered (as the so-called Peierls argument) for the Ising model and further developed to apply to rather general classical spin systems (now known as the Pirogov-Sinai contour method, cf. [48]). Its quantum modification was firstly implemented to the study of phase transition in the $(\varphi^4)_2-$model of Euclidean field theory (cf. [33]) and then in [14] to its lattice approximation (2) with $V(q_k) = \lambda(q_k^2 - \lambda^{-1})^2 + \lambda q_k^2/2$, where $\lambda > 0$ is the strength of the interaction. Following the idea of [33], [14], one defines a "*collective spin variable*" $\sigma_k :=\mathrm{sign}\int_{S_\beta} \omega_k(\tau)d\tau$ taking values ± 1 and a long-range parameter as the correlation function $< \sigma_k\sigma_{k'} >:= \lim_{|\Lambda|\to\infty} E_{\mu_\Lambda^{\mathrm{per}}}\sigma_k\sigma_{k'}$. Then, for any fixed λ, the existence of long-range behaviour, and hence phase transition, follows from the estimate $< \sigma_k\sigma_{k'} >\geq 1/2$ valid for large enough values of $\mathfrak{m}$ and β.

VI. Euclidean ground states. Of special interest for quantum systems is the case of *zero absolute temperature*, i.e., $\beta = \infty$, which is technically more complicated and less studied in the literature. In particular, it involves an important problem of the operator realization of the formal Hamiltonian (2) in quantum mechanics (cf. [1], [16]). The corresponding Gibbs measures $\mu \in \mathcal{G}_{gr}$ on the *"path lattice"* $[C(\mathbb{R})]^{\mathbb{Z}^d}$, so-called *Euclidean ground states*, also allow the DLR-description, but through a family of local specifications $\pi_{I\times\Lambda}$ indexed by "time-space" windows $I \times \Lambda$ with $I \Subset \mathbb{R}$, $\Lambda \Subset \mathbb{Z}^d$, cf. [41]. A principal difference with the previous case $0 < \beta < \infty$ is that now there is not available any such (independent from boundary conditions ξ) reference measure so that all $\pi_{I\times\Lambda}(d\omega|\xi)$ are defined as its Gibbs modifications. So far, there are very few rigorous results about Gibbs measures on the path space $[C(\mathbb{R})]^{\mathbb{Z}^d}$, which all are mainly related to the existence problem. A recent progress in this direction was achieved in the series of papers [41], [42], [39], where the limit measures $\lim_{\Lambda\nearrow\mathbb{Z}^d} \lim_{I\nearrow\mathbb{R}} \pi_{I\times\Lambda} \in \mathcal{G}_{gr}$ for the $P(\phi)-$lattice models (2) have been constructed through cluster expansions w.r.t. the small mass parameter $\mathfrak{m} << 1$. At the same time, for fixed $\Lambda \Subset \mathbb{Z}^d$, the corresponding unique Gibbs measures $\mu_{gr,\Lambda} := \lim_{I\nearrow\mathbb{R}} \pi_{I\times\Lambda}$ on the path space $[C(\mathbb{R})]^{\Lambda}$ are well-known as the $P(\phi)_1-$processes and can be looked upon as a special case of Euclidean field theory in space-dimension zero (cf. [43], [18]). Besides, the Gibbs measures on the path space $[C(\mathbb{R})]^{\mathbb{Z}^d}$ also appear in a natural way as weak solutions for SDE's in $\mathbb{Z}^d$ ([22], [41]). Within the Schwerpunktprogramm, related Gibbs measures on path spaces have detailed been studied by J. Loerinczi and H. Spohn (see their survey in this volume).

In this respect it should be mentioned that in the recent preprint [34] some (deterministic) version of integration by parts for local specifications has been used to prove existence of Gibbs measures relative to Brownian motion on the path space $C(\mathbb{R})$. Finally, let us note that our method based on the (IbP)-formula (39) can also be modified to apply to the case of zero absolute temperature, i.e., $\beta = \infty$, and corresponding symmetrizing measures on $[C(\mathbb{R})]^{\mathbb{Z}^d}$. This case is under present investigation.

VI. Random lattice systems. At this stage the case of spin systems with *random interactions* as those studied within the Schwerpunktprogramm by A. Bovier and Ch. Külske (see their contribution to this volume) has not yet been considered. We think, however, that our method can also be applied to some of such situations, at least in a modified way.

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Albeverio, S. : Institut für Angewandte Mathematik, Universität Bonn, D-53155 Bonn, Germany; BiBoS Research Centre, Bielefeld, Germany; and CERFIM, Locarno, Switzerland.

Kondratiev, Yu.G..: Fakultät für Mathematik and BiBoS Research Centre, Bielefeld Universität, D-33615 Bielefeld, Germany; and Institute of Mathematics, NASU, Kiev, Ukraine.

Pasurek, T. : Fakultät für Mathematik and BiBoS Research Centre, Bielefeld Universität, D-33615 Bielefeld, Germany.

Röckner, M. : Fakultät für Mathematik and BiBoS Research Centre, Bielefeld Universität, D-33615 Bielefeld, Germany.

E-mail: albeverio@uni-bonn.de; kondrat@mathematik.uni-bielefeld.de;
pasurek@physik.uni-bielefeld.de roeckner@mathematik.uni-bielefeld.de