The Law of Entropy Increase and Generalized Relativistic Billiards

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Abstract

A mechanism for the occurrence of irreversibility is proposed. It is based on the allowance for the relativistic factor in a mechanical system consisting of a finite number of particles interacting with one other particle in accordance with the law of elastic collision and moving in a vessel with a natural law of reflection from the vessel walls. The investigated model, a modification of Poincaré model, is the generalized billiard and describes the motion of three-dimensional gas particles in a parallelepiped. It is proved that for general conditions the Gibbs entropy and thermodynamic entropy for this system of particles increase with time when the relativistic factor is taken into account, whereas in Newtonian case the Gibbs entropy is a constant.

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1 Introduction

In this paper we give a proof of the law of entropy increase for a three-dimensional gas consisting of finitely many identical mass point particles $P_1, \ldots, P_N$ moving in a vessel having the form of a parallelepiped, and the influence of the boundary of the vessel on the gas is periodic. Proofs were given in [6] in the case of one-dimensional gas where the vessel is a segment, but the result for three-dimensional was announced there without detailed proof.

General setting is due to Poincaré who introduced a similar model in paper [5] and considered the case when the force caused by an external body (a hot body) acts on the particles. The feature peculiar to our models considered here is that the influence of the external body is modeled by periodic actions of the walls of the volume. We assume that the volume has the form of a parallelepiped defined by the inequalities

$a_1 \leq q_1 \leq b_1$, $a_2 \leq q_2 \leq b_2$, $a \leq q \leq b$, where q_1, q_2, q is the set of orthogonal coordinates, particles $P_1, \ldots, P_N$ move in the vessel, and, on colliding with each other, interact by the law of elastic collision, and we assume that the following boundary conditions are imposed. After reflection from the parallelepiped boundaries given by the equations $q = a$ ($q = b$) at the time t the particle moves in the same way as if it had reflected at this time from a wall moving in the direction q by a law $q = f_1(t)$ (by law $q = f_2(t)$); after reflection from boundaries given by the equations $q = a_1$ ($q = b_1$) at the time t the particle moves in the same way as if it had reflected at this time from a wall moving in the direction q_1 by law $q_1 = g_1(t)$ (by law $q_1 = g_2(t)$); after reflection from boundaries given by the equation $q = a_2$ ($q = b_2$) at the time t the particle moves in the same way as if it had reflected at this time from a wall moving in the direction q_2 by law $q_2 = h_1(t)$ (by law $q_2 = h_2(t)$). We suppose that the functions $f_1(t)$ and $f_2(t)$ are smooth and of period 1 with respect t, the functions $g_1(t)$, $g_2(t)$, $h_1(t)$, $h_2(t)$ are constants and motion of particles is specified by equations of special theory of relativity.

From physical point of view functions $f_i(t)$, $g_i(t)$, $h_i(t)$ ($i = 1, 2$) simulate fields of small amplitudes of oscillations, caused by atoms of the vessel walls.

The system under consideration is a generalized billiards in a parallelepiped which introduced for general closed region in [7] and was studied in [1], [2] and [6] in relativistic case. The main results are the following: the Gibbs , entropy (constructed with respect to the Newtonian invariant measure) and the thermodynamic entropy (constructed with respect to the phase volume) increases as the time t increases if we take the relativistic effect into account. If one considers this system in the framework of Newtonian mechanics then the Gibbs entropy will be a constant [6]. The growth of entropy is not monotonic (this corresponds to physical presentation), but completely irreversible. These results resolve the well-known reversibility paradox, according to which, if the entropy increases then, changing signs of velocities, one obtains that the entropy should decrease, because the trajectories are preserved and the motion goes in the opposite direction. In our case such a phenomena is not possible because the general conditions for the growth of the entropy depends only on the absolute values of the velocity, and if it is bigger then some constant the entropy will increase. This is the reason why in the relativistic case the generalized billiard is irreversible and dissipative. Another proof of dissipativity of generalized relativistic billiards is given in [1] and [2]. I would like to note that as stressed in [4], to resolve reversibility paradox and to justify the second law of thermodynamics in the framework of Newtonian mechanics is not possible.

2 Introducing a measure on the phase space for the distribution function and entropy

Denoting the space coordinates q_1, q_2, q of the particle P_s by $q_1^{(s)}$, $q_2^{(s)}$, $q^{(s)}$, and the components of the momentum vector of the particle P_s along the directions q_1, q_2, q by

$p_1^{(s)}, p_2^{(s)}, p^{(s)}$, we introduce a statistical distribution function

$$\rho(t) = \rho(q_1^{(1)}, q_2^{(1)}, q^{(1)}, p_1^{(1)}, p_2^{(1)}, p^{(1)}, \ldots, q_1^{(N)}, q_2^{(N)}, q^{(N)}, p_1^{(N)}, p_2^{(N)}, p^{(N)}, t) \geq 0$$

for particles $P_1, \ldots, P_N$ at time t. Since the measure

$$d\Gamma = \frac{d\vec{q}d\vec{p}}{|v_1^{(1)}v_2^{(1)}v^{(1)} \ldots v_1^{(N)}v_2^{(N)}v^{(N)}|} \tag{1}$$

is invariant with respect to classical dynamics for the gas model under consideration ([5]), where

$$d\vec{q} = dq_1^{(1)}dq_2^{(1)}dq^{(1)} \ldots dq_1^{(N)}dq_2^{(N)}dq^{(N)},$$
$$d\vec{p} = dp_1^{(1)}dp_2^{(1)}dp^{(1)} \ldots dp_1^{(N)}dp_2^{(N)}dp^{(N)},$$
$$v_1^{(s)} = v_1^{(s)}(t), \quad v_2^{(s)} = v_2^{(s)}(t), \quad v^{(s)} = v^{(s)}(t)$$

are components of the velocity vector of the particle P_2 at time t along the directions q_1, q_2, q, we see that the entropy

$$H(t) = -\int_K \rho(t) \ \ln \ \rho(t) \ d\Gamma \tag{2}$$

plays the role of the Gibbs entropy for the model of the three-dimensional gas. Here the integration is taken over the phase space K given by the direct product

$$K = \underbrace{\Pi \times \ldots \times \Pi}_{N} \times \underbrace{\mathbf{R}_3 \times \ldots \times \mathbf{R}_3}_{N} \ ,$$

where

$$\Pi = \{q_1, q_2, q : a_1 \leq q_1 \leq b_1, a_2 \leq q_2 \leq b_2, a \leq q \leq b\} \ ,$$

and $\mathbf{R}_3$ is the three-dimensional space. In addition, we consider the thermodynamic entropy

$$\tilde{H}(t) = -\int_K \tilde{\rho}(t) \ \ln \ \tilde{\rho}(t) \ d\vec{q} \ d\vec{p}$$

for the distribution function

$$\tilde{\rho}(t) = \frac{\rho(t)}{|v_1^{(1)}v_2^{(1)}v^{(1)} \ldots v_1^{(N)}v_2^{(N)}v^{(N)}|}$$

with respect to the phase volume. The normalization condition and the definitions of $d\Gamma$ and $\tilde{\rho}(t)$ yield the relations

$$\int_K \rho(t) \ d\Gamma = 1, \quad \int_K \tilde{\rho}(t) \ d\vec{q} \ d\vec{p} = 1, \quad \int_K \rho(t) \ d\vec{q} \ d\vec{p} < \infty,$$

as well as the following condition, which characterizes the conservation mass law:

$$\rho(t)d\Gamma(t) = \rho(t_0)d\Gamma(t_0), \quad t > t_0 \ ,$$

and using the equality (2) we obtain the relation

$$\tilde{H}(t) = H(t) + \int_K \tilde{\rho}(t) \ \ln \ |v_1^{(1)}v_2^{(1)}v^{(1)} \ldots v_1^{(N)}v_2^{(N)}v^{(N)}| \ d\vec{q} \ d\vec{p} \ . \tag{3}$$

3 Relativistic model of a three-dimensional gas and formulation of main results

We now use the system of units in which the rest-mass of the particle is equal to 1 and the light velocity is $c = 1$. Then in the three-dimensional case the energy $E^{(s)}$, the velocity vector $\vec{v}^{(s)}$, and the momentum vector $\vec{p}^{(s)}$ of the particle P_s are connected by the relations

$$\vec{p}^{(s)} = \frac{\vec{v}^{(s)}}{\sqrt{1 - |\vec{v}^{(s)}|^2}}, \quad E^{(s)} = \frac{1}{\sqrt{1 - |\vec{v}^{(s)}|^2}} = \sqrt{|\vec{p}^{(s)}|^2 + 1}.$$

Theorem 1. *Suppose that the relativistic model of a three-dimensional gas satisfies the following conditions:*
1) the functions $f_i(t)$ $(i = 1, 2)$ have the period 1 with respect to t, and the functions $g_i(t)$, $h_i(t)$ $(i = 1, 2)$ do not depend on t and have the form

$$g_1(t) \equiv a_1, \quad g_2(t) \equiv b_1, \quad h_1(t) \equiv a_2, \quad h_2(t) \equiv b_2;$$

2) the height $l = b - a$ is an irrational number;
3) the inequality

$$\delta = \delta(f_1(t), f_2(t)) = \int_0^1 \ln \frac{(1 + \dot{f}_1(t))(1 - \dot{f}_2(t - l))}{(1 - \dot{f}_1(t))(1 + \dot{f}_2(t - l))} dt > 0 \tag{4}$$

is valid;
4) at the initial time t_0 the statistical distribution fuction $\rho(t_0)$ vanishes in the region

$$|p_1^{(s)}| > p_0^{(s)}, \quad |p_2^{(s)}| > p_0^{(s)}, \quad |p^{(s)}| < p^*,$$

where $s = 1, \cdots, N$; $p_i^{(s)} = p_i^{(s)}(t_0)$, $(i = 1, 2)$, $p^{(s)} = p^{(s)}(t_0)$.

If the constant p^* is sufficiently large for fixed constants $p_0^{(s)}$ $(s = 1, \ldots, N)$ then for any time $t \geq t_0$ the following inequalities hold

$$\begin{aligned} H(t) - H(t_0) &> C_1^* + C_2^*(t - t_0), \\ \widetilde{H}(t) - \widetilde{H}(t_0) &> C_1^* + C_2^*(t - t_0), \end{aligned}$$

where $C_2^* > 0$ and the constants C_1^* and C_2^* do not dependent of t and t_0.

Remark 1. As follows from [5] and [1] the physical sense of inequality (4) is the walls of the vessel are hot relative to the gas because the energy transmitted to particles in a sufficiently large time from upper and lower boundaries of the vessel is positive in the relativistic case. The inequality (4) holds if

$$\begin{aligned} f_1(t) &= f_1^*(t) = \varepsilon(Q_1 \sin(2\pi K t) + Q_2 \sin(4\pi K t)) + c^*, \\ f_2(t) &= f_2^*(t) \equiv b, \end{aligned}$$

where $\frac{dc^*}{dt} \equiv 0$, $KQ_2 > 0$, K is integer, $Q_1 \neq 0$, $\varepsilon > 0$, ε is a small parameter, and for this case the quantity $\delta(f_1^*(t), f_2^*(t)) = \delta^*$ introduced in the theorem 2, has the form

$$\delta^* = 8\pi^3 K^3 Q_1^2 Q_2 \varepsilon^3 + O(\varepsilon^5) \ ,$$

and $\delta^* > 0$ for small ε.

Remark 2. The assertions of Theorem 1 are valid if the functions $g_i(t)$ and $h_i(t)$ $(i = 1, 2)$ are not constants, but modules of their derivatives are sufficiently small relative to the constant δ.

Remark 3. Using arguments of the paper [1] one can change the condition 2) of Theorem 1 by the requirement according to which the number l is not equal to a rational number with denominator less than some positive constant.

The proof of Theorem 1 is given in Section 5. It use essentially auxiliary Lemmas and Theorem 2 which are formulated and proved in Section 4.

4 Auxiliary Lemmas and Theorem

Let $N = 1$. We assume that after collision at time t with the lower boundary $q = a$ the particle has the momentum vector $\vec{p} = (p_1, p_2, p)$ and the velocity vector $\vec{v} = (v_1, v_2, v)$, and the component v of $\vec{v}$ satisfies the inequality $v > 0$ and is directed to the upper boundary $q = b$. After the first collision with the upper boundary at time $\vec{t}$ the particle has the momentum vector $\vec{\bar{p}} = (\bar{p}_1, \bar{p}_2, \bar{p})$ and the velocity vector $\vec{\bar{v}} = (\bar{v}_1, \bar{v}_2, \bar{v})$, whose component $\bar{v}$ is directed to the lower boundary $q = a$. After the first collision with the lower boundary at time t' the particle has the momentum vector $\vec{p}\,' = (p_1', p_2', p')$. We define the transformations

$$A : (t, p) \to (t', p'), \qquad \bar{A} : (t, p) \to (\bar{t}, -\bar{p}) \tag{5}$$

depending on parameters p_1, p_2, and the transformations

$$\hat{A} : (t, p) \to (\hat{t}, \hat{p}), \qquad \bar{\bar{A}} : (t, p) \to (\bar{\bar{t}}, \bar{\bar{p}})$$

by the relations

$$\bar{\bar{t}} = t + l, \qquad \bar{\bar{p}} = p \frac{1 - \dot{f}_2(\bar{\bar{t}})}{1 + \dot{f}_2(\bar{\bar{t}})} \ , \tag{6}$$

$$\hat{t} = t + 2l, \qquad \hat{p} = p \frac{(1 + \dot{f}_1(\hat{t}))(1 - \dot{f}_2(\bar{\bar{t}}))}{(1 - \dot{f}_1(\hat{t}))(1 + \dot{f}_2(\bar{\bar{t}}))}. \tag{7}$$

Lemma 1. *The transformations $\bar{A}$ and A are given by the following relations:*

$$\begin{aligned} \bar{t} &= t + \frac{l\sqrt{1 + |\vec{p}|^2}}{p}, \\ -\bar{p} &= p \frac{1 - \dot{f}_2(\bar{t})}{1 + \dot{f}_2(\bar{t})} - \frac{2\dot{f}_2(\bar{t})}{1 - \dot{f}_2^2(\bar{t})} \sqrt{p^2 + \Delta} - p), \end{aligned} \tag{8}$$

$$
\begin{aligned}
t' &= \bar{t} + \frac{l\sqrt{1+|\vec{\bar{p}}|^2}}{-\bar{p}}, \\
p' &= (-\bar{p})\frac{1+\dot{f}_1(t')}{1-\dot{f}_1(t')} + \frac{2\dot{f}_1(t')}{1-\dot{f}_1^2(t')}(\sqrt{\bar{p}^2+\bar{\Delta}}-(-\bar{p}))
\end{aligned} \tag{9}
$$

where $\Delta = p_1^2 + p_2^2 + 1$, $\bar{\Delta} = \bar{p}_1^2 + \bar{p}_2^2 + 1$, *and the values of* $\sqrt{p^2+\Delta}$ *and* $\sqrt{\bar{p}^2+\Delta}$ *are assumed to be positive.*

Proof. The equalities for $\bar{t}$ and t' follow from the definitions of $\overline{A}$ and A obviously.

Here we prove the equality for $-\overline{p}$ in (8). The equality for p' in (9) is proved completely analogously. We first assume that the rest-mass of the upper wall is finite and equal to M. Then we pass to the limit in the relations as $M \to \infty$. By the momentum and energy of conservation law we obtain:

$$
p + PM = \overline{p} + \overline{P}M, \tag{10}
$$

$$
\sqrt{|\vec{p}|^2+1} + M\sqrt{P^2+1} = \sqrt{|\vec{\overline{p}}|^2+1} + M\sqrt{\overline{P}^2+1}\ , \tag{11}
$$

where $P = \frac{V}{\sqrt{1-V^2}}$, $\overline{P} = \frac{\overline{V}}{\sqrt{1-\overline{V}^2}}$, V is the velocity of the upper wall at the time $\overline{t}$ before the collision with the particle, $\overline{V}$ is the velocity of the upper wall at time $\overline{t}$ after the collision with the particle.

Solving equation (11) we obtain:

$$
\begin{gathered}
M\overline{P} = M\Big\{\Big(\frac{\sqrt{1+|\vec{p}|^2}-\sqrt{1+|\vec{\overline{p}}|^2}}{M} + \sqrt{1+P^2}\Big)^2 - 1\Big\}^{\frac{1}{2}} = \\
MP + \frac{\sqrt{1+P^2}}{P}(\sqrt{1+|\vec{p}|^2} - \sqrt{1+|\vec{\overline{p}}|^2}) + O\left(\frac{1}{M}\right).
\end{gathered}
$$

We substitute this equality in (10). Then in the limit as $M \to \infty$ we have $\sqrt{|\vec{\overline{p}}|^2+1} - \sqrt{|\vec{p}|^2+1} = \dot{f}_2(\overline{t})(\overline{p}-p)$, which implies that

$$
\overline{p}^2 - p^2 = \dot{f}_2(\overline{t})(\overline{p}-p)(\sqrt{|\vec{\overline{p}}|^2+1} + \sqrt{|\vec{p}|^2+1}),
$$

$$
\overline{p} + p = \dot{f}_2(\overline{t})(\sqrt{|\vec{\overline{p}}|^2+1} + \sqrt{|\vec{p}|^2+1}), \tag{12}
$$

$$
-\dot{f}_2^2(\overline{t})(\overline{p}-p) = -\dot{f}_2(\overline{t})(\sqrt{|\vec{\overline{p}}|^2+1} - \sqrt{|\vec{p}|^2+1}). \tag{13}
$$

Combining equalities (12) and (13), we obtain

$$
\begin{aligned}
-\overline{p}(1-\dot{f}_2^2(\overline{t})) &= p(1+\dot{f}_2^2(\overline{t})) - 2\dot{f}_2(\overline{t})\sqrt{|\vec{p}|^2+1} \\
&= p(1-\dot{f}_2(\overline{t}))^2 - 2\dot{f}_2(\overline{t})(\sqrt{|\vec{p}|^2+1}-p)\ ,
\end{aligned}
$$

from which the equality for $-\overline{p}$ in (8) follows.

Lemma 1 is proved.

Remark 2. From definitions of $\widehat{A}$ and $\overline{\overline{A}}$ and from Lemma 1 it follows that for fixed values of p_1 and p_2 and for large value of p the transformations A and $\widehat{A}$ differ by a quantity $O(\frac{1}{p})$ and the transformations $\overline{A}$ and $\overline{\overline{A}}$ differ by a quantity $O(\frac{1}{p})$.

We now introduce the new variable $\eta = \ln p$, and define the transformations $\mathcal{D}, \overline{\mathcal{D}}$, $\widehat{\mathcal{D}}$, $\overline{\overline{\mathcal{D}}}$ as follows:

$$\mathcal{D} : (t, \eta) \to (t' \bmod 1, \eta'), \quad \overline{\mathcal{D}} : (t, \eta) \to (\bar{t} \bmod 1, \overline{\eta}),$$
$$\widehat{\mathcal{D}} : (t, \eta) \to (\widehat{t} \bmod 1, \widehat{\eta}), \quad \overline{\overline{\mathcal{D}}} : (t, \eta) \to (\bar{\bar{t}} \bmod 1, \overline{\overline{\eta}}),$$

where $\eta' = \ln p'$, $\overline{\eta} = \ln(-\overline{p})$, $\widehat{\eta} = \ln \widehat{p}$, $\overline{\overline{\eta}} = \ln \overline{\overline{p}}$.

Lemma 2. *We assume that the coordinates p_1, p_2, of the vector $\vec{p} = (p_1, p_2, p)$ satisfy by the inequalities $|p_i| \leq p_0$ $(i = 1, 2)$. Then there exists a number $\Delta_0 > 0$, depending only on the functions $f_1(t), f_2(t)$ and p_0, such that if $p \geq e^{\Delta_0}$, then the transformations $\overline{A}$ and A are defined, and if $\eta \geq \Delta_0$, then the transformations $\overline{\mathcal{D}}$ and $\mathcal{D}$ are defined.*

The proof of Lemma 2 follows from the equalities (6), (7), from Lemma 1 and from definitions of transformations $\mathcal{D}$, $\overline{\mathcal{D}}$, $\widehat{\mathcal{D}}$, $\overline{\overline{\mathcal{D}}}$.

Lemma 3. *Let*

$$F(t) = \ln \frac{(1 + \dot{f}_1(t))(1 - \dot{f}_2(t - l))}{(1 - \dot{f}_1(t))(1 + \dot{f}_2(t - l))} \tag{14}$$

and $(\widehat{t}_k, \widehat{p}_k) = \widehat{A}^k(t, p)$, where $\widehat{A}^k$ is the kth power of $\widehat{A}$. Suppose that l is irrational number and

$$\int_0^1 F(t)dt = \delta > 0. \tag{15}$$

If $\widetilde{\delta}$ is an arbitrary number such that $0 < \widetilde{\delta} < \delta$, then there is a natural number $\widetilde{m} = \widetilde{m}(\widetilde{\delta})$ such that for all t, $p > 0$, and integer $m \geq \widetilde{m}$, the following inequalities hold:

$$\sum_{k=1}^{m} F(\widehat{t}_k) \geq m\widetilde{\delta}, \quad \ln \widehat{p}_m \geq \ln p + m\widetilde{\delta}. \tag{16}$$

Proof. We consider the circle mapping $B : t \to \widehat{t} = t + 2l \bmod 1$. Since l is irrational, we see that the transformation B is uniformly ergodic [3], that is the ergodic theorem for a continuous function holds everywhere, and therefore, by Birknoff's ergodic theorem ([3]), for any t there exists the limit

$$I = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F(B^k t) = \int_0^1 F(t)dt,$$

where B^k is the kth power of B. By (15) we have $I = \delta > 0$, therefore

$$\sum_{k=1}^{m} F(B^k t) = m\delta + o(m), \tag{17}$$

where $\lim\limits_{m\to\infty} \frac{o(m)}{m} = 0$, and the convergence is inform with respect to all t. Now the conclusion of Lemma 3 follows from the equalities (17), (14) and the definition $\widehat{A}$. This completes the proof of Lemma 3.

Theorem 2. *We assume that the number of particles $N=1$, for this case the conditions* 1), 2), 3) *of Theorem* 1 *hold and the coordinates p_1, p_2 of vector $\vec{p} = (p_1, p_2, p)$ satisfy the inequalities $|p_1| < p_0$, $|p_2| < p_0$. Let $(t_*^{(m)}, p_*^{(m)}) = A^m(t,p)$ and $(\overline{t}_*^{(m)}, \overline{p}_*^{(m)}) = \overline{A}A^{m-1}(t,p)$ (A^m is the mth power of A; $m = 1, 2, \cdots$). Then for any $\widetilde{\delta} > 0$ such that $0 < \widetilde{\delta} < \delta$ there exists a number $\widetilde{p} = \widetilde{p}(p_0, \widetilde{\delta})$ such that if $p \ge \widetilde{p}$, then for all t $p_*^{(m)} \to \infty$, $\overline{p}_*^{(m)} \to \infty$ as $m \to \infty$, and the following inequalities hold:*

$$p_*^{(m)} > \widetilde{C} p e^{\widetilde{\delta} m}, \quad \overline{p}_*^{(m)} > \widetilde{C} p e^{\widetilde{\delta} m},$$

$$\prod_{k=1}^{m} \left\{ \frac{\partial p_*^{(k)}}{\partial \overline{p}_*^{(k)}}(t_*^{(k)}, \overline{p}_*^{(k)}) \frac{\partial \overline{p}_*^{(k)}}{\partial p_*^{(k-1)}}(\overline{t}_*^{(k)}, p_*^{(k-1)}) \right\} > \widetilde{C} e^{\widetilde{\delta} m}, \tag{18}$$

$$\frac{\partial \overline{p}_*^{(m)}}{\partial p_*^{(m-1)}}(t_*^{(m)}, p_*^{(m-1)}) \prod_{k=1}^{m-1} \left\{ \frac{\partial p_*^{(k)}}{\partial \overline{p}_*^{(k)}}(t_*^{(k)}, \overline{p}_*^{(k)}) \frac{\partial \overline{p}_*^{(k)}}{\partial p_*^{(k-1)}}(\overline{t}_*^{(k)}, p_*^{(k-1)}) \right\} > \widetilde{C} e^{\widetilde{\delta} m}, \tag{19}$$

where $p_*^{(0)} = p$, $\widetilde{C}$ is a positive constant, independent of parameters t, p_1, p_2, p, m, the quantities $\frac{\partial p_*^{(k)}}{\partial \overline{p}_*^{(k)}}(t_*^{(k)}, \overline{p}_*^{(k)})$, $\frac{\partial \overline{p}_*^{(k)}}{\partial p_*^{(k-1)}}(\overline{t}_*^{(k)}, p_*^{(k-1)})$ on the left-hand side of (18), (19) are the functions of $t_*^{(k)}, \overline{p}_*^{(k)}$ and $\overline{t}_*^{(k)}, p_*^{(k-1)}$ respecively, and the product on the left-hand side of (19) is replaced by 1 *when $m = 1$.*

Proof of Theorem 2. Let Δ_0 be the number satisfying the condition of Lemma 2. By Lemmas 2 and 1 and the definitions of A, $\overline{A}$, $\widehat{A}$, $\overline{\overline{A}}$, $\mathcal{D}$, $\overline{\mathcal{D}}$, $\widehat{\mathcal{D}}$, $\overline{\overline{\mathcal{D}}}$ it follows that if $|p_1| \le p_0$, $|p_2| \le p_0$, then transformations $\mathcal{D}$, $\overline{\mathcal{D}}$, $\widehat{\mathcal{D}}$, $\overline{\overline{\mathcal{D}}}$ are well defined in the region $\Gamma = \{t, \eta : 0 \le t \le 1, \eta \ge \Delta_0\}$, and if $(t, \eta) \in \Gamma$, then the following estimates hold:

$$|t' - \hat{t}| + |\overline{t} - \overline{\overline{t}}| < \frac{C_1}{p}, \quad |\eta' - \hat{\eta}| + |\overline{\eta} - \overline{\overline{\eta}}| < \frac{C_1}{p}, \tag{20}$$

$$\left| \ln \left\{ \frac{\partial p'}{\partial(-\overline{p})}(t', -\overline{p}) \frac{\partial(-\overline{p})}{\partial p}(\overline{t}, p) \right\} - \ln \left\{ \frac{\partial \widehat{p}}{\partial \overline{\overline{p}}}(\hat{t}, \overline{\overline{p}}) \frac{\partial \overline{\overline{p}}}{\partial p}(\overline{\overline{t}}, p) \right\} \right| < \frac{C_1}{p}, \tag{21}$$

where $(t', \eta') = \mathcal{D}(t, \eta)$, $(\overline{t}, \overline{\eta}) = \overline{\mathcal{D}}(t, \eta)$, $(\hat{t}, \hat{\eta}) = \widehat{\mathcal{D}}(t, \eta)$, $(\overline{\overline{t}}, \overline{\overline{\eta}}) = \overline{\overline{\mathcal{D}}}(t, \eta)$, C_1 is a constant independent of t, η, p_1, p_2, and the functions $\frac{\partial p'}{\partial(-\overline{p})}(t', -\overline{p})$, $\frac{\partial(-\overline{p})}{\partial p}(\overline{t}, p)$, $\frac{\partial \widehat{p}}{\partial \overline{\overline{p}}}(\hat{t}, \overline{\overline{p}})$, $\frac{\partial \overline{\overline{p}}}{\partial p}(\overline{\overline{t}}, p)$ are the derivatives of the functions $p' = p'(t', -\overline{p})$, $-\overline{p} = -\overline{p}(\overline{t}, p)$, $\widehat{p} = \widehat{p}(\hat{t}, \overline{\overline{p}})$, $\overline{\overline{p}} = \overline{\overline{p}}(\overline{\overline{t}}, p)$, which are defined by equalities (6)–(13).

By the definitions of $\widehat{A}$, $\overline{\overline{A}}$ we have: $\hat{\eta} = \overline{\overline{\eta}} + F_1(\hat{t})$, $\overline{\overline{\eta}} = \eta + F_2(\overline{\overline{t}})$, where the functions $F_1(t) = \ln \frac{1+\dot{f}_1(t)}{1-\dot{f}_1(t)}$, $F_2(t) = \ln \frac{1-\dot{f}_2(t)}{1+\dot{f}_2(t)}$ satisfy the inequalities

$$\left| \frac{dF_1}{dt}(t) \right| < C_2, \quad \left| \frac{dF_2}{dt}(t) \right| < C_2, \tag{22}$$

and C_2 is a constant independent of t.

By Lemma 3, for every $\widetilde{\delta}$ such that $0 < \widetilde{\delta} < \delta$ there exists a natural number $\widetilde{m} = \widetilde{m}(\widetilde{\delta})$ such that the inequalities (16) are valid for all t, $p > 0$ and integer $m \geq \widetilde{m}$.

We introduce the following notations:

$$\begin{gathered}(x_0, y_0) = \widehat{A}^{\widetilde{m}}(t,p),\ (x_n, y_n) = A^n(x_0, y_0), \quad (\overline{x}_n, \overline{y}_n) = \overline{A}A^{n-1}(x_0, y_0),\\ (\widehat{x}_n, \widehat{y}_n) = \widehat{A}^n(x_0, y_0),\ (\overline{\overline{x}}_n, \overline{\overline{y}}_n) = \overline{\overline{A}}\widehat{A}^{n-1}(x_0, y_0),\ (t_0, \eta_0) = (x_0, \ln y_0),\\ (t_n, \eta_n) = \mathcal{D}^n(t_0, \eta_0),\quad (\overline{t}_n, \overline{\eta}_n) = \overline{\mathcal{D}}\mathcal{D}^{n-1}(t_0, \eta_0),\quad (\widehat{t}_n, \widehat{\eta}_n) = \widehat{\mathcal{D}}^n(t_0, \eta_0),\\ (\overline{\overline{t}}_n, \overline{\overline{\eta}}_n) = \overline{\overline{\mathcal{D}}}\widehat{\mathcal{D}}^{n-1}(t_0, \eta_0),\end{gathered}$$

where $n = 1, 2, \cdots$ and $\mathcal{D}^n$ and $\widehat{\mathcal{D}}^n$ are the nth powers of $\mathcal{D}$ and $\widehat{\mathcal{D}}$, respectively.

By (20), (21) and (22) if $|p_1| \leq p_0$, $|p_2| \leq p_0$, then the following inequalities hold for $n = 1, 2, \cdots$:

$$|t_n - \widehat{t}_n| + |\overline{t}_n - \overline{\overline{t}}_n| < C_1 \sum_{s=1}^{n} \frac{1}{y_{s-1}}, \tag{23}$$

$$|\eta_n - \widehat{\eta}_n| < C_2 \sum_{s=1}^{n} (|t_s - \widehat{t}_s| + |\overline{t}_s - \overline{\overline{t}}_s|) + C_1 \sum_{s=1}^{n} \frac{1}{y_{s-1}},$$

$$\begin{aligned}&\left| \sum_{k=1}^{n} \ln \left\{ \frac{\partial y_k}{\partial \overline{y}_k}(x_k, \overline{y}_k) \frac{\partial \overline{y}_k}{\partial y_{k-1}}(\overline{x}_k, y_{k-1}) \right\} \right.\\ &\quad \left. - \sum_{k=1}^{n} \ln \left\{ \frac{\partial \widehat{y}_k}{\partial \overline{\overline{y}}_k}(\widehat{x}_k, \overline{\overline{y}}_k) \frac{\partial \overline{\overline{y}}_k}{\partial \widehat{y}_{k-1}}(\overline{\overline{x}}_k, \widehat{y}_{k-1}) \right\} \right|\\ &< C_2 \sum_{s=1}^{n} (|t_s - \widehat{t}_s| + |\overline{t}_s - \overline{\overline{t}}_s|) + C_1 \sum_{s=1}^{n} \frac{1}{y_{s-1}},\end{aligned}$$

provided that all points (t_k, η_k) $(k = 0, 1, \ldots, n-1)$ belong to Γ. Therefore, if we assume that for $n = 1, 2, \ldots$

$$|t_n - \widehat{t}_n| + |\overline{t}_n - \overline{\overline{t}}_n| < d, \quad C_1 \left| \sum_{s=0}^{n} \frac{1}{y_s} \right| < d \tag{24}$$

and the constant d can be chosen arbitrarily small and independent of n for large p, then we obtain:

$$\eta_n \geq \widehat{\eta}_n - |\eta_n - \widehat{\eta}_n| > \ln p + (n + \widetilde{m})(\widetilde{\delta} - (C_2 + 1)d), \tag{25}$$

$$\begin{aligned}&\sum_{k=1}^{n} \ln \left\{ \frac{\partial y_k}{\partial \overline{y}_k}(x_k, \overline{y}_k) \frac{\partial \overline{y}_k}{\partial y_{k-1}}(\overline{x}_k, y_{k-1}) \right\}\\ &\geq \sum_{k=1}^{n} \ln \left\{ \frac{\partial \widehat{y}_k}{\partial \overline{\overline{y}}_k}(\widehat{x}_k, \overline{\overline{y}}_k) \frac{\partial \overline{\overline{y}}_k}{\partial \widehat{y}_{k-1}}(\overline{\overline{x}}_k, \widehat{y}_{k-1}) \right\} -\\ &\quad - \left| \sum_{k=1}^{n} \ln \left\{ \frac{\partial y_k}{\partial \overline{y}_k}(x_k, \overline{y}_k) \frac{\partial \overline{y}_k}{\partial y_{k-1}}(\overline{x}_k, y_{k-1}) \right\} \right.\\ &\qquad \left. - \ln \left\{ \frac{\partial \widehat{y}_k}{\partial \overline{\overline{y}}_k}(\widehat{x}_k, \overline{\overline{y}}_k) \frac{\partial \overline{\overline{y}}_k}{\partial \widehat{y}_{k-1}}(\overline{\overline{x}}_k, \widehat{y}_{k-1}) \right\} \right|\\ &\qquad\qquad > (n + \widetilde{m})(\widetilde{\delta} - (C_2 + 1)d).\end{aligned} \tag{26}$$

However, this assumption implies inequality

$$\left|\sum_{s=0}^{n}\frac{1}{y_s}\right| < \frac{1}{p}\sum_{s=1}^{n}\exp(-(s+\widetilde{m})(\widetilde{\delta}-(C_2+1)d))$$

$$< \frac{\exp(-\widetilde{m}(\widetilde{\delta}-(C_2+1)d))}{p(1-\exp(-(\widetilde{\delta}-(C_2+1)d)))}\ ,$$

where the right-hand side of the inequality can be made arbitrarily small by choosing sufficiently large p. Therefore from (23)–(26), definitions of A, $\overline{A}$ and Lemma 1 it follows that for every $\widetilde{\delta}$ such that $0<\widetilde{\delta}<\delta$ there exist constants $\widetilde{\widetilde{p}}$ and $\widetilde{m}$ such that the condition $p\geq\widetilde{\widetilde{p}}$ implies that $(t_n,\eta_n)\in\Gamma$ for all $n=0,1,\cdots$, and following inequalities hold:

$$y_n > C_3pe^{(n+\widetilde{m})\widetilde{\delta}},\quad \overline{y}_n > C_3pe^{(n+\widetilde{m})\widetilde{\delta}}, \tag{27}$$

$$\prod_{k=1}^{n}\left\{\frac{\partial y_k}{\partial\overline{y}_k}(x_k,\overline{y}_k)\frac{\partial\overline{y}_k}{\partial y_{k-1}}(\overline{x}_k,y_{k-1})\right\} > C_3pe^{(n+\widetilde{m})\widetilde{\delta}}, \tag{28}$$

$$\frac{\partial\overline{y}_n}{\partial y_{n-1}}(\overline{x}_n,y_{n-1})\prod_{k=1}^{n}\left\{\frac{\partial y_k}{\partial\overline{y}_k}(x_k,\overline{y}_k)\frac{\partial\overline{y}_k}{\partial y_{k-1}}(\overline{x}_k,y_{k-1})\right\} > C_3e^{(n+\widetilde{m})\widetilde{\delta}}, \tag{29}$$

where C_3 is a positive constant independent of the parameters t,p,n,p_1, p_2, and the product on the left-hand side of (29) is replased by 1 when $n=1$. Introducing the notation

$$\beta=\max_{0\leq t\leq 1}\left|\frac{(1+\dot{f}_1(t))(1-\dot{f}_2(t-l))}{(1-\dot{f}_1(t))(1+\dot{f}_2(t-l))}\right|,\quad \widetilde{p}=\widetilde{\widetilde{p}}\beta^{\widetilde{m}}\ \text{ and }\ \widetilde{C}=C_3\beta^{-\widetilde{m}},$$

we see that the conclusion of Theorem 2 follows from inequalities (27)-(29). Theorem 2 is proved.

5 Proof of Theorem 1

Assume that at time t the particle P_s has the coordinate vector $\vec{q}^{(s)}=\vec{q}^{(s)}(t)=(q_1^{(s)},q_2^{(s)},q^{(s)})$, the momentum vector $\vec{p}^{(s)}=\vec{p}^{(s)}(t)=(p_1^{(s)},\ p_2^{(s)},\ p^{(s)})$, the velocity vector $\vec{v}^{(s)}=\vec{v}^{(s)}(t)=(v_1^{(s)},v_2^{(s)},v^{(s)})$ and suppose also that at time $\widetilde{t}>t$ it has the coordinate vector $\vec{\widetilde{q}}^{(s)}=\vec{\widetilde{q}}^{(s)}(\widetilde{t})=(\widetilde{q}_1^{(s)},\widetilde{q}_2^{(s)},\widetilde{q}^{(s)})$, the momentum vector $\vec{\widetilde{p}}^{(s)}=\vec{\widetilde{p}}^{(s)}(\widetilde{t})=(\widetilde{p}_1^{(s)},\widetilde{p}_2^{(s)},\widetilde{p}^{(s)})$, and the velocity vector $\vec{\widetilde{v}}^{(s)}=\vec{\widetilde{v}}^{(s)}(\widetilde{t})=(\widetilde{v}_1^{(s)},\widetilde{v}_2^{(s)},\widetilde{v}^{(s)})$. We introduce differentials $d\vec{q}=dq_1^{(1)}dq_2^{(1)}dq^{(1)}\cdots dq_1^{(N)}dq_2^{(N)}dq^{(N)}$, $d\vec{p}=dp_1^{(1)}dp_2^{(1)}\ dp^{(1)}\cdots dp_1^{(N)}\ dp_2^{(N)}dp^{(N)}$, $d\vec{\widetilde{q}}=d\widetilde{q}_1^{(1)}d\widetilde{q}_2^{(1)}d\widetilde{q}^{(1)}\cdots d\widetilde{q}_1^{(N)}d\widetilde{q}_2^{(N)}d\widetilde{q}^{(N)}$, $d\vec{\widetilde{p}}=d\widetilde{p}_1^{(1)}d\widetilde{p}_2^{(1)}d\widetilde{p}^{(1)}\cdots d\widetilde{p}_1^{(N)}\ d\widetilde{p}_2^{(N)}d\widetilde{p}^{(N)}$, and the quantities $\rho=\rho(q_1^{(1)},\cdots,p^{(N)},t)$, $\widetilde{\rho}=\rho(\widetilde{q}_1^{(1)},\cdots,\widetilde{p}^{(N)},\widetilde{t})$.

Since the number of particles in the elements of the phase space K of measures

$$d\Gamma=\frac{d\vec{q}d\vec{p}}{|v_1^{(1)}v_2^{(1)}v^{(1)}\cdots v_1^{(N)}v_2^{(N)}v^{(N)}|}$$

and

$$d\widetilde{\Gamma} = \frac{d\vec{\widetilde{q}}d\vec{\widetilde{p}}}{|\widetilde{v}_1^{(1)}\widetilde{v}_2^{(1)}\widetilde{v}^{(1)}\cdots\widetilde{v}_1^{(N)}\widetilde{v}_2^{(N)}\widetilde{v}^{(N)}|}$$

is the same, we have

$$\rho d\Gamma = \widetilde{\rho} d\widetilde{\Gamma}. \tag{30}$$

Putting

$$\tau^{(s)}(\vec{q}^{(s)}, \vec{p}^{(s)}, \widetilde{t}^{(s)}) = \frac{\mathcal{D}(\widetilde{q}_1^{(s)}, \widetilde{q}_2^{(s)}, \widetilde{q}^{(s)}, \widetilde{p}_1^{(s)}, \widetilde{p}_2^{(s)}, \widetilde{p}^{(s)})}{\mathcal{D}(q_1^{(s)}, q_2^{(s)}, q^{(s)}, p_1^{(s)}, p_2^{(s)}, p^{(s)})} \ ,$$

we obtain

$$d\vec{q}\ d\vec{p}\ \prod_{s=1}^{N} \tau^{(s)}(\vec{q}^{(s)}, \vec{p}^{(s)}, \widetilde{t}) = d\vec{\widetilde{q}}d\vec{\widetilde{p}}.$$

Therefore, by (30) we have

$$\begin{aligned} \ln\widetilde{\rho} \ &= \ \ln\rho + \sum_{s=1}^{n}(\ln|\widetilde{v}_1^{(s)}\widetilde{v}_2^{(s)}\widetilde{v}^{(s)}| - \ln|v_1^{(s)}v_2^{(s)}v^{(s)}|) - \\ &\quad - \sum_{s=1}^{N}\ln|\tau^{(s)}(\vec{q}^{(s)}, \vec{p}^{(s)}, \widetilde{t})|. \end{aligned} \tag{31}$$

We estimate the quantities $\tau^{(s)}(\vec{q}^{(s)}, \vec{p}^{(s)}, \widetilde{t})$, $(s = 1, \cdots, N)$. Without loss of generality one can assume that at initial time t the particle P_s is on the boundary $q = a$ of the vessel and the coordinate $v^{(s)}$ of its velocity vector $\vec{v}^{(s)}$ satisfies the inequality $v^{(s)} > 0$. The case $v^{(s)} < 0$ is reduced to the case $v^{(s)} > 0$, if to interchange the lower and upper boundaries of the vessel. We denote the kth (after t) time t of the collision of the particle P_s with the boundary $q = a$ by $t_k^{(s)}$, and the momentum vector and velocity vector of the particle P_s at time $t_k^{(s)}$ after the collision with the boundary $q = a$ by $\vec{p}_k^{(s)} = (p_{k1}^{(s)}, p_{k2}^{(s)}, p_{k0}^{(s)})$, $\vec{v}_k^{(s)} = (v_{k1}^{(s)}, v_{k2}^{(s)}, v_{k0}^{(s)})$, respectively. We also denote the kth (after t) time t of particle P_s with the boundary $q = b$ by $\overline{t}_k^{(s)}$, and the momentum vector and velocity vector of the particle P_s at time $\overline{t}_k^{(s)}$ after collision with the boundary $q = b$ by $\vec{\overline{p}}^{(s)} = (\overline{p}_{k,1}^{(s)}, \overline{p}_{k,2}^{(s)}, \overline{p}_{k,0}^{(s)})$, and $\vec{\overline{v}}_k^{(s)} = (\overline{v}_{k,1}^{(s)}, \overline{v}_{k,2}^{(s)}, \overline{v}_{k0}^{(s)})$, respectively. Two following cases are possible:

$$\begin{array}{c} 1)\ \ t_{n_s}^{(s)} \le \widetilde{t} < \overline{t}_{n_s+1}^{(s)}, \\ 2)\ \ \overline{t}_{n_s}^{(s)} \le \widetilde{t} < t_{n_s}^{(s)}, \end{array}$$

where n_s is an natural number. Putting $p_{0,0}^{(s)} = p^{(s)}(t)$, in the case 1) we have:

$$\begin{aligned} &\tau^{(s)}(\vec{q}^{(s)}\vec{p}^{(s)}, \widetilde{t}) = \\ &\frac{\mathcal{D}(\overline{t}_1^{(s)}, p^{(s)})}{\mathcal{D}(q^{(s)}, p^{(s)})}\prod_{k=1}^{n_s}\left\{\frac{\partial p_{k,0}^{(s)}}{\partial(-\overline{p}_{k,0}^{(s)})}(t_k^{(s)}, -\overline{p}_{k,0}^{(s)})\frac{\partial(-\overline{p}_{k,0}^{(s)})}{\partial(p_{k-1,0}^{(s)})}(\overline{t}_k^{(s)}, p_{k-1,0}^{(s)})\right\} \times \\ &\ \frac{\mathcal{D}(\widetilde{q}^{(s)}, \widetilde{p}^{(s)})}{\mathcal{D}(t_{n_s}^{(s)}, \widetilde{p}^{(s)})} = \\ &\quad = \frac{\widetilde{v}^{(s)}}{v^{(s)}}\prod_{k=1}^{n_s}\left\{\frac{\partial p_{k,0}^{(s)}}{\partial(-\overline{p}_{k,0}^{(s)})}(t_k^{(s)}, -\overline{p}_{k,0}^{(s)})\frac{\partial(-\overline{p}_{k,0}^{(s)})}{\partial(p_{k-1,0}^{(s)})}(\overline{t}_k^{(s)}, p_{k-1,0}^{(s)})\right\}, \end{aligned} \tag{32}$$

and in the case 2) we have

$$\tau^{(s)}(\vec{q}^{(s)}\vec{p}^{(s)},\widetilde{t}) = \frac{|\widetilde{v}^{(s)}|}{v^{(s)}}\frac{\partial(-\overline{p}^{(s)}_{n_s,0})}{\partial(p^{(s)}_{n_s-1,0})}(\overline{t}^{(s)}_{n_s},p^{(s)}_{n_s-1,0})\times$$
$$\times\prod_{k=1}^{n_s-1}\left\{\frac{\partial p^{(s)}_{k,0}}{\partial(-\overline{p}^{(s)}_{k,o})}(t^{(s)}_k,-\overline{p}^{(s)}_{k,0})\frac{\partial(-\overline{p}^{(s)}_{k,0})}{\partial p^{(s)}_{k-1,0}}(\overline{t}^{(s)}_k,p^{(s)}_{k-1,0})\right\}. \tag{33}$$

In the case $n_s = 1$ the product on the right-hand side is replased by 1. Applying the results of Theorem 2 to the equalities (32), (33), we see, that for every $\widetilde{\delta}$ such that $0<\widetilde{\delta}<\delta$ (the quantity δ is defined in (4)) there exists a constant $\widehat{p}=\widehat{p}(p_0,\widetilde{\delta})$ such that if $p^{(s)}(t)\geq\widehat{p}$ $(s=1,\cdots,N)$, then

$$\tau^{(s)}(\vec{q}^{(s)},\vec{p}^{(s)},\widetilde{t}) > C_4 e^{\widetilde{\delta}n_s},$$

for all $\widetilde{t}>t$ and $s=1,\cdots,N$, where $C_4>0$ is positive constant independent of t and n_s. Hence

$$\ln\tau^{(s)}(\vec{q}^{(s)}\vec{p}^{(s)},\widetilde{t}) > \widetilde{C}_1+\widetilde{C}_2(\widetilde{t}-t), \tag{34}$$

where $\widetilde{C}_2>0$ and the constants $\widetilde{C}_1$ and $\widetilde{C}_2$ do not depend on the parameters $t,\widetilde{t},\vec{q}^{(s)},\vec{p}^{(s)}$, s. Using equalities (1), (2), (30), (31) we obtain:

$$\begin{aligned}H(\widetilde{t}) &= -\int_K\frac{\widetilde{\rho}\ \ln\widetilde{\rho}\ d\vec{\widetilde{q}}\ d\vec{\widetilde{p}}}{\prod_{s=1}^N|v_1^{(s)}(\widetilde{t})v_2^{(s)}(\widetilde{t})v^{(s)}(\widetilde{t})|}=\\ &= -\int_K\frac{\rho\ d\vec{q}\ d\vec{p}}{\prod_{s=1}^N|v_1^{(s)}(t)v_2^{(s)}(t)v^{(s)}(t)|}\times\\ &\quad\left\{\ln\rho+\sum_{s=1}^N(\ln(|\widetilde{v}_1^{(s)}\widetilde{v}_2^{(s)}\widetilde{v}^{(s)}|-\ln|v_1^{(s)}v_2^{(s)}v^{(s)}|)\right.\\ &\quad\left.-\sum_{s=1}^N\ln\tau^{(s)}(\vec{q}^{(s)},\vec{p}^{(s)},\widetilde{t})\right\},\end{aligned} \tag{35}$$

$$H(t) = -\int\limits_K\frac{\rho\ \ln\rho\ d\vec{q}\ d\vec{p}}{\prod\limits_{s=1}^N|v_1^{(s)}(t)v_2^{(s)}(t)v^{(s)}(t)|}.$$

Now, applying estimate (34) to equality (35), and Theorem 2 to equality (3) we obtain the assertion of Theorem 1.

Theorem 1 is proved.

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