Energy Levels of Quantum Periodic Systems and Quantum Chaos

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Abstract

A concept of energy levels of quantum systems periodically depending on time and having a discrete or a continuous spectrum is proposed. A natural concept of adjacent and effective energy levels as well as a notion of distance between the levels are introduced. The results of the theory presented are applied to justify the quantum chaos conjecture for a class of systems including, as a special case, the “kicked rotator” model.

1 The concepts of energy levels of quantum systems and distances between them

We consider a quantum system given by an Hamiltonian operator $\hat{H} = \hat{H}(t)$ depending periodically on time $t$, i.e., $\hat{H}(t) = \hat{H}(t + T)$, where $T > 0$ is the operator’s period. The corresponding Schrödinger equation is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi ,$$

where $\Psi = \Psi(q,t)$ is the wave function, namely a function of $q$ for fixed $t$ belonging to Hilbert space $L^2$. Let $\Psi(q,t)$ be the solution of equation (1) for $t \geq t_0$ satisfying the initial condition $\psi(q) = \Psi(q,t_0) \in L^2$. We define the (Floquet) monodromy operator $U = U_{t_0} : \Psi(q,t_0) \rightarrow \Psi(q,t_0 + T)$. It is well-known that $U$ is a unitary operator and for distinct values of $t_0$ the corresponding operators $U_{t_0}$ are unitarily equivalent to each other [1]. Hence, its spectrum is a set of complex numbers of absolute value 1. First, we assume that the spectrum of the operator $U$ is discrete and is represented by the sequence of eigenvalues $\lambda_n$ such that $\lambda_n = e^{i\alpha_n}$ where $n \in \mathbb{Z}$ is an integer and $\alpha_n$ is a real number. Let $\psi_{\lambda_n}(q)$ be the eigenfunction corresponding to the eigenvalue $\lambda_n$. 


such that \( U\psi_{\lambda_n}(q) = \lambda_n\psi_{\lambda_n}(q) \). Then solution \( \Psi_{\alpha_n}(q, n) \) of equation (1) with the initial condition \( \Psi_{\alpha_n}(q, t_0) = \psi_{\lambda_n}(q) \) satisfies

\[
\Psi_{\alpha_n}(q, t_0 + T) = e^{-i\alpha_n} \Psi_{\alpha_n}(q, t_0) .
\]

Such a solution \( \Psi_{\alpha_n}(q, t) \) is called quasistationary, and the corresponding value \( E_n = \frac{\alpha_n}{2\pi} \) introduced in [13] is called quasienergy. In this article we will call the value \( \alpha_n \) the energy level.

We assume now that the spectrum of operator \( U \) is continuous (i.e., there are no eigenvalues) and that \( U \) has the following structure: The Hilbert space \( L^2 \) has a basis \( \psi_n(q)(n \in \mathbb{Z}) \) satisfying, for each \( n \in \mathbb{Z} \),

\[
U\psi_n(q) = e^{-i\mu_n(q)}\psi_n(q) .
\]

In (2) \( \mu_n(q) \) is a real function such that for any pair \((n', n'')\) of integers, the function \( \Delta_{n', n''}(q) = \mu_{n'}(q) - \mu_{n''}(q) \) takes only finitely or countably many different values. The functions \( \Delta_{n', n''}(q) \) play the role of distances between the energy levels \( \mu_n(q) \) which exhibited in quantum mechanics over passing from one energy level to another. Thus, despite the fact that the set of the energy levels \( \mu_n(q) \) is not discrete, the set of all possible values of the distances between them is discrete; this set treats the physical meaning of \( \mu_n(q) \).

We consider a special important case for which operator \( U = U_2 \cdot U_1 \) is the composition of two unitary operators \( U_1 \) and \( U_2 \) such that operator \( U_1 \) is represented by an infinite diagonal matrix with the diagonal entries \( \lambda_n = e^{-i\alpha_n} \), and \( U_2 \) is the operator of multiplication by the function \( \lambda(q) = e^{-\mu(q)} \), i.e., for any \( n \in \mathbb{Z} \) the following equalities hold:

\[
U\psi_n(q) = \lambda^{(n)}(q)\Psi_n(q) , \quad \lambda^{(n)}(q) = e^{-i(\mu(q) + \alpha_n)} .
\]

The functions \( \mu_n(q) = \mu(q) + \alpha_n \) are the energy levels, and the distances \( \Delta_{n', n''}(q) \) do not depend on the basis \( \psi_n(q) \). By (3), this statement is equivalent to the statement that the spectrum of operator \( U_* = \frac{1}{\lambda^{(q)}(q)} U \) is discrete and is invariant. Consequently, the eigenvalue \( \frac{\lambda^{(n)}(q)}{\lambda^{(n')}_{(q)}(q)} \) of operator \( U_* \) and the distances \( \Delta_{n', n''}(q) = i \left( \ln \frac{\lambda^{(n)}(q)}{\lambda^{(n')}(q)} - \ln \frac{\lambda^{(n)}(q)}{\lambda^{(n'')}(q)} \right) \) do not depend on the basis \( \psi_n(q) \).

2 Adjacent, effective, and noneffective energy levels of quantum systems

Let \( n' \) and \( n'' \) be two distinct integers. The energy levels \( \mu_{n'}(q) \) and \( \mu_{n''}(q) \) are called adjacent if for all \( q \) does not exist on integer \( n, n \neq n', n'' \), such that \( \mu_n(q) \) belongs to the closed interval \([\mu_{n'}(q), \mu_{n''}(q)]\).

\[
\min(\mu_{n'}(q), \mu_{n''}(q)) \leq \mu_n(q) \leq \max(\mu_{n'}(q), \mu_{n''}(q)) .
\]

The Hilbert space \( L^2 \) is the space of \( 2\pi \)-periodic square integrable functions and assume that the energy levels are defined with respect to its orthogonal basis \( \psi_n(q) = e^{inq}(n \in \mathbb{Z}) \).
We represent the energy level \( \mu_n(q) \) as follows:

\[
\mu_n(q) = 2\pi m_n(q) + 2\pi\beta_n(q),
\]

where \( m_n(q) \) is an integer and function \( \beta_n(q) \) satisfies \( 0 \leq \beta_n(q) < 1 \). It follows from (5) that \( m_n(q) = \left\lfloor \frac{\mu_n(q)}{2\pi} \right\rfloor \) is the integer part of the number \( \frac{\mu_n(q)}{2\pi} \) and \( \beta_n(q) = \left\{ \frac{\mu_n(q)}{2\pi} \right\} = \frac{\mu_n(q)}{2\pi} - \left\lfloor \frac{\mu_n(q)}{2\pi} \right\rfloor \) is its fractional part. From equalities (2) and (5) it follows that the first term \( 2\pi m_n(q) \) in (5) does not affect to the wave functions \( \psi_n(q) \). Therefore, we call the function \( 2\pi m_n(q) \) the noneffective energy level. On the contrast, we call the second term, \( 2\pi\beta_n(q) \), the effective energy level. For two energy levels, \( \mu_{n'}(q) \) and \( \mu_{n''}(q) \) with \( \mu_{n'}(q) \leq \mu_{n''}(q) \), we define the distance \( \rho(\mu_{n'}(q), \mu_{n''}(q)) \) between them by

\[
\rho(\mu_{n'}(q), \mu_{n''}(q)) = \beta_{n''}(q) - \beta_{n'}(q).
\]

### 3 Justification of quantum chaos conjecture for some class of quantum systems

Quantum chaos theory studies the distribution of the distances between the adjacent energy levels of a quantum system. There are two main conjectures based on numerical simulations concerning distribution laws of these distances ([2],[6],[7],[9]). The first conjecture concerns quantum systems that are quantum analogues of classical integrable systems. The conjecture states that the distribution law of distances for such a system is close to the Poisson distribution with the density \( \exp(-\sigma) \) and coincides with it asymptotically as \( \sigma \to 0 \). The second conjecture states that for the quantum analogue of a classical strong nonintegrable system, the distribution law of distances is close to the distribution with the density const \( \sigma \) as \( \sigma \to 0 \). In the present article, the quantum chaos conjecture is justified for a special class of quantum system. This class includes, as a special case, a “kicked rotator” model ([1],[3],[4],[5],[8],[9]).

To describe the quantum model, first we introduce the corresponding classical model. We consider a one-dimensional nonlinear oscillator associated to the Hamiltonian function \( H = H(q, I, t) = H_0(I) + H_1(q, t) \), where \( I, q \) are the 'action-angle' variables, \( t \) is an independent variable, and function \( H_1(q, t) \) has period \( 2\pi \) in \( q \), period \( T > 0 \) in \( t \), and is represented in the form

\[
H_1(q, t) = F(q) \sum_{k = -\infty}^{\infty} \delta(t - kT).
\]

Here \( F(q) \) is a smooth \( 2\pi \)-periodic function, \( \delta = \delta(t) \) is the Dirac measure, and the summation is taken over all integers \( k \). The first rigorous results on behavior of the system’s solutions with the Hamiltonian function \( H = H_0(I) + H_1(q, t) \), where function \( H_0(I) \) is that of a general form, have been established in [8]. We assume here that

\[
H_0(I) = \sum_{s=0}^{\infty} b_s I^s
\]

is an entire function (in particular, a polynomial) with coefficients \( b_s = \frac{1}{\hbar^s}, s = 0, 1, \ldots, \) where \( \hbar \) is Planck’s constant and \( a_s \) are real numbers. In a special case, when \( a_s = 0 \) for \( s \neq 2 \), \( F(q) = c \cos q \), \( c \) is a constant, this system is nothing else than a “kicked rotator.”

Z).
Getting onto the quantum model, we introduce the Hilbert space $L^2$ of complex $2\pi$-periodic in $q$ square integrable functions as the space of states of the quantum system and also introduce impulse operator $\hat{I} = \frac{\hbar}{i} \frac{\partial}{\partial q}$. The wave function $\Psi = \Psi(q,t) \in L^2$ satisfied the Schrödinger equation (1), where $\hat{H} = \hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$, $\hat{H}_0 = \sum_{s=0}^{\infty} b_s \hat{I}^s$ and operator $\hat{H}_1(t)$ is the limit, as $\epsilon \to 0$ ($\epsilon > 0$), of the operators of multiplication by function $\hat{H}^{(c)}_1$ obtained from function $\hat{H}_1$ in (7) after replacing the delta-function by a smooth function $\delta_\epsilon$ with support on the interval $[0, \epsilon]$ with $\int_0^\epsilon \delta_\epsilon = 1$.

Let $\Psi_+(q,nT)$ denote the solution of equation (1) immediately after the instant $t = nT$ ($n \in \mathbb{Z}$). We define the monodromy operator $U : \Psi_+(q,nT) \to \Psi_+(q,(n+1)T)$ to be the limit as $\epsilon \to 0$, of the monodromy operators $U^{(c)}$ corresponding to equation (1) with operator $\hat{H}(t)$ on the right hand side replaced by operator $\hat{H}_0 + \hat{H}^{(c)}_1$, where $\hat{H}^{(c)}_1$ is the operator of multiplication by the function $\hat{H}^{(c)}_1$. It has been proven in [1], [5] and [12] that this limit exists and has the following form: $U = \exp \left( -i \frac{T}{\hbar} \right) \exp \left( i \frac{TH_0}{\hbar} \right)$.

Moreover, if $\psi(q) = \exp(inq)$, then $U \psi_n(q) = \lambda_n(q) \psi_n$, where

$$\lambda_n(q) = \exp(-i \mu_n(q)), \quad \mu_n(q) = \left( F(q) + T \sum_{s=0}^{\infty} a_s q^n \right) / \hbar .$$

(8)

The equalities (8) show that the functions $\mu_n(q)$ are the energy levels in the sense of the definition given in Section 1. In particular, if $F(q) = \text{const}$, then the spectrum of $U$ is discrete, $\psi_n(q)$ are the corresponding eigenfunctions, and the $\lambda_n(q)$’s are the corresponding eigenvalues.

Assume that the real function $G(x) = \frac{T}{2\pi} \sum_{s=0}^{\infty} a_s x^s$ of satisfies the following condition:

(i) all zeros of $G(x)$ (if they exist) lie in a bounded region of the real line;

(ii) $\lim_{n \to \infty} |G(n+1) - G(n)| = \infty$;

(iii) for any real numbers $\sigma_1$ and $\sigma_2$ satisfying $0 < \sigma_\nu \leq 1$, $\nu = 1, 2$, the number $D_N(\sigma_1, \sigma_2)$ of two-dimensional vectors $\vec{\kappa}_n = \{G(n)\} \{G(n+1)\}$ in the sequence $\vec{\kappa}_1, \ldots, \vec{\kappa}_N$ that belong to rectangle $\Pi = \{y = (y_1, y_2) : 0 \leq y_1 < \sigma_1, 0 \leq y_2 < \sigma_2\}$ satisifies $\lim_{N \to \infty} \frac{D_N(\sigma_1, \sigma_2)}{N} = \sigma_1 \sigma_2$.

Condition (iii) means that the joint distribution of two adjacent fractional parts of function $G(x)$ is uniform. All the three conditions hold for polynomials $G(x) = \sum_{s=0}^{\ell} a_s x^s$ of degree $\ell \geq 2$, for which at least one of the coefficients $a_2, a_3, \ldots, a_\ell$ is an irrational number ([10]). By (8), if the conditions (i) and (ii) hold, then there is a number $n_0 \geq 0$ for which the energy levels $\mu_n(q)$ and $\mu_{n+1}(q)$ are adjacent whenever $n > n_0$. The adjacent energy levels correspond to the adjacent quantum states $\psi_n(q)$ and $\psi_{n+1}(q)$ with the adjacent frequences $\frac{n}{2\pi}$ and $\frac{n+1}{2\pi}$. It follows from (iii) that for $0 < \sigma \leq 1$ and for the number $D_N^{(\sigma)}(\sigma,q)$ of values $n$, $n \in \{1, \ldots, N\}$, for which $0 \leq \left\{ \frac{\mu_n(q)}{2\pi} \right\} - \left\{ \frac{\mu_{n+1}(q)}{2\pi} \right\} < \sigma$, the following holds:

$$P^*(\sigma) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{D_N^{(\sigma)}(\sigma,q)}{N} = |\Pi^*| = \sigma - \frac{\sigma^2}{2} .$$

(9)
Here, $\Pi^*$ stands for the set

$$\Pi^* = \{ y = (y_1, y_2) : 0 \leq y_1 < 1, 0 \leq y_2 < 1, 0 \leq y_2 - y_1 < \sigma \}$$

and $|\Pi^*|$ stands for the area of $\Pi^*$. In view of (6) and (9), the distribution function $P^*(\sigma)$ of the distances between the adjacent energy levels differs from the Poisson’s law distribution function $1 - \exp(-\sigma)$ with density $\exp(-\sigma)$ by terms of third order in $\sigma$, as $\sigma \to 0$. Thus, the quantum chaos conjectures holds for the class of quantum systems in question. In the special case, when $H_0(I)$ is a general polynomial, this result has been obtained in [11] and [12] from pure mathematical point of view.

References


