Strong Solutions of Stochastic Generalized Porous Media Equations: Existence, Uniqueness and Ergodicity *

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Abstract

Explicit conditions are presented for the existence, uniqueness and ergodicity of the strong solution to a class of generalized stochastic porous media equations. Our estimate of the convergence rate is sharp according to the known optimal decay for the solution of the classical (deterministic) porous medium equation.

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1 Main Results

Let $(E, \mathscr{M}, \mathbf{m})$ be a separable probability space and $(L, \mathscr{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ having discrete spectrum with eigenvalues

$$0 > -\lambda_1 \geq -\lambda_2 \geq \cdots \to -\infty$$

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and $L^2(m)$-normalized eigenfunctions $\{e_i\}$ such that $e_i \in L^{r+1}(\mathbf{m})$ for any $i \geq 1$, where $r > 1$ is a fixed number throughout this paper. We assume that L^{-1} is bounded in $L^{r+1}(\mathbf{m})$, which is e.g. the case if L is a Dirichlet operator (cf. e.g. [11]) since in this case the interpolation theorem or simply Jensen's inequality implies $\|\mathrm{e}^{tL}\|_{r+1} \leq \mathrm{e}^{-\lambda_1 t 2/(r+1)}$ for all $t \geq 0$. A classical example of L is the Laplacian operator on a smooth bounded domain in a complete Riemannian manifold with Dirichlet boundary conditions.

In this paper we consider the following stochastic differential equation:

$$\mathrm{d}X_t = (L\Psi(X_t) + \Phi(X_t))\mathrm{d}t + Q\mathrm{d}W_t, \tag{1.1}$$

where Ψ and Φ are (non-linear) continuous functions on $\mathbb{R}$, and Q is a densely defined linear operator on $L^2(\mathbf{m})$ with $Qe_i := \sum_{j=1}^\infty q_{ji} e_j$ $(i \geq 1)$ such that $q_i^2 := \sum_{j=1}^\infty q_{ij}^2$ satisfies

$$q := \sum_{i=1}^\infty \frac{q_i^2}{\lambda_i} < \infty.$$

An appropriate Hilbert space H as state space for the solutions to (1.1) is given as follows. Let

$$H^1 := \Big\{ f \in L^2(m) \,:\, \sum_{i=1}^\infty \lambda_i \mathbf{m}(f e_i)^2 < \infty \Big\}.$$

Define H to be its topological dual with inner product $\langle\, , \,\rangle_H$. Identifying $L^2(\mathbf{m})$ with its dual we get the continuous and dense embeddings

$$H^1 \subset L^2(\mathbf{m}) \subset H.$$

We denote the duality between H and H^1 by $\langle\, , \,\rangle$. Obviously, when restricted to $L^2(\mathbf{m}) \times H^1$ this coincides with the natural inner product in $L^2(\mathbf{m})$, which we therefore also denote by $\langle\, , \,\rangle$, and it is also clear that

$$\langle f, g\rangle_H = \sum \lambda_i^{-1} \langle f, e_i\rangle \langle g, e_i\rangle, \quad f, g \in H.$$

Furthermore, in (1.1) $W_t = (b_t^i)_{i\in\mathbb{N}}$ is a cylindrical Brownian motion on $L^2(\mathbf{m})$ where $\{b_t^i\}$ are independent one-dimensional Brownian motions on a complete probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_t$ be the natural filtration of W_t. Then

$$QW_t := \sum_{i=1}^\infty \Big(\sum_{j=1}^\infty q_{ij} b_t^j \Big) e_i, \quad t \geq 0,$$

is a well-defined process taking values in H which is a martingale.

Recall that the classical porous medium equation reads

$$\mathrm{d}X_t = \Delta X_t^m \mathrm{d}t$$

on a domain in $\mathbb{R}^d$, see e.g. [1] and the references therein. So, we may call (1.1) the generalized stochastic porous medium equation. Recently, the existence and uniqueness of weak solutions as well as the existence of invariant probability measures for the stochastic porous medium equation, i.e. (1.1) with $L := \Delta$ on a bounded domain in $\mathbb{R}^d$ with Dirichlet boundary conditions, were proved in [5, 6, 4, 3]. In this paper we aim to prove the existence and uniqueness of strong solutions of (1.1), in particular, describe the convergence rate of the solution, for $t \to \infty$.

To solve (1.1), we assume that there exist some constants $c \geq 0$, $\eta, \sigma \in \mathbb{R}$ such that

$$\begin{aligned} &|\Psi(s)| + |\Phi(s)| \leq c(1+|s|^r), \\ &(s-t)(\Psi(s)-\Psi(t)) \geq \eta|s-t|^{r+1} + \sigma(s-t)^2, \quad s,t \in \mathbb{R}. \end{aligned} \tag{1.2}$$

Since by the Cauchy-Schwarz inequality one has

$$\begin{aligned} \frac{2^{3-r}|s-t|^{r+1}}{(r+1)^2} &\leq \frac{4}{(r+1)^2}\big(|s|^{(r+1)/2}\operatorname{sgn}(s) - |t|^{(r+1)/2}\operatorname{sgn}(t)\big)^2 \\ &= \bigg(\int_t^s |u|^{(r-1)/2}\mathrm{d}u\bigg)^2 \leq (s-t)\int_t^s |u|^{r-1}\mathrm{d}u, \end{aligned}$$

(1.2) holds if $\Psi(0) = 0$ and there exists $\kappa > 0$ such that (cf. [4])

$$\sigma + \frac{(r+1)^2}{4}\eta|s|^{r-1} \leq \Psi'(s) \leq \kappa(1+|s|^{r-1}), \quad s \in \mathbb{R}.$$

Next, assume that there exist $\theta < \eta$ and $\delta \leq \sigma$ such that

$$-\mathbf{m}\big((\Phi(x)-\Phi(y))L^{-1}(x-y)\big) \leq \theta\|x-y\|_{r+1}^{r+1} + \delta\|x-y\|_2^2, \quad x,y \in L^{r+1}(\mathbf{m}), \tag{1.3}$$

where here and in the sequel, $\|\cdot\|_p$ denotes the L^p-norm with respect to $\mathbf{m}$ for any $p \geq 1$. We note that since L^{-1} is bounded on $L^{r+1}(\mathbf{m})$ and $r > 1$, if there exist constants $c_1, c_2 \geq 0$ such that

$$|\Phi(s) - \Phi(t)| \leq c_1|s-t|^r + c_2|s-t|, \quad s,t \in \mathbb{R},$$

then

$$-\mathbf{m}\big((\Phi(x)-\Phi(y))L^{-1}(x-y)\big) \leq c_1\|L^{-1}\|_{r+1}\|x-y\|_{r+1}^{r+1} + c_2\lambda_1^{-1}\|x-y\|_2^2,$$

hence (1.3) holds for $\theta := c_1\|L^{-1}\|_{r+1}$ and $\delta := c_2\lambda_1^{-1}$.

Definition 1.1. Let $\nu(\mathrm{d}t) := \mathrm{e}^{-t}\mathrm{d}t$. An H-valued continuous $(\mathscr{F}_t)$-adapted process X_t is called a solution to (1.1), if $X \in L^{r+1}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$ such that

$$\langle X_t, e_i\rangle = \langle X_0, e_i\rangle + \int_0^t \mathbf{m}\big(\Psi(X_s)Le_i + \Phi(X_s)e_i\big)\mathrm{d}s + q_iB_t^i, \quad i \geq 1, t > 0, \tag{1.4}$$

where $B_t^i := \frac{1}{q_i}\sum_{j=1}^\infty q_{ij}b_t^j$ (:= 0 if $q_i = 0$) is an $(\mathscr{F}_t)$-Brownian motion on $\mathbb{R}$ (provided it is non-trivial).

Remark 1.1. (i) We note that (1.4) indeed makes sense, since by the first inequality in (1.2) we have

$$\Psi(X), \Phi(X) \in L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E,\, \nu \times P \times \mathbf{m}).$$

(ii) We emphasize that for each solution of (1.4) there also exists a vector-valued version of the equation. More precisely, the integral comes from an H-valued random vector with a natural integrand which, however, takes values in a larger Banach space $\mathbb{B}'$. To describe this in detail, we need some preparations.

Consider the separable Banach space $\mathbb{B} := L^{r+1}(\mathbf{m})$. Then we can obtain a presentation of its dual space $\mathbb{B}'$ through the embeddings

$$\mathbb{B} \subset H \equiv H' \subset \mathbb{B}',$$

where H is identified with its dual through the Riesz-isomorphism. In other words $\mathbb{B}'$ is just the completion of H with respect to the norm

$$\|f\|_{\mathbb{B}'} := \sup_{\|g\|_{r+1} \le 1} \langle f, g\rangle_H, \quad f \in H.$$

Since H is separable, so is $\mathbb{B}'$. We note that this is different from the usual representation of $\mathbb{B} = L^{r+1}(\mathbf{m})$ through the embedding

$$\mathbb{B} \subset L^2(\mathbf{m}) \equiv L^2(\mathbf{m})',$$

which, of course, gives $L^{(r+1)/r}(\mathbf{m})$ as dual. But it is easy to identify the isomorphism between $L^{(r+1)/r}(\mathbf{m})$ and $\mathbb{B}'$. Below ${}_{\mathbb{B}'}\langle\, ,\, \rangle_{\mathbb{B}}$ denotes the duality between $\mathbb{B}$ and $\mathbb{B}'$. Clearly, ${}_{\mathbb{B}'}\langle\, ,\, \rangle_{\mathbb{B}} = \langle\, ,\, \rangle_H$ on $\mathbb{B} \times H$.

Proposition 1.1. *The linear operator*

$$Lf := -\sum_{i=1}^{\infty} \lambda_i\, \mathbf{m}(fe_i)e_i, \quad f \in L^2(\mathbf{m}),$$

defines an isometry from $L^{(r+1)/r}(\mathbf{m})$ to $\mathbb{B}'$ with dense domain. Its (unique) continuous extension $\bar{L}$ to all of $L^{(r+1)/r}(\mathbf{m})$ is an isometric isomorphism from $L^{(r+1)/r}(\mathbf{m})$ onto $\mathbb{B}'$ such that

$$(1.5) \qquad {}_{\mathbb{B}'}\langle -\bar{L}f, g\rangle_{\mathbb{B}} = \mathbf{m}(fg) \quad \textit{for all } f \in L^{(r+1)/r}(\mathbf{m}),\ g \in L^{r+1}(\mathbf{m}).$$

Proof. Let $f \in L^2(\mathbf{m})$, $N > n \ge 1$. Then

$$\Big\|\sum_{i=n}^{N} \lambda_i\, \mathbf{m}(fe_i)e_i\Big\|_{\mathbb{B}'} = \sup_{\|g\|_{r+1}\le 1}\Big|\mathbf{m}\Big(g\sum_{i=n}^{N}\mathbf{m}(fe_i)e_i\Big)\Big|$$
$$= \Big\|\sum_{i=n}^{N}\mathbf{m}(fe_i)e_i\Big\|_{(r+1)/r}.$$

Since $f = \sum_{i=1}^{\infty} \mathbf{m}(fe_i)e_i$ with the series converging in $L^2(\mathbf{m})$, hence in $L^{\frac{r+1}{r}}(\mathbf{m})$ (because $r > 1$), the first part of the assertion follows, and $\bar{L}$ is an isometry from $L^{(r+1)/r}(\mathbf{m})$ into $\mathbb{B}'$. Now let $T \in \mathbb{B}'$. Then there exists $f \in L^{(r+1)/r}(\mathbf{m})$ such that for all $g \in L^{r+1}(\mathbf{m})$

$$
\begin{aligned}
{}_{\mathbb{B}'}\langle T, g\rangle_{\mathbb{B}} &= \mathbf{m}(fg) \\
&= \lim_{n\to\infty} \mathbf{m}(f_n g)
\end{aligned}
\tag{1.6}
$$

for some $f_n \in D(L)$ such that $\lim_{n\to\infty} f_n = f$ in $L^{(r+1)/r}(\mathbf{m})$. Hence for all $g \in L^{r+1}(\mathbf{m})$

$$
\begin{aligned}
{}_{\mathbb{B}'}\langle T, g\rangle_{\mathbb{B}} &= \lim_{n\to\infty} \mathbf{m}\big(f_n\, L(L^{-1}g)\big) \\
&= \lim_{n\to\infty} \mathbf{m}(Lf_n\, L^{-1}g) \\
&= \lim_{n\to\infty} \langle -Lf_n, g\rangle_H = -\, {}_{\mathbb{B}'}\langle \bar{L}f, g\rangle_{\mathbb{B}}
\end{aligned}
\tag{1.7}
$$

and the second assertion is proved. Since any $f \in L^{(r+1)/r}(\mathbf{m})$ defines a $T \in \mathbb{B}'$, the last assertion follows from (1.6) and (1.7). □

Since L^{-1} is bounded on $L^{r+1}(\mathbf{m})$ by our assumptions on L (which was not used so far), we obtain the following consequence.

Corollary 1.2. *Let* $(L^{-1})' : L^{(r+1)/r}(\mathbf{m}) \to L^{(r+1)/r}(\mathbf{m})$ *be the dual operator of* $L^{-1} : L^{r+1}(\mathbf{m}) \to L^{r+1}(\mathbf{m})$*. Then the operator*

$$J : \bar{L} \circ (L^{-1})' : L^{(r+1)/r}(\mathbf{m}) \to \mathbb{B}'$$

extends the natural inclusion $L^2(\mathbf{m}) \subset H \subset \mathbb{B}'$ *and for all* $f \in L^{(r+1)/r}(\mathbf{m})$

$$
{}_{\mathbb{B}'}\langle Jf, g\rangle_{\mathbb{B}} = -\mathbf{m}(f\, L^{-1}g) \quad \text{for all } g \in L^{r+1}(\mathbf{m}).
\tag{1.8}
$$

Proof. If $f \in L^2(\mathbf{m})$, then $(L^{-1})'\, f = L^{-1}f$, hence $Jf = f \in L^2(\mathbf{m}) \subset H \subset \mathbb{B}'$. The last assertion follows by (1.5). □

Multiplied by λ_i^{-1}, (1.4) by the above now reads

$${}_{\mathbb{B}'}\langle X_t, e_i\rangle_{\mathbb{B}} = {}_{\mathbb{B}'}\langle X_0, e_i\rangle_{\mathbb{B}} + \int_0^t {}_{\mathbb{B}'}\big\langle \bar{L}\Psi(X_s) + J\Phi(X_s),\, e_i\big\rangle_{\mathbb{B}}\, \mathrm{d}s + {}_{\mathbb{B}'}\langle QW, e_i\rangle_{\mathbb{B}}.$$

By Remark 1.1, Proposition 1.1 and Corollary 1.2 the Bochner integrals

$$\int_0^t \big(\bar{L}\Psi(X_s) + J\Phi(X_s)\big)\, \mathrm{d}s, \quad t \geq 0,$$

exist in $\mathbb{B}'$. So, (1.4) can always be rewritten equivalently in vector form as an equation in $\mathbb{B}'$ as

$$
X_t = X_0 + \int_0^t \big(\bar{L}\Psi(X_s) + J\Phi(X_s)\big)\, \mathrm{d}s + QW_t, \quad t \geq 0.
\tag{1.9}
$$

Note that by Definition 1.1, $X_t \in H$ and also $QW_t \in H$, hence the integral in (1.9) is necessarily a continuous H-valued process.

Now we can state our main results.

Theorem 1.3. *Assume* (1.2) *and* (1.3) *with* $\sigma \geq \delta$ *and* $\eta > \theta$. *We have:*

(1) For any $\mathscr{F}_0/\mathscr{B}(H)$*-measurable* $\xi : \Omega \to H$ *with* $\mathbb{E}\|\xi\|_H^2 < \infty$ *there exists a unique solution* X *to (1.1) such that* $X_0 = \xi$. *Furthermore, there exists* $C > 0$ *such that*

$$\mathbb{E}\|X_t\|_H^2 \leq C(1 + t^{-2/(r-1)}), \quad t > 0. \tag{1.10}$$

In particular, for any $x \in H$ *there exists a unique solution* $X_t(x)$ *to* (1.1) *with initial value* x, *whose distributions form a continuous strong Markov process on* H.

(2) For any two solutions X *and* Y *of (1.1) we have for all* $t \geq s \geq 0$

$$\begin{aligned} \|X_t - Y_t\|_H^2 &\leq \left\{\|X_s - Y_s\|_H^{1-r} + (r-1)(\eta - \theta)\lambda_1^{(r+1)/2}(t-s)\right\}^{-2/(r-1)} \\ &\leq \|X_s - Y_s\|_H^2 \wedge \left\{(r-1)(\eta-\theta)\lambda_1^{(r+1)/2}(t-s)\right\}^{-2/(r-1)}. \end{aligned} \tag{1.11}$$

Consequently, setting $P_tF(x) := \mathbb{E}F(X_t(x))$ *for* $F : H \to \mathbb{R}$, *Borel measurable, so that the expectation makes sense, we have that* $(P_t)_{t>0}$ *is a Feller semigroup on* $C_b(H)$ *and, in addition, for Lipschitz continuous* F

$$\left|P_tF(x) - P_tF(y)\right| \leq \mathscr{L}(F)\|x - y\|_H, \quad x, y \in H, \tag{1.12}$$

where $\mathscr{L}(F)$ *is the Lipschitz constant of* F.

(3) P_t *has a unique invariant probability measure* μ *and for some constant* $C > 0$, μ *satisfies*

$$\sup_{x \in H} |P_tF(x) - \mu(F)| \leq C\mathscr{L}(F)t^{-1/(r-1)}, \quad t > 0, \tag{1.13}$$

for any Lipschitz continuous function F *on* H. *Moreover,* $\mu(\|\cdot\|_{r+1}^{r+1}) < \infty$.

(4) If $\sigma > \delta$ *then for any two solutions* X *and* Y *of (1.1) we have for all* $t \geq s \geq 0$

$$\|X_t - Y_t\|_H \leq \|X_s - Y_s\|_H \, \mathrm{e}^{-(\sigma-\delta)(t-s)} \tag{1.14}$$

and there exists $C > 0$ *such that*

$$\|X_t - Y_t\|_H \leq C\mathrm{e}^{-(\sigma-\delta)t}, \quad t \geq 1. \tag{1.15}$$

Consequently, for some constant $C > 0$,

$$\sup_{x \in H} |P_tF(x) - \mu(F)| \leq C\mathscr{L}(F)\mathrm{e}^{-(\sigma-\delta)t}, \quad t \geq 1, \tag{1.16}$$

for any Lipschitz continuous function F *on* H.

Remark 1.2. (1) When $Q = 0$, the Dirac measure δ_0 is the unique invariant measure. Thus, (1.13) with $F(x) := \|x\|_H$ implies

$$\sup_x \|X_t(x)\|_H \leq Ct^{-1/(r-1)}, \quad t > 0.$$

This coincides with the optimal decay of the solution to the classical porous medium equation obtained by Aronson and Peletier (see [2, Theorem 2]).

(2) In the case where $\Phi = 0$ and $\Psi(r) = \alpha r + r^m$ for $\alpha \geq 0$ and $m \geq 3$ odd, and $L := \Delta$ on a regular domain in $\mathbb{R}^d$, in [3] and [5] much stronger integrability results for the invariant measure have been proved, namely, if either $m = 3$ or $\alpha > 0$ then $\mu\big(\big|\nabla(\operatorname{sign} x\,|x|^n)\big|^2\big) < \infty$ for any $n \geq 1$.

(3) In the case where $L := \Delta$ on a bounded smooth domain in $\mathbb{R}^d$, the existence of an invariant measure μ was proved in [4] under the conditions that $\kappa_0|s|^{r-1} \leq \Psi'(s) \leq C\kappa_1|s|^{r-1}$ and $|\Phi(s)| \leq C+\delta|s|^r$ for some constants $C, \kappa_0, \kappa_1 > 0, \delta \in (0, 4\kappa_0\lambda_1(r+1)^{-2}$ and all $s \in \mathbb{R}$. Also in [4] stronger integrability properties for μ have been proved, namely that $\mu\big(\big|\nabla(\operatorname{sign} x\,|x|^{\ell})\big|^2\big) < \infty$ for all $\ell \in [(r+1)/2, r]$.

Finally, we note that in this paper the coefficient in front of the noise is constant (i.e. so-called additive noise). Under the usual Lipschitz assumptions, however, properly reformulated versions of our results also hold for non-constant diffusion coefficients. Details on this will be contained in a forthcoming paper.

2 Some preliminaries

We shall make use of a finite-dimensional approximation argument to construct the solution of (1.1). For any $n \geq 1$, let $r_t^{(n)} := (r_{t,1}^{(n)}, \cdots, r_{t,n}^{(n)})$ solve the following SDE on $\mathbb{R}^n$:

$$\mathrm{d}r_{t,i}^{(n)} = q_i \mathrm{d}B_t^i - \lambda_i \mathbf{m}\Big(e_i \Psi\Big(\sum_{k=1}^n r_{t,k}^{(n)} e_k\Big)\Big)\mathrm{d}t + \mathbf{m}\Big(e_i \Phi\Big(\sum_{k=1}^n r_{t,k}^{(n)} e_k\Big)\Big)\mathrm{d}t \tag{2.1}$$

with $r_{0,i}^{(n)} = \langle X_0, e_i\rangle$, $1 \leq i \leq n$, where $X_0 : \Omega \to H$ is a fixed $\mathscr{F}_0/\mathscr{B}(H)$-measurable map such that $\mathbb{E}\|X_0\|_H^2 < \infty$. Here and below for a topological space S we denote its Borel σ-algebra by $\mathscr{B}(S)$. By [8, Theorem 1.2] there exists a unique solution to (2.1) for all $t \geq 0$.

Lemma 2.1. *Under the assumptions of Theorem 1.3, there exists a constant $C > 0$ independent of n and X_0 such that $X_t^{(n)} := \sum_{i=1}^n r_{t,i}^{(n)} e_i$ satisfies*

$$\mathbb{E}\int_0^T \mathbf{m}(|X_t^{(n)}|^{r+1})\mathrm{d}t \leq C(\|X_0\|_H^2 + T), \quad T > 0, \tag{2.2}$$

and

$$\mathbb{E}\|X_t^{(n)}\|_H^2 \le C(1+t^{-2/(r-1)}), \quad t>0. \tag{2.3}$$

Proof. By (1.2) we have

$$s\Psi(s) \ge s\Psi(0) + \eta|s|^{r+1} + \sigma s^2, \quad s,t\in\mathbb{R}, \tag{2.4}$$

and by (1.3) we have

$$-\mathbf{m}\big(\Phi(x)\, L^{-1}x\big) \le -\Phi(0)\,\mathbf{m}(L^{-1}x) + \theta\|x\|_{r+1}^{r+1} + \delta\|x\|_2^2, \quad x\in L^{r+1}(\mathbf{m}).$$

Hence for all $x \in \operatorname{span}\{e_i : i\in\mathbb{N}\}$

$$\begin{aligned} &-\mathbf{m}\big(\Psi(x)\,x\big) - \mathbf{m}\big(\Phi(x)\,L^{-1}x\big)\\ &\qquad \le \big(|\Psi(0)| + |\Phi(0)|\,\lambda_1^{-1}\big)\|x\|_2 - (\eta-\theta)\|x\|_{r+1}^{r+1} - (\sigma-\delta)\|x\|_2^2. \end{aligned}$$

Combining this with (2.1) and using Itô's formula, we obtain

$$\frac{1}{2}\mathrm{d}\|X_t^{(n)}\|_H^2 \le \mathrm{d}M_t^{(n)} + \frac{c_1}{2}\mathrm{d}t - \frac{c_2}{2}\mathbf{m}(|X_t^{(n)}|^{r+1})\mathrm{d}t \tag{2.5}$$

for some local martingale $M_t^{(n)}$ and constants $c_1, c_2 > 0$ independent of n. This implies (2.2). Moreover, since $\mathbf{m}(|X_t^{(n)}|^{r+1}) \ge \lambda_1^{(r+1)/2}\|X_t^{(n)}\|_H^{r+1}$, it follows from (2.5) that

$$\mathbb{E}\|X_t^{(n)}\|_H^2 - \mathbb{E}\|X_s^{(n)}\|_H^2 \le c_1(t-s) - c_2\int_s^t \big(\mathbb{E}\|X_u^{(n)}\|_H^2\big)^{\frac{r+1}{2}}\,\mathrm{d}u, \quad 0\le s\le t. \tag{2.6}$$

To prove (2.3), let h solve the equation

$$h'(t) = -c_2 h(t)^{(r+1)/2} + c_1, \quad t\ge 0, \qquad h(0) = \mathbb{E}\|X_0\|_H^2 + (4c_1/c_2)^{2/(r+1)}. \tag{2.7}$$

Then it is easy to see that (2.6) implies

$$\mathbb{E}\|X_t^{(n)}\|_H^2 \le h(t), \quad t\ge 0. \tag{2.8}$$

Let $\phi_t := h(t) - \mathbb{E}\|X_t^{(n)}\|_H^2$ and

$$\tau := \inf\{t\ge 0 : \phi_t \le 0\}.$$

Suppose $\tau<\infty$, then by continuity $\phi_\tau \le 0$ and by the mean-value theorem and (2.6), (2.7) we obtain

$$\phi_t \ge \phi_0 - c_2\int_0^t \Big(h_\varepsilon(u)^{(r+1)/2} - \big(\mathbb{E}\|X_u^{(n)}\|_H^2\big)^{(r+1)/2}\Big)\mathrm{d}u \ge (4c_1/c_2)^{2/(r+1)} - c\int_0^t \phi_u \mathrm{d}u, \;\; 0\le t\le \tau,$$

where $c := c_2\frac{r+1}{2}\max_{t\in[0,\tau]} t^{(r-1)/2} = c_2\frac{r+1}{2}\tau^{(r-1)/2}$. By Gronwall's lemma we arrive at $\phi_\tau \geq (4c_1/c_2)^{2/(r+1)}\mathrm{e}^{-c\tau} > 0$. This contradiction proves (2.8).

To estimate $h(t)$, let

$$\tau := \inf\{t \geq 0 : h(t)^{(r+1)/2} \leq 2c_1/c_2\}.$$

Since $h(0)^{(r+1)/2} \geq 4c_1/c_2 > 2c_1/c_2$, $\tau \geq t_0$ for some $t_0 > 0$ independent of n. Indeed, we may define t_0 as τ above with h replaced by the solution to (2.7) with initial condition $h(0) := (4c_1/c_2)^{2/(r+1)}$. By (2.7) we have

$$h'(t) \leq -\frac{c_2}{2}h(t)^{(r+1)/2}, \quad 0 \leq t \leq \tau.$$

Therefore, for some constant $c > 0$ independent of n,

$$h(t) \leq ct^{-2/(r-1)}, \quad 0 \leq t \leq \tau. \tag{2.9}$$

Clearly, $h'(t) \leq 0$ for all $t \geq 0$, since by an elementary consideration we have $h \geq (c_1/c_2)^{2/(r+1)}$, consequently

$$h(t) \leq h(\tau) \leq c\tau^{-2/(r-1)} \leq ct_0^{-2/(r-1)}, \quad t > \tau.$$

Therefore, (2.3) holds. □

According to (2.2) in Lemma 2.1, $X^{(n)}$ is bounded in $L^{r+1}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$, where $\nu(\mathrm{d}t) := \mathrm{e}^{-t}\mathrm{d}t$. Thus, there exists a subsequence $n_k \to \infty$ and a process X such that $X^{(n_k)} \to X$ weakly in $L^{r+1}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$. To prove that this limit provides a solution of (1.1), we shall make use of Theorem 3.2 in Chapter 1 of [10]. We state this result in detail for the reader's convenience specialized to our situation.

Theorem 2.2. ([10, Theorem I.3.2]) *Consider three maps $v : \mathbb{R}_+ \times \Omega \to \mathbb{B}$, $\tilde{v} : \mathbb{R}_+ \times \Omega \to \mathbb{B}'$, $h : \mathbb{R}_+ \times \Omega \to H$ such that*

(i) v is $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}/\mathscr{B}(\mathbb{B})$-measurable and $v_t := v(t, \cdot)$ is $\mathscr{F}_t/\mathscr{B}(\mathbb{B})$-measurable for all $t \geq 0$.

(ii) $\tilde{v}$ is $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}/\mathscr{B}(\mathbb{B}')$-measurable and $\tilde{v}_t := \tilde{v}(t, \cdot)$ is $\mathscr{F}_t/\mathscr{B}(\mathbb{B}')$-measurable. Moreover, $\int_0^T \|\tilde{v}_t\|_{\mathbb{B}'}\,\mathrm{d}t < \infty$ P-a.s. for all $T > 0$.

(iii) h is an H-valued $(\mathscr{F}_t)$-adapted continuous local semi-martingale.

Set

$$\tilde{h}_t := \int_0^t \tilde{v}_s \mathrm{d}s + h_t.$$

If $\tilde{h}_t(\omega) = v_t(\omega)$ for $\nu \times P$-a.e. (t, ω), then $\tilde{h}_t$ is an H-valued continuous $(\mathscr{F}_t)$-adapted process satisfying the following Itô formula for the square of the norm:

$$\|\tilde{h}_t\|_H^2 = \|\tilde{h}(0)\|_H^2 + 2\int_0^t {}_{\mathbb{B}'}\langle \tilde{v}_s, v_s\rangle_{\mathbb{B}}\mathrm{d}s + 2\int_0^t \langle \tilde{h}_s, \mathrm{d}h_s\rangle_H + [h]_t. \tag{2.10}$$

where $[h]$ denotes the quadratic variation process of h.

3 Proof of the existence

a) By Lemma 2.1 and (1.2), $\{\Psi(X_t^{(n)})\}$ and $\{\Phi(X_t^{(n)})\}$ are bounded in $L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$, where $\nu(\mathrm{d}t) := \mathrm{e}^{-t}\mathrm{d}t$. Hence there exist a subsequence $n_k \to \infty$ and processes $U, V \in L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$ such that

$$\Psi(X^{(n_k)}) \to U, \ \Phi(X^{(n_k)}) \to V \text{ weakly in } L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m}). \tag{3.1}$$

Moreover, by Lemma 2.1, we may also assume that

$$X^{(n_k)} \to \bar{X} \text{ weakly in } L^{r+1}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m}). \tag{3.2}$$

E.g. by [7, Chap. 3, § 7] we may also assume that the Cesaro means of the sequences in (3.1) converge strongly in $L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E;\ \nu \times P \times \mathbf{m})$ so the limits have $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{M}$-measurable versions. Furthermore, as continuous processes the approximants are all progressively measurable as $L^{(r+1)/r}(\mathbf{m})$-valued processes, hence so are their limits. In particular, these are adapted. The same holds for the sequence in (3.2) respectively its limit with $(r+1)/r$ replaced by $r+1$. Below we always consider versions of U, V, $\bar{X}$ with all these measurability properties and denote them by the same symbols. Since for $t \geq 0$

$$\mathbf{m}(X_t^{(n_k)} e_i) = \langle X_0, e_i \rangle + \int_0^t \big\{\mathbf{m}(\Psi(X_s^{(n_k)}) L e_i) + \mathbf{m}(e_i \Phi(X_s^{(n_k)})\big\} \mathrm{d}s + q_i B_t^i, \quad 1 \leq i \leq n_k,$$

it follows from (3.1) and (3.2) that, for any real-valued bounded measurable process φ,

$$\mathbb{E}\int_0^T \varphi_t \mathbf{m}(\bar{X}_t e_i)\nu(\mathrm{d}t) = \mathbb{E}\int_0^T \varphi_t \Big\{\langle X_0, e_i\rangle + \int_0^t \big\{\mathbf{m}(U_s L e_i) + \mathbf{m}(V_s e_i)\big\}\mathrm{d}s + q_i B_t^i\Big\}\nu(\mathrm{d}t)$$

for all $T > 0$. Thus,

$$\mathbf{m}(\bar{X}_t e_i) = \langle X_0, e_i\rangle + \int_0^t \mathbf{m}(U_s L e_i + V_s e_i)\mathrm{d}s + q_i B_t^i, \quad \text{for } \nu \times P\text{-a.e. } (t,\omega), i \geq 1. \tag{3.3}$$

b) To apply Theorem 2.2, let

$$\tilde{v}_s := \bar{L} U_s + J V_s.$$

By (1.2), Lemma 2.1, Proposition 1.1 and Corollary 1.2 we have $\mathbb{E}\int_0^T \|\bar{v}_s\|_{\mathbb{B}'}\mathrm{d}s < \infty$ for any $T > 0$. So, we see that in Theorem 2.2 conditions (i), (ii) with $v = \bar{X}$ and also (iii) with $h := QW$ are satisfied and by Proposition 1.1 and Corollary 1.2, (3.3) with e_i replaced by $\lambda_i^{-1} e_i$ implies

$${}_{\mathbb{B}'}\langle \bar{X}_t, e_i\rangle_{\mathbb{B}} = {}_{\mathbb{B}'}\langle X_0, e_i\rangle_{\mathbb{B}} + \int_0^t {}_{\mathbb{B}'}\langle \tilde{v}_s, e_i\rangle_{\mathbb{B}}\,\mathrm{d}s + {}_{\mathbb{B}'}\langle QW_t, e_i\rangle_{\mathbb{B}}, \quad i \geq 1, \text{ for } \nu \times P\text{-a.e. } (t,\omega).$$

Hence defining

$$X_t := X_0 + \int_0^t \tilde{v}_s \mathrm{d}s + QW_t, \quad t \geq 0 \tag{3.4}$$

we see that

$$\bar{X} = X \quad \nu \times P\text{-a.e.} \tag{3.5}$$

Therefore, by Theorem 2.2, X is an H-valued continuous $(\mathscr{F}_t)$-adapted process and (2.10) holds with X replacing $\tilde{h}$. Therefore, to prove that X solves (1.1), by Proposition 1.1 and Corollary 1.2 it suffices to show that

$$\mathbf{m}\big(e_i[V_s - \lambda_i U_s]\big) = \mathbf{m}\big(e_i[\Phi(\bar{X}_s) - \lambda_i \Psi(\bar{X}_s)]\big), \quad i \geq 1, \text{ for } \nu \times P\text{-a.e. } (s, \omega). \tag{3.6}$$

This will be proved by the following two steps.

c) We claim that for any $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded, Borel measurable with compact support,

$$\liminf_{k\to\infty} \int_0^\infty \psi(t)\mathbb{E}\|X_t^{(n_k)}\|_H^2 \mathrm{d}t \geq \int_0^\infty \psi(t)\mathbb{E}\|\bar{X}_t\|_H^2 \mathrm{d}t. \tag{3.7}$$

Since $X^{(n_k)} \to \bar{X}$ weakly in $L^2(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$, by Fatou's lemma we have

$$\begin{aligned}
\int_0^\infty \psi(t)\mathbb{E}\|\bar{X}_t\|_H^2 \mathrm{d}t &= \sum_{i=1}^\infty \lambda_i^{-1}\mathbb{E}\int_0^\infty \psi(t)\mathbf{m}(e_i\bar{X}_t)^2\mathrm{d}t \\
&= \sum_{i=1}^\infty \lambda_i^{-1} \lim_{k\to\infty} \mathbb{E}\int_0^\infty \psi(t)\mathbf{m}(e_i X_t^{(n_k)})\mathbf{m}(e_i\bar{X}_t)\mathrm{d}t \\
&\leq \frac{1}{2}\sum_{i=1}^\infty \lambda_i^{-1} \liminf_{k\to\infty} \mathbb{E}\int_0^\infty \psi(t)\mathbf{m}(e_i X_t^{(n_k)})^2\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i=1}^\infty \lambda_i^{-1}\mathbb{E}\int_0^\infty \psi(t)\mathbf{m}(e_i\bar{X}_t)^2\mathrm{d}t \\
&\leq \frac{1}{2}\liminf_{k\to\infty}\mathbb{E}\int_0^\infty \psi(t)\|X_t^{(n_k)}\|_H^2\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^\infty \psi(t)\|\bar{X}_t\|_H^2\mathrm{d}t.
\end{aligned}$$

Since $\bar{X} \in L^2(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$ so that $\int_0^\infty \psi(t)\mathbb{E}\|\bar{X}_t\|_H^2\mathrm{d}t < \infty$, this implies (3.7) immediately.

d) By (2.10) in Theorem 2.2, (1.5) and (1.8) we have

$$\mathbb{E}\|X_t\|_H^2 = \mathbb{E}\|X_0\|_H^2 - 2\int_0^t \mathbb{E}\big(\mathbf{m}(\bar{X}_s U_s) + \mathbf{m}(L^{-1}(\bar{X}_s)V_s)\big)\mathrm{d}s + \sum_{i=1}^\infty \lambda_i^{-1} q_i^2 t. \tag{3.8}$$

On the other hand, by Itô's formula,

$$\mathbb{E}\|X_t^{(n)}\|_H^2 = \mathbb{E}\|X_0\|_H^2 - 2\mathbb{E}\int_0^t \mathbf{m}\big(X_s^{(n)}\Psi(X_s^{(n)}) + L^{-1}(X_s^{(n)})\Phi(X_s^{(n)})\big)\mathrm{d}s + \sum_{i=1}^n \lambda_i^{-1} q_i^2 t. \tag{3.9}$$

Then for any $\varphi \in L^{r+1}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$, we obtain from (1.2), (1.3) and (3.9) that

$$\begin{aligned}
0 \le I_k(t) &:= 2\mathbb{E}\int_0^t \mathbf{m}\Big([X_s^{(n_k)} - \varphi_s]\big[\Psi(X_s^{(n_k)}) - \Psi(\varphi_s)\big] \\
&\qquad + L^{-1}(X_s^{(n_k)} - \varphi_s)\big[\Phi(X_s^{(n_k)}) - \Phi(\varphi_s)\big]\Big)\mathrm{d}s \\
&= 2\mathbb{E}\int_0^t \mathbf{m}\Big(X_s^{(n_k)}\Psi(X_s^{(n_k)}) + L^{-1}(X_s^{(n_k)})\Phi(X_s^{(n_k)})\Big)\mathrm{d}s \\
&\quad - 2\mathbb{E}\int_0^t \mathbf{m}\Big(\varphi_s\big[\Psi(X_s^{(n_k)}) - \Psi(\varphi_s)\big] + X_s^{(n_k)}\Psi(\varphi_s) \\
&\qquad + L^{-1}(X_s^{(n_k)})\Phi(\varphi_s) + L^{-1}(\varphi_s)\big[\Phi(X_s^{(n_k)}) - \Phi(\varphi_s)\big]\Big)\mathrm{d}s \\
&= -\mathbb{E}\|X_t^{(n_k)}\|_H^2 + \|X_0\|_H^2 + \sum_{i=1}^{n_k} \lambda_i^{-1} q_i^2 t \\
&\quad - 2\mathbb{E}\int_0^t \mathbf{m}\Big(\varphi_s\big[\Psi(X_s^{(n_k)}) - \Psi(\varphi_s)\big] + X_s^{(n_k)}\Psi(\varphi_s) \\
&\qquad + L^{-1}(X_s^{(n_k)})\Phi(\varphi_s) + L^{-1}(\varphi_s)\big[\Phi(X_s^{(n_k)}) - \Phi(\varphi_s)\big]\Big)\mathrm{d}s,
\end{aligned} \tag{3.10}$$

Since $\Psi(X^{(n_k)}) \to U$ and $\Phi(X^{(n_k)}) \to V$ weakly in $L^{(r+1)/r}(\mathbb{R}_+ \times \Omega \times E; \nu \times P \times \mathbf{m})$ and $X^{(n_k)} \to \bar{X}$ weakly in $L^{r+1}(\mathbb{R}_+ \times \Omega \times E,\ \nu \times P \times \mathbf{m})$, for any $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, bounded, Borel-measurable with compact support, we obtain from (3.10) and (3.7) that

$$\begin{aligned}
0 \le &\int_0^\infty \psi(t)\mathrm{d}t\Big\{ - \mathbb{E}\|\bar{X}_t\|_H^2 + \mathbb{E}\|X_0\|_H^2 + \sum_{i=1}^\infty \lambda_i^{-1} q_i^2 t \\
&- 2\mathbb{E}\int_0^t \mathbf{m}\Big(\varphi_s\big[U_s - \Psi(\varphi_s)\big] + \bar{X}_s\Psi(\varphi_s) + L^{-1}(\bar{X}_s)\Phi(\varphi_s) + L^{-1}(\varphi_s)\big[V_s - \Phi(\varphi_s)\big]\Big)\mathrm{d}s\Big\}.
\end{aligned}$$

Combining this with (3.8) we arrive at

$$0 \le \int_0^\infty \psi(t)\mathrm{d}t\mathbb{E}\int_0^t \mathbf{m}\Big([\bar{X}_s - \varphi_s]\big[U_s - \Psi(\varphi_s)\big] + L^{-1}(\bar{X}_s - \varphi_s)\big[V_s - \Phi(\varphi_s)\big]\Big)\mathrm{d}s. \tag{3.11}$$

By first taking $\varphi_s := \bar{X}_s - \varepsilon\tilde{\varphi}_s e_i$ for given $\varepsilon > 0$ and $\tilde{\varphi} \in L^\infty(\mathbb{R}_+ \times \Omega; \nu \times P)$, then dividing by ε and letting $\varepsilon \to 0$, we obtain from the continuity of Φ, Ψ and the dominated convergence theorem, which is valid due to (1.2), (1.3) and because $\bar{X} \in L^{r+1}([0,\infty) \times \Omega \times E; \nu \times P \times \mathbf{m})$, that

$$\int_0^\infty \psi(t)\mathrm{d}t\mathbb{E}\int_0^t \tilde{\varphi}_s \mathbf{m}\big(e_i[U_s - \Psi(\bar{X}_s) - \lambda_i^{-1}(V_s - \Phi(\bar{X}_s))]\big)\mathrm{d}s \geq 0.$$

Since $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ bounded, Borel-measurable with compact support, and $\tilde{\varphi} \in L^\infty(\mathbb{R}_+ \times \Omega; \nu \times P)$ are arbitrary, this implies (3.6).

4 Proof of the other assertions

a) Proofs of the uniqueness, (1.10), (1.11) and (1.13).
(1.10) is an immediate consequence of (2.3) and (3.7), since X is P-a.e. continuous in t. Let X and Y be solutions of (1.1). Then by (1.9) $Z := X - Y$ solves the equation

$$Z_t = Z_0 + \int_0^t \Big[\bar{L}\big(\Psi(X_t) - \Psi(Y_t)\big) + J\big(\Phi(X_t) - \Phi(Y_t)\big)\Big]\mathrm{d}t, \quad t \geq 0.$$

Thus, by Theorem 2.2 with $h = 0$, (1.5), (1.8) and finally by (1.2), (1.3), we obtain

(4.1)
$$\|Z_t\|_H^2 = \|Z_0\|_H^2 - 2\int_0^t \mathbf{m}\Big([X_s - Y_s]\big[\Psi(X_s) - \Psi(Y_s)\big] + L^{-1}(X_s - Y_s)\big[\Phi(X_s) - \Phi(Y_s)\big]\Big)\mathrm{d}s \leq 0.$$

If $X_0 = Y_0$, this implies $Z_t = 0$ for all $t \geq 0$. By a slight modification of a standard argument one obtains as usual that the uniqueness also implies the stated Markov property and hence the semigroup property of P_t. The strong Markov property then follows from the Feller property of P_t proved below, since all solutions of (1.1) have continuous sample paths in H.

Similarly to (4.1), we have for $0 \leq s \leq t$ that

(4.2)
$$\begin{aligned}&\|X_t - Y_t\|_H^2 - \|X_s - Y_s\|_H^2 \\ &\leq -2\int_s^t \big\{(\sigma - \delta)\|X_u - Y_u\|_2^2 + (\eta - \theta)\|X_u - Y_u\|_{r+1}^{r+1}\big\}\mathrm{d}u.\end{aligned}$$

Noting that $\sigma \geq \delta, \eta > \theta$, $\|\cdot\|_2^2 \geq \lambda_1\|\cdot\|_H^2$, $\|\cdot\|_{r+1}^{r+1} \geq \lambda_1^{(r+1)/2}\|\cdot\|_H^{r+1}$ and that for $\varepsilon > 0$ the function $h_{\varepsilon,t} := \{(\varepsilon + \|X_s - Y_s\|_H)^{1-r} + (r-1)(\eta - \theta)\lambda_1^{(r+1)/2}(t-s)\}^{-2/(r-1)}$, $t \geq s$, solves for $s \geq 0$ fixed the equation

(4.3)
$$h_t' = -2(\eta - \theta)\lambda_1^{(r+1)/2}h_t^{(r+1)/2}, \quad t \geq s(\geq 0), \quad h_0 = (\|X_s - Y_s\|_H + \varepsilon)^2$$

due to the same comparison argument as in the proof of Lemma 2.1, it follows that $\|X_t - Y_t\|_H^2 \leq h_{\varepsilon,t}\ \forall\ t \geq s$. Letting $\varepsilon \to 0$ this implies (1.11) and the Feller property of P_t. By (1.10) it follows that $P_t|F|(x) < \infty$ for all Lipschitz continuous $F : H \to \mathbb{R}$. Now (1.12) is obvious by (1.11).

b) Proof of (3).
Let δ_0 be the Dirac measure at $0 \in H$. Set

$$\mu_n := \frac{1}{n}\int_0^n \delta_0 P_t \mathrm{d}t, \quad n \geq 1.$$

We intend to show the tightness of $\{\mu_n\}$. Then by the Feller property of P_t the weak limit of a subsequence provides an invariant probability measure of P_t. By Lemma 2.1 and the weak convergence of $X^{(n_k)}$ to X, we have

$$\int_H \mathbf{m}(|x|^2)\mu_n(\mathrm{d}x) = \frac{1}{n}\int_0^n \mathbb{E}\mathbf{m}(|X_t(0)|^2)\mathrm{d}t \leq C$$

for some constant $C > 0$ and all $n \geq 1$, where we set $\mathbf{m}(|x|^2) = \infty$ if $x \notin L^2(\mathbf{m})$. Since the function $x \mapsto \mathbf{m}(|x|^2)$ is compact, that is, $\{x \in H : \mathbf{m}(|x|^2) \leq r\}$ is relatively compact in H for any $r \geq 0$, we conclude that $\{\mu_n\}$ is tight.

Next, let μ be an invariant probability measure. For any bounded Lipschiz function F on H, (1.11) implies that there exists $C > 0$ such that

$$|P_t F(x) - \mu(F)| \leq \int_H \mathbb{E}|F(X_t(x)) - F(X_t(y))|\mu(\mathrm{d}y) \leq C\mathscr{L}(F)t^{-1/(r-1)}$$

for all $x \in H, t > 0$. Thus, (1.13) holds and hence P_t has a unique invariant measure.

Let μ be the invariant probability measure of P_t. It remains to show that $\mu(\|\cdot\|_{r+1}^{r+1}) < \infty$.
By (1.10), since μ is P_t-invariant, we have

$$\int_H \|x\|_H^2 \mu(\mathrm{d}x) = \int \mathbb{E}\big\|X_1(x)\big\|_H^2 \, \mu(\mathrm{d}x) \leq 2C < \infty. \tag{4.4}$$

Next, since X is the weak limit of $X^{(n_k)}$ in $L^{r+1}([0,\infty) \times \Omega \times E; \nu \times P \times \mathbf{m})$, by Hölder's inequality we have

$$\begin{aligned}
\int_1^2 \mathbb{E}\,\mathbf{m}(|X_t(x)|^{r+1})\mathrm{d}t &= \lim_{k\to\infty}\int_1^2 \mathbb{E}\mathbf{m}\big(X_t^{(n_k)}(|X_t|^r \mathrm{sgn}(X_t))\big)\mathrm{d}t \\
&\leq \liminf_{k\to\infty}\left(\int_1^2 \mathbb{E}\mathbf{m}(|X_t^{(n_k)}|^{r+1})\mathrm{d}t\right)^{1/(r+1)}\left(\int_1^2 \mathbb{E}\mathbf{m}(|X_t|^{r+1})\mathrm{d}t\right)^{r/(r+1)}.
\end{aligned}$$

Therefore, by (2.2), for some $C_1 > 0$

$$\int_1^2 \mathbb{E}\,\mathbf{m}(|X_t(x)|^{r+1})\mathrm{d}t \leq C_1(1 + \|x\|_H^2), \quad x \in H.$$

Then by (4.4) we obtain

$$\mu(\|\cdot\|_{r+1}^{r+1}) = \int_H \mu(\mathrm{d}x)\int_1^2 \mathbb{E}\,\mathbf{m}(|X_t(x)|^{r+1})\mathrm{d}t < \infty.$$

c) Exponential ergodicity, i.e. proof of (4).
If $\sigma > \delta$ then (4.2) implies

$$\|X_t - Y_t\|_H^2 \leq \|X_s - Y_s\|_H^2 \, \mathrm{e}^{-2(\sigma-\delta)(t-s)}, \quad t \geq s \geq 0.$$

So, (1.14) holds. Combining this with (1.11) we arrive at

$$\|X_{t+1} - Y_{t+1}\|_H^2 \leq \|X_1 - Y_1\|_H^2 \, \mathrm{e}^{-2(\sigma-\delta)t} \leq C\mathrm{e}^{-2(\sigma-\delta)t}, \quad t \geq 0.$$

This implies (1.15) and (1.16) immediately. □

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