Large Deviations in Discrete Rotation Maps

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Abstract

The qualitative and experimental investigation of dynamical system representing a discretization of rotation was carried out. For different values of angle and different initial data we found the first periodic point, period, maximum and minimum values of coordinates of trajectory points. The main experimental result is the discovery of such a value of angle for which all trajectories with the different initial points go very far from their initial points. In this case the first periodic point occurs after appearing more than 1500000 non-periodic points of the trajectory and the maximum value of coordinates is not less than 1023 for all initial points being close to the origin. The value of period which is 2049 and the maximum and minimum values of coordinates of the trajectory points do not depend on the initial points. The angle for which such a phenomena takes place differs from the value $\pi/2$ in only $14-th$ digit in the computer representation of the number $\pi/2$.

1 Setting of Problem

The investigation of a dynamical system representing a discretization of classical rotation on two-dimensional plane is carried out in this paper. Let $\mathbb{R}^2$ be a plane with rectangular coordinates $x,\ y$ and $\mathbb{Z}^2 \subset \mathbb{R}^2$ be the two-dimensional lattice, such that any point $(n,\ m) \in \mathbb{Z}^2$ of the lattice has integer coordinates n and m. For any real number a we shall denote here by $[a]$ the integer part of a, i. e. the greatest integer number which does not exceed the number a. We introduce a map $A = A(\varphi) : (n,m) \to (n',m')$ of the lattice $\mathbb{Z}^2$ into itself depending on a real number φ, which sends a point $(n,\ m)$ to a point $(n',\ m')$ such that $n' = [x'],\ m' = [y']$ where $x' = n\cos\varphi - m\sin\varphi,\ y' = n\sin\varphi + m\cos\varphi$. Since a map $(n,m) \to (x',y')$ is a rotation of a vector $(n,\ m)$ by the angle φ, we call the map $A(\varphi)$

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a discrete rotation of the angle φ. Our task is to study the behaviour of trajectories $(n_k,\ m_k) = A^k(n,\ m)$ $(k = 1, 2, \dots)$ of a point $(n,\ m)$ and their dependence on the angle φ and the initial point $(n,\ m)$. In more general cases where the angle φ is not a constant but a function $\varphi = \varphi(r)$ of a distance $r = \sqrt{n^2 + m^2}$ between a point $(x,\ y)$ and the origin $(0,\ 0)$ the map $A = A_r = A_r(\varphi)$ is defined in the following way: $A_r = A_r(\varphi) : (n,\ m) \to (n^{(r)},\ m^{(r)})$ where $n^{(r)} = [x^{(r)}],\quad m^{(r)} = [y^{(r)}],\quad r = \sqrt{n^2 + m^2},\quad x^{(r)} = n \cos\varphi(r) - m \sin\varphi(r),\quad y^{(r)} = n \sin\varphi(r) + m \cos\varphi(r)$. This case with the function satisfying some general condition was qualitatively studied in [1] and [2] and it was shown that any trajectory $(n_k^{(r)},\ m_k^{(r)}) = A_r^k(n,\ m)$ $(k = 1, 2, \dots)$ becomes periodic, i. e. there exist natural numbers k_0 and t such that the equalities

$$n_k^{(r)} = n_{k+t}^{(r)},\ \ m_k^{(r)} = m_{k+t}^{(r)} \tag{1}$$

hold for all integer $k \geq k_0$. We call the first point $(n_{k_0}^{(r)},\ \ m_{k_0}^{(r)})$ the beginning of period and the smallest t for which the equalities (1) are valid is called a period. As a corollary of periodicity, we obtained that the trajectory can not go to infinity. The statement of periodicity was proved in [1] and [2] for any function $\varphi(r)$ being infinitely differentiable and such that the inequality

$$\left|\frac{d\varphi}{dr}(r)\right| > \frac{c^*}{r} \tag{2}$$

holds for some constant $c^* > 0$. In the case $\varphi = \varphi(r) = const$ the inequality (2) is clearly not valid. Nevertheless, qualitative but not rigorous arguments show that the periodicity property holds for a constant φ too. These arguments are presented in Section 1. However, in view of the fact that the periodicity occurs, the following problem can be set up: whether the trajectory goes very far from the initial point $(n,\ m)$ and what are the beginning of period and the period itself in such a case. It is not possible to solve these problems by using only rigorous mathematical methods. In view of that, an experimental work was carried out in which on the base of numerical analysis for different values of the angle φ and initial points $(n,\ m)$ the following parameters were found:

- maximum and minimum of coordinates of trajectory points;
- the beginning of period;
- period.

The main result following from the experiments is the discovery of such a value of the angle φ for which the trajectory of initial point $(n,\ m)$ with $10 \leq m, n \leq 50$ goes very far from the origin (the distance is greater than $2^{10} - 1$) even though the initial point is close enough to the origin. In this case the period is 2049, the beginning of the period is greater than 1500000 and in cases we have considered maximal and minimal values of coordinates and periods of the trajectory points do not depend on the initial point $(n,\ m)$. It is an unexpected result, because for any other values of the angle φ the maximal and minimal deviation of trajectory points is almost equal to the initial distance $r = \sqrt{n^2 + m^2}$ and this is natural because the discrete rotation is close to usual rotation by the angle φ, (see section 2) for which the distance between any trajectory point and the origin is a constant.

2 Qualitative Justification of Periodicity

We express the map $A(\varphi)$ in polar coordinates α, r, where $\alpha = \arctan \frac{m}{n} \mod 2\pi$ is the angle, $r = \sqrt{n^2 + m^2}$ is the radius. Then supposing $\alpha' = \arctan \frac{m'}{n'} \mod 2\pi,\ r' = \sqrt{(n')^2 + (m')^2}$ we obtain the map $B_\varphi : (\alpha, r) \to (\alpha', r')$, where

$$\alpha' = \alpha + \varphi + f(\alpha, r) \mod 2\pi,\ r' = r + g(\alpha, r). \tag{3}$$

From the definition of the map $A(\varphi)$ it follows that the functions $f(\alpha, r)$ and $g(\alpha, r)$ satisfy the inequalities

$$|f(\alpha, r)| < \frac{c}{r}\ ,\ \ |g(\alpha, r)| < c\ , \tag{4}$$

where c is a constant independent of α, r and φ. Now we introduce the new variable $\rho = \frac{1}{r}$. The map B_φ goes over as a result of this change of variable into the transformation $D_\varphi(\alpha, \rho) \to (\alpha', \rho')$, where $\rho' = \frac{1}{r'}$,

$$\alpha' = \alpha + \varphi + f_1(\alpha, \rho) \mod 2\pi,\ \rho' = \rho + g_1(\alpha, \rho)\ , \tag{5}$$

$$f_1(\alpha, \rho) = f(\alpha, \frac{1}{\rho})\ ,\ \ g_1(\alpha, \rho) = -\frac{\rho^2 g(\alpha, \frac{1}{\rho})}{1 + \rho g(\alpha, \frac{1}{\rho})}\ , \tag{6}$$

and on account of (3)-(6) the inequalities

$$|f_1(\alpha, \rho)| < c\rho\ ,\ \ |g_1(\alpha, \rho)| < 2c\rho^2\ , \tag{7}$$

hold in the region $|\rho| \leq \varepsilon$ for some $\varepsilon > 0$. Further we introduce a new variables $\Delta = \frac{\rho}{\varepsilon}\ ,\ \ \Delta' = \frac{\rho'}{\varepsilon}$ and express the transformation D_φ in terms of the variables Δ and Δ'. As a result we obtain the transformation $U_\varphi : (\alpha, \Delta) \to (\alpha', \Delta')$, where

$$\alpha' = \alpha + \varphi + f_2(\alpha, \Delta) \mod 2\pi \ , \quad \Delta' = \Delta + g_2(\alpha, \Delta) \tag{8}$$

and by virtue of (5)-(7) we have:

$$f_2(\alpha, \Delta) = f_1(\alpha, \Delta\varepsilon) = f(\alpha, \frac{1}{\varepsilon\Delta}) \ , \quad g_2(\alpha, \Delta) = \frac{1}{\varepsilon} g_1(\alpha, \varepsilon\Delta) \ ,$$

$$|f_2(\alpha, \Delta)| < c\varepsilon\Delta \ , \quad |g_2(\alpha, \Delta)| < 2c\varepsilon\Delta^2 \ . \tag{9}$$

It follows from (8) and (9) that if the positive numbers Δ_0 and ε are sufficiently small then in the region $|\Delta| \leq \Delta_0$ the transformation U_φ is close to the rotation

$$U^*_\varphi : (\alpha, \Delta) \to (\alpha^*, \Delta^*) = (\alpha + \varphi \mod 2\pi, \Delta).$$

Therefore it is natural to apply the Moser theorem [3] to the map U_φ, according to which for any sufficiently small $\varepsilon > 0$ the transformation (8) satisfying (9) has a simple closed curve in the region $|\Delta| \leq \Delta_0$ surrounding the point $\Delta = 0$ which is invariant under the transformation U_φ. The statement of periodicity follows from Moser's theorem, because the region inside this invariant curve is invariant under the transformation U_φ and passing back from the variable Δ to the variable $r = \frac{1}{\Delta\varepsilon}$ we obtain that any trajectory of the transformation $A(\varphi)$ never leaves a certain disk with center at the point $r = 0$. Since in any disk there are only a finite number of points of the lattice $\mathbb{Z}^2$, then any trajectory of the map $A(\varphi)$ becomes periodic. To apply Moser's theorem to the transformation U_φ one needs to justify the following conditions:

1. U_φ is a sufficiently smooth map (for example, infinite differentiable);
2. The following intersection property holds: every closed curve surrounding the point $\Delta = 0$ intersects its image under the transformation U_φ;
3. The angle of the rotation $\varphi = \varphi(r)$ is not a constant but a function satisfying the inequality (2) with some constant c^*.

In our case, by virtue of the definition of the map A_φ, the transformation U_φ is discontinuous. To reduce the problem to Moser's theorem in [1] and [2] the smooth approximations of the map $A(\varphi)$ were proposed. They approximate

A_φ from above and from below and the estimate (9) and the intersection property 2 are valid. But the condition (2) was assumed to be true. Here we assume that the angle is a constant and therefore the inequality (2) is not satisfied, and we fail to prove rigorously the periodicity of the trajectories. However many numerical experiments show that periodicity property takes place in the case of a constant φ and moreover the deviations of trajectories of the discrete rotation are small for almost all φ and all initial points $(n,\ m)$.

3 Method and Results of Numerical Experiments

Input data of the numerical experiments are a real number φ (radian measured in) and two integer numbers n, m, and the trajectory $(n_k,\ m_k) = A^k(n,\ m)$ $(k = 1, 2, \dots; A = A(\varphi))$ is found with the help of the algorithm described in section 1. In addition given some natural number R we define a collection Ω_R of integer vectors (n, m) in the following way: for any integer n in the interval $-R \le n \le R$ we found the unique number $m = [\sqrt{R^2 - n^2}]$. We apply the algorithm above to every such a vector$(n, m) \in \Omega_R$ and calculate the trajectory $(n_k,\ m_k)$. The sense of such approach is that for fixed R all integer vectors $(n, m) \in \Omega_R$ are lying near of the circle S_R of radius R centered at the point $(0, 0)$ and this circumstance allows us to study a statistics of parameters of a trajectory both for fixed and different values of R.

Output data representing results of experiments are following: the beginning of period, period, $\min x$ and $\max x$, which are respectively minimal and maximal values of the coordinate x of points of trajectory, $\min y$ and $\max y$, which are respectively minimal and maximal values of the coordinate y of trajectory points. We display here three examples of our experiments.

In the first one $R = 50$, $\varphi = 1,0471975521965979$ is very close to $\pi/3$,$-50 \le n \le 50, m = [\sqrt{2500 - n^2}]$. The analysis of the results shows that there are only three values of periods $6, 12$ and 18 in 101 variants and there are 19 different values, of the beginning of period. The minimal and maximal values for coordinates of trajectory points satisfy the following inequalities:

$$-51 \le \min x \le -43, \quad 43 \le \max x \le 52, \tag{10}$$

$$-54 \le \min y \le -44, \quad 43 \le \max y \le 53. \tag{11}$$

According to inequalities (10) and (11) minimal and maximal deviations of coordinates of all trajectories points of the transformation A differ from minimal values -50 and maximal values 50 very little.

In the second example $R = 20$, the angle $\varphi = 1,5707963267948966$ is close to $\pi/2$, $-50 \leq n \leq 50, m = [\sqrt{400 - n^2}]$. As the result of simulations we have got that there are two values of period 4 and 61 and 30 different values of the beginning of period. The minimal and maximal values of coordinates of trajectories points satisfy the inequalities

$$-20 \leq \min x \leq -14, \quad 14 \leq \max x \leq 20, \tag{12}$$

$$-20 \leq \min y \leq -14, \quad 14 \leq \max y \leq 20. \tag{13}$$

In the third example $R = 30$ and the value of φ is the same as in the second one. Here we obtain two values of period 4 and 246 and there are 40 different values of the beginning of period. The minimal and maximal values of coordinates x and y of trajectories points satisfy the inequalities:

$$-32 \leq \min x \leq -21, \quad 21 \leq \max x \leq 32, \tag{14}$$

$$-32 \leq \min y \leq -21, \quad 21 \leq \max y \leq 32. \tag{15}$$

According to inequalities (12)-(15) the minimal and maximal deviations of coordinates of all trajectories points are very close to values $-R$ and R respectively. From results of examples 2 and 3 we see that besides the main value 4 of period there are some other periods which depend on R for the same φ.

4 Large Deviation of Trajectories

Here we describe the main result of simulations in which one discovers a value of the angle $\pi/2$ such that trajectories of the map $A = A(\varphi)$ have large deviations from initial points. It is worth noting that in such situations the period of trajectories is always equal to 2049, minimal values of coordinates x and y are -1024, maximal value of x is equal to 1024, maximal value of y is equal to 1023, and the beginning of period is greater than 1500000. The angle for which these results are valid is close to $\pi/2$ and equals to $1,5707963267948$. These results are obtained for $R = 10, 20, 30, 40, 50$.

5 Conclusion

The value of the angle for which large deviations take place differs from the angle, for which there are no large deviations only in 14-th digit in computer representation of number $\pi/2$. This very small perturbation of rotation angle leads to enormous change of behavior of the trajectories with universal characteristics. In the case of large deviations the value of period and minimal and maximal values of coordinates of trajectory points are the same for different initial points.

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References

[1] M.L. Blank, T.Kruger, L.D.Pustyl'nikov, A KAM type theorem for systems with round-off errors//BiBoS- preprint,No.765/3/97, BiBoS, Universitat Bielefeld, 1997.

[2] M.L. Blank, T.Kruger, L.D. Pustyl'nikov, A KAM type theorem for systems with round-off errors, in "15-th IMACS world congress on scientific computation, modelling and applied mathematics" (Berlin, August 24-29, 1997), V.2, 6p.

[3] J.K.Moser, On invariant curves of area-preserving mappings of an annulus//Nachr.Akad.Wiss.Gottingen II,Math.-Phys.K1.,1(1962),1-20.