Global gradient bounds for dissipative diffusion operators

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Abstract

Let L be a second order elliptic operator on $\mathbb{R}^d$ with a constant diffusion matrix and a dissipative (in a weak sense) drift $b \in L^p_{loc}$ with some $p > d$. We assume that L possesses a Lyapunov function, but no local boundedness of b is assumed. It is known that then there exists a unique probability measure μ satisfying the equation $L^*\mu = 0$ and that the closure of L in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geq 0}$ with the resolvent $\{G_\lambda\}_{\lambda>0}$. We prove that, for any Lipschitzian function $f \in L^1(\mu)$ and all $t, \lambda > 0$, the functions $T_t f$ and $G_\lambda f$ are Lipschitzian and $\sup_{x,t} |\nabla T_t f(x)| \leq \sup_x |\nabla f(x)|$ and $\sup_x |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} \sup_x |\nabla f(x)|$. In addition, we show that for every bounded Lipschitzian function g, the function $G_\lambda g$ is the unique bounded solution of the equation $\lambda f - Lf = g$ in the Sobolev class $H^{2,2}_{loc}(\mathbb{R}^d)$.

Estimations globales de gradient pour les opérateurs de diffusion dissipatifs

Résumé

Soit L un opérateur elliptique sur $\mathbb{R}^d$ tel que son terme du premier ordre $b \in L^p_{loc}$, $p > d$, est dissipatif (mais pas necessairement localement borné) et il existe une fonction de Liapounoff. Il est connue qu'il existe une probabilité unique μ telle que $L^*\mu = 0$ au sense faible et la fermeture de L dans $L^1(\mu)$ est le générateur d'un semigroupe markovien $\{T_t\}_{t\geq 0}$ à résolvante $\{G_\lambda\}_{\lambda>0}$. Nous montrons que pour chaque fonction lipschitzienne $f \in L^1(\mu)$ et tous $t, \lambda > 0$ les fonctions $T_t f$ et $G_\lambda f$ sont lipschitziennes et on a $\sup_{x,t} |\nabla T_t f(x)| \leq \sup_x |\nabla f(x)|$ et $\sup_x |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} \sup_x |\nabla f(x)|$. De plus, nous montrons que pour chaque fonction bornée lipschitzienne g la fonction $G_\lambda g$ est la solution unique bornée de l'équation $\lambda f - Lf = g$ dans la classe de Sobolev $H^{2,2}_{loc}(\mathbb{R}^d)$.

Version française abrégée

Soit L un opérateur elliptique sur $\mathbb{R}^d$ de la forme

$$Lf = \sum_{i,j\leq d} a^{ij}\partial_{x_i}\partial_{x_j} f + \sum_{i\leq d} b^i \partial_{x_i} f,$$

où $A = (a^{ij})$ est une matrice constante symétrique positive et $b = (b^i)\colon \mathbb{R}^d \to \mathbb{R}^d$ est une application mesurable telle que $|b| \in L^p_{loc}(\mathbb{R}^d)$ avec $p > d$. Nous considérons l'équation

$$\lambda f - Lf = g, \quad \lambda > 0,$$

et obtenons une condition qui fournit l'inégalité

$$\sup_x |\nabla f(x)| \leq \lambda^{-1} \sup_x |\nabla g(x)|$$

pour chaque fonction bornée lipschitzienne g. Cette inégalité est connue (par exemple, elle a été établie dans [8], [9] même en dimension infinie) sous les hypothèses suivantes: $\bigl(b(x)-b(y), x-y\bigr) \le -\omega|x-y|^2$, oú $\omega > 0$, et b est lipschitzienne (ou continue et $|b(x)||x|^2 \in L^2(\mu)$). Ces conditions sont assez réstrictives pour applications dans l'analyse stochastique. Supposons que l'application b soit dissipative au sense plus faible: pour chaque $h \in \mathbb{R}^d$ on a $\bigl(b(x+h)-b(x), h\bigr) \le 0$ p.p. Suppposons aussi qu'il existe une fonction $V \ge 0$ de la classe C^2 (une fonction de Liapounoff) telle que $\lim\limits_{|x|\to\infty} V(x) = \infty$ et $\lim\limits_{|x|\to\infty} LV(x) = -\infty$. Par exemple, si $\bigl(b(x), x\bigr) \le c < 0$ en dehors d'une boule, on peut prendre $V(x) = (x,x)^m$ avec m suffisemment grand. D'après [5], il existe une mesure de probabilité μ telle que $L^*\mu = 0$ au sense suivant:

$$\int_{\mathbb{R}^d} L\varphi \, d\mu = 0 \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d).$$

La mesure μ possède une densité continue positive ϱ de la classe $H^{p,1}_{loc}(\mathbb{R}^d)$. De plus, il existe (et est unique) un semigroupe fortement continue markovien $(T_t)_{t\ge0}$ dans $L^1(\mu)$ tel que μ est invariante pour $(T_t)_{t\ge0}$, c'est à dire que

$$\int_{\mathbb{R}^d} T_t f \, d\mu = \int_{\mathbb{R}^d} f \, d\mu \quad \forall\, f \in L^1(\mu),$$

et le générateur $\overline{L}$ de ce semigroupe est une extention de $\bigl(L, C_0^\infty(\mathbb{R}^d)\bigr)$. Soit G_λ la résolvante correspondante. Notons $D(\overline{L})$ le domaine de $\overline{L}$ dans $L^1(\mu)$. Étant donné $\lambda > 0$, pour chaque $g \in L^1(\mu)$ il existe une fonction unique $f \in \overline{L}$ telle que $\lambda f - \overline{L} f = g$. Si $g \in L^2(\mu)$, on a $f \in L^2(\mu)$ et $f \in H^{2,2}_{loc}(\mathbb{R}^d)$, et notre équation s'écrit comme $\lambda f - Lf = g$ p.p.

Théorème. *Pour chaque fonction lipschitzienne $f \in L^1(\mu)$ et tous $t, \lambda > 0$ les fonctions $T_t f$ et $G_\lambda f$ sont lipschitziennes et on a*

$$\sup_{x,t} |\nabla T_t f(x)| \le \sup_x |\nabla f(x)|, \quad \sup_x |\nabla G_\lambda f(x)| \le \frac{1}{\lambda} \sup_x |\nabla f(x)|.$$

De plus, étant donné une fonction bornée lipschitzienne g, la fonction $G_\lambda g$ est la solution unique bornée de l'équation $\lambda f - Lf = g$ dans la classe de Sobolev $H^{2,2}_{loc}(\mathbb{R}^d)$.

Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$Lu = \sum_{i,j\le d} a^{ij} \partial_{x_i} \partial_{x_j} u + \sum_{i\le d} b^i \partial_{x_i} u, \tag{1}$$

where $A = (a^{ij})$ is a constant strictly positive definite symmetric matrix and $b = (b^i)$ is a measurable vector field which satisfy the following hypotheses.

(Ha) $|b| \in L^p_{loc}(\mathbb{R}^d)$ with some $p > d$, $p \ge 2$.

(Hb) b is *dissipative* i.e., for every $h \in \mathbb{R}^d$, there exists a measure zero set $N_h \subset \mathbb{R}^d$ such that $\bigl(b(x+h)-b(x), h\bigr) \le 0$ for all $x \in \mathbb{R}^d \setminus N_h$.

(Hc) there exists a *Lyapunov function* V for L, i.e., a nonnegative C^2-function V such that $V(x) \to +\infty$ and $LV(x) \to -\infty$ as $|x| \to \infty$.

Hypotheses (Ha) and (Hc) imply (see [3] and [5]) that there exists a unique probability measure μ on $\mathbb{R}^d$ such that μ has a strictly positive continuous weakly differentiable density ϱ, $|\nabla\varrho| \in L^p_{loc}(\mathbb{R}^d)$, and $L^*\mu = 0$ in the following weak sense: $\int Lu \, d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. The closure $\overline{L}$ of L with domain $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geq 0}$ for which μ is invariant. Let $D(\overline{L})$ denote the domain of $\overline{L}$ in $L^1(\mu)$ and let $\{G_\lambda\}_{\lambda>0}$ denote the corresponding resolvent, i.e., $G_\lambda = (\lambda - \overline{L})^{-1}$. The restrictions of T_t and G_λ to $L^2(\mu)$ are contractions on $L^2(\mu)$. In particular, if $v \in D(\overline{L})$ is such that $\lambda v - \overline{L}v = g \in L^2(\mu)$, then $v \in L^2(\mu)$. Moreover, it follows by [5, Theorem 2.8] (for bounded g this follows also from [11, Lemma 2.1]) that one has $v \in H^{2,2}_{loc}(\mathbb{R}^d)$ and $\overline{L}v = Lv$ a.e., so that one has a.e.

$$\lambda v - Lv = g. \tag{2}$$

In fact, due to our assumptions on the coefficients of L one has even $v \in H^{p,2}_{loc}(\mathbb{R}^d)$ (see [7]). The main result of this note is the following theorem.

Theorem 1. *Let A and b satisfy* (Ha), (Hb) *and* (Hc). *Then, for any Lipschitzian function $f \in L^1(\mu)$ and all $t, \lambda > 0$, $T_t f$ and $G_\lambda f$ have Lipschitzian versions such that*

$$\sup_{x,t} |\nabla T_t f(x)| \leq \sup_x |\nabla f(x)| \quad \text{and} \quad \sup_x |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} \sup_x |\nabla f(x)|. \tag{3}$$

In addition, for every bounded Lipschitzian function g on $\mathbb{R}^d$ and any $\lambda > 0$, the function $G_\lambda g$ is the unique bounded solution of equation (2) *in the Sobolev class $H^{2,2}_{loc}(\mathbb{R}^d)$.*

Such estimates have been established by probabilistic methods in [9] (even in the infinite dimensional case) under the assumption that $\big(b(x) - b(y), x - y\big) \leq -\alpha|x - y|^2$ (*strong dissipativity*) with some $\alpha > 0$, b is m-dissipative and one has $(1 + |b|^2)(1 + |x|^4) \in L^1(\mu)$; see also [8] for the case where the last assumption is replaced by the one that b is globally Lipschitzian. Note that strong dissipativity implies (Hc) with $V(x) = |x|^2$. In fact a weaker assumption $\big(b(x), x\big) \leq c < 0$ outside a ball, implies that one can take $V(x) = |x|^m$ for m big enough.

Let $\widehat{L}$ be the elliptic operator with the same second order part as L, but with drift is $\widehat{b} = 2A\nabla\varrho/\varrho - b$. Then by the integration by parts formula

$$\int \psi L\varphi \, d\mu = \int \varphi \widehat{L}\psi \, d\mu \quad \text{for all } \psi, \varphi \in C_0^\infty(\mathbb{R}^d).$$

In addition, for any $\lambda > 0$, the ranges of $\lambda - L$ and $\lambda - \widehat{L}$ on $C_0^\infty(\mathbb{R}^d)$ are dense in $L^1(\mu)$. The operator $\widehat{L}$ also generates a Markov semigroup on $L^1(\mu)$ with respect to which μ is invariant. The corresponding resolvent is denoted by $\widehat{G}_\lambda$. For the proofs we refer to [4, Proposition 2.9] or [12, Proposition 1.10(b)] (see also [5, Theorem 3.1]).

Lemma 1. *Suppose that b is infinitely differentiable, Lipschitzian, and strongly dissipative, so for some $\alpha > 0$, one has $\big(b(x + h) - b(x), h\big) \leq -\alpha(h, h)$ for all $x, h \in \mathbb{R}^d$. Let $\lambda > 0$ and let $v \in L^2(\mu)$ satisfy the inequality $(\lambda - L)v \leq 0$ in the μ-weak sense, i.e., $\int v(\lambda - \widehat{L})\varphi \, d\mu \leq 0$ for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then $v \leq 0$.*

Proof. Letting $V(x) = (x, x)$ and denoting the matrix trace by tr, we obtain

$$LV(x) = 2\mathrm{tr}\, A + 2\big(b(x), x\big) \leq 2\mathrm{tr}\, A - 2\alpha(x, x) + 2\big(b(0), x\big) \leq 2\mathrm{tr}\, A + \alpha^{-1}|b(0)|^2 - \alpha(x, x).$$

According to [3], μ has all moments, hence $|b| \in L^2(\mu)$. As shown in [1], this implies $|\nabla\varrho/\varrho| \in L^2(\mu)$. Let $\zeta_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \leq \zeta_0 \leq 1$ and $\zeta_0(x) = 1$ whenever $|x| \leq 1$. Let $\zeta_k(x) = \zeta_0(x/k)$, $k \in \mathbb{N}$. Then $0 \leq \zeta_k \leq 1$, $|\nabla\zeta_k|$ is bounded uniformly in k, $\zeta_k \to 1$ pointwise and $|\nabla\zeta_k|, L\zeta_k, \widehat{L}\zeta_k \to 0$ in $L^2(\mu)$ as $k \to \infty$.

Let $\eta \in C_0^\infty(\mathbb{R}^d)$, $\eta \geq 0$ and $u := \widehat{G}_\lambda\eta$. Then u is bounded nonnegative, by the Markovian property, and smooth, by the elliptic regularity, since $\widehat{b}$ is smooth. It is known that $|\nabla u| \in L^2(\mu)$. This follows by [12, Theorem 1.5(c)] or by [11, Lemma 2.1]), but can be verified directly as follows. For all $\zeta \in C_0^\infty(\mathbb{R}^d)$ one has $\int u(\lambda u - \widehat{L}u)\zeta\, d\mu = \int \zeta u\eta\, d\mu$. Since $u\widehat{L}u = \frac{1}{2}\widehat{L}u - (A\nabla u, \nabla u)$, it follows that

$$\int u^2(\lambda - \frac{1}{2}L)\zeta\, d\mu + \int (A\nabla u, \nabla u)\zeta\, d\mu = \int u\eta\zeta\, d\mu.$$

Choosing ζ_k as defined above, we get $|\nabla u| \in L^2(\mu)$.

Let now $\varphi_k := \zeta_k u$. Then $\varphi_k \in C_0^\infty(\mathbb{R}^d)$, $\varphi_k \geq 0$ and

$$(\lambda - \widehat{L})\varphi_k = \zeta_k\eta + u\,\widehat{L}\zeta_k + 2(A\nabla\zeta_k, \nabla u) \to \eta \quad \text{in } L^2(\mu) \text{ as } k \to \infty$$

by the dominated convergence theorem, since u is bounded and $|\nabla u| \in L^2(\mu)$. Hence

$$\int v\eta\, d\mu = \lim_{k\to\infty} \int v(\lambda - \widehat{L})\varphi_k\, d\mu \leq 0,$$

which yields that $v \leq 0$, since η is an arbitrary smooth function with compact support. □

In the following lemma we also consider the case of smooth, Lipschitzian, strongly dissipative b. This lemma follows, of course, from [8], [9], where probabilistic arguments are given, but for the reader's convenience we include an alternative analytic proof.

Lemma 2. *Let b be the same as in the previous lemma. Then, for any $\lambda > 0$ and any smooth bounded Lipschitzian function f, one has pointwise*

$$|\nabla G_\lambda f| \leq G_\lambda |\nabla f|.$$

In particular, $\sup_x |\nabla G_\lambda f(x)| \leq \lambda^{-1} \sup_x |\nabla f(x)|$.

Proof. Let $u = G_\lambda f$. As explained in the proof of the previous lemma, $u \in C^\infty(\mathbb{R}^d)$ and $|\nabla u| \in L^2(\mu)$. Observe that $(\lambda - \overline{L})G_\lambda|\nabla f| = (\lambda - L)G_\lambda|\nabla f| = |\nabla f|$, because $|\nabla f|$ is bounded Lipschitzian and $G_\lambda|\nabla f| \in H^{p,2}_{loc}(\mathbb{R}^d)$. Hence it suffices to show that $v := |\nabla u|$ is a μ-weak sub-solution of the equation $(\lambda - L)v = |\nabla f|$, i.e.,

$$\int v(\lambda\varphi - \widehat{L}\varphi)\, d\mu \leq \int |\nabla f|\varphi\, d\mu \tag{4}$$

for every nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$, because then Lemma 1 implies the assertion. Let $\varepsilon > 0$ and $v_\varepsilon := \big(|\nabla u|^2 + \varepsilon\big)^{1/2}$. It is readily verified that v_ε satisfies the equation $\lambda v_\varepsilon - Lv_\varepsilon - w_\varepsilon v_\varepsilon = f_\varepsilon$, where

$$w_\varepsilon := (Db\nabla u, \nabla u)v_\varepsilon^{-2} \leq 0,$$

$$f_\varepsilon := \lambda\varepsilon v_\varepsilon^{-1} + (\nabla u, \nabla f)v_\varepsilon^{-1} - v_\varepsilon^{-1}\Big[\mathrm{tr}(D^2uAD^2u) - v_\varepsilon^{-2}(\nabla u, D^2uAD^2u\nabla u)\Big].$$

Let $\xi := v_\varepsilon^{-1}\nabla u$ and $S := D^2u$. Noting that $|\xi| \leq 1$ and the matrix SAS is symmetric and nonnegative definite, we obtain $(\xi, \nabla f) \leq |\nabla f|$ and $\mathrm{tr}(SAS) - (\xi, SAS\xi) \geq 0$. Therefore, $(\lambda - L)v_\varepsilon \leq \lambda\varepsilon v_\varepsilon^{-1} + |\nabla f|$ pointwise, in particular, in the μ-weak sense. Letting $\varepsilon \to 0$ we obtain (4). □

Proof of Theorem 1. We recall that if a sequence of functions on $\mathbb{R}^d$ is uniformly Lipschitzian with constant L and bounded at a point, then it contains a subsequence that converges uniformly on every ball to a function that is Lipschitzian with the same constant. Therefore, approximating f in $L^1(\mu)$ by a sequence of bounded smooth functions f_j with $\sup_x |\nabla f_j(x)| \leq \sup_x |\nabla f(x)|$, it suffices to prove our estimates for smooth bounded f.

Moreover, due to Euler's formula $T_t f = \lim_n \left(\frac{t}{n} G_{\frac{t}{n}}\right)^n f$, it suffices to establish the resolvent estimate. First we construct a suitable sequence of smooth strongly dissipative Lipschitzian vector fields b_k such that $b_k \to b$ in $L^p(U)$ on every ball U as $k \to \infty$. Let $\sigma_j(x) = j^{-d}\sigma(x/j)$, where σ is a smooth compactly supported probability density. Let $\beta_j := b * \sigma_j$. Then β_j is smooth and dissipative and $\beta_j \to b$, $j \to \infty$, in $L^p(U)$ on every ball U. For every $\alpha > 0$, the mapping $I - \alpha\beta_j$ is a homeomorphism of $\mathbb{R}^d$ and the inverse mapping $(I - \alpha\beta_j)^{-1}$ is Lipschitzian with constant α^{-1} (see [6]). Let us consider the Yosida approximations

$$F_\alpha(\beta_j) := \alpha^{-1}\big((I - \alpha\beta_j)^{-1} - I\big) = \beta_j \circ (I - \alpha\beta_j)^{-1}.$$

It is known (see [6, Ch. II]) that $|F_\alpha(\beta_j)(x)| \le |\beta_j(x)|$, the mappings $F_\alpha(\beta_j)$ converge locally uniformly to β_j as $\alpha \to 0$, and one has $\big(F_\alpha(\beta_j)(x) - F_\alpha(\beta_j)(y), x - y\big) \le 0$.

Thus, the sequence $b_k := F_{\frac{1}{k}}(b * \sigma_k) - \frac{1}{k}I$, $k \in \mathbb{N}$, is the desired one. For every $k \in \mathbb{N}$, let L_k be the elliptic operator defined by (1) with the same constant matrix A and drift b_k in place of b. Let $\mu_k = \varrho_k\, dx$ be the corresponding invariant probability measure and let $G_\lambda^{(k)}$ denote the associated resolvent family on $L^1(\mu_k)$. Since b_k is smooth, Lipschitzian and strongly dissipative, $v_k := G_\lambda^{(k)} f$ is smooth, bounded, Lipschitzian and

$$\sup_x |v_k(x)| \le \frac{1}{\lambda} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v_k(x)| \le \frac{1}{\lambda} \sup_x |\nabla f(x)|$$

by Lemma 2. Moreover, for every ball $U \subset \mathbb{R}^d$, the functions v_k are uniformly bounded in the Sobolev space $H^{2,2}(U)$, since the mappings $|b_k|$ are bounded in $L^p(U)$ uniformly in k and f is bounded. Thus, the sequence $\{v_k\}$ contains a subsequence, again denoted by $\{v_k\}$, that converges locally uniformly to a bounded Lipschitzian function $v \in H^{2,2}_{loc}(\mathbb{R}^d)$ such that

$$\sup_x |v(x)| \le \lambda^{-1} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v(x)| \le \lambda^{-1} \sup_x |\nabla f(x)|,$$

and, in addition, the restrictions of v_k to any ball U converge to $v|_U$ weakly in $H^{2,2}(U)$. Now we show that $v = G_\lambda f$. Note that $\varrho_k \to \varrho$ uniformly on balls according to [3], [2]. Hence, given $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in a ball U, we have

$$\int [\lambda v - Lv - f]\varphi\varrho\, dx = \lim_{k\to\infty} \int [\lambda v_k - L_k v_k - f]\varphi\varrho_k\, dx = 0$$

by weak convergence of v_k to v in $H^{2,2}(U)$ combined with convergence of b_k to b in $L^p(U, \mathbb{R}^d)$. Therefore, by the integration by parts formula

$$\int v(\lambda\varphi - \widehat{L}\varphi)\, d\mu = \int f\varphi\, d\mu$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The function $G_\lambda f$ is bounded and satisfies the same relation, so it remains to recall that if a bounded function u satisfies $\int u(\lambda\varphi - \widehat{L}\varphi)\, d\mu = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $u = 0$ a.e., since $(\lambda - \widehat{L})\big(C_0^\infty(\mathbb{R}^d)\big)$ is dense in $L^1(\mu)$. The same reasoning proves also the last claim. □

Remark 1. Apart from weaker assumptions on b, the main novelty of the result is uniqueness in the class of bounded solutions of the class $H^{2,2}_{loc}(\mathbb{R}^d)$ that are not supposed in advance to be in the domain of generator $\overline{L}$. We emphasize that the assumption of existence of a Lyapunov function has been only used in the uniqueness statement. Our reasoning without that assumption (i.e., only with (Ha) and (Hb)) shows that given a bounded Lipschitzian function g, there exists some bounded solution $v \in H^{2,2}_{loc}(\mathbb{R}^d)$ of the equation $(\lambda - L)v = g$ satisfying (3). Indeed, we obtain uniformly bounded and uniformly Lipschitzian functions $v_k = G_\lambda^{(k)}$ satisfying the equations $\lambda v_k - L_k v_k = g$. Then we find a subsequence in $\{v_k\}$ that converges weakly in $H^{2,2}(U)$ for every ball U. The limit is a desired solution. We do not know whether such a solution is unique in this case. Note also that if b is locally Hölder continuous, then, by the classical theory (see, e.g., [10]), the second derivative of f is locally Hölder continuous.

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