One-Point Gibbs States

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In the approach to equilibrium statistical mechanics initiated by Dobrushin [1], [2], [3], [4], and Lanford and Ruelle [8], [9] the equilibrium measures (Gibbs states) are defined to be those probability measures having certain conditional probabilities specified by a family of kernels. For lattice models this family of kernels is indexed by the non-empty finite subsets of the underlying set S of sites (which is usually $\mathbb{Z}^d$ for some d) and where the σ-algebra associated with such a finite set Λ consists of the events which are observable from outside of Λ.

Now there are situations in which it is known that a probability measure has the specified conditional probabilities, but *a priori* only for singleton sets, i.e., only for the sets $\{t\}$, $t \in S$. Such a measure could perhaps be called a *one-point Gibbs state*. For example, there is a technique of Dobrushin [5] for establishing the existence of Gibbs states which essentially constructs one-point Gibbs states. The question thus arises: Is a one-point Gibbs state actually a Gibbs state? If all the conditional probabilities are strictly positive then the answer is in the affirmative. A precise formulation of this statement is given in Theorem 1.33 of Georgii [6], which is a direct extension of a remark of Dobrushin [5].

Without the assumption of strict positivity things are more complicated. The purpose of this note is to rework Georgii's proof to obtain Theorem 1 below, in which the assumption of strict positivity can be relaxed somewhat.

The assumptions in Theorem 1 are satisfied in the set-up considered by Kutoviy [7] in his thesis. Kutoviy employs Dobrushin's technique to show the existence of Gibbs states for continuous models (point processes). This generalises results of Ruelle [9] as well as providing alternative proofs of these results.

The following framework will be employed: Given is a countably infinite set S and a set X. Moreover, for each $A \subset S$ there is a σ-algebra $\mathcal{F}^A$ of subsets of X such that

(A1) $\mathcal{F}^{A\cup B} = \mathcal{F}^A \vee \mathcal{F}^B$ for all $A, B \in \mathcal{S}$,

(A2) if $A, B \subset S$ with $A \cap B = \varnothing$ then $E \cap F \neq \varnothing$ whenever $E \in \mathcal{F}^A$, $F \in \mathcal{F}^B$ with $E \neq \varnothing$ and $F \neq \varnothing$, where in (A1) $\mathcal{F}^A \vee \mathcal{F}^B$ denotes the smallest σ-algebra containing both $\mathcal{F}^A$ and $\mathcal{F}^B$. Of course, (A1) implies that $\mathcal{F}^A \subset \mathcal{F}^B$ whenever $A, B \in \mathcal{S}$ with $A \subset B$. Condition (A2) (which is only needed in Proposition 1) will always be satisfied when there some kind of underlying product structure.

The set S should be thought of as a set of *sites*, X as a basic set of *configurations*, and for each $A \subset S$ the σ-algebra $\mathcal{F}^A$ should be thought of as consisting of those subsets of configurations which can be determined by conditions only involving subsets of the set A.

Let L denote the set of non-empty finite subsets of S and for each $\Lambda \in L$ denote the σ-algebra $\mathcal{F}^{S\setminus\Lambda}$ by $\mathcal{F}_\Lambda$; this should thus be thought of as consisting of those subsets of configurations which can be determined by conditions only involving subsets of the complement (i.e. the outside) of the set Λ.

Put $\mathcal{F} = \mathcal{F}^S$; this gives a measurable space $(X, \mathcal{F})$, a non-empty index set L for which the inclusion order $\subset$ is directed and countably generated, and a decreasing family $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in L}$ of sub-σ-algebras of $\mathcal{F}$.

Now for each $\Lambda \in L$ let $\pi_\Lambda : X \times \mathcal{F} \to \mathbb{R}^+$ be a quasi probability kernel (i.e., π_Λ is a kernel such that $\pi_\Lambda(x, X)$ is either 0 or 1 for each $x \in X$). The family $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in L}$ is called an $\mathbb{F}$-*specification* if for every $\Lambda \in L$

(S1) $\pi_\Lambda(\cdot, F)$ is $\mathcal{F}_\Lambda$-measurable for each $F \in \mathcal{F}$,

(S2) $\pi_\Lambda(\cdot, F' \cap F) = I_{F'}\pi_\Lambda(\cdot, F)$ for each $F \in \mathcal{F}$, $F' \in \mathcal{F}_\Lambda$,

and if

(S3) $\pi_{\Lambda'} = \pi_{\Lambda'}\pi_\Lambda$ whenever $\Lambda, \Lambda' \in L$ with $\Lambda \subset \Lambda'$,

where the kernel $\pi_{\Lambda'}\pi_\Lambda$ is given by

$$(\pi_{\Lambda'}\pi_\Lambda)(x, F) = \int \pi_\Lambda(y, F)\pi_{\Lambda'}(x, dy)$$

for each $x \in X$, $F \in \mathcal{F}$. (See, for example Preston [10], [11].)

Let $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in L}$ be an $\mathbb{F}$-specification; then a probability measure $\mu \in \mathrm{P}(X, \mathcal{F})$ is called a *Gibbs state with specification* $\mathcal{V}$ if μ is compatible with the kernels $\{\pi_\Lambda\}_{\Lambda\in L}$ in the sense that

$$\mathrm{E}_\mu(I_F|\mathcal{F}_\Lambda) = \pi_\Lambda(\cdot, F) \ \mu\text{-a.e. for each } F \in \mathcal{F},\ \Lambda \in L.$$

Writing this out, it follows that $\mu \in \mathrm{P}(X, \mathcal{F})$ is a Gibbs state if and only if

$$\mu(F' \cap F) = \int_{F'} \pi_\Lambda(x, F)\, d\mu(x)$$

for all $F \in \mathcal{F}$, $F' \in \mathcal{F}_\Lambda$, $\Lambda \in L$, and thus by (S2) μ is a Gibbs state if and only if $\mu = \mu\pi_\Lambda$ for each $\Lambda \in L$, where $\mu\pi_\Lambda$ is the measure defined by

$$(\mu\pi_\Lambda)(F) = \int \pi_\Lambda(x, F)\, d\mu(x) .$$

The set of Gibbs states with specification $\mathcal{V}$ will be denoted by $\mathcal{G}(\mathcal{V})$, hence

$$\mathcal{G}(\mathcal{V}) = \{\mu \in \mathrm{P}(X, \mathcal{F}) : \mu = \mu\pi_\Lambda \text{ for all } \Lambda \in L\} .$$

As already mentioned, this definition of Gibbs states originates from Dobrushin, Lanford and Ruelle. The equations defining the elements of $\mathcal{G}(\mathcal{V})$ (i.e., $\mu = \mu\pi_\Lambda$ for all $\Lambda \in L$, or these equations written in some other form) are thus often referred to as the *DLR-equations*.

Let $\mathcal{V} = \{\pi_\Lambda\}_{\Lambda\in L}$ be an $\mathbb{F}$-specification, which is considered to be fixed in what follows; for $t \in S$ it is convenient to just write π_t instead of $\pi_{\{t\}}$. The question to be discussed here is: If $\mu \in \mathrm{P}(X, \mathcal{F})$ with $\mu\pi_t = \mu$ for each $t \in S$, when can it be concluded that $\mu \in \mathcal{G}(\mathcal{V})$? A sufficient condition for this to hold is given in Theorem 1.

Denote by $\mathrm{M}^+(X, \mathcal{F})$ the set of all $\mathcal{F}$-measurable mappings from X to $\mathbb{R}^+$. If $\pi : X \times \mathcal{F} \to \mathbb{R}^+$ is a kernel and $f \in \mathrm{M}^+(X, \mathcal{F})$ then $\pi f : X \to \mathbb{R}^+_\infty$ is given by

$$(\pi f)(x) = \int f(y)\pi(x, dy)$$

for each $x \in X$; this mapping is $\mathcal{F}$-measurable and

$$\int (\pi f)\, d\mu = \int f\, d(\mu\pi)$$

for all $\mu \in \mathrm{P}(X, \mathcal{F})$. If $\{\tau_\Lambda\}_{\Lambda\in L}$ is an $\mathbb{F}$-specification then by (S1) it follows that $\tau_\Lambda f$ is $\mathcal{F}_\Lambda$-measurable for all $f \in \mathrm{M}^+(X, \mathcal{F})$ and by (S2) $\tau_\Lambda(gf) = g\tau_\Lambda f$ whenever g is $\mathcal{F}_\Lambda$-measurable.

It will be assumed that the kernels π_Λ, $\Lambda \in L$, are locally absolutely continuous with respect to something which can be considered as a product measure or a free field. In the present set-up this involves the following concept:

A probability measure $\lambda \in \mathrm{P}(X, \mathcal{F})$ is said to be *independent* if

$$\lambda(F_1 \cap F_2) = \lambda(F_1)\lambda(F_2) \quad \text{for all } F_1 \in \mathcal{F}^{\Lambda_1},\ F_2 \in \mathcal{F}^{\Lambda_2}$$

for any $\Lambda_1, \Lambda_2 \in L$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$.

Proposition 1 *Let $\lambda \in \mathrm{P}(X, \mathcal{F})$ be an independent measure; then there exists an $\mathbb{F}$-specification $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda\in L}$ such that for each $\Lambda \in L$*

$$\lambda_\Lambda(x, F' \cap F) = I_{F'}(x)\lambda(F)$$

for all $x \in X$, $F' \in \mathcal{F}_\Lambda$ and $F \in \mathcal{F}^\Lambda$. Moreover, $\mathcal{G}(\mathcal{U}) = \{\lambda\}$.

Proof A proof can be found in Chapter 3 of Preston [11]. (It is for the existence of the kernel λ_Λ that condition (A2) is needed.) $\square$

In what follows fix an independent measure $\lambda \in \mathrm{P}(X,\mathcal{F})$ and let $\mathcal{U} = \{\lambda_\Lambda\}_{\Lambda\in L}$ be the $\mathbb{F}$-specification given in Proposition 1. It will be assumed that there is given a family $\{w_\Lambda\}_{\Lambda\in L}$ of elements from $\mathrm{M}^+(X,\mathcal{F})$ such that for each $\Lambda \in L$

$$\pi_\Lambda(x,F) = \left(\int w_\Lambda(y)\lambda_\Lambda(x,dy)\right)^\diamond \int_F w_\Lambda(y)\lambda_\Lambda(x,dy)$$

for all $x \in X$, $F \in \mathcal{F}$, where $x^\diamond = x^{-1}$ if $x \in (0,\infty)$ and $0^\diamond = \infty^\diamond = 0$. Note that for any $w_\Lambda \in \mathrm{M}^+(X,\mathcal{F})$ this always defines a quasi probability kernel π_Λ satisfying (S1) and (S2). Of course, the family $\{w_\Lambda\}_{\Lambda\in L}$ has to be chosen appropriately for (S3) to hold, but this will always be the case for the families arising in models from statistical mechanics. Let $\Lambda \in L$; then for all $f \in \mathrm{M}^+(X,\mathcal{F})$

$$\pi_\Lambda f = (\lambda_\Lambda w_\Lambda)^\diamond \lambda_\Lambda(f w_\Lambda) = \lambda_\Lambda(f(\lambda_\Lambda w_\Lambda)^\diamond w_\Lambda)\ ,$$

since $(\lambda_\Lambda w_\Lambda)^\diamond$ is $\mathcal{F}_\Lambda$-measurable and the kernel λ_Λ satisfies (S2). It is therefore convenient to work with the normalised family $\{v_\Lambda\}_{\Lambda\in L}$, where $v_\Lambda = (\lambda_\Lambda w_\Lambda)^\diamond w_\Lambda$ for each $\Lambda \in L$. Thus $v_\Lambda \in \mathrm{M}^+(X,\mathcal{F})$ and

$$\pi_\Lambda f = \lambda_\Lambda(f v_\Lambda)$$

holds for all $f \in \mathrm{M}^+(X,\mathcal{F})$.

Let $\Lambda \in L$ and $\mu \in \mathrm{P}(X,\mathcal{F})$; then for each $f \in \mathrm{M}^+(X,\mathcal{F})$

$$\int f\, d(\mu\pi_\Lambda) = \int \pi_\Lambda f\, d\mu = \int \lambda_\Lambda(f v_\Lambda)\, d\mu = \int f v_\Lambda\, d(\mu\lambda_\Lambda)$$

and thus $\mu = \mu\pi_\Lambda$ holds if and only if μ is absolutely continuous with respect to $\mu\lambda_\Lambda$ with density v_Λ. Equivalently, $\mu = \mu\pi_\Lambda$ holds if and only if

$$\int f\, d\mu = \int \lambda_\Lambda(f v_\Lambda)\, d\mu$$

for all $f \in \mathrm{M}^+(X,\mathcal{F})$.

For each $\Lambda \in L$ put $N_\Lambda = \{x \in X : v_\Lambda(x) = 0\}$; thus

$$N_\Lambda = \{x \in X : w_\Lambda(x) = 0 \text{ or } \lambda_\Lambda w_\Lambda\,(x) = 0 \text{ or } \lambda_\Lambda w_\Lambda\,(x) = \infty\}\ .$$

Note that for all $\mu \in \mathrm{P}(X,\mathcal{F})$

$$(\mu\pi_\Lambda)(N_\Lambda) = \int I_{N_\Lambda} v_\Lambda\, d(\lambda_\Lambda\mu) = 0$$

and so in particular $\mu(N_\Lambda) = 0$ whenever $\mu = \mu\pi_\Lambda$. Let

$$\mathrm{P}_\mathcal{V}(X,\mathcal{F}) = \{\mu \in \mathrm{P}(X,\mathcal{F}) : (\mu\lambda_\Lambda)(N_{\Lambda'}) = 0 \text{ for all } \Lambda,\, \Lambda' \in L \text{ with } \Lambda' \subset \Lambda\}\ .$$

Theorem 1 $\{\mu \in \mathrm{P}_{\mathcal{V}}(X,\mathcal{F}) : \mu\pi_t = \mu$ *for all* $t \in S\} \subset \mathcal{G}(\mathcal{V})$.

Proof This proceeds via a number of lemmas.

Lemma 1 *Let* $\{\tau_\Lambda\}_{\Lambda \in L}$ *be any* $\mathbb{F}$*-specification; then*

$$(\tau_\Lambda\lambda_\Lambda)(x,F) = \tau_\Lambda(x,X)\lambda_\Lambda(x,F) \ .$$

for all $\Lambda \in L$, $x \in X$, *and* $F \in \mathcal{F}$. *(Thus* $\tau_\Lambda\lambda_\Lambda = \lambda_\Lambda$ *almost holds, except for the factor* $\tau_\Lambda(\cdot,X)$, *and note that this factor can only take on the values* 1 *and* 0.*) Equivalently, this says that*

$$(\tau_\Lambda\lambda_\Lambda)f = I_{D_\Lambda}\lambda_\Lambda f$$

for all $\Lambda \in L$, $f \in \mathrm{M}^+(X,\mathcal{F})$, *where* $D_\Lambda = \{x \in X : \tau_\Lambda(x,X) = 1\}$.

Proof Let $x \in X$; then for all $F \in \mathcal{F}^\Lambda$, $F' \in \mathcal{F}_\Lambda$

$$\begin{aligned}(\tau_\Lambda\lambda_\Lambda)(x,F'\cap F) &= \int \lambda_\Lambda(y,F'\cap F)\tau_\Lambda(x,dy) = \int I_{F'}(y)\lambda(F)\tau_\Lambda(x,dy)\\ &= \lambda(F)\int I_{F'}(y)\tau_\Lambda(x,dy) = \lambda(F)\tau_\Lambda(x,F')\\ &= \lambda(F)I_{F'}(x)\tau_\Lambda(x,X) = \tau_\Lambda(x,X)\lambda_\Lambda(x,F'\cap F)\end{aligned}$$

and thus $(\pi_\Lambda\lambda_\Lambda)(x,F) = \tau_\Lambda(x,X)\lambda_\Lambda(x,F)$ for all $F \in \mathcal{F}$, since $\mathcal{F} = \mathcal{F}_\Lambda \vee \mathcal{F}^\Lambda$ and $\{F'\cap F : F' \in \mathcal{F}_\Lambda,\ F \in \mathcal{F}^\Lambda\}$ is closed under taking finite intersections. □

For each $\Lambda \in L$ put

$$\mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F}) = \{\mu \in \mathrm{P}(X,\mathcal{F}) : (\mu\lambda_\Lambda)(N_{\Lambda'}) = 0 \text{ for all } \Lambda' \in L \text{ with } \Lambda' \subset \Lambda\} \ .$$

Lemma 2 *For each* $\mu \in \mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F})$

$$\mu(\{x \in X : \pi_\Lambda(x,\cdot) \in \mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F})\}) = 1 \ .$$

Proof For $\Lambda' \subset L$ put $A_{\Lambda'} = \{x \in X : (\pi_\Lambda(x,\cdot)\lambda_{\Lambda'})(N_{\Lambda'}) = 0\}$. Then

$$\{x \in X : \pi_\Lambda(x,\cdot) \in \mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F})\} = \bigcap_{\Lambda'\subset\Lambda} A_{\Lambda'}$$

and so it is enough to show that $\mu(A_{\Lambda'}) = 1$ for all $\Lambda' \subset \Lambda$, $\mu \in \mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F})$.
Let $\Lambda' \in L$ with $\Lambda' \subset \Lambda$ and $\mu \in \mathrm{P}^\Lambda_{\mathcal{V}}(X,\mathcal{F})$; then

$$\begin{aligned}\int \lambda_{\Lambda'}(\cdot,N_{\Lambda'})\,d(\mu\lambda_\Lambda) = \int \lambda_\Lambda(\lambda_{\Lambda'}(\cdot,N_{\Lambda'}))\,d\mu &= \int \lambda_\Lambda(\cdot,N_{\Lambda'})\,d\mu\\ &= (\mu\lambda_\Lambda)(N_{\Lambda'}) = 0 \ ,\end{aligned}$$

i.e., $\lambda_{\Lambda'}(\cdot, N_{\Lambda'}) = 0$ $\mu\lambda_\Lambda$-a.e.. Thus also $\lambda_{\Lambda'}(\cdot, N_{\Lambda'})v_\Lambda = 0$ $\mu\lambda_\Lambda$-a.e. and so

$$0 = \int \lambda_{\Lambda'}(\cdot, N_{\Lambda'})v_\Lambda \, d(\mu\lambda_\Lambda) = \int \lambda_\Lambda(\lambda_{\Lambda'}(\cdot, N_{\Lambda'})v_\Lambda) \, d\mu \, .$$

This implies that $\mu(A_{\Lambda'}) = 1$, since

$$(\pi_\Lambda(x,\cdot)\lambda_{\Lambda'})(N_{\Lambda'}) = \pi_\Lambda(\lambda_{\Lambda'}(\cdot, N_{\Lambda'}))(x) = \lambda_\Lambda(\lambda_{\Lambda'}(\cdot, N_{\Lambda'})v_\Lambda)(x)$$

and hence $\mu(A_{\Lambda'}) = 1$ if and only if $\int \lambda_\Lambda(\lambda_{\Lambda'}(\cdot, N_{\Lambda'})v_\Lambda) \, d\mu = 0$. □

Now for each $\Lambda \in L$ let

$$\mathcal{O}_\Lambda = \{\mu \in \mathrm{P}^\Lambda_\mathcal{V}(X, \mathcal{F}) : \mu\pi_t = \mu \text{ for all } t \in \Lambda\} \, .$$

Lemma 3 *Let $\Lambda \in L$ and suppose there exists $u_\Lambda \in \mathrm{M}^+(X, \mathcal{F})$ such that*

$$\int f \, d\mu = \int \lambda_\Lambda(fu_\Lambda) \, d\mu$$

for all $f \in \mathrm{M}^+(X, \mathcal{F})$, $\mu \in \mathcal{O}_\Lambda$. Then $\mu\pi_\Lambda = \mu$ for all $\mu \in \mathcal{O}_\Lambda$.

Remark: As with v_Λ, the condition involving u_Λ is equivalent to having

$$\mu(F) = \int_F u_\Lambda \, d(\mu\lambda_\Lambda)$$

for each $F \in \mathcal{F}$ for all $\mu \in \mathcal{O}_\Lambda$, meaning that μ is absolutely continuous with respect to $\mu\lambda_\Lambda$ with density u_Λ for each $\mu \in \mathcal{O}_\Lambda$.

Proof Since $\mathcal{V}$ is an $\mathbb{F}$-specification $\pi_\Lambda\pi_t = \pi_\Lambda$ for each $t \in \Lambda$, which implies that $\pi_\Lambda(x,\cdot)\pi_t = \pi_\Lambda(x,\cdot)$ for each $x \in X$. Let $B_\Lambda = \{x \in X : \pi_\Lambda(x,\cdot) \in \mathrm{P}^\Lambda_\mathcal{V}(X, \mathcal{F})\}$; thus $\pi_\Lambda(x,\cdot) \in \mathcal{O}_\Lambda$ for each $x \in B_\Lambda$ and hence by Lemma 1

$$\begin{aligned}(\pi_\Lambda f)(x) = \int f \, d\pi_\Lambda(x,\cdot) &= \int \lambda_\Lambda(fu_\Lambda) \, d\pi_\Lambda(x,\cdot) \\ &= \int fu_\Lambda \, d(\pi_\Lambda(x,\cdot)\lambda_\Lambda) = \int f(y)u_\Lambda(y)(\pi_\Lambda\lambda_\Lambda)(x, dy) \\ &= \int f(y)u_\Lambda(y)\lambda_\Lambda(x, dy) = \lambda_\Lambda(fu_\Lambda)\,(x)\end{aligned}$$

for all $x \in B_\Lambda$, $f \in \mathrm{M}^+(X, \mathcal{F})$. Let $\mu \in \mathcal{O}_\Lambda$, thus by Lemma 2 $\mu(B_\Lambda) = 1$ and therefore

$$\begin{aligned}\int f \, d(\mu\pi_\Lambda) = \int \pi_\Lambda f \, d\mu &= \int_{B_\Lambda} \pi_\Lambda f \, d\mu \\ &= \int_{B_\Lambda} \lambda_\Lambda(fu_\Lambda) \, d\mu = \int \lambda_\Lambda(fu_\Lambda) \, d\mu = \int f \, d\mu\end{aligned}$$

for all $f \in \mathrm{M}^+(X, \mathcal{F})$, which implies that $\mu\pi_\Lambda = \mu$. □

The following two simple facts about the kernels $\{\lambda_\Lambda\}_{\Lambda \in L}$ will be needed:

Lemma 4 *(1) Let $\Lambda \in L$; then for all $f,\ g \in \mathrm{M}^+(X,\mathcal{F})$*

$$\lambda_\Lambda(f\lambda_\Lambda g) = \lambda_\Lambda(g\lambda_\Lambda f)\ .$$

(2) Let $\Lambda_1,\ \Lambda_2 \in L$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$; then for all $f \in \mathrm{M}^+(X,\mathcal{F})$

$$\lambda_{\Lambda_1\cup\Lambda_2} f = \lambda_{\Lambda_2}(\lambda_{\Lambda_1} f)\ .$$

Proof (1) If $F,\ G \in \mathcal{F}^\Lambda$ and $F',\ G' \in \mathcal{F}_\Lambda$ then

$$\begin{aligned}\lambda_\Lambda(I_{F'\cap F}\lambda_\Lambda I_{G'\cap G}) = \lambda_\Lambda(I_{F'\cap F} I_{G'}\lambda(G)) &= I_{F'\cap G'}\lambda(F)\lambda(G)\\ &= I_{G'\cap F'}\lambda(G)\lambda(F) = \lambda_\Lambda(I_{G'\cap G}\lambda_\Lambda I_{F\cap F'})\end{aligned}$$

and it thus follows that $\lambda_\Lambda(I_F\lambda_\Lambda I_G) = \lambda_\Lambda(I_G\lambda_\Lambda I_F)$ for all $F,\ G \in \mathcal{F}$. This in turn implies that $\lambda_\Lambda(f\lambda_\Lambda g) = \lambda_\Lambda(g\lambda_\Lambda f)$ for all $f,\ g \in \mathrm{M}^+(X,\mathcal{F})$.

(2) Let $F_1 \in \mathcal{F}^{\Lambda_1}$, $F_2 \in \mathcal{F}^{\Lambda_2}$ and $F \in \mathcal{F}_{\Lambda_1\cap\Lambda_2}$. Then

$$\begin{aligned}\lambda_{\Lambda_2}(\lambda_{\Lambda_1} I_{F\cap F_1\cap F_2}) = \lambda_{\Lambda_2}(I_{F\cap F_2}\lambda(F_1) &= I_F\lambda(F_2)\lambda(F_1)\\ &= I_F\lambda(F_1\cap F_2) = \lambda_{\Lambda_1\cup\Lambda_2} I_{F\cap F_1\cap F_2}\end{aligned}$$

which implies as usual that $\lambda_{\Lambda_1\cup\Lambda_2} I_F = \lambda_{\Lambda_2}(\lambda_{\Lambda_1} I_F)$ for all $F \in \mathcal{F}$ and hence that $\lambda_{\Lambda_1\cup\Lambda_2} f = \lambda_{\Lambda_2}(\lambda_{\Lambda_1} f)$ for all $f \in \mathrm{M}^+(X,\mathcal{F})$. □

Lemma 5 *Let $\Lambda_1,\ \Lambda_2 \in L$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$ and put $\Lambda = \Lambda_1 \cup \Lambda_2$. Suppose $\mu\pi_{\Lambda_1} = \mu\pi_{\Lambda_2} = \mu$ for all $\mu \in \mathcal{O}_\Lambda$. Then $\mu\pi_\Lambda = \mu$ for all $\mu \in \mathcal{O}_\Lambda$.*

Proof Note that $v^\diamond_{\Lambda_1} v_{\Lambda_1} = I_{X\setminus N_{\Lambda_1}}$ and $v^\diamond_{\Lambda_2} v_{\Lambda_2} = I_{X\setminus N_{\Lambda_2}}$. Let $\mu \in \mathcal{O}_\Lambda$; then

$$\int \lambda_{\Lambda_1}(f v_{\Lambda_1})\, d\mu = \int f\, d\mu = \int \lambda_{\Lambda_2}(f v_{\Lambda_2})\, d\mu$$

for all $f \in \mathrm{M}^+(X,\mathcal{F})$, since $\mu\pi_{\Lambda_1} = \mu = \mu\pi_{\Lambda_2}$. Now making use of Lemma 4 and the fact that $(\mu\lambda_{\Lambda_1})(N_{\Lambda_1}) = (\mu\lambda_{\Lambda_2})(N_{\Lambda_2}) = 0$, it follows that

$$\begin{aligned}\int \lambda_\Lambda f\, d\mu &= \int \lambda_{\Lambda_2}(\lambda_{\Lambda_1} f)\, d\mu\\ &= \int \lambda_{\Lambda_2}(v^\diamond_{\Lambda_2}(\lambda_{\Lambda_1} f) v_{\Lambda_2})\, d\mu = \int v^\diamond_{\Lambda_2}\lambda_{\Lambda_1} f\, d\mu\\ &= \int \lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1}\lambda_{\Lambda_1} f)\, d\mu = \int \lambda_{\Lambda_1}(f\lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1}))\, d\mu\\ &= \int \lambda_{\Lambda_1}(f v^\diamond_{\Lambda_1}\lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1}) v_{\Lambda_1})\, d\mu = \int f v^\diamond_{\Lambda_1}\lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1})\, d\mu\\ &= \int f h\, d\mu\end{aligned}$$

for all $f \in \mathrm{M}^+(X, \mathcal{F})$, where $h = v^\diamond_{\Lambda_1} \lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1})$. In particular, putting $f = 1$ shows that $\mu(\{x \in X : h(x) = \infty\}) = 0$. Let $B = \{x \in X : \lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1})\,(x) = 0\}$; then $B \in \mathcal{F}_{\Lambda_1}$ and (since $\mu\pi_{\Lambda_1} = \mu$)

$$0 = \int I_B \lambda_{\Lambda_1}(v^\diamond_{\Lambda_2} v_{\Lambda_1})\, d\mu = \int \lambda_{\Lambda_1}(I_B v^\diamond_{\Lambda_2} v_{\Lambda_1})\, d\mu = \int I_B v^\diamond_{\Lambda_2}\, d\mu \; .$$

It follows that $\mu(B) = 0$, since $\{x \in X : v^\diamond_{\Lambda_2}(x) = 0\} = \{x \in X : v_{\Lambda_2}(x) = 0\}$ and $\mu(\{x \in X : v_{\Lambda_2}(x) = 0\}) = 0$. Moreover,

$$\mu(\{x \in X : v^\diamond_{\Lambda_1}(x) = 0\} = \mu(\{x \in X : v_{\Lambda_1}(x) = 0\}) = 0$$

and hence $\mu(\{x \in X : h(x) = 0\} = 0$, i.e., $\mu(\{x \in X : 0 < h(x) < \infty\}) = 1$.

Put $g = (hI_B)^\diamond$, where $B = \{x \in X : 0 < h(x) < \infty\}$; thus $g \in \mathrm{M}^+(X, \mathcal{F})$ and the above implies that $\int \lambda_\Lambda(fg)\, d\mu = \int f\, d\mu$ for all $f \in \mathrm{M}^+(X, \mathcal{F})$ and all $\mu \in \mathcal{O}_\Lambda$. Therefore by Lemma 3 $\mu\pi_\Lambda = \mu$ for all $\mu \in \mathcal{O}_\Lambda$. □

Theorem 1 now follows by induction on $|\Lambda|$ from Lemma 5. □

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