L^1–THEORY FOR THE KOLMOGOROV OPERATORS OF STOCHASTIC GENERALIZED BURGERS EQUATIONS

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Abstract. This paper contains supplementary results to the recent paper [21] by the two authors. It focuses on the L^1–theory of a class of Kolmogorov operators L in infinitely many variables which e.g. are associated to stochastic generalized Burgers equations. Their L^1–theory is developed with respect to a whole class of reference measures identified in this paper, which contains in particular infinitesimally invariant measures for L. Essential maximal dissipativity for L with initial domain given by C^2–smooth bounded cylinder functions is proved to hold on $L^1(\nu)$ for all measures ν in this class. The obtained respective C_0–semigroup on $L^1(\nu)$ is proved to come from the semigroup of kernels constructed in [21]. Finally, a measure is constructed in this class which is of full topological support, i.e. charges every non–empty open set of the underlying infinite dimensional space, which here is $L^2(0,1)$.

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1. Introduction

Consider the following stochastic partial differential equation on $X := L^2(0,1) = L^2((0,1),dr)$ (where dr denotes Lebesgue measure)

$$\begin{aligned} dx_t &= (\Delta x_t + F(x_t))\,dt + \sqrt{A}\,dw_t \\ x_0 &= x \in X. \end{aligned} \tag{1.1}$$

Here $A : X \to X$ is a nonnegative definite symmetric operator of trace class, $(w_t)_{t\geqslant 0}$ a cylindrical Brownian motion on X, Δ denotes the Dirichlet Laplacian (i.e. with Dirichlet boundary conditions) on $(0,1)$, and $F : H_0^1 \to X$ is a measurable vector field satisfying certain conditions specified below. Here $H_0^1 := H_0^1(0,1)$ denotes the Sobolev space of order 1 in $L^2(0,1)$ with Dirichlet boundary conditions. As a special case SPDE (1.1) contains so–called stochastic generalized Burgers equations (cf. [21], see also [20], where $(0,1)$ is replaced by an open set in $\mathbb{R}^2$).

A heuristic (i.e. not worrying about existence of solutions) application of Itô's formula to (1.1) implies that the corresponding generator or Kolmogorov operator L on smooth cylinder functions $u : X \to \mathbb{R}$, i.e.

$$u \in \mathcal{D} := \mathcal{F}C_b^2 := \big\{u = g \circ P_N \;\big|\; N \in \mathbb{N}, g \in C_b^2(E_N)\big\},$$

is of the following form:

$$\begin{aligned} Lu(x) &:= \frac{1}{2}\operatorname{Tr}\big(AD^2u(x)\big) + \big(\Delta x + F(x), Du(x)\big) \\ &= \frac{1}{2}\sum_{i,j=1}^{\infty} A_{ij}\partial_{ij}^2 u(x) + \sum_{k=1}^{\infty}\big(\Delta x + F(x), \eta_k\big)\partial_k u(x), \quad x \in H_0^1. \end{aligned} \tag{1.2}$$

Here $\eta_k(r) := \sqrt{2}\sin(\pi k r)$, $k \in \mathbb{N}$, is the eigenbasis of Δ in $L^2(0,1)$, equipped with the usual inner product $(\ ,\ )$, $E_N := \operatorname{span}\{\eta_k | 1 \leqslant k \leqslant N\}$, P_N is the corresponding orthogonal projection, and $A_{ij} := (\eta_i, A\eta_j)$, $i,j \in \mathbb{N}$. Finally, Du, D^2u denote the first and second Fréchet derivatives, $\partial_k := \partial_{\eta_k}$, $\partial_{ij}^2 := \partial_{\eta_i}\partial_{\eta_j}$ with $\partial_y :=$ directional derivative in direction $y \in X$ and $(\Delta x, \eta_k) := (x, \Delta\eta_k)$ for $x \in X$.

Hence the Kolmogorov equations corresponding to SPDE (1.1) are given by

$$\begin{aligned} \frac{dv}{dt}(t,x) &= \bar{L}v(t,x), \quad x \in X, \\ v(0,\,\cdot\,) &= f, \end{aligned} \tag{1.3}$$

where the function $f : X \to \mathbb{R}$ is a given initial condition for this parabolic PDE with variables in the infinite dimensional space X. We emphasize that (1.3) is only reasonable for some extension $\bar{L}$ of L (whose construction is an essential part of the entire problem) since even for $f \in \mathcal{D}$, it will essentially never be true that $v(t,\,\cdot\,) \in \mathcal{D}$.

Because of the lack of techniques to solve PDE's in infinite dimensions in situations as described above, the "classical" approach to solve (1.3) was to first solve (1.1) and then show in what sense the transition probabilities of the solution solve (1.3) (cf. e.g. [12], [2], [8], [14], [15], [18], [22], [5] and the references therein). Since about 1998, however, a substantial part of recent work in this area (cf. e.g. [9] [23], [24], and one of the initiating papers, [19]) is based on the attempt to solve Kolmogorov equations in infinitely many variables (as (1.3) above) directly and, reversing strategies, use the solution to construct weak solutions, i.e., solutions in the sense of a martingale problem as formulated by Stroock and Varadhan (cf. [25]), for SPDE's as (1.1) above, even for very singular coefficients (naturally appearing in many applications).

In [21] a new method was presented to solve (1.3) for all $x \in X$ (or an explicitly described subset thereof). It is based on finite dimensional approximation, obtaining a solution which despite of the lack of (elliptic and) parabolic regularity results on infinite dimensional spaces will nevertheless have regularity properties. More precisely, setting $X_p := L^p((0,1), dr)$, we shall construct a semigroup of Markov probability kernels $p_t(x, dy)$, $x \in X_p$, $t > 0$, on X_p such that for all $u \in \mathcal{D}$ we have $t \mapsto p_t(|Lu|)(x)$ is locally Lebesgue integrable on $[0, \infty)$ and

$$p_t u(x) - u(x) = \int_0^t p_s(Lu)(x)\,ds \quad \forall\, x \in X_p. \tag{1.4}$$

Here as usual for a measurable function $f : X_p \to \mathbb{R}$ we set

$$p_t f(x) := \int f(y)\, p_t(x, dy), \quad x \in X_p,\ t > 0, \tag{1.5}$$

if this integral exists. p has to be large enough compared to the growth of F. As a second step in [21] a conservative strong Markov process with weakly continuous paths was constructed, which is unique under a mild growth condition and which solves the martingale problem given by L as in (1.2), and hence also (1.1) weakly, for every starting point $x \in X_p$. Also an invariant measure for this process was constructed in [21].

The present paper can be considered as a supplement to [21], focussing on the $L^1(\nu)$–theory of (1.3) with respect to suitably chosen reference measures ν on X, one of which is the mentioned invariant measure. We shall thus concentrate on (1.3) and refer for treating SPDE (1.1) to [21]. The advantage of an $L^1(\nu)$–theory for (1.3) is that one really gets solutions of (1.3) in its original differential formulation rather than merely its integral (or mild) formulation (1.4). The disadvantage is that statements are considerably weakened to merely ν–a.e. statements, i.e. allowing ν–zero sets as exceptional sets of points in X where the equation does not hold. Nevertheless, one gets useful information. Such $L^1(\nu)$ (or even $L^p(\nu)$)–theory has been developed in [23], [24], [19] and more recently in [9], [1], [3], [6], [7], [10], [11], [16], in partly more special cases than ours or in other situations. But ν was always chosen to be an infinitesimally invariant measure of L, i.e. ν is a probability measure on X solving the equation $L^*\nu = 0$ (cf. Theorem 3.2(i) below). Our main aim in this paper is to specify a large class $\mathfrak{M}$ of probability measures ν on X for which the results in [21] imply that (L, D) is well–defined and closable on $L^1(X,\nu)$, and that its closure $(\overline{L}_1^\nu, D(\overline{L}_1^\nu))$ generates a C_0–semigroup on $L^1(X,\nu)$ which then by definition gives a solution for (1.3) in $L^1(X,\nu)$. Because there is no Lebesgue measure on X, it is particularly important to find substitutes. One important feature of such a substitute should be that this measure should have full topological support, i.e. it should be strictly positive on every non–empty open set, to have that continuous representatives of $L^1(X,\nu)$–classes are unique. This issue has hardly been addressed in the above mentioned literature. In this paper for the first time we construct a measure ν_0 in the above class $\mathfrak{M}$ with full topological support on X.

We emphasize that the methods to establish an L^1-theory for Kolmogorov operators of type (1.2) developed in this paper work in general and are not restricted to the case of underlying domains which are in $\mathbb{R}$ as $(0,1)$ above, but also extend to d-dimensional domains.

The organization of this paper is as follows. In Section 2 we recall the framework from [21], keeping the notation introduced there, and describe examples. In Section 3 we summarize those results from [21] used subsequently. In Section 4 we define the said class $\mathfrak{M}$ of probability measures and show well–definedness, closability of $(L,\mathcal{D})$ on $L^1(X,\nu)$ as well as that its closure generates a Markov C_0–semigroup on $L^1(X,\nu)$ for all $\nu \in \mathfrak{M}$. Section 5 is devoted to the construction of the specific reference measure ν_0 of full topological suppor mentioned above.

This paper in connection with [21] (see also [20]) covers a major part of the contents of the lecture series given by the first named author during the conference "Quantum Information and Complexity" held at Meijo University, Nagoya, in January 2003. We refer to the references, quoted in the text below, for other material touched upon in the lectures.

It is a great pleasure for the first named author to thank Professor Takeyuki Hida for organizing the above mentioned conference and for creating such a nice scientifically stimulating atmosphere among the participants. Thanks also go to all Japanese colleagues and friends who supported him, in particular Professor K. Saito, and also Meijo University and the staff involved. We hope that also in the future Professor Hida will find all necessary support for such conferences to be able to provide both his extraordinary scientific input as well as his warmhearted hospitality to the participating mathematical community.

2. Framework and main examples

Let us recall the framework and notation from [21].

For a σ-algebra $\mathcal{B}$ on an arbitrary set E we denote the space of all bounded resp. positive real-valued $\mathcal{B}$-measurable functions by $\mathcal{B}_b$, $\mathcal{B}^+$ respectively. If E is equipped with a topology, then $\mathcal{B}(E)$ denotes the corresponding Borel σ-algebra. The spaces $X = L^2(0,1)$ and H_0^1 are as in the introduction and they are equipped with their usual norms $|\cdot|_2$ and $|\cdot|_{1,2}$; so we define for $x : (0,1) \to \mathbb{R}$, measurable,

$$|x|_p := \Big(\int_0^1 |x(r)|^p \, dr\Big)^{1/p} \quad (\in [0,\infty]), \quad p \in [1,\infty),$$

$$|x|_\infty := \operatorname{ess\,sup}_{r\in(0,1)}|x(r)|,$$

and define $X_p := L^p((0,1), dr)$, $p \in [2,\infty]$, so $X = X_2$. If $x, y \in H_0^1$, set

$$|x|_{1,2} := |x'|_2, \quad (x,y)_{1,2} := (x', y'),$$

where $x' := \frac{d}{dr}x$ is the weak derivative of x. We shall use this notation from now on and we also write $x'' := \frac{d^2}{dr^2}x = \Delta x$.

Let H^{-1} with norm $|\cdot|_{-1,2}$ be the dual space of H_0^1. We always use the continuous and dense embeddings

$$H_0^1 \subset X \equiv X' \subset H^{-1}, \tag{2.1}$$

so ${}_{H_0^1}\langle x, y\rangle_{H^{-1}} = (x,y)$ if $x \in H_0^1$, $y \in X$. The terms "Borel-measurable" or "measure on X, H_0^1, H^{-1} resp." will below always refer to their respective Borel σ-algebras, if it is clear on which space we work. We note that since $H_0^1 \subset X \subset H^{-1}$ continuously, by Kuratowski's Theorem $H_0^1 \in \mathcal{B}(X)$, $X \in \mathcal{B}(H^{-1})$ and $\mathcal{B}(X) \cap H_0^1 = \mathcal{B}(H_0^1)$, $\mathcal{B}(H^{-1}) \cap X = \mathcal{B}(X)$. Furthermore, the Borel σ–algebras on X and H_0^1 corresponding to the respective weak topologies coincide with $\mathcal{B}(X)$, $\mathcal{B}(H_0^1)$ respectively.

For a function $V : X \to (0,\infty]$ having weakly compact level sets $\{V \le c\}$, $c \in \mathbb{R}_+$, we define:

$$\begin{aligned} WC_V := \Big\{ f : \{V<\infty\} \to \mathbb{R} \,\Big|\, & f \text{ is continuous on each } \{V \leqslant R\},\ R \in \mathbb{R}, \\ & \text{in the weak topology inherited from } X, \\ & \text{and } \lim_{R\to\infty} \sup_{\{V \geqslant R\}} V^{-1}|f| = 0 \Big\}, \end{aligned} \tag{2.2}$$

equipped with the norm $\|f\|_V := \sup_{\{V<\infty\}} V^{-1}|f|$. Obviously, WC_V is a Banach space with this norm. We are going to consider various choices of V, distinguished by respective subindices, namely we define for $\kappa \in (0,\infty)$

$$\begin{aligned} V_\kappa(x) &:= e^{\kappa|x|_2^2}, \quad x \in X, \\ \Theta_\kappa(x) &:= V_\kappa(x)(1+|x'|_2^2), \quad x \in H_0^1, \end{aligned} \tag{2.3}$$

and for $p > 2$

$$\begin{aligned} V_{p,\kappa}(x) &:= e^{\kappa|x|_2^2}\big(1+|x|_p^p\big), \quad x \in X, \\ \Theta_{p,\kappa}(x) &:= V_{p,\kappa}(x)(1+|x'|_2^2) + V_\kappa(x)\Big|\big(|x|^{\frac{p}{2}}\big)'\Big|_2^2, \quad x \in H_0^1. \end{aligned} \tag{2.4}$$

Clearly, $\{V_{p,\kappa} < \infty\} = X_p$ and $\{\Theta_{p,\kappa} < \infty\} = H_0^1$. Each $\Theta_{p,\kappa}$ is extended to a function on X by defining it to be equal to $+\infty$ on $X \setminus H_0^1$. Abusing notation, for $p = 2$ we also set $V_{2,\kappa} := V_\kappa$ and $\Theta_{2,\kappa} := \Theta_\kappa$. For abbreviation, for $\kappa \in (0,\infty)$, $p \in [2,\infty)$ we set

$$WC_{p,\kappa} := WC_{V_{p,\kappa}}, \quad W_1C_{p,\kappa} := W_1C_{\Theta_{p,\kappa}}, \tag{2.5}$$

and we also abbreviate the norms correspondingly,

$$\|\cdot\|_{p,\kappa} := \|\cdot\|_{V_{p,\kappa}}, \quad \text{and} \quad \|\cdot\|_{1,p,\kappa} := \|\cdot\|_{\Theta_{p,\kappa}}. \tag{2.6}$$

All these norms are, of course, well defined for any function on X with values in $[-\infty,\infty]$. And therefore we shall apply them below not just for functions in $WC_{p,\kappa}$ or $W_1C_{p,\kappa}$. For $p' \geqslant p$ and $\kappa' \geqslant \kappa$ by restriction $WC_{p,\kappa}$ is continuously and densely embedded into $WC_{p',\kappa'}$ and into $W_1C_{p,\kappa}$ (see Korollary 5.6 in [21]), as well is the latter into $W_1C_{p',\kappa'}$. V_κ and $V_{p,\kappa}$ will serve as convenient Lyapunov functions for L and Θ_κ, $\Theta_{p,\kappa}$ naturally appear as parts of functions bounding LV_κ, $LV_{p,\kappa}$, respectively (cf. condition (F2) and Example 2.1 below, as well as Lemma 4.6 in [21]). Note that the level sets of $\Theta_{p,\kappa}$ are even strongly compact in X.

For a function $V \colon X \to (1,\infty]$, we also define spaces $Lip_{l,p,\kappa}$, $p \geqslant 2$, $\kappa > 0$, consisting of functions on X which are locally Lipschitz continuous in the norm

$|(-\Delta)^{-l/2}\cdot|_2$, $l\in\mathbb{Z}_+$. The respective semi-norms are defined as follows:

$$(2.7)\quad (f)_{l,p,\kappa} := \sup_{y_1,y_2\in X_p} \big(V_{p,\kappa}(y_1)\vee V_{p,\kappa}(y_2)\big)^{-1} \frac{|f(y_1)-f(y_2)|}{|(-\Delta)^{-l/2}(y_1-y_2)|_2} \quad (\in\ [0,\infty]).$$

For $l\in\mathbb{Z}_+$ we define

$$(2.8)\qquad Lip_{l,p,\kappa} := \big\{f : X_p\to\mathbb{R}\ \big|\ \|f\|_{Lip_{l,p,\kappa}}<\infty\big\},$$

where $\|f\|_{Lip_{l,p,\kappa}} := \|f\|_{p,\kappa} + (f)_{l,p,\kappa}$. When X is of finite dimension, $(f)_{l,p,\kappa}$ is a weighted norm of the generalised gradient of f (cf. Lemma 3.6 in [21]). Also, $(Lip_{l,p,\kappa}, \|\cdot\|_{Lip_{l,p,\kappa}})$ is a Banach space (cf. Lemma 5.7 in [21]) and $Lip_{l,p,\kappa}\subset Lip_{l',p',\kappa'}$ if $l'\leqslant l$, $p'\geqslant p$, $\kappa'\geqslant\kappa$. In this paper we shall only deal with the case $l=0$.

Obviously, each $f\in Lip_{0,p,\kappa}$ is uniformly $|\cdot|_2$–Lipschitz continuous on every $|\cdot|_p$-bounded set and by restriction for $p'\in[p,\infty)$, $\kappa'\in[\kappa,\infty)$

$$(2.9)\qquad \mathcal{B}_b(X_p)\cap Lip_{0,p,\kappa}\subset W_1C_{p',\kappa'}.$$

Besides the space $\mathcal{D} := \mathcal{F}C_b^2$ defined in the introduction, other test function spaces $\mathcal{D}_{p,\kappa}$ on X will turn out to be convenient. They are for $\kappa\in(0,\infty)$ defined as follows:

$$(2.10)\quad \mathcal{D}_\kappa := \big\{u = g\circ P_N\ \big|\ N\in\mathbb{N},\ g\in C^2(\mathbb{R}^N),\\ \|u\|_{2,\kappa} + \big\||Du|_2\big\|_{2,\kappa} + \big\|Tr(AD^2u)\big\|_{2,\kappa}<\infty\big\}.$$

Obviously, $\mathcal{D}_\kappa\subset WC_{2,\kappa}$ and $\mathcal{D}_\kappa\subset\mathcal{D}_{\kappa'}$ if $\kappa'\in[\kappa,\infty)$. We extend the definition (1.2) of the Kolmogorov operator L for all $u\in\mathcal{F}C^2 := \{u = g\circ P_N\,|\,N\in\mathbb{N}, g\in C^2(\mathbb{R}^{\mathbb{N}})\}$. So, L can be considered with domain $\mathcal{D}_\kappa$.

Now let us collect our precise hypotheses on the Kolmogorov operator (1.2). First we recall that in the entire paper $\Delta = x''$ is the Dirichlet Laplacian on $(0,1)$. Consider the following condition on the map $A : X\to X$:

(A) A is a nonnegative symmetric linear operator from X to X of trace class such that $A_N := P_NAP_N$ is an invertible operator represented by a diagonal matix on E_N for all $N\in\mathbb{N}$.

Here E_N, P_N are as defined in the introduction. Furthermore, we set

$$(2.11)\qquad a_0 := \sup_{x\in H_0^1\setminus\{0\}} \frac{(x,Ax)}{|x'|_2^2} = |A|_{H_0^1\to H^{-1}},$$

where $|\cdot|_{H_0^1\to H^{-1}}$ denotes the usual operator norm on bounded linear operators from H_0^1 into its dual H^{-1}.

Consider the following condition for a map $F : H_0^1\to X$.

(F2) For every $k\in\mathbb{N}$ the map $F^{(k)} := (F,\eta_k) : H_0^1\to\mathbb{R}$ is $|\cdot|_2$-continuous on $|\cdot|_{1,2}$-balls and there exists a sequence $F_N : E_N\to E_N$, $N\in\mathbb{N}$, of bounded locally Lipschitz continuous vector fields satisfying the following conditions:

(F2a) There exist $\kappa_0\in(0,\frac{1}{4a_0}]$ and a set $Q_{\rm reg}\subset[2,\infty)$ such that $2\in Q_{\rm reg}$ and for all $\kappa\in(0,\kappa_0)$, $q\in Q_{\rm reg}$ there exist $m_{q,\kappa}>0$ and $\lambda_{q,\kappa}\in\mathbb{R}$ such that for

$$(2.12)\quad L_Nu(x) := \frac{1}{2}\,\mathrm{Tr}\big(A_ND^2u\big)(x) + \big(x''+F_N(x), Du(x)\big),\\ u\in W^{2,1}_{loc}(E_N), x\in E_N, N\in\mathbb{N},$$

we have for all $N\in\mathbb{N}$

$$(2.13)\qquad L_NV_{q,\kappa} := L_N\big(V_{q,\kappa}\restriction_{E_N}\big)\leqslant\lambda_{q,\kappa}V_{q,\kappa} - m_{q,\kappa}\Theta_{q,\kappa}\quad\text{on } E_N.$$

(F2b) For all $\varepsilon \in (0,1)$ there exists $C_\varepsilon \in (0,\infty)$ such that for all $N \in \mathbb{N}$ and dx-a.e. $x \in E_N$ (where dx denotes Lebesgue measure on E_N)

$$\big(DF_N(x)y, y\big) \leqslant |y'|_2^2 + \big(\varepsilon|x'|_2^2 + C_\varepsilon\big)|y|_2^2 \quad \forall\, y \in E_N.$$

(F2c) $\lim_{N\to\infty} \big|P_N F - F_N \circ P_N\big|_2(x) = 0 \quad \forall\ x \in H_0^1$, and $\lim_{N\to\infty} \big|(\eta_k, F_N) - F^{(k)}\big| = 0$ uniformly on H_0^1-balls for all $k \in \mathbb{N}$.

(F2d) For κ_0 and Q_{reg} as in (F2a), there exist $\kappa \in (0,\kappa_0)$, $p \in Q_{\mathrm{reg}}$ such that for some $C_{p,\kappa} > 0$ and some $\omega : [0,\infty) \to [0,1]$ vanishing at infinity

$$\big|F_N \circ P_N\big|_2(x) \leqslant C_{p,\kappa}\Theta_{p,\kappa}(x)\omega\big(\Theta_{p,\kappa}(x)\big) \quad \forall\, x \in H_0^1,\ N \in \mathbb{N}.$$

We note that the second part of (F2c) was not assumed to hold in [21], except for the last part of its Appendix.

Although condition (F2) looks abstract and at first sight artificial, it is exactly what is needed for our analysis and what is satisfied in many situations as the following example shows.

Example 2.1. Consider the following condititons on the map $F : H_0^1 \to X$:

(F1)

$$F(x) = \frac{d}{dr}\big(\Psi \circ x\big)(r) + \Phi\big(r, x(r)\big), \quad x \in H_0^1(0,1),\ r \in (0,1), \tag{2.14}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$, $\Phi : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(Ψ) $\Psi \in C^{1,1}(\mathbb{R})$ (i.e. Ψ is differentiable with locally Lipschitz derivative) and there exist $C \in [0,\infty)$ and a bounded, Borel-measurable function $\omega : [0,\infty) \to [0,\infty)$ vanishing at infinity such that

$$|\Psi_{xx}|(x) \le C + \sqrt{|x|}\ \omega(|x|) \quad \text{for } dx\text{-a.e. } x \in \mathbb{R}.$$

(Φ1) Φ is Borel-measurable in the first and continuous in the second variable and there exist $g \in L^{q_1}(0,1)$ with $q_1 \in [2,\infty]$ and $q_2 \in [1,\infty)$ such that

$$|\Phi(r,x)| \le g(r)(1 + |x|^{q_2}) \quad \text{for all } r \in (0,1),\ x \in \mathbb{R}.$$

(Φ2) There exist $h_0, h_1 \in L_+^1(0,1)$, $|h_1|_1 < 2$, such that for a.e. $r \in (0,1)$

$$\Phi(r,x)\operatorname{sign} x \le h_0(r) + h_1(r)|x| \quad \text{for all } x \in \mathbb{R}.$$

(Φ3) There exist $\rho_0 \in (0,1]$, $g_0 \in L_+^1(0,1)$, $g_1 \in L_+^{p_1}(0,1)$ for some $p_1 \in [2,\infty]$, and a function $\omega : [0,\infty) \to [0,\infty)$ as in (Ψ) such that with $\sigma : (0,1) \times \mathbb{R} \to \mathbb{R}$, $\sigma(r,x) := \frac{|x|}{\sqrt{r(1-r)}}$ for a.e. $r \in (0,1)$

$$\Phi(r,y) - \Phi(r,x) \le \Big[g_0(r) + g_1(r)\big|\sigma(r,x)\big|^{2-\frac{1}{p_1}}\omega\big(\sigma(r,x)\big)\Big](y-x)$$

for all $x, y \in \mathbb{R}$, $0 \leqslant y - x \leqslant \rho_0$.

Then we have that (F1) implies (F2). More precisely, (F2a) holds with $\kappa_0 := \frac{2-|h_1|_1}{8a_0}$, $Q_{\mathrm{reg}} := [2,\infty)$ and (F2d) holds with $p \in [2,\infty) \cap \big(q_2 - 3 + \frac{2}{q_1}, \infty\big)$ and any $\kappa \in (0,\kappa_0)$. The proof of this fact is extremely involved. All details can be found in Section 4 in [21]. It is obviously very easy to find plenty of examples for functions Ψ, Φ as in (F1). For instance, one can take $\Psi(x) := \alpha x^2$, $\alpha \in [0,\infty)$, and $\Phi(r,x) := -\beta x^m$, $\beta \in [0,\infty)$, $m \in \mathbb{N}$, m odd. If $\alpha := \frac{1}{2}$, $\beta := 0$, SPDE (1.1) is just the classical stochastic Burgers equation, and if $\alpha := 0$, $\beta > 0$, it is just a classical stochastic reaction diffusion equation of Ginsburg–Landau type.

3. Main background results

In this section we list a number of results from [21], which we shall use below, but in appropriately shortened form. For the complete formulations and detailed proofs we refer to [21].

Theorem 3.1. *("Pointwise solutions of the Kolmogorov equations"). Suppose (A) and (F2) hold. Let κ_0, Q_{reg} be as in (F2a), $\kappa \in (0, \kappa_0)$ and $p \in Q_{reg}$ be as in (F2d). Let $\kappa^* \in (\kappa, \kappa_0)$, $\kappa_1 \in (0, \kappa^* - \kappa]$. Then there exists a semigroup $(p_t)_{t>0}$ of probability kernels on X_p, independent of κ^*, having the following properties:*

(i) ("Existence") *Let $u \in \mathcal{D}_{\kappa_1}$. Then $t \mapsto p_t(|Lu|)(x)$ is locally Lebesgue integrable on $[0, \infty)$ and*

$$p_t u(x) - u(x) = \int_0^t p_s(Lu)(x)\, ds \quad \text{for all } x \in X_p, \tag{3.1}$$

where as usual

$$p_t u(x) := \int u(y)\ p_t(x, dy)\ . \tag{3.2}$$

In particular, for all $s \in [0, \infty)$

$$\lim_{t \to 0} p_{s+t} u(x) = p_s u(x) \quad \text{for all } x \in X_p.$$

(ii) There exists $\lambda_{\kappa^*} \in (0, \infty)$ such that

$$\int_0^\infty e^{-\lambda_{\kappa^*} s} p_s(\Theta_{p,\kappa^*})(x)\, ds < \infty \quad \text{for all } x \in X_p. \tag{3.3}$$

(iii) ("Uniqueness") *Let $(q_t)_{t>0}$ be a semigroup of probability kernels on X_p satisfying (i) with $(p_t)_{t>0}$ replaced by $(q_t)_{t>0}$. If in addition,* (3.3) *holds with $(q_t)_{t>0}$ replacing $(p_t)_{t>0}$ for some $\kappa \in (0, \kappa_0)$ replacing κ^*, then $p_t(x, dy) = q_t(x, dy)$ for all $t > 0$, $x \in X_p$.*

Proof. See Proposition 6.7 in [21]. □

Theorem 3.2. *("Invariant measure"). Assume that (A) and (F2) hold. Let p, κ^* be as in Theorem 3.1.*

(i) There exists a probability measure μ on H_0^1 which is "L-infinitesmally invariant", i.e. $Lu \in L^1(H_0^1, \mu)$ and

$$\int Lu\, d\mu = 0 \quad \text{for all } u \in \mathcal{D} \tag{3.4}$$

($L^*\mu = 0$ for short). Furthermore,

$$\int \Theta_{p,\kappa^*}\, d\mu < \infty. \tag{3.5}$$

(ii) μ, extended by zero to all of X_p, is $(p_t)_{t>0}$-invariant, i.e. for all $f : X \to \mathbb{R}$, bounded, measurable, and all $t > 0$,

$$\int p_t f\ d\mu = \int f\ d\mu.$$

(with $(p_t)_{t>0}$ from Theorem 3.1).

Proof. See Appendix in [21]. The method is taken from [4] which we also refer to as a general reference for infinitesimally invariant measures. □

Now for $(p_t)_{t>0}$ as above consider the corresponding resolvent of kernels $g_\lambda(x,dy)$ defined for $\lambda \in (0,\infty)$, $x \in X_p$ as follows: for $f \in \mathcal{B}_b(X_p)$

$$\begin{aligned} g_\lambda f(x) &:= \int f(y)\ g_\lambda(x,dy) \\ &:= \int_0^\infty e^{-\lambda t}\ p_t f(x) dt\ . \end{aligned} \tag{3.6}$$

We extend the measures $g_\lambda(x,dy)$, $x \in X_p$, $\lambda \in (0,\infty)$, by zero to all of X.

Proposition 3.3. *Suppose (A), (F2) hold and let p, κ^*, κ_1 be as in Theorem 3.1. Then there exists $\lambda_* \in (0,\infty)$ such that for $\lambda > \lambda_*$*

$$g_\lambda\big((\lambda - L)u\big) = u \text{ for all } u \in \mathcal{D}_{\kappa_1}\ .$$

Furthermore, there exists $m_{p,\kappa^*} \in (0,\infty)$ such that

$$g_\lambda \Theta_{p,\kappa^*} \le \frac{1}{m_{p,\kappa^*}}\ V_{p,\kappa^*}\ . \tag{3.7}$$

In particular, $g_\lambda(x, X \setminus H_0^1) = 0$ for all $x \in X_p$.

Proof. Theorem 6.4 and Proposition 6.7(i) in [21]. □

Proposition 3.4. *Suppose (A), (F2) hold and let p, κ^*, κ_1 be as in Theorem 3.1. Let $N \in \mathbb{N}$. For L_N as in (2.12) let $(\mathcal{R}_\lambda^{(N)})_{\lambda>0}$ be the corresponding Markovian pseudo–resolvent on $L^\infty(E_N)$ (which exists according to Proposition 3.1 in* [21]*. For $f \in \mathcal{B}_b(X)$ define*

$$G_\lambda^{(N)} f := \big(\mathcal{R}_\lambda^{(N)}(f\restriction_{E_N})\big) \circ P_N\ .$$

Then there exists $\lambda_* \in (0,\infty)$ such that for all $f \in Lip_{0,2,\kappa_1} \cap \mathcal{B}_b(X_p)$ the following assertions hold

(i) $\lambda g_\lambda f = \lim_{m\to\infty} \lambda G_\lambda^{(m)} f$ weakly in $WC_{p,\kappa*}$ (hence pointwise on X_p) uniformly in $\lambda \in [\lambda_*,\infty)$.

(ii) For all $\lambda > \lambda_*$, $G_\lambda^{(N)} f \in \cap_{\varepsilon>0} \mathcal{D}_{\kappa_1+\varepsilon}$ and, provided $f \in \mathcal{D}$, $G_\lambda^{(N)} f \in \bigcap_{\varepsilon>0} \mathcal{D}_\varepsilon$.

(iii) For all $\lambda > \lambda_*$ and all $x \in H_0^1$

$$\begin{aligned} &\big|(\lambda - L) G_\lambda^{(N)} f(x) - \big(f \circ P_N\big)(x)\big| \\ &\qquad \leqslant \frac{1}{\lambda - \lambda_*} \big|P_N F - F_N \circ P_N\big|_2(x)\, V_{2,\kappa_1}(x)\, (f)_{0,2,\kappa_1} \end{aligned}$$

and

$$\lim_{N\to\infty} \sup_{\lambda \ge \lambda_*} \lambda \big|(\lambda - L) G_\lambda^{(N)} f - f\big|(x) = 0\ .$$

Proof. See Theorem 6.4 and Corollary 4.2, in particular, (4.5), in [21]. □

Corollary 3.5. *Consider the situation of Proposition 3.4 and let $f \in Lip_{0,2,\kappa_1} \cap \mathcal{B}_b(X_p)$. Let $x \in \bigcup_{n\in\mathbb{N}} P_N X$. Then*

$$\lim_{\lambda\to\infty} \lambda g_\lambda f(x) = f(x)\ .$$

Proof. We have $x = P_N x$ for some $N \in \mathbb{N}$. Hence for all $m \ge N$, $x = P_m x$, and

$$\begin{aligned} &\big|\lambda g_\lambda f(x) - f(x)\big| \\ &\qquad \le \big|\lambda g_\lambda f(x) - \lambda G_\lambda^{(m)} f(x)\big| + \big|\lambda \mathcal{R}_\lambda^{(m)}(f\restriction_{E_m})(P_m x) - f(P_m x)\big|. \end{aligned} \tag{3.8}$$

By Proposition 3.4(i) the first term in the right hand side of (3.8) converges to zero as $m \to \infty$ uniformly in $\lambda \in [\lambda_*,\infty)$. But the second term converges to zero as $\lambda \to \infty$ by Proposition 3.4(i) in [21] for all $m \ge N$. Hence the assertion follows. □

4. A class of natural reference measures

In all of this section we assume (A) and (F2) to hold. Let p, κ be as in (F2d) and let $\kappa^* \in (\kappa, \infty)$. Let $\mathfrak{M}$ denote the set of all probability measures ν on X such that

$$\int \Theta_{p,\kappa^*}\, d\nu < \infty \ , \tag{4.1}$$

and ν is *"infinitesimally L–excessive"*, i.e. for some $\lambda_\nu \in (0, \infty)$

$$\int Lu\, d\nu \le \lambda_\nu \int u\, d\nu \quad \text{for } u \in \mathcal{D} \text{ such that } u \geqslant 0\ \nu\text{-a.e.} \tag{4.2}$$

Note that by (4.1) we have that $\nu(H_0^1) = 1$ for every $\nu \in \mathfrak{M}$ and by (F2d) that $Lu \in L^1(X, \mu)$ for all $u \in \mathcal{D}$. Obviously, an L-infinitesimally invariant measure satisfying (4.1) is in $\mathfrak{M}$. We also emphasize that for $\nu \in \mathfrak{M}$ there might exist a non–empty open set $U \subset X$ such that $\nu(U) = 0$, so $u \in \mathcal{D}$ with $u = 0$ ν–a.e. might not be identically equal to zero. So, the following proposition is crucial to consider L as an operator on $L^1(X, \nu)$.

Proposition 4.1. *Assume that (A), (F2) hold with p, κ as in (F2d) and let $\nu \in \mathfrak{M}$. Then*

$$u \in \mathcal{D}, u = 0\ \nu\text{–a.e.} \quad \Rightarrow \quad Lu = 0\ \nu\text{–a.e.} \tag{4.3}$$

Proof. First we note that for $u, v \in \mathcal{D}$

$$L(uv) = uLv + vLu + 2\big(A^{1/2}Du, A^{1/2}Dv\big) \ . \tag{4.4}$$

So, if $u = 0$ ν–a.e., then by (4.2) and (4.4) with $v := u$ it follows that

$$0 \leqslant 2 \int \big|A^{1/2}Du\big|_2^2\, d\nu = \int Lu^2\, d\nu \leqslant \lambda_\nu \int u^2\, d\nu = 0 \ ,$$

so $A^{1/2}Du = 0$ ν–a.e. Applying (4.4) again with $v \in \mathcal{D}$ arbitrary we find, since $u \cdot (\pm 1) \cdot v = 0$ ν–a.e., that

$$\int (\pm 1) \cdot v\, Lu\, d\nu = \int L(uv)\, d\nu \leqslant \lambda_\nu \int uv\, d\nu = 0,$$

so,

$$\int v\, Lu\, d\nu = 0 \quad \text{for all } v \in \mathcal{D}.$$

Since $\mathcal{D}$ is closed under multiplication and generates $\mathcal{B}(X)$, by a monotone class argument it follows that $Lu = 0$ ν–a.e. □

Let $\nu \in \mathfrak{M}$. We define $\mathcal{D}^\nu$ to be the set of all ν–equivalence classes determined by $\mathcal{D}$. $\mathcal{D}^\nu_\kappa$ for $\kappa \in (0, \infty)$ is defined correspondigly. By Proposition 4.1 $\big(L, \mathcal{D}^\nu_{(\kappa_1)}\big)$ is a well-defined operator on $L^1(X, \nu)$. Our aim in this section is to prove that $(L, \mathcal{D}^\nu)$ is essentially maximal dissipative on $L^1(X, \nu)$.

Lemma 4.2. *Assume that (A) and (F2) hold with p, κ as in (F2d). Let $\kappa_1 \in (0, \kappa^* - \kappa]$ and $\nu \in \mathfrak{M}$. Then:*

- *(i) $Lu \in L^1(X, \nu)$ for all $u \in \mathcal{D}_{\kappa_1}$.*
- *(ii) $(L - \lambda_\nu, \mathcal{D}^\nu)$ is dissipative, hence $(L, \mathcal{D}^\nu)$ is closable on $L^1(V, \nu)$.*
- *(iii) For the closure $(\overline{L}_1^\nu, \mathcal{D}(\overline{L}_1^\nu))$ it follows that $\mathcal{D}^\nu_{\kappa_1} \subset D(\overline{L}_1^\nu)$ and $\overline{L}_1^\nu u$ is given by formula (1.2). In particular,* (4.2) *holds for all $u \in \mathcal{D}_{\kappa_1}$.*

Proof.

(i): Since $\kappa^* > \kappa$, the assertion follows immediately by (4.1).

(ii): (L, D^ν) is a diffusion operator in the sense of Appendix B in [13]. So, the assertion follows by Appendix B, Lemma 1.8 in [13].

(iii): Let $u \in \mathcal{D}_{\kappa_1}$ and $N \in \mathbb{N}$ be such that $u = u \circ P_N$. Choose $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi' \le 0$, $0 \le \varphi \le 1$, $\varphi = 1$ on $[0,1]$ and $\varphi = 0$ on $(2,\infty)$. For $n \in \mathbb{N}$ let $\varphi_n(x) := \varphi\big(\frac{|P_N x|_2^2}{n^2}\big)$, $x \in X$, $u_n := \varphi_n u$. Then $u_n \in \mathcal{D}$ and

$$Lu_n = \varphi_n Lu + uL\varphi_n + 2(Du, A_N D\varphi_n).$$

Note that for $i, j = 1, \dots, N$ there are $c_j, c_{ij} \in (0,\infty)$ such that

$$|\partial_j \varphi_n| \le \frac{c_j}{n} 1_{\{|P_N x|_2 < 2n\}},\ |\partial^2_{ij}\varphi_n| \le \frac{c_{ij}}{n^2} 1_{\{|P_N x|_2 < 2n\}}.$$

Then $0 \le \varphi_n \uparrow 1$ as $n \to \infty$, $|A_N D\varphi_n| \le \frac{\max c_j}{n}$, and $|L\varphi_n(x)| \le \frac{c}{n}\big(|x'|_2 + |P_N F|_2\big) \le \frac{2c}{n}\Theta_{p,\kappa^*}(x)$ for all $x \in H_0^1$ and some $c \in (0,\infty)$ independent of x and n by (F2c) and (F2d). So $u_n \to u$ and $Lu_n \to Lu$ pointwise on H_0^1, up to a constant uniformly bounded by Θ_{p,κ^*}. Hence the assertion follows by (4.1) and Lebesgue's dominated convergence theorem.

□

Theorem 4.3. *Assume that (A) and (F2) hold with p, κ as in (F2d) and let λ_* be as in Proposition 3.3. Let $\nu \in \mathfrak{M}$. Then:*

(i) Let $\lambda > \lambda_*$. Then $(\lambda - L)(\mathcal{D}^\nu)$ is dense in $L^1(X,\nu)$. In particular, the closure $(\overline{L}_1^\nu, \mathcal{D}(\overline{L}_1^\nu))$ of $(L, \mathcal{D}^\nu)$ generates a C_0–(i.e. strongly continuous) semigroup $(e^{t\overline{L}_1^\nu})_{t>0}$ on $L^1(X,\nu)$ and $(\overline{L}_1^\nu, \mathcal{D}(\overline{L}_1^\nu))$ is the only closed extension of $(L, \mathcal{D}^\nu)$ with this property.

(ii) $(e^{t\overline{L}_1^\nu})_{t>0}$ is Markov.

(iii) Let $(p_t)_{t>0}$ be as in Theorem 3.1. Then for all $t > 0$, $f \in \mathcal{B}^+(X)$

$$\int p_t f\, d\nu \le e^{\lambda_\nu t} \int f\, d\nu \tag{4.5}$$

and $p_t f$ is a ν–version of $e^{t\overline{L}_1^\nu} f$ for all $t > 0$, $f \in \mathcal{B}_b(X)$.

Proof. (i) Let $u \in \mathcal{D}$. By Proposition 3.4(ii) we know that $G_\lambda^{(N)} u \in \mathcal{D}_{\kappa_1}$ for all $N \in \mathbb{N}$. Furthermore, Proposition 3.4(iii), (F2d), and (4.1) imply that $(\lambda - L)G_\lambda^{(N)} f \to f$ as $N \to \infty$ in $L^1(X,\nu)$ by Lebesgue's dominated convergence theorem. Since $\mathcal{D}$ is dense in $L^1(X,\nu)$, the first part of the assertion follows by Lemma 4.2(ii). The rest is then a consequence of the classical Lumer–Phillips theorem.

(ii) The assertion follows by (i) and Lemma 1.9 in Appendix B of [13].

(iii) Let $t > 0$. To prove (4.5), since $\mathcal{D}$ is dense in $L^1(X,\nu)$, we may assume that $f \in \mathcal{D}$, $f \ge 0$. Then for large enough $\lambda > 0$ and all $N \in \mathbb{N}$ by Proposition 3.4(ii) we have $G_\lambda^{(N)} f \in \mathcal{D}_{\kappa_1}$ and hence by Proposition 3.3 it follows that

$$\begin{aligned}
&\int \lambda g_{\lambda+\lambda_\nu}\big((\lambda + \lambda_\nu - L)G_\lambda^{(N)} f\big)\, d\nu \\
&= \lambda \int G_\lambda^{(N)} f\, d\nu \\
&\le \int (\lambda + \lambda_\nu - L) G_\lambda^{(N)} f\, d\nu,
\end{aligned}$$

where the last step follows by the last part of Lemma 4.2(iii). Letting $N \to \infty$ we conclude by Proposition 3.4(iii), (F2d), (3.7), (4.1), and Lebesgue's dominated convergence theorem that

$$\int \lambda g_{\lambda+\lambda_\nu} f \, d\nu \leq \int f \, d\nu \ .$$

Hence by (3.6) and Fubini's theorem

$$\int_0^\infty e^{-\lambda t} \int e^{-\lambda_\nu t} p_t f \, d\nu \, dt \leq \int_0^\infty e^{-\lambda t} \int f \, d\nu \, dt$$

for all $\lambda > \lambda_0$ with $\lambda_0 \in (0, \infty)$ sufficiently large. So, for $g(t) := \int f \, d\nu - \int e^{-\lambda_\nu t} p_t \, f \, d\nu$ we have that

$$h(\lambda) := \int_0^\infty e^{-\lambda t} g(t) \, dt \geq 0 \text{ for all } \lambda > \lambda_0 \ .$$

Since h is obviously completely monotone, by the Hausdorff–Bernstein–Widder theorem (see e.g. Theorem 3.1 in the Appendix of [17]) it follows that $g \geq 0$ dt-a.e., hence $g \geq 0$ every where by right–continuity (cf. Theorem 3.1(i) above). Hence (4.5) is proved. In particular, p_t extends to a bounded linear operator on $L^1(X, \nu)$ with operator norm less than $e^{\lambda \nu t}$ for all $t > 0$, which we denote by T_t. Clearly, $(T_t)_{t>0}$ is a semigroup of operators and (by the last part of Theorem 3.1(i)):

$$\lim_{t \to \infty} T_t f = f \quad \text{in } L^1(X, \nu)$$

if $f \in \mathcal{D}$. But then a 2ε-argument proves that $(T_t)_{t>0}$ is strongly continuous on $L^1(X, \nu)$. Let $\tilde{L}$ be the corresponding generator. We have to show that

$$T_t = e^{t \bar{L}_1^\nu} \quad \text{tor all } t > 0. \tag{4.6}$$

Let us consider the corresponding resolvents

$$G_\lambda^{(1)} f := \int e^{-\lambda t} T_t f \, dt,$$

and

$$G_\lambda^{(2)} f := \int e^{-\lambda t} e^{t \bar{L}_1^\nu} f \, dt = \left(\lambda - \bar{L}_1^\nu\right)^{-1} f$$

for $\lambda > \lambda_\nu$, $f \in L^1(X, \nu)$. It is easy to check (again using the uniqueness of the Laplace transform) that $g_\lambda f$ is a ν-version of $G_\lambda^{(1)} f$ for all $\lambda > \lambda_\nu$, $f \in L^1(X, \nu)$. Hence for all $u \in \mathcal{D}^\nu$ and λ large enough we have by Proposition 3.3

$$G_\lambda^{(1)}\big((\lambda - L)u\big) = u = G_\lambda^{(2)}\big((\lambda - L)u\big).$$

So, by continuity and since $(\lambda - L)(\mathcal{D}^\nu)$ is dense in $L^1(X, \nu)$ by assertion (i), it follows that $G_\lambda^{(1)} = G_\lambda^{(2)}$ for large enough λ. Hence (4.6) follows. □

Remark 4.4. Consider the situation of Theorem 4.3. By definition for $t \geqslant 0$ and $f \in D(\bar{L}_1^\nu)$ we have

$$\frac{d}{dt} e^{t \bar{L}_1^\nu} f = \bar{L}_1^\nu e^{t \bar{L}_1^\nu} f = e^{t \bar{L}_1^\nu} \bar{L}_1^\nu f, \tag{4.7}$$

where $\frac{d}{dt}$ is taken in the norm in $L^1(X, \nu)$. Since $p_t f$ is a ν-version of $e^{t \bar{L}_1^\nu} f$ by Theorem 4.3(iii), we see that (4.7) is an infinitesimal version of (3.1), however, only in an $L^1(X, \nu)$-sense, in particular, valid only outside a ν-zero-set of points in X.

5. A suitable reference measure

Let (A), (F2) hold with p, κ as in (F2d). In this section we shall construct a measure ν_0 in $\mathfrak{M}$ such that $\nu_0(U) > 0$ for every non-empty open set $U \subset X$.

Let $\{x_n \,|\, n \in \mathbb{N}\} \subset \bigcup_{N\in\mathbb{N}} P_N X$ be a dense subset of X. Let $\lambda_0 \in (0,\infty)$ bigger than the λ_* in Propositions 3.3 and 3.4. Set

$$\beta_n := \lambda_0\big(2^n + \lambda_0 g_{\lambda_0}\Theta_{p,\kappa^*}(x_n)\big)^{-1}, \quad n \in \mathbb{N},$$

and define a measure ν_0 on X by

$$\nu_0(B) := Z^{-1}\sum_{n=1}^{\infty}\beta_n g_{\lambda_0}(x_n, B), \quad B \in \mathcal{B}(X), \tag{5.1}$$

where $Z \in (0,\infty)$ is chosen so that ν_0 is a probability measure.

Theorem 5.1. *Let (A), (F2) hold with p, κ as in (F2d) and let ν_0 be defined as in* (5.1)*. Then $\nu_0 \in \mathfrak{M}$ with $\lambda_{\nu_0} = \lambda_0$ and $\nu_0(U) > 0$ for every non-empty open set $U \subset X$.*

Proof. Let $u \in \mathcal{D}$, $u \geqslant 0$. Then by Proposition 3.3

$$\begin{aligned}\int(\lambda_0 - L)u\, d\nu_0 &= Z^{-1}\sum_{n=1}^{\infty}\beta_n g_{\lambda_0}\big((\lambda_0 - L)u\big)(x_n)\\ &= Z^{-1}\sum_{n=1}^{\infty}\beta_n u(x_n) \quad \geqslant \quad 0,\end{aligned}$$

and by construction

$$\int \Theta_{p,\kappa^*}\, d\nu_0 < \infty.$$

So, both (4.1) and (4.2) are satisfied, hence $\nu_0 \in \mathfrak{M}$. To show the last part of the assertion, let $\emptyset \neq U \subset X$, U open such that $\nu_0(U) = 0$. We may assume that U is a ball in X of radius $r > 0$ and centre y_0. Define

$$f(x) := \frac{\mathrm{dist}(x, U^c)}{r}$$

with $\mathrm{dist}(x, U^c) := \inf\{|x - y|_2 \,|\, y \in U^c\}$. Then f is a bounded Lipschitz function, hence $f \in Lip_{0,2,\kappa_1}$, $0 \leqslant f \leqslant 1$, and $f(y_0) = 1$. Then

$$0 = \nu_0(U) \geqslant \int f\, d\nu_0 \geqslant 0,$$

so for all $n \in \mathbb{N}$, $\lambda \geqslant \lambda_0$,

$$0 = g_{\lambda_0} f(x_n) \geqslant g_\lambda f(x_n) \geqslant 0.$$

Therefore, by Corollary 3.5

$$0 = \lim_{\lambda\to\infty}\lambda g_\lambda f(x_n) = f(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $\{\kappa_n \,|\, n \in \mathbb{N}\}$ is dense in X, it follows that $f = 0$ on X, in contradiction to $f(y_0) = 1$. Hence such a ball U does not exist. □

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