Optimal Distributed Dynamic Advertisement

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Abstract

We extend the classical dynamic advertising model of M. Nerlove and J. Arrow [11] adding a spatial dimension. We consider the general mixed problem of maximizing a utility function of product image at a fixed time and minimizing the corresponding cumulative cost of advertising. After an abstract treatment allowing for general utility and cost functions, as well as state and control constraints, we give a more explicit (although in general only suboptimal) solution of the problem through the linear-quadratic regulator in infinite dimensions. We also study, by similar methods, the problem of reaching a target level of awareness of the advertised product by a given deadline.

Key Words: optimal advertising, new product introduction, distributed control, infinite dimensional analysis, linear-quadratic control, partial differential equations.

AMS Classification: 90B60, 91B72, 49K27, 93C25, 93C30

1 Introduction

M. Nerlove and J. Arrow introduced in their classical paper [11] the following model for the dynamics of goodwill under the influence of advertising:

$$\frac{dx(t)}{dt} = u(t) - \rho x(t), \tag{1}$$

where $x(t)$ is the goodwill level at time $t \geq 0$, u is the rate of advertisement spending, and $\rho > 0$ is a constant deterioration factor in absence of advertising. Here x is assumed to be only a function of time. It would be interesting to add a spatial dimension as well, to model fluctuations in the awareness of

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a product with respect to geographic location. It is natural to assume, for instance, that advertisement reaches some areas better than others. In contrast to the classical case of goodwill depending only on time, the treatment of more general problems of optimal advertising in the space-time setting is new to the best of our knowledge.

In the first part of this work we address a modeling question, namely how to describe the dynamics of goodwill depending on space and time. In particular, we propose to model the controlled dynamics of goodwill through the following partial differential equation (PDE)

$$\frac{\partial x}{\partial t}(t,\xi) = -\rho x(t,\xi) + \Delta_\xi x(t,\xi) + b(\xi)u(t,\xi) \tag{2}$$

where $x : [0,T] \times \Xi \to \mathbb{R}$, $\Xi \subset \mathbb{R}^2$ (see below for more details about the exact hypotheses of the model). This model is able to capture a diffusion effect, through the term $\Delta_\xi x(t,\xi)$, and clearly reduces to the Nerlove-Arrow model (1) if x does not depend on the spatial coordinate ξ and $b(\xi) \equiv 1$.

In the second part, we formulate and solve the space–time analogs of some of the optimal control problems studied when x depends on time only. In particular, we study two types of problems: in the first case our aim is to maximize a functional of the product image $x(T,\cdot)$ at a given time $T > 0$, while minimizing the cumulative cost advertising. We begin by approaching the problem in the abstract setting of control of nonlinear infinite dimensional systems via Pontryagin's maximum principle. This way we are able to obtain quite general results on existence of optimal policies and their characterization, imposing only very mild assumptions on the data, but we cannot get an explicit representation of optimal controls. Instead, strengthening our assumptions and using the linear-quadratic (LQ) regulator in infinite dimensions, we can give much more explicit results. The LQ approach is also used to solve the second related problem, i.e. the minimization of a weighted sum of the distance of $x(T,\cdot)$ from a target level of product awareness and the total quadratic cost to reach the target.

We proceed as follows: first we show that the controlled PDE (2) can be written as an abstract linear (differential) control system equation of the type

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3}$$

in a suitable Hilbert space of functions X. Then we show that the control problem for the PDE (2) is equivalent to a control problem on X for the abstract differential equation (3). The general case is solved (although only in an abstract way) by the weak maximum principle in infinite dimensions, the theory of which can be found, e.g., in V. Barbu [2]. The simpler case of quadratic utility and cost functions is reduced to the study of a Riccati equation by an application of the dynamic programming principle. The solution of such an equation yields in particular the optimal policy for the original problem in feedback form. For the infinite dimensional LQ problem we refer to A. Bensoussan, G. Da Prato, M. Delfour and S. Mitter [4], [5] and R. Curtain and H. Zwart [6] for the standard case of positive definite costs (see also I. Lasiecka and R. Triggiani [8], [9]), and to X. Li and J. Yong [10] for the general case with indefinite costs.

2 Preliminaries and Notation

Let X be a real separable Hilbert space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$. Then $L^2([0,T];X)$ shall denote the Hilbert space of functions $f : [0,T] \to X$ with finite norm $\|f\|$,

$$\|f\|^2 = \int_0^T |f(t)|^2 \, dt < \infty.$$

When there will be no possibility of confusion, we shall still denote by $\langle \cdot, \cdot \rangle$ the inner product of spaces like $L^2([0,T];X)$. The space of linear bounded operators between the Hilbert spaces X and Y will be denoted with $\mathcal{L}(X,Y)$, or simply with $\mathcal{L}(X)$ when $X = Y$.

Given an operator $A : D(A) \subset X \to X$, we shall say that A is *uniformly positive definite* ($A \gg 0$) if there exists $\delta > 0$ such that $A - \delta I \geq 0$, i.e.

$$\langle Ax, x \rangle \geq \delta |x|^2 \quad \forall x \in D(A).$$

We shall make use of some functional spaces: for a bounded open set $\Omega \subset \mathbb{R}^d$, $H^k(\Omega)$ is the Sobolev space of functions with (generalized) derivatives up of order k in $L^2(\Omega)$, and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the topology of $H^1(\Omega)$.

3 Model and problem formulation

Let Ξ be a bounded open set of $\mathbb{R}^2$ with regular boundary $\partial\Xi$. This set will be the model for the geographic region of interest for the advertising campaign. Let $x : [0,T] \times \Xi \to \mathbb{R}$ be the goodwill level (or density) of a given product, specified as a function of time t and location $\xi \in \Xi$. The model for the controlled dynamics of x will be given by the following equation:

$$\begin{cases} \dfrac{\partial x(t,\xi)}{\partial t} = (-\rho + \Delta_\xi) x(t,\xi) + b(\xi) u(t,\xi), & (t,\xi) \in]0,T] \times \Xi \\ x(t,\xi) = 0, & (t,\xi) \in]0,T] \times \partial\Xi \\ x(0,\xi) = x_0(\xi), & \xi \in \Xi, \end{cases} \tag{4}$$

where $T > 0$ is a fixed time (which could be thought, for instance, as the time of introduction to the market of a new product), $\rho > 0$ is a natural deterioration factor of the product image in absence of advertisement, $\Delta_\xi = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$ is the Laplacian with respect to the spatial variable ξ, $u : [0,T] \times \Xi \to \mathbb{R}_+$ is the rate of advertisement spending, which also depends on time and location, and $b(\cdot)$ is a (bounded) factor of advertising effectiveness. Moreover, $x_0 : \Xi \to \mathbb{R}$ represent the initial level of goodwill as a function of location in the region of interest (for example, one may assume $x_0 = 0$ if at time $t = 0$ the advertised product is unknown), and Ξ is assumed big enough, so that we do not need to care about the goodwill x "far out" from the interior of Ξ, hence the boundary condition in (4). Clearly, one could impose other boundary conditions, such as $x(t, \xi) = f(t, \xi)$, for a given fixed $f : [0,T] \times \partial\Xi \to \mathbb{R}$, or $\partial x/\partial\nu = 0$, whose meaning will become clear in a moment ($\partial/\partial\nu$ is the outward normal derivative).

The diffusion term $\Delta_\xi x(t, \xi)$ is motivated by heuristic arguments similar to those leading to the heat equation. Let $G(t)$ denote the total goodwill at time t on a region $\Omega \subseteq \Xi$. Then $G(t)$ is obtained by integrating x over Ω, i.e.

$$G(t) = \int_\Omega x(t, \xi)\, d\xi,$$

which also implies

$$\frac{dG}{dt} = \int_\Omega \frac{\partial x}{\partial t}(t, \xi)\, d\xi.$$

We assume that changes of total goodwill in the region Ω are due to three factors: the effect of advertisement, a forgetting effect, and interactions with the region $\Xi \backslash \Omega$. More precisely, we assume that transfer of goodwill through the boundary $\partial\Omega$ of Ω is (locally) proportional to the goodwill gradient and to a function of "goodwill conductivity" $\kappa(\cdot)$ (a proxy for the intensity of the interaction among people, such as population density, for instance). We assume then

$$\frac{dG}{dt} = \int_{\partial\Omega} \nu \cdot \kappa(\xi) D_\xi x(t, \xi)\, d\xi - \int_\Omega \rho x(t, \xi)\, d\xi + \int_\Omega b(\xi) u(t, \xi)\, d\xi \tag{5}$$

where ν is the outward normal vector. Then one can use Gauß' divergence theorem (equivalently, integrate by parts) and write

$$\int_{\partial\Omega} \nu \cdot \kappa(\xi) D_\xi x(t, \xi)\, d\xi = \int_\Omega D_\xi \Big(\kappa(\xi) D_\xi x(t, \xi) \Big)\, d\xi.$$

Since (5) must hold for any region $\Omega \subseteq \Xi$, then

$$\frac{\partial x}{\partial t}(t, \xi) = \frac{\partial}{\partial \xi} \Big(\kappa(\xi) \frac{\partial x}{\partial \xi}(t, \xi) \Big) - \rho x(t, \xi) + b(\xi) u(t, \xi)$$

holds almost everywhere in Ξ. If κ is constant, then we also have

$$\frac{\partial x}{\partial t}(t, \xi) = \kappa \Delta_\xi x(t, \xi) - \rho x(t, \xi) + b(\xi) u(t, \xi).$$

Remark 1 The assumption $\kappa(\xi) \equiv \kappa$, even if not very realistic, does not invalidate our approach to the problem. In fact, we would be able to obtain completely similar results also in the general case, but with less explicit formulas. This is due to the fact that the Laplace operator Δ is very well studied and plenty of explicit results are known, while much less is available for operators of the type $D_\xi \kappa(\xi) D_\xi$.

Remark 2 The model does not feature any spatial interactions, or in other words its evolution is only local. That is to say, what happens around a point ξ_1 does not have any influence on what happens around $\xi_2 \neq \xi_1$. Of course one could generalize such a situation, including a "potential" term to introduce interactions. This interesting extension, however, is left for future work.

A rather general optimal advertising problem one would like to solve can be described as follows:

(P) maximize the functional

$$J_c(u) = \int_\Xi \phi_0(x(T,\xi))\, d\xi - \int_0^T \int_\Xi h_0(u(t,\xi))\, d\xi\, dt \tag{6}$$

over all controls $u(t,\xi) \in [0,R]$ (t,ξ)-a.e., subject to the dynamics (4) and the additional constraint $x(t,\xi) \geq 0$ (t,ξ)-a.e.. Here ϕ_0, $h_0 : \mathbb{R} \to \mathbb{R}$ are such that the integrals in (6) are finite, and can be thought as utility of final goodwill and cost of advertising, respectively.

We shall also solve the two simpler problems:

(P1) minimize the functional

$$J_i(u) = -\gamma \int_\Xi |x(T,\xi)|^2\, d\xi + \int_0^T \int_\Xi |u(t,\xi)|^2\, d\xi\, dt \tag{7}$$

over all controls $u \in L^2([0,T] \times \Xi; \mathbb{R})$, subject to the dynamics (4);

(P2) minimize the functional

$$J_h(u) = \gamma \int_\Xi |x(T,\xi) - k(\xi)|^2\, d\xi + \int_0^T \int_\Xi |u(t,\xi)|^2\, d\xi\, dt, \tag{8}$$

over all controls $u \in L^2([0,T] \times \Xi; \mathbb{R})$, subject to the dynamics (4),

where $\gamma > 0$ can be considered as the weight of the first objective with respect to the second in the corresponding optimizations. In (P2), $k : \Xi \to \mathbb{R}$ is the target level of goodwill to be reached at time T.

Let us mention that in the original problem of M. Nerlove and J. Arrow [11] the horizon T was infinite. While we plan to treat this problem elsewhere, we would like to observe that one could consider the finite and the infinite horizon problems as complementary: if T is the time at which a product will be introduced into the market, then our optimal state at launch time $x^*(T,\cdot)$ could be taken as initial condition for a spatial version of the problem of maximizing profits net of advertising expenditures from T to infinity.

4 Reformulation as control problems in infinite dimensions

Let X be the Hilbert space of square integrable functions defined on the domain Ξ, i.e. $X = L^2(\Xi)$, equipped with the natural inner product

$$\langle f, g \rangle := \int_\Xi f(\xi) g(\xi)\, d\xi$$

and norm

$$|f| := \left(\int_\Xi f^2(\xi)\, d\xi \right)^{1/2}.$$

Denote by A the following linear operator in X:

$$\begin{cases} Ay = (\Delta_\xi - \rho) y \\ D(A) = H^2(\Xi) \cap H_0^1(\Xi). \end{cases} \tag{9}$$

Setting (with a slight abuse of notation) $x(t) = x(t,\cdot)$ and $u(t) = u(t,\cdot)$, we can write (4) as an abstract linear system on the Hilbert space X:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \in X \end{cases} \tag{10}$$

for $t \in [0,T]$, with $A : D(A) \subset X \to X$ as in (9) and $B : X \to X$ is the linear bounded operator defined by

$$B : y(\xi) \mapsto b(\xi) y(\xi).$$

Note that A is the infinitesimal generator of a strongly continuous semigroup on X, which we shall denote e^{tA}, hence the unconstrained Cauchy problem (10) admits a unique mild solution x given by the variation of constants formula:

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds$$

(see, e.g., Pazy [12]). Existence and uniquess for (10) with the constraint $x \geq 0$ is more delicate and will be discussed in the following section.

Problem (P) can now be written as

$$\inf_{u\in\mathcal{U}} \left(\phi(x(T)) + \int_0^T h_1(u(t))\,dt \right) \tag{11}$$

subject to the dynamics (10), where ϕ, $h_1 : X \to \mathbb{R}$ are defined as

$$\begin{aligned} \phi(x) &= -\int_\Xi \phi_0(x(\xi))\,d\xi \\ h_1(u) &= \int_\Xi h_0(u(\xi))\,d\xi, \end{aligned}$$

and

$$\mathcal{U} = \left\{ u : [0,T] \to X \,\middle|\, u(\cdot)(\xi) \in [0,R],\ x(\cdot)(\xi) \geq 0\ \xi\text{-a.e.} \right\}.$$

Similarly, the objective functionals (7) and (8) can be respectively written as

$$J_i(u) = -\gamma |x(T)|^2 + \int_0^T |u(t)|^2\,dt$$

and

$$J_h(u) = \gamma |x(T) - k|^2 + \int_0^T |u(t)|^2\,dt.$$

The aim is to find an optimal control, i.e. a function $u^* \in \mathcal{U}_{ad}$, such that

$$J(u^*) \leq J(u) \quad \forall u \in \mathcal{U}_{ad}.$$

Here J is either J_c, J_i, or J_h, and $\mathcal{U}_{ad}$ is the class of admissible controls: $\mathcal{U}_{ad} = \mathcal{U}$ for problem (P), and $\mathcal{U}_{ad} = L^2([0,T];X)$ for problems (P1) and (P2). The couple (x^*, u^*), where x^* is the solution of (10) with $u \equiv u^*$, is often called an *optimal pair* for the corresponding optimal control problem.

Remark 3 Problem (P1) is an indefinite linear-quadratic (LQ) optimal control problem with indefinite costs, while problem (P2) is an LQ problem with positive costs similar to those encountered in the study of target tracking. While problem (P2) is always well-posed and always admits an optimal control, problem (P1) will be in general only locally well-posed, and global well-posedness will follow from additional assumptions on the parameters of the problem. For more details on LQ problems with indefinite costs in infinite dimensions, see X. Li and J. Yong [10].

Remark 4 The objective functional of problem (P1) (the following considerations apply unchanged to problem (P2) as well) could be taken, more generally, as

$$J_i(u) = -\langle P_0 x(T), x(T) \rangle + \int_0^T |u(t)|^2\,dt,$$

with $P_0 : X \to X$. For example, if the goodwill is more valued in a region $\Xi_1 \subset \Xi$, P_0 could be an operator such that $P_0 = \gamma_1 I$ on Ξ_1, and $P_0 = \gamma I$ on $\Xi \setminus \Xi_1$, with $\gamma < \gamma_1$. Even more generally, one could fix a bounded function $p : \Xi \to \mathbb{R}^+$ modelling the importance of goodwill at each point of Ξ, and define $P_0 = pI$, I being the identity function on X.

5 Solution of the general constrained problem

The basic idea is to embed the control and state constraints into the structure of the problem. This is obtained by rewriting the state equation (10) subject to the constraint $x \geq 0$ as a nonlinear evolution equation, and by assigning infinite cost to the controls that do not satisfy the constraint $0 \leq u \leq R$. Once this is done, we show that the problem admits a solution, i.e. that an optimal exists, and finally we write a weak maximum principle that gives a (abstract) characterization of the sought optimal advertising policy.

Let us define $h : X \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$h(u) = \begin{cases} h_1(u) & u \in [0, R] \\ +\infty & u \notin [0, R] \end{cases}$$

Proposition 5 *Assume that*

(i) $h_0 : \mathbb{R} \to \mathbb{R}$ *is a convex lower semicontinuous (lsc) function such that*

$$h_1(u) \geq c_1 |u|^2 - c_2$$

for all $u \in X$ *and for some* $c_1 > 0$ *and* $c_2 \in \mathbb{R}$*;*

(ii) ϕ_0 *is continuous and concave;*

(iii) x_0 *is nonnegative and* $x_0 \in H_0^1(\Xi)$.

Then there exists at least one optimal pair (x^*, u^*) *for problem (P). Moreover, for any such optimal pair there exists a function* $p \in L^\infty([0,T]; X)$ *such that* $p' + (\Delta - \rho)p \in L^\infty([0,T] \times \Xi)'$ *satisfying the following system of equations*

$$\begin{cases} (p' + \Delta p - \rho p)_{ac} = 0 & a.e. \ in \ \{x^* > 0\} \\ pBu^* = 0 & a.e. \ in \ \{x^* = 0\} \\ p(T) + \partial\phi(x^*(T)) \ni 0 & in \ \Xi \\ B^* p(t) \in \partial h(u^*(t)) & a.e. \ t \in]0, T[. \end{cases} \tag{12}$$

In the statement of the proposition $\cdot'$ stands for the (topological) dual, and μ_{ac} denotes the absolutely continuous part of the measure μ with respect to Lebesge measure in $[0,T] \times \Xi$. Finally, ∂f denotes the subdifferential of f. For more informations on these notions see e.g. T. Rockafellar [13] or V. Barbu and Th. Precupanu [3].

Proof. The linear constrained control system can be equivalently written as the following controlled parabolic variation inequality (also known as obstacle problem)

$$\begin{cases} \dot{x}(t) - Ax(t) + \partial I_K(x(t)) \ni Bu(t) \\ x(0) = x, \end{cases} \tag{13}$$

where

$$K = \Big\{ y \in X : \; y(\xi) \geq 0 \,\text{a.e.} \Big\},$$

$I_K(y) := 0$ for $y \in K$ and $I_K = \infty$ otherwise, hence

$$\partial I_K(r) = \begin{cases} 0, & r > 0 \\ \mathbb{R}_-, & r = 0 \\ \emptyset, & r < 0. \end{cases}$$

If $x_0(\xi) \geq 0$ a.e. and $x_0 \in H_0^1(\Xi)$, then (13) has a unique solution $x \in C([0,T];X)$. In fact, $A + \partial I_K$ generates a contraction semigroup, $x_0 \in D(A) \cap K$, and $u \in \mathcal{U}$ implies $u \in L^2([0,T];X)$, hence by Corollary 3.2, p. 280 in [2] the result follows. Proposition 1.1, p. 319 in [2] also guarantees that problem (P) admits at least a solution u^* with corresponding optimal trajectory x^* (all the technical conditions under which this result holds are satistied in our case). The characterization of the so-called dual extremal arc p given by (12) is a straightforward adaptation of Theorem 1.2, p. 332, in [2]. ■

6 Solution of the indefinite LQ problem

We shall solve the optimal control problems (P1) and (P2) through the dynamic programming approach, that is, first we solve the associated operator Riccati equations, and then we show that the optimal control u^* can be written as a linear feedback of the optimal trajectory x^*. In this and the following section we also assume $B = I$, for simplicity. The slightly more general case of $B \in \mathcal{L}(X)$ is not significantly different.

We can give an abstract solution of problem (P1) by applying results of chapter 9 of X. Li and J. Yong [10]. Before stating the main results, we need some preparation. Namely, let us introduce the linear operators

$$\begin{aligned} L_1 : L^2([0,T];X) &\to X \\ u(\cdot) &\mapsto \int_0^T e^{(T-s)A} u(s)\, ds \end{aligned}$$

and

$$\begin{aligned} \Phi : L^2([0,T];X) &\to L^2([0,T];X) \\ u(\cdot) &\mapsto (I + L_1^* P_0 L_1)u = (I - \gamma L_1^* L_1)u, \end{aligned}$$

where L_1^* is the adjoint of L_1 and P_0 is the weight of $x(T)$ in the objective function J_i, i.e. $-P_0 = \gamma I \gg 0$. Note that $L_1^* : X \to L^2([0,T];X)$ can be explicitly written as

$$(L_1^* y)(s) = e^{(T-s)A} y,$$

where we have used the fact that A is self-adjoint.

Proposition 6 *Suppose that the parameters of problem (P1) satisfy the inequality*

$$1 - \frac{\gamma}{2\rho}(1 - e^{-2\rho T}) > 0.$$

Then for any $x_0 \in X$, J_i admits a unique minimizer u^* given by

$$u^* = -\Phi^{-1}\Theta x_0,$$

where $\Theta = -\gamma L_1^* e^{-AT}$. Moreover, the value function is a bounded bilinear form on X explicitly given by

$$V(y) = \inf_u J(y;u) = \langle \Gamma y, y\rangle - \langle \Phi^{-1}\Theta y, \Theta y\rangle,$$

for all $y \in X$, where $\Gamma = e^{AT} P_0 e^{AT} = -\gamma e^{2AT}$.

Proof. The operator $L_1^* P_0 L_1 = -\gamma L_1^* L_1$ is clearly self-adjoint. Since A is also self-adjoint, by proposition 9.2.8(iii) in [10], it follows that the (linear) operator $L_1^* P_0 e^{-AT}$ is bounded. Let us now determine under which conditions Φ is a positive operator. Write

$$\begin{aligned} \langle \Phi v, v\rangle &= \|v\|^2 - \gamma\langle L_1^* L_1 v, v\rangle = \|v\|^2 - \gamma\langle L_1 v, L_1 v\rangle \\ &= \|v\|^2 - \gamma |L_1 v|^2. \end{aligned}$$

We have

$$\begin{aligned} |L_1 v| &= \left| \int_0^T e^{-\rho(T-s)} e^{\Delta(T-s)} v(s,\xi)\, ds \right| \\ &\le \int_0^T e^{-\rho(T-s)} |e^{\Delta(T-s)} v(s,\xi)|\, ds \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T e^{\rho(T-s)} |v(s,\xi)|\, ds \\
&\leq \left[\int_0^T e^{-2\rho(T-s)}\, ds\right]^{1/2} \left[\int_0^T |v(s,\xi)|^2\, ds\right]^{1/2} \\
&= C_{\rho,T}^{1/2} \|v\|.
\end{aligned}
$$

We have used (in order) a standard estimate, the contractivity of the heat semigroup, the Cauchy-Schwarz inequality, and Fubini's theorem for positive integrands. By

$$
C_{\rho,T} = \int_0^T e^{-2\rho(T-s)}\, ds = \frac{1}{2\rho}(1 - e^{-2\rho T})
$$

the result follows immediately, setting $\delta = 1 - \frac{\gamma}{2\rho}(1 - e^{-2\rho T})$. ■

This optimal control suffers of two major drawbacks: it is of the open-loop type, and it is difficult to write explicitly (in particular, it does not seem straightforward to compute Φ^{-1}). Appealing to the dynamic programming principle, one can obtain a more explicit feedback characterization of the optimal policy.

Proposition 7 *If*

$$
1 - \frac{\gamma}{2\rho}(1 - e^{-2\rho T}) > 0, \tag{14}
$$

then there exists $P \in C([0,T]; \mathcal{L}(X))$, $P(t) = P(t)^*$ *for all* $t \in [0,T]$, *which solves the operator Riccati equation*

$$
\begin{cases} P' = 2AP - P^2 \\ P(0) = P_0 = -\gamma I. \end{cases} \tag{15}
$$

Moreover, the optimal control is given by

$$
u^*(t) = -P(T-t)x^*(t),
$$

x^* *being the unique (mild) solution of the closed loop equation*

$$
\begin{cases} \dot{x}(t) = Ax(t) - P(T-t)x(t) \\ x(0) = x_0, \end{cases}
$$

and the value function can be written as

$$
V(t,y) = J_i(u^*) = \langle P(T-t)y, y\rangle.
$$

Proof. The proof is mainly an application of the proof of Theorem 9.4.3 of X. Li and J. Yong [10]. In fact, by assumption we have that the weight of the control in the value function J_i is the identity, hence it is uniformly positive definite. Moreover, by the assumption (14), Proposition 6 guarantees that $\Phi \gg 0$. In order to apply the above mentioned result of X. Li and J. Yong, we need to verify that their conditions (H1), (H3) and (H4) (ibid., p. 374) are satisfied. Using their notation, since the case of distributed control corresponds to $\alpha = 0$, we have that $L_1 \in \mathcal{L}(L^2([0,T];X);X)$, hence $L_1^* P_0 L_1$ is self-adjoint, i.e. (H1) is verified. (H3) requires that

$$R(P_0) \subseteq D((A^*)^\beta)$$

for some $\beta > \alpha - 1/2$. But then it is enough to choose $\beta = 0$. (H4) is trivially verified, since the running weight on x in our problem is zero.
The differential operator Riccati equation (15) follows by the integral Riccati equation of Theorem 9.4.3 in ibid. by a time reversal $t \mapsto T - t$ and by rewriting it as a mild evolution equation (as it is done, for example, in G. Da Prato [7]). ■

Remark 8 The integral equation of X. Li and J. Yong [10] is

$$\begin{aligned} P(t)x &= e^{-A(T-t)} Q_1 e^{-A(T-t)} x + \int_t^T e^{-A(s-t)} Q e^{-A(s-t)} x \, ds \\ &\quad - \int_t^T e^{-A(s-t)} P(s)^* B R^{-1} B^* P(s) e^{-A(s-t)} x \, ds. \end{aligned}$$

with terminal condition $\langle P(T)x, z \rangle = \langle Q_1 x, z \rangle \ \forall x, \, z \in E$.

The Riccati equation of the above proposition can actually be solved explicitly in our case. In fact, it is well known (see e.g. S. Agmon [1]) that there exists a complete orthonormal (ONC) system $(e_k)_{k\in\mathbb{N}}$ in X and a sequence of positive numbers $(\lambda_k)_{k\in\mathbb{N}} \uparrow +\infty$ such that

$$Ae_k = -\lambda_k e_k \quad \forall k \in \mathbb{N}.$$

Then we can "project" the Riccati equation on this ONC system, obtaining the infinite set of Cauchy problems

$$\begin{cases} p_k'(t) = -2\lambda_k p_k(t) - p_k^2(t) \\ p_k(0) = -\gamma, \end{cases} \tag{16}$$

where $p_k(\cdot) := P(\cdot)e_k$. It is immediate that $q_k(t) \equiv -2\lambda_k$ is a particular solution for the k-th problem. Then set

$$z_k(t) = \frac{1}{p_k(t) - q(t)} = \frac{1}{p_k(t) + 2\lambda_k}.$$

One easily obtains that z_k satisfies the linear equation

$$z_k'(t) = -2\lambda_k z_k(t) + 1,$$

whose general solution is given by

$$z_k(t) = \frac{1}{2\lambda_k} + C_k e^{-2\lambda_k t}.$$

This in turn implies that the general solution for (16) is given by

$$p_k(t) = -2\lambda_k + \frac{1}{(2\lambda_k)^{-1} + C_k e^{-2\lambda_k t}}, \tag{17}$$

and by the initial condition $C_k = \frac{\gamma}{2\lambda_k(2\lambda_k - \gamma)}$.

An explicit epression for the optimal trajectory can also be obtained, again by projecting on the orthonormal system $(e_k)_{k\in\mathbb{N}}$. In particular, one has

$$x_k'(t) = -\lambda_k x_k(t) + p_k(T-t)x_k(t)$$

for each k, hence

$$x_k^*(t) = x_k(0)e^{-\lambda_k t}e^{\int_0^t p_k(T-s)\,ds}x_k(0), \tag{18}$$

and

$$x^*(t,\xi) = \sum_{k=0}^{\infty} x_k^*(t)e_k(\xi).$$

We can now write explicitly, in terms of the basis $(e_k)_{k\in\mathbb{N}}$, the optimal distributed control. Namely,

$$u_k^*(t) = p_k(T-t)x_k^*(t),$$

with p_k as in (16), and x_k^* is given by (18), hence

$$u^*(t,\xi) = \sum_{k=1}^{\infty} p_k(T-t)x_k^*(t)e_k(\xi).$$

In general it is not possible to determine explicitly the eigenvalues and the corresponding eigenfunctions of A for a generic bounded domain $\Xi \subset \mathbb{R}^2$. However, for particular shapes of Ξ they are known, e.g. for Ξ being a rectangle. If $\Xi = [0,L]\times[0,H]$, one has

$$\Delta e_{m,n} = -\lambda_{m,n}e_{m,n}$$

with

$$e_{m,n}(\xi_1,\xi_2) = \sin\frac{m\pi\xi_1}{L}\sin\frac{n\pi\xi_1}{H}$$

and

$$\lambda_{m,n} = \Big(\frac{m^2}{L^2} + \frac{n^2}{H^2}\Big)\pi^2,$$

hence $Ae_{m,n} = -(\lambda_{m,n}+\rho)e_{m,n}$.

7 Solution of the targeting problem

In the Hilbert space setting introduced in section 4, let us define the distance from the target $y : [0,T] \to X$ as

$$y(t,\xi) = h(\xi) - x(t,\xi),$$

where $h \in L^2(\Xi)$ is the desired configuration of goodwill to reach at time T. Then y is the unique mild solution of the following non-homogeneous linear Cauchy problem in X:

$$\begin{cases} \dot{y}(t) = Ay(t) - u(t) + f(t) \\ y(0) = h - x_0, \end{cases} \tag{19}$$

with $f(\xi) = -Ah(\xi)$.

Problem (P2) can be rewritten as

$$\inf_{u \in L^2([0,T];E)} \gamma|y(T)|^2 + \int_0^T |u(t)|^2 \, dt, \tag{20}$$

subject to (19). Appealing again to the dynamic programming principle, we can write the Riccati equation associated to problem (20) as follows:

$$\begin{cases} \dot{P}(t) = 2AP(t) - P^2(t) \\ P(0) = \gamma I, \end{cases} \tag{21}$$

and its adjoint backward equation as

$$\begin{cases} \dot{r}(t) + (A - P(t))r(t) + P(t)f = 0 \\ r(T) = 0. \end{cases} \tag{22}$$

Then one can uniquely solve the tracking problem in terms of the solution to (21) and (22).

Proposition 9 *If the target function h is such that $Ah \in L^2(\Xi)$, then the optimal control problem (P2) admits a unique optimal control u^* given by the feedback law*

$$u^*(t) = P(T-t)y^*(t) + r(t),$$

where y^* is the unique mild solution of the closed-loop equation

$$\begin{cases} \dot{y}(t) = (A - P(T-t))y(t) - r(t) + f \\ y(0) = h - x_0, \end{cases} \tag{23}$$

and $P(\cdot)$, $r(\cdot)$ solve, respectively, equations (21) and (22). Moreover, the value function J^* is given by

$$\begin{aligned} J^* = J(u^*) &= \langle P(T)y(0), y(0)\rangle + 2\langle r(0), y(0)\rangle \\ &\quad + \int_0^T [2\langle r(t), f\rangle - |r(t)|^2] \, dt. \end{aligned}$$

Proof. The assumption $Ah \in L^2(\Xi)$ guarantees that $f \in X = L^2(\Xi)$. Moreover, A generates a strongly continuous semigroup on X, $B = I$ is a bounded linear operator on X, and $P_0 = \gamma I$ is hermitian and positive definite. Then Theorem 7.1 of Bensoussan et al. [4], p. 172, yields the result. ■

Remark 10 In contrast to (P1), there exists always an optimal solution for problem (P2), for any choice of the parameters and initial condition.

Let us obtain an expression for the optimal trajectory and the optimal control in terms of a basis of $L^2(\Xi)$, as we have done for (P1). In particular, by projecting the Riccati equation (21) on the system $(e_k)_{k\in\mathbb{N}}$, we obtain the infinite set of Cauchy problems

$$\begin{cases} \dot{p}_k(t) = -2\lambda_k p_k(t) - p_k^2(t) \\ p_k(0) = \gamma, \end{cases} \tag{24}$$

each of which admits the explicit solution

$$p_k(t) = -2\lambda_k + \frac{1}{(2\lambda_k)^{-1} + C_k e^{-2\lambda_k t}},$$

with $C_k = -\gamma(2\lambda_k(2\lambda_k + \gamma))^{-1}$. As before, we have set $p_k(\cdot) := P(\cdot)e_k$.

The adjoint backward Cauchy problem (22) can be solved similarly: projecting on the system $(e_k)_{k\in\mathbb{N}}$ we get

$$\begin{cases} \dot{r}_k(t) - \lambda_k r_k(t) - p_k(T-t) r_k(t) + p_k(T-t) f_k = 0 \\ r_k(T) = 0, \end{cases} \tag{25}$$

where $r_k(\cdot) := r(\cdot)e_k$ and $f_k := fe_k$. Setting $\eta_k(t) := r_k(T-t)$, one has

$$\begin{cases} \dot{\eta}_k(t) = \lambda_k \eta_k(t) - p_k(t)\eta_k(t) + p_k(t) f_k \\ \eta_k(0) = 0, \end{cases} \tag{26}$$

These Cauchy problems can be solved explicitly, yielding

$$\eta_k(t) = \gamma f_k e^{\lambda_k t + \int_0^t p_k(s)\,ds} + f_k \int_0^t e^{\lambda_k(t-\tau) - \int_\tau^t p_k(s)\,ds} p_k(\tau),$$

and finally $r_k(t) = \eta_k(T-t)$.

Let us now write explicitly the optimal trajectory by computing the solution of the closed-loop equation. We project again the equation on the system $(e_k)_{k\in\mathbb{N}}$, obtaining

$$\begin{cases} \dot{y}_k(t) = (-\lambda_k - p_k(T-t)) y_k(t) - r_k(t) + f_k \\ y_k(0) = h_k - x_k(0), \end{cases} \tag{27}$$

hence

$$y_k^*(t) = (h_k - x_k(0))e^{\int_0^t a_k(s)\,ds} + \int_0^t e^{\int_\tau^t a_k(s)\,ds}(f_k - r_k(s))\,ds,$$

where we have set $a_k(s) := -\lambda_k - p_k(T-s)$.
The optimal trajectory can now be written as

$$y^*(t,\xi) = \sum_{k=0}^{\infty} y_k^*(t)e_k(\xi).$$

Similarly, the optimal policy is given by

$$u^*(t,\xi) = \sum_{k=0}^{\infty} u_k^*(t)e_k(\xi) = \sum_{k=0}^{\infty}(p_k(T-t)y_k^*(t) + r_k(t))e_k(\xi).$$

8 Further problems

We have assumed for simplicity that the cost of advertisement is not discounted. This could be easily generalized, obtaining a linear time-dependent control system. Under mild additional assumptions, very similar results could be derived.

Instead of considering distributed control, one could consider pointwise control, or a combination of them. Such a generalization seems to be particularly meaningful, as it can be seen as a model for the situation where advertisement can be done only in certain locations of a region of interest. The main difference with the case treated here is that the control operator B is unbounded, but a solution of the problem is still possible, although with some extra technical complications.

Several other problems, such as investigating the sensitivity of the value function and of the optimal control with respect to the intensity of the diffusion effect, extending the class of tracking targets, or studying the case of infinite horizon, could also be considered.

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