Stochastic integrals and stochastic differential equations with respect to compensated Poisson random measures in infinite dimensional Hilbert spaces

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Introduction

The purpose of this paper is to give a complete proof of the existence of a mild solution of a stochastic differential equation with respect to a compensated Poisson random measure by a fixpoint argument in the spirit of [DaPrZa 96]. This will be done within the following framework.

Let \((H, \langle \cdot, \cdot \rangle)\) be an infinite dimensional, separable Hilbert space, \((U, \mathcal{B}, \nu)\) a \(\sigma\)-finite measure space and \((\Omega, \mathcal{F}, P)\) a complete probability space with filtration \(\mathcal{F}_t, t \geq 0\) such that \(\mathcal{F}_0\) contains all \(P\)-nullset of \(\mathcal{F}\). Consider the following stochastic differential equation in \(H\) on the intervall \([0, T], T > 0:\)

\[
\begin{align*}
\text{(1)} \quad dX(t) &= [AX(t) + F(X(t))] \, dt + B(X(t), y) \, q(dt, dy) \\
X(0) &= \xi
\end{align*}
\]

where

- \(A : D(A) \subset H \rightarrow H\) is the infinitesimal generator of a \(C_0\)-semigroup \(S(t), t \geq 0,\) of linear, bounded operators on \(H,\)

- \(F : H \rightarrow H\) is \(\mathcal{B}(H)/\mathcal{B}(H)\)-measurable,

- \(B : H \times U \rightarrow H\) is \(\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)\)-measurable,

- \(q(dt, dy) := \Pi(dt, dy) - \lambda(dt) \otimes \nu(dy),\) is a compensated Poisson random measure on \((\{0, \infty\} \times U, \mathcal{B}(\{0, \infty\}) \otimes \mathcal{B})\) where \(\Pi\) is a Poisson random measure on \((\{0, \infty\} \times U, \mathcal{B}(\{0, \infty\}) \otimes \mathcal{B})\) with intensity measure \(\lambda(ds) \otimes \nu(dy),\)

- \(\xi\) is an \(H\)-valued, \(\mathcal{F}_0\)-measurable random variable.

A mild solution of equation (1) is an \(H\)-valued predictable process such that
\[ X(t) = S(t) \xi + \int_0^t S(t-s) F(X(s)) \, ds \]
\[ + \int_0^{t^+} \int_U S(t-s) B(X(s), y) \, q(ds, dy) \quad P\text{-a.s.} \]

for all \( t \in [0, T] \).

The organization of this paper is as follows. In Chapter 1 we present the definition of that type of stochastic integral with respect to a compensated Poisson random measure which we use in this paper. For this end, in Section 1 and 2 we first repeat the notions of Poisson random measures and Poisson point processes where we refer to the book [IkWa 81].

In Section 3, the construction of the stochastic integral of Hilbert space valued predictable processes with respect to a compensated Poisson random measure with intensity measure \( \lambda(ds) \otimes \nu(dy) \) will be done by an isometric formula in the style of the definition of the stochastic integral with respect to the Wiener process in [DaPrZa 92] or square integrable martingales in [Me 82]. For real valued processes this can be found in [BeLi 82]. Independently, this definition was done in [Rue 2003].

Denote by \( E \) the space of elementary processes where an \( H \)-valued process \( \Phi(t) : \Omega \times U \rightarrow H, t \in [0, T] \), on \( (\Omega \times U, \mathcal{F} \otimes \mathcal{B}, P \otimes \nu) \) is said to be elementary if there exist \( 0 = t_0 < t_1 < \ldots < t_k = T \) and for \( m \in \{0, \ldots, k-1\} \) exist \( B_{t_0}^m, \ldots, B_{t_{m+1}}^m \in \Gamma_p, I(m) \in \mathbb{N} \), pairwise disjoint, such that

\[
\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m}(\omega) F_{t_m, t_{m+1}}(\omega) B_{t_i^m}^m
\]

where \( x_i^m \in H \) and \( F_i^m \in \mathcal{F}_{t_m}, 1 \leq i \leq I(m), 0 \leq m \leq k-1 \).

Define

\[
\text{Int}(\Phi)(t, \omega) := \int_0^{t^+} \int_U \Phi(s, y) q(ds, dy)(\omega) := \int_0^T \int_U 1_{[0,t]}(s) \Phi(s, y) q(ds, dy)(\omega)
\]

\[
:= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m}(\omega) q(\omega)(t_{m+1} \wedge t, B_{t_i^m}^m) - q(\omega)(t_m \wedge t, B_{t_i^m}^m),
\]

\( t \in [0, T] \) and \( \omega \in \Omega \).

Then, if \( \Phi \in \mathcal{E} \), \( \text{Int}(\Phi) \in \mathcal{M}^2_T(H) \) which denotes the space of all square inte-
grable $H$-valued martingales and we obtain the following isometric formula
\[
\|\text{Int}(\Phi)\|_{\mathcal{M}_T^2} := \sup_{t \in [0,T]} E[\|\int_0^t \Phi(s, y) q(ds, dy)\|^2]
\]
\[=
E[\int_0^T \int_U \|\Phi(s, y)\|^2 \nu(dy) ds] =: \|\Phi\|_T,
\]
i.e. \(\text{Int}: (\mathcal{E}, \|\cdot\|_T) \to (\mathcal{M}_T^2(H), \|\cdot\|_{\mathcal{M}_T^2})\) is an isometric transformation and can therefore be extended to the space \(\mathcal{E}'\|\cdot\|_T\). \(\mathcal{E}'\|\cdot\|_T\) can be characterized by
\[
\mathcal{N}_q^2(T, U, H) = L^2([0, T] \times \Omega \times U, P_T(U), P \otimes \lambda \otimes \nu; H).
\]
The main emphasis is on the Chapter 2 where we prove the existence of the mild solution
\[
X(\xi) \in \mathcal{H}^2(T, H) := \{Y(t), t \in [0, T] | Y \text{ is an } H\text{-predictable process s.t.} \}
\]
\[
\|Y\|_{\mathcal{H}^2} := \sup_{t \in [0,T]} E[\|Y(t)\|^2] < \infty
\]
of problem (1) and the continuity of the mapping \(X : L^2(\Omega, \mathcal{F}_0, P, H) \to \mathcal{H}^2(T, H)\).
A mild solution of the stochastic differential equation (1) is defined implicitly by \(X(\xi) = \mathcal{F}(\xi, X(\xi))\), where \(\mathcal{F} : L^2(\Omega, \mathcal{F}_0, P, H) \times \mathcal{H}^2(T, H) \to \mathcal{H}^2(T, H)\) is given by
\[
\mathcal{F}(\xi, Y)(t) = S(t)\xi + \int_0^t S(t-s)F(Y(s)) \, ds
\]
\[+ \int_0^{t+} \int_U S(t-s)B(Y(s), y) \, q(ds, dy), \quad t \in [0, T].\]
To obtain the existence of the solution, first, we have to show that \(\mathcal{F}(\xi, Y)\) is well defined for all \(\xi \in L^2(\Omega, \mathcal{F}_0, P, H)\) and \(Y \in \mathcal{H}^2(T, H)\) and is an element of \(\mathcal{H}^2(T, H)\). In particular, this includes the proof of the existence of a predictable version of the stochastic integral denoted by
\[
\int_0^{t-} \int_U S(t-s)B(Y(s), y) \, q(ds, dy), \quad t \in [0, T].
\]
Secondly, to apply a fixpoint argument, we have to prove that \(\mathcal{F}\) is a contraction in the second variable.
In a future paper the differential dependence of the mild solution on the initial data will be examined and it will be proved that
\[
X : L^2(\Omega, \mathcal{F}_0, P, H) \to \mathcal{H}^2(T, H)
\]
is Gâteaux differentiable.
Chapter 1

The Stochastic Integral with Respect to Poisson Point Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(U, \mathcal{B})$ a measurable space.

1.1 Poisson random measures

Let $\mathbb{M}$ be the space of non-negative (possibly infinite) integral-valued measures on $(U, \mathcal{B})$ and

$$
\mathcal{B}_\mathbb{M} := \sigma(\mathbb{M} \to \mathbb{Z}_+ \cup \{+\infty\}, \mu \mapsto \mu(B) \mid B \in \mathcal{B})
$$

**Definition 1.1 (Poisson random measure).** A random variable $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_\mathbb{M})$ is called Poisson random measure on $(U, \mathcal{B})$ if the following conditions hold:

(i) For all $B \in \mathcal{B}$: $\Pi(B) : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is Poisson distributed with parameter $E(\Pi(B))$, i.e.:

$$
P(\Pi(B) = n) = \exp(-E(\Pi(B)))(E(\Pi(B)))^n/n!, \; n \in \mathbb{N} \cup \{0\}
$$

If $E(\Pi(B)) = +\infty$ then $\Pi(B) = +\infty$ $P$-a.s.

(ii) If $B_1, \ldots, B_m \in \mathcal{B}$ are pairwise disjoint then $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.
Remark 1.2. If $\Pi$ is a Poisson random measure then the mapping $\Omega \rightarrow Z_+ \cup \{+\infty\}$, $\omega \mapsto \Pi(\omega)(B)$, $B \in \mathcal{B}$, is $\mathcal{F}$-measurable since the mapping $\Omega \rightarrow M$, $\omega \mapsto \Pi(\omega)$ is $\mathcal{F}/\mathcal{B}_M$-measurable by Definition 1.1 and since the mapping $M \rightarrow Z_+ \cup \{+\infty\}$, $\mu \mapsto \mu(B)$ is $\mathcal{B}_M$-measurable by the definition of $\mathcal{B}_M$.

Lemma 1.3. Let $m \in \mathbb{N}$ and $\mu$ and $\nu$ be two probability measures on $[0, \infty[^m$. If for all $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$

$$\int_{[0, \infty[^m} e^{-\langle \alpha, x \rangle} \mu(dx) = \int_{[0, \infty[^m} e^{-\sum_{j=1}^{m} \alpha_j x_j} \mu(dx_1, \ldots, x_m))$$

$$= \int_{[0, \infty[^m} e^{-\sum_{j=1}^{m} \alpha_j x_j} \nu(dx_1, \ldots, x_m)) = \int_{[0, \infty[^m} e^{-\langle \alpha, x \rangle} \nu(dx).$$

then $\mu = \nu$.

Proof. Denote by $\mathcal{H}$ the space of all $\mathcal{B}(\mathbb{R}^m_+)$-measurable functions $f : \mathbb{R}^m_+ \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^m_+} f \mu = \int_{\mathbb{R}^m_+} f \nu$. Then $\mathcal{H}$ is a monotone vector space. Moreover define

$$\mathcal{A} := \{\mathbb{R}^m_+ \rightarrow \mathbb{R}, x \mapsto \exp(-\sum_{j=1}^{m} \alpha_j x_j) | \alpha_j \in \mathbb{Q}_+, 1 \leq j \leq m\}.$$ 

Then $\mathcal{A}$ is a class of bounded, measurable functions, which is closed under multiplication and which is a subset of $\mathcal{H}$ by assumption. By the monotone class theorem it follows that $\sigma(\mathcal{A}) \subset \mathcal{H}$.

Moreover, $\mathcal{A} \subset \{f : \mathbb{R}^m_+ \rightarrow \mathbb{R} | f \text{ is } \mathcal{B}(\mathbb{R}^m_+)-\text{measurable}\}$ is countable and separates the points of $\mathbb{R}^m_+$. Thus, we obtain that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^m_+)$ and $\mathcal{B}(\mathbb{R}^m_+)$ is countable and separates the points of $\mathbb{R}^m_+$. In particular, we get for $A \in \mathcal{B}(\mathbb{R}^m_+)$ that $\mu(A) = \nu(A)$. □

Lemma 1.4. Let $X$ be a Poissonian random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with parameter $c > 0$, i.e. $X : \Omega \rightarrow \mathbb{N}_+ \cup \{+\infty\}$ such that for all $n \in \mathbb{N} \cup \{0\}$:

$$P(X = n) = c^n \exp(-c).$$

Then

$$E(e^{\alpha X}) = \int_0^{\infty} e^{\alpha x} P(X^{-1}(dx)) = \sum_{n=0}^{\infty} e^{\alpha n} \frac{c^n}{n!} = \exp(c(e^\alpha - 1)) \forall \alpha \in \mathbb{R}$$

Theorem 1.5. Given a $\sigma$-finite measure $\nu$ on $(U, \mathcal{B})$ there exists a Poisson random measure $\Pi$ on $(U, \mathcal{B})$ with $E(\Pi(B)) = \nu(B)$ for all $B \in \mathcal{B}$. $\nu$ is then called the mean measure or intensity measure of the Poisson random measure $\Pi$. 
Proof. [IkWa 81, Theorem 8.1, p.42]

**Step 1.** \( \nu(U) < \infty \)

Let \( N \) be a Poissonian random variable with parameter \( c := \nu(U) \).
Moreover let \( \xi_1, \xi_2, \ldots \) be independent \( U \)-valued random variables with distribution \( \frac{1}{c} \nu \), also independent of \( N \).
Define \( \Pi := \sum_{k=1}^{N} \delta_{\xi_k} \).

**Claim 1.** Let \( B \in \mathcal{B} \). Then \( \Pi(B) \) is Poisson distributed with parameter \( \nu(B) \).

Let \( s \leq 0 \), then

\[
E(e^{s\Pi(B)}) = \begin{cases} 
E\left[\exp(s \sum_{k=1}^{N} \delta_{\xi_k}(B))\right] & \text{if } N = 0 \text{ then } \sum_{k=1}^{N} \delta_{\xi_k}(B) = 0 \\
\sum_{n=0}^{\infty} E\left[\prod_{k=1}^{n} \exp(s1_{B}(\xi_k))\right]1_{\{N=n\}} & \text{if } N = n \\
E\left[\prod_{k=1}^{n} \exp(s1_{B}(\xi_k))\right]P(N = n) \\
\sum_{n=0}^{\infty} \left(E\left[\exp(s1_{B}(\xi_1))\right]\right)^n e^{-c} \frac{c^n}{n!} \\
= \exp\left(c (E[\exp(s1_{B}(\xi_1))] - 1)\right) \\
= \exp\left(c \nu(B) e^s + c (1 - \frac{\nu(B)}{c}) - c\right) \\
= \exp\left(\nu(B) (e^s - 1)\right) 
\end{cases}
\]

By Lemma 1.4 and Lemma 1.3 the assertion follows.

**Claim 2.** Let \( B_1, \ldots, B_m \in \mathcal{B} \) pairwise disjoint. Then \( \Pi(B_1), \ldots, \Pi(B_m) \) are independent.

Let \( s_1, \ldots, s_m \in \mathbb{R}_- \), then:

\[
\int_{[0, \infty]^m} \exp\left(\sum_{j=1}^{m} s_j x_j\right) P \circ (\Pi(B_1), \ldots, \Pi(B_m))^{-1} d(x_1, \ldots, x_m)
\]
\begin{align*}
&= E \left[ \exp \left( \sum_{j=1}^{m} s_j \Pi(B_j) \right) \right] \\
&= E \left[ \sum_{n=0}^{\infty} \exp \left( \sum_{j=1}^{m} s_j \sum_{k=1}^{n} 1_{B_j}(\xi_k) \right) 1_{\{N=n\}} \right] \\
&= \sum_{n=0}^{\infty} E \left[ \prod_{k=1}^{n} \exp \left( \sum_{j=1}^{m} s_j 1_{B_j}(\xi_k) \right) \right] e^{-c^n/n!} \\
&= \sum_{n=0}^{\infty} \left( E \left( \exp \left( \sum_{j=1}^{m} s_j 1_{B_j}(\xi_1) \right) \right) \right)^n e^{-c^n/n!} \\
&= \exp \left( c \left( E \left( \exp \left( \sum_{j=1}^{m} s_j 1_{B_j}(\xi_1) \right) \right) - 1 \right) \right) \\
&= \exp \left( c \left( \sum_{j=1}^{m} 1_{\{\xi_1 \in B_j\}} e^{s_j} + 1_{\{\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c \}} - 1 \right) \right) \\
&= \exp \left( c \left( \sum_{j=1}^{m} P(\xi_1 \in B_j) e^{s_j} + P(\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c) - 1 \right) \right) \\
&= \exp \left( c \left( \sum_{j=1}^{m} \frac{\nu(B_j)}{c} e^{s_j} + (1 - \sum_{j=1}^{m} \frac{\nu(B_j)}{c}) - 1 \right) \right) \\
&= \exp \left( \sum_{j=1}^{m} \nu(B_j)(e^{s_j} - 1) \right) = \prod_{j=1}^{m} \exp(\nu(B_j)(e^{s_j} - 1)) \\
&= \prod_{j=1}^{m} \int_{0}^{\infty} \exp(s_j x_j) P \circ \Pi(B_j)^{-1}(dx_j) \\
&= \int_{[0,\infty]^m} \exp \left( \sum_{j=1}^{m} s_j x_j \right) P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1} d(x_1, \ldots, x_m)
\end{align*}

Hence, by Proposition 1.3, we can conclude that

\[ P \circ (\Pi(B_1), \ldots, \Pi(B_m))^{-1} = P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1} \]

which implies the required independence.
Step 2. \( \nu \) is \( \sigma \)-finite

There exist \( U_i \in \mathcal{B} \), \( i \in \mathbb{N} \), pairwise disjoint such that \( \nu(U_i) < \infty \) for all \( i \in \mathbb{N} \) and \( U = \bigcup_{i=1}^{\infty} U_i \). Set \( \nu_i := \nu(\cdot \cap U_i) \), \( i \in \mathbb{N} \).

For \( i \in \mathbb{N} \) let \( N_i \) be a Poissonian random variable with parameter \( c_i := \nu(U_i) \) and \( \xi_1, \xi_2, \ldots \) independent \( U_i \)-valued random variables with distribution \( \frac{1}{\nu_i} \nu_i \), also independent of \( N_i \). Moreover the families of random variables \( \{N_i, \xi_1, \xi_2, \ldots \} \) are independent.

Let \( \Pi_i \) be the Poisson random measure on \( U_i \) associated with \( N_i \) and \( \xi_1, \xi_2, \ldots \) with intensity measure \( \nu_i \) as defined in Step 1.

Define \( \Pi := \sum_{i=1}^{\infty} \Pi_i := \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{\xi_k} \). Then one has for \( B \in \mathcal{B} \) that

\[
\Pi(B) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_B(\xi_k) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_{B \cap U_i}(\xi_k)
\]

Claim 1. Let \( B \in \mathcal{B} \) with \( E[\Pi(B)] < \infty \) then

\[
\nu(B) = \sum_{i=1}^{\infty} \nu(B \cap U_i) = \sum_{i=1}^{\infty} E[\Pi_i(B \cap U_i)], \text{ by Step 1, Claim 1}
\]

Then \( \Pi(B) \) is Poisson distributed with parameter \( \nu(B) \).

Let \( s \leq 0 \), then:

\[
E[e^{s \Pi(B)}] = \lim_{m \to \infty} E\left[\exp\left(s \sum_{i=1}^{m} \Pi_i(B \cap U_i)\right)\right] = \lim_{m \to \infty} \prod_{i=1}^{m} E\left[\exp(s \Pi_i(B \cap U_i))\right],
\]

since the families of random variables \( \{N_i, \xi_1, \xi_2, \ldots \} \) are independent,

\[
= \lim_{m \to \infty} \prod_{i=1}^{m} \exp(\nu(B \cap U_i)(e^s - 1)), \text{ by Step 1}
\]

\[
= \exp(\nu(B)(e^s - 1))
\]

By Lemma 1.4 and Lemma 1.3 the assertion follows.

Claim 2. Let \( B \in \mathcal{B} \) with \( \nu(B) = E[\Pi(B)] = +\infty \). Then \( \Pi(B) = +\infty \) \( P \)-a.s.,

\[
P(\Pi(B) = +\infty) = P\left(\bigcap_{m \in \mathbb{N}} \bigcup_{i \geq m} \{\Pi_i(B \cap U_i) > 0\}\right)
\]
Since

$$P\left(\bigcap_{i\geq m} \{\Pi_i(B \cap U_i) > 0\}^c\right) = P\left(\bigcap_{i\geq m} \{\Pi_i(B \cap U_i) = 0\}\right)$$

$$= \lim_{n \rightarrow \infty} P\left(\bigcap_{i=m}^{m+n} \{\Pi_i(B \cap U_i) = 0\}\right) = \lim_{n \rightarrow \infty} \prod_{i=m}^{m+n} e^{-\nu(B \cap U_i)}$$

$$= \lim_{n \rightarrow \infty} \exp\left(-\sum_{i=m}^{m+n} \nu(B \cap U_i)\right) = 0$$

it follows that $P(\bigcup_{i\geq m} \{\Pi_i(B \cap U_i) > 0\}) = 1$ for all $m \in \mathbb{N}$ and therefore $P(\Pi(B) = +\infty) = 1$.

**Claim 3.** Let $B_1, \ldots, B_m \in \mathcal{B}$ pairwise disjoint. Then $\Pi(B_1), \ldots, \Pi(B_m)$ are independent.

If $E[\Pi(B_j)] < \infty$ for all $j \in \{1, \ldots, m\}$ then one gets for all $s_1, \ldots, s_m \in \mathbb{R}_-$ that

$$E\left[\exp\left(\sum_{j=1}^{m} s_j \Pi(B_j)\right)\right] = E\left[\exp\left(\sum_{i=1}^{\infty} \sum_{j=1}^{m} s_j \Pi_i(B_j \cap U_i)\right)\right]$$

$$= \lim_{n \rightarrow \infty} E\left[\exp\left(\sum_{i=1}^{n} \sum_{j=1}^{m} s_j \Pi_i(B_j \cap U_i)\right)\right]$$

$$= \lim_{n \rightarrow \infty} \prod_{i=1}^{n} \prod_{j=1}^{m} E\left[\exp\left(s_j \Pi_i(B_j \cap U_i)\right)\right]$$

$$= \lim_{n \rightarrow \infty} \prod_{i=1}^{n} \prod_{j=1}^{m} \exp\left(\nu(B_j \cap U_i)(e^{s_j} - 1)\right)$$

$$= \prod_{j=1}^{m} \exp\left(\nu(B_j)(e^{s_j} - 1)\right)$$

If there exists $i \in \{1, \ldots, m\}$ with $E[\Pi(B_i)] = \infty$, then, by Step 2, Claim 2, $\Pi(B_i) = \infty$ $P$-a.s. Let $\{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$, then the independence of $\Pi(B_{i_1}), \ldots, \Pi(B_{i_n})$ follows from the case $E[\Pi(B_j)] < \infty$ for all $j \in \{1, \ldots, m\}$ and the above statement.

$\square$
1.2 Point processes and Poisson point processes

Definition 1.6 (Point function on U). A point function \( p \) on \( U \) is a mapping \( p : D_p \subset (0, \infty) \rightarrow U \) where the domain \( D_p \) is a countable subset of \( (0, \infty) \).

\( p \) defines a measure \( N_p(dt, dy) \) on \( ((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B}) \) in the following way:

Define \( \bar{p} : (0, \infty) \rightarrow (0, \infty) \times U, t \mapsto (t, p(t)) \) and denote by \( c \) the counting measure on \( (D_p, \mathcal{P}(D_p)) \), i.e. \( c(A) := |A| \) for all \( A \in \mathcal{P}(D_p) \).

For \( \bar{B} \in \mathcal{B}((0, \infty)) \otimes \mathcal{B} \) define \( N_p(\bar{B}) := c(\bar{p}^{-1}(\bar{B})) \).

Then, in particular, we have for all \( A \in \mathcal{B}((0, \infty)) \) and \( B \in \mathcal{B} \)

\[
N_p(A \times B) := \#\{t \in D_p | t \in A, p(t) \in B\}.
\]

Notation: \( N_p(t, B) := N_p([0, t] \times B), t \geq 0, B \in \mathcal{B} \)

Let \( \mathcal{P}_U \) be the space of all point functions on \( U \) and

\[
\mathcal{B}_{\mathcal{P}_U} := \sigma(\mathcal{P}_U \rightarrow \mathbb{Z}_+ \cup \{+\infty\}, p \mapsto N_p([0, t] \times B) | t > 0, B \in \mathcal{B})
\]

Definition 1.7 (Point process). (i) A point process on \( U \) is a random variable \( p : (\Omega, \mathcal{F}) \rightarrow (\mathcal{P}_U, \mathcal{B}_{\mathcal{P}_U}) \).

(ii) A point process \( p \) is called stationary if for every \( t > 0 \) \( p \) and \( \theta_t p \) have the same probability law, where \( \theta_t p \) is defined by \( D_{\theta_t p} := \{s \in (0, \infty) | s + t \in D_p\} \) and \( (\theta_t p)(s) := p(s + t) \).

(iii) A point process is called Poisson point process if there exists a Poisson random measure \( \Pi \) on \( (0, \infty) \times U \) such that there exists \( N \in \mathcal{F}, P(N) = 0 \), such that for all \( \omega \in N^c \) and for all \( \bar{B} \in \mathcal{B}((0, \infty)) \otimes \mathcal{B} \): \( N_{\omega}(\bar{B}) = \Pi(\omega)(\bar{B}) \).

(iv) A point process \( p \) is called \( \sigma \)-finite if there exist \( U_i \in \mathcal{B}, i \in \mathbb{N}, U_i \uparrow U, i \rightarrow \infty, \) and \( E[N_p(t, U_i)] < \infty \) for all \( t > 0 \) and \( i \in \mathbb{N} \).

The statement of the following proposition about stationary Poisson point processes can be found in [IkWa 81, I.9 Point processes and Poisson point processes, p.43]
Proposition 1.8. Let $p$ be a $\sigma$-finite Poisson point process. Then $p$ is stationary if and only if there exists a $\sigma$-finite measure $\nu$ on $(U, B)$ such that
\[
E[N_p(dt, dy)] = \lambda(dt) \otimes \nu(dy)
\]
where $\lambda$ denotes the Lebesgue-measure on $(0, \infty)$. $\nu$ is called characteristic measure of $p$.

Theorem 1.9. Given a $\sigma$-finite measure $\nu$ on $(U, B)$ there exists a stationary Poisson point process on $U$ with characteristic measure $\nu$.

Proof. Let $\Pi$ be a Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ with intensity measure $\lambda \otimes \nu$ where $\lambda$ denotes the Lebesgue-measure on $(0, \infty)$, $\mathcal{B}((0, \infty)))$. Remember the construction of $\Pi$ in the proof of Theorem 1.5:
There exist $U_i$, $i \in \mathbb{N}$, pairwise disjoint such that $U = \bigcup_{i \in \mathbb{N}} U_i$ and $c_i := \nu(U_i) < \infty$. For $i \in \mathbb{N}$ let

- $N_i$ be a Poissonian random variable with parameter $c_i$,
- $\xi^i_k = (t^i_k, x^i_k)$, $k \in \mathbb{N}$, i.i.d. $[i-1,i] \times U_i$-valued random variables with distribution $\lambda \otimes (\frac{1}{c_i} \nu(\cdot \cap U_i))$, also independent of $N_i$.

Moreover the families of random variables $\{N_i, \xi^i_1, \xi^i_2, \ldots\}$, $i \in \mathbb{N}$, are independent.

Then
\[
\Pi := \sum_{i=1}^{\infty} \Pi_i := \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{(t^i_k, x^i_k)}
\]
is a Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ with intensity measure $\lambda \otimes \nu$ and for $\bar{B} \in \mathcal{B}((0, \infty)) \otimes \mathcal{B}$ holds
\[
(1.1) \quad \Pi(\bar{B}) = \sum_{i=1}^{\infty} \Pi_i(\bar{B} \cap ([i-1,i] \times U_i))
\]
Then there exists a $P$-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$:
$\Pi(\omega)\{\{t\} \times U\} = 1$ or $0$ for all $t > 0$, since
\[
P(\bigcup_{t>0} \{\Pi(\{t\} \times U) > 1\}) = P(\bigcup_{i=1}^{\infty} \bigcup_{t \in [i-1,i]} \{\Pi(\{t\} \times U) > 1\})
\]
\[
\leq \sum_{i=1}^{\infty} P\left( \bigcup_{t \in [i-1, i]} \{ \Pi(\{t\} \times U_i) > 1 \} \right)
\]
\[
\leq \sum_{i=1}^{\infty} P\left( \bigcup_{n \neq m} \bigcup_{t \in [i-1, i]} \{ \delta_{\xi_n}^i(\{t\} \times U_i) = 1 \} \cap \{ \delta_{\xi_m}^i(\{t\} \times U_i) = 1 \} \right)
\]
\[
\leq \sum_{i=1}^{\infty} \sum_{n \neq m} P\left( \bigcup_{t \in [i-1, i]} \{ t_n = t_m = t \} \right)
\]
\[
= \sum_{i=1}^{\infty} \sum_{n \neq m} \lambda \otimes \lambda (\{(t, t) \mid t \in [i-1, i]\})
\]
\[
= 0
\]

If \( \omega \in N^c \) and \( t \in [i-1, i] \), then
\[
\Pi(\omega(\{t\} \times U)) = 1
\]
\[
\iff \sum_{k=1}^{N_i(\omega)} \delta(t_k^i(\omega), x_k^i(\omega))\{(t) \times U_i\} = \Pi_i(\omega(\{t\} \times U_i)
\]
\[
= \Pi(\omega(\{t\} \times U)) , \text{ by equation (1.1),}
\]
\[
= 1
\]
\[
\iff \exists! k \in \{1, \ldots, N_i(\omega)\} \text{ such that } t = t_k^i(\omega)
\]

In this case we set
\[
p(\omega)(t) := x_k^i(\omega) \text{ and } D_{p(\omega)} := \{ t \in (0, \infty) \mid \Pi(\omega(\{t\} \times U) \neq 0 \}
\]

If \( \omega \in N \) then define \( p_0 \in \mathcal{P}_U \) by \( D_p := \{ t_0 \} \subset (0, \infty) \) and \( p_0(t_0) = x_0 \in U \) and set \( p(\omega) = p_0 \).

**Claim 1.** \( N_{p(\omega)} = \Pi(\omega) \) for all \( \omega \in N^c \).

Let \( \omega \in N^c \), \( A \in \mathcal{B}((0, \infty)) \) and \( B \in \mathcal{B} \) then:
\[
\Pi(\omega)(A \times B)
\]
\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{N_i(\omega)} \delta(t_k^i(\omega), x_k^i(\omega))(A \cap [i-1, i] \times B \cap U_i)
\]
\[
= \sum_{i=1}^{\infty} \#\{ s \in [i-1, i] \mid s \in A, \exists k \in \{1, \ldots, N_i(\omega)\} \text{ such that } s = t_k^i(\omega) \text{ and } x_k^i(\omega) \in B \cap U_i \}
\]
\[
= \sum_{i=1}^{\infty} \#\{ s \in [i-1, i] \mid s \in A, \exists k \in \{1, \ldots, N_i(\omega)\} \text{ such that } s = t_k^i(\omega) \text{ and } x_k^i(\omega) \in B \cap U_i \},
\]
since $\Pi(\omega)(\{s\} \times U) \in \{0, 1\}$ for all $s \in [0, \infty[$,

$= \#\{s \in D_{p(\omega)} | s \in A, p(\omega)(s) \in B\}$,

by the definition of $p$,

$= N_{p(\omega)}(A \times B)$

Claim 2. For all $\bar{B} \in \mathcal{B}((0, \infty)) \otimes \mathcal{B}$ the mapping $N_p(\bar{B})$ is $\mathcal{F}$-measurable and $E[N_p(dt, dx)] = \lambda(dt) \otimes \nu(dx)$.

Since $N_p(\bar{B}) = \Pi(\bar{B})$ $P$-a.s. the measurability is obvious by Remark 1.2 and the completeness of $(\Omega, \mathcal{F}, P)$. Now $E[N_p(\bar{B})]$ is well defined and we obtain that $E[N_p(\bar{B})] = E[\Pi(\bar{B})] = \lambda \otimes \nu(\bar{B})$, since $\Pi$ is a Poisson random measure with intensity measure $\lambda(dt) \otimes \nu(dx)$.

Claim 3. $p : \Omega \to \mathcal{P}_U$ is $\mathcal{F}/\mathcal{B}_{\mathcal{P}_U}$-measurable.

$\mathcal{B}_{\mathcal{P}_U} = \sigma(\mathcal{P}_U \to \mathbb{Z}_+ \cup \{+\infty\}, p \mapsto N_p([0, t] \times B) | t > 0, B \in \mathcal{B})$

$= \sigma(\{p \in \mathcal{P}_U | N(t, B) = m \} | t > 0, B \in \mathcal{B}, m \in \mathbb{Z}_+)$

and for $t > 0$, $B \in \mathcal{B}$, $m \in \mathbb{Z}_+$ one gets by Claim 2 that

$\{p \in \{N(t, B) = m\} \} = \{N_p(t, B) = m\} \in \mathcal{F}.$

By Claim 1 - 3 it follows that $p$ is a Poisson point process with characteristic measure $\nu$. By Proposition 1.8 $p$ is stationary.

### 1.3 Stochastic integrals with respect to Poisson point processes

Let $\mathcal{F}_t$, $t \geq 0$, be a filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all $P$-nullsets of $\mathcal{F}$.

**Definition 1.10.** A point process $p$ is called $(\mathcal{F}_t)$-*adapted* if for every $t > 0$ and $B \in \mathcal{B}$ $N_p(t, B)$ is $\mathcal{F}_t$-measurable.

For an arbitrary point process $p$ define the following set $\Gamma_p := \{B \in \mathcal{B} | E[N_p(t, B)] < \infty \text{ for all } t > 0\}$.

**Definition 1.11.** An $(\mathcal{F}_t)$-adapted point process $p$ on $U$ is said to be of class (QL) *(quasi-left-continuous)* with respect to $\mathcal{F}_t$, $t \geq 0$, if it is $\sigma$-finite and there exists for all $B \in \mathcal{B}$ a process $\hat{N}_p(t, B)$, $t \geq 0$, such that
(i) for $B \in \Gamma_p$ $t \mapsto \hat{N}_p(t, B)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process,

(ii) for all $t \geq 0$ and $P$-a.e. $\omega \in \Omega$: $\hat{N}_p(\omega)(t, \cdot)$ is a $\sigma$-finite measure on $(U, \mathcal{B})$,

(iii) for $B \in \Gamma_p$ $q(t, B) := N_p(t, B) - \hat{N}_p(t, B)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale.

$\hat{N}_p$ is called the compensator of the point process $p$ and $q$ the compensated Poisson random measure of $p$.

**Definition 1.12.** A point process $p$ is called an $(\mathcal{F}_t)$-Poisson point process if it is an $(\mathcal{F}_t)$-adapted, $\sigma$-finite Poisson point process such that $\left\{ N_p([t, t+h] \times B) \mid h > 0, B \in \mathcal{B} \right\}$ is independent of $\mathcal{F}_t$ for all $t \geq 0$.

**Remark 1.13.** Let $p$ be a $\sigma$-finite Poisson point process on $U$. Then there exists a filtration $\mathcal{F}_t, t \geq 0$, on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all $P$-nullsets of $\mathcal{F}$ and $p$ is an $(\mathcal{F}_t)$-Poisson point process.

**Proof.** Define $\mathcal{N} := \left\{ N \in \mathcal{F} \mid P(N) = 0 \right\}$ and for $t \geq 0$

$$\mathcal{F}_t := \sigma(N_p(t, B) \mid B \in \mathcal{B}) \cup \mathcal{N}.$$  

Then $p$ is an $(\mathcal{F}_t)$-adapted, $\sigma$-finite Poisson point process. Moreover $\sigma(N_p(t, B) \mid B \in \mathcal{B}) \cup \mathcal{N} = \sigma(\Pi([0, t] \times \mathcal{B}) \mid B \in \mathcal{B}) \cup \mathcal{N}$ is independent of $\sigma(\Pi([t, t+h] \times \mathcal{B}) \mid h > 0, B \in \mathcal{B}) \cup \mathcal{N}$ by Definition 1.1 (ii) since $[0,t] \times B$ and $[t, t+h] \times \hat{B}$ are disjoint for all $h > 0$ and $B, \hat{B} \in \mathcal{B}$. Since

$$\sigma(\Pi([t, t+h] \times B) \mid h > 0, B \in \mathcal{B}) \cup \mathcal{N} = \sigma(N_p([t, t+h] \times B) \mid h > 0, B \in \mathcal{B}) \cup \mathcal{N}$$

the assertion follows.

For the rest of this section fix a $\sigma$-finite measure $\nu$ on $(U, \mathcal{B})$ and a stationary $(\mathcal{F}_t)$-Poisson point process $p$ on $U$ with characteristic measure $\nu$.

**Proposition 1.14.** $p$ is of class (QL) with compensator $\hat{N}_p(t, B) = t\nu(B)$, $t \geq 0$, $B \in \mathcal{B}$.

**Proof.** Set for $t \geq 0$ and $B \in \mathcal{B}$: $\hat{N}_p(t, B) := t\nu(B)$. Then condition (i) and (ii) of Definition 1.11 are fulfilled. Moreover, for $B \in \Gamma_p$ $q(t, B) := N_p(t, B) - \hat{N}_p(t, B)$, $t \geq 0$, is $(\mathcal{F}_t)$-adapted. It remains to
check that for $B \in \Gamma_p$, $q(t, B) \geq 0$, has the martingale property.
For this end let $0 \leq s < t < \infty$ and $F_s \in \mathcal{F}_s$, then
\[
E[q(t, B)1_{F_s}] = E[(N_p(t, B) - \hat{N}_p(t, B))1_{F_s}]
= E[N_p(t, B)1_{F_s}] - t\nu(B)P(F_s)
= E[(N_p(t, B) - N_p(s, B))1_{F_s}] + E[N_p(s, B)1_{F_s}] - t\nu(B)P(F_s)
= E[N_p(t, B) - N_p(s, B)]P(F_s) + E[N_p(s, B)1_{F_s}] - (t - s)\nu(B)P(F_s)
- s\nu(B)P(F_s)
= E[(N_p(s, B)1_{F_s}) - s\nu(B)P(F_s)
= E[(N_p(s, B) - \hat{N}_p(s, B))1_{F_s}]
= E[q(s, B)1_{F_s}]
\]

Remark 1.15. If $t \in [0, \infty[$ and

$B \in \Gamma_p = \{ B \in \mathcal{B} \mid E[N_p(t, B)] < \infty \text{ for all } t > 0 \} = \{ B \in \mathcal{B} \mid \nu(B) < \infty \}$

then $q(t, B) \in \mathbb{R}$ $P$-a.s. since $q(t, B) = N_p(t, B) - t\nu(B)$ where $N_p(t, B) < \infty$ $P$-a.s. as $E[N_p(t, B)] < \infty$.
If $0 \leq s \leq t < \infty$ and $B \in \Gamma_p$ then
\[
q(t, B) - q(s, B) = N_p(t, B) - N_p(s, B) - (t - s)\nu(B)
= N_p([s, t] \times B) - (t - s)\nu(B) \quad P\text{-a.s.}
\]

Notation: In the following we will use the following notation:
$q([s, t] \times B) := N_p([s, t] \times B) - (t - s)\nu(B)$, $0 \leq s \leq t < \infty$, $B \in \mathcal{B}$.

Proposition 1.16. For $A \in \Gamma_p$ $(q(t, A), t \geq 0)$ is an element of $\mathcal{M}^2$ and we have for $A_1, A_2 \in \Gamma_p$ that
\[
\langle q(\cdot, A_1), q(\cdot, A_2) \rangle(t) = \hat{N}_p(t, A_1 \cap A_2), \quad t \geq 0.
\]
In particular, this means that for all $A \in \Gamma_p$ the following holds:
$M(t) := q(t, A)^2 - \hat{N}_p(t, A)$, $t \geq 0$, is an $(\mathcal{F}_t)_{t \geq 0}$-martingale and in this case:
$E[M(t)] = E[M(0)] = 0$ for all $t \geq 0$.

Proof. [Ikeda, Watanabe, Theorem 3.1, p.60; Lemma 3.1, p.60] □

Step 1. Definition of the stochastic integral for elementary processes
Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and fix $T > 0$.
The class $\mathcal{E}$ of all elementary processes is determined by the following definition
Definition 1.17. An $H$-valued process $\Phi(t) : \Omega \times U \to H$, $t \in [0, T]$, on $(\Omega \times U, \mathcal{F} \otimes \mathcal{B}, P \otimes \nu)$ is said to be elementary if there exist $0 = t_0 < t_1 < \cdots < t_k = T$, $k \in \mathbb{N}$, and for $m \in \{0, \ldots, k - 1\}$ exist $B_{t_1}^m, \ldots, B_{t_{l(m)}}^m \in \Gamma_p$, pairwise disjoint, $I(m) \in \mathbb{N}$, such that

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} x_i^m 1_{|t_m, t_{m+1}|} \times B_i^m$$

where $x_i^m \in H$ and $F_i^m \in \mathcal{F}_{t_m}$, $1 \leq i \leq I(m)$, 0 $\leq m \leq k - 1$.

Proposition 1.18. If $\Phi \in \mathcal{E}$ then $\left( \int_0^{t+} \int_U \Phi(s, y) q(ds, dy) \right)(t) \in \mathcal{M}_2^2(H)$ and

$$\|\int(\Phi)(t, \omega)\|_{\mathcal{M}_2^2} = \sup_{t \in [0, T]} \mathbb{E}\left[ \int_0^{t+} \int_U \Phi(s, y) q(ds, dy) \right]$$

Proof.

Claim 1. $\int(\Phi)$ is $(\mathcal{F}_t)$-adapted.

Let $t \in [0, T]$ then:

$$\int(\Phi)(t)$$

$$= \sum_{m \in \{0, \ldots, k-1\} \atop t_m \leq t} \sum_{i=1}^{l(m)} x_i^m 1_{|t_m, t_{m+1}|} \times B_i^m \left( N_p(t_{m+1} \wedge t, B_i^m) - N_p(t_m, B_i^m) - (t_{m+1} \wedge t - t_m) \nu(B_i^m) \right)$$

which is $\mathcal{F}_t$-measurable since $p$ is $(\mathcal{F}_t)$-adapted.
Claim 2. For all $t \in [0, T]$:

$$E[\|\text{Int}(\Phi)(t)\|^2] = E\left[ \int_0^t \int_U \|\Phi(s, y)\|^2 \nu(dy) ds \right] < \infty :$$

$$E[\|\text{Int}(\Phi)(t)\|^2] = E\left[ \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_m^i} q([t_m \wedge t, t_{m+1} \wedge t] \times B_i^m) \right]^2$$

$$= E\left[ \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} \|x_i^m 1_{F_m^i} q([t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2 \right]$$

$$+ 2 \sum_{0 \leq m < n \leq k-1} \sum_{l_m \leq t, l_n \leq t} \langle x_i^m \Delta_i^m, x_j^n \Delta_j^n \rangle$$

where $\Delta_i^l := q([t_l \wedge t, t_l+1 \wedge t] \times A_i^l)$, $0 \leq l \leq k-1$, $1 \leq h \leq I(l)$.

1. For $m \in \{0, \ldots, k-1\}$, $t_m \leq t$, $i \in \{1, \ldots, I(m)\}$ holds:

$$E[\|x_i^m 1_{F_m^i} q([t_m \wedge t, t_{m+1} \wedge t] \times B_i^m)\|^2] \leq E[\|x_i^m \Delta_i^m\|^2] < \infty :$$

For this purpose let $0 \leq s \leq t \leq T$ and $B \in \Gamma_p$, then:

$$E[q([s, t] \times B)^2] = E[(q(t, B) - q(s, B))^2]$$

$$= E\left[ q(t, B)^2 - 2 q(t, B) q(s, B) + q(s, B)^2 \right]$$

(a) By Proposition 1.16 and Proposition 1.14 it follows that

$$E[q(t, B)^2] = E[\hat{N}_p(t, B)] = t \nu(B) < \infty.$$

(b) Since $|q([s, t] \times B)|$ and $|q(s, B)|$ are independent we get that

$$E[|q(t, B) q(s, B)|] \leq E[|q([s, t] \times B)| q(s, B)|] + E[q(s, B)^2]$$

$$= E[|q([s, t] \times B)|] E[|q(s, B)|] + E[q(s, B)^2]$$

$$< \infty.$$

From (a) and (b) it follows that $E[q([s, t] \times B)^2] < \infty$. Moreover we obtain that
(1.2) \[ E[q([s, t] \times B)^2] \]
\[ = E[q(t, B)^2] - 2E[q(s, B)] + E[q(s, B)^2] \]
\[ = \nu(B) - 2E[q(s, t) \times B]E[q(s, B)] - \nu(B) \]
\[ = (t - s)\nu(B), \quad \text{as } E[q(s, t)] = [N_p([0, s] \times B) - \nu(B)] = 0 \]

2.: For \( m, n \in \{0, \ldots, k - 1\}, m < n, t_n \leq t, i \in \{1, \ldots, I(m)\}, j \in \{1, \ldots, I(n)\} \) holds:
\[ E[|\langle x_i^m 1_{F_i^m} \Delta_i^m, x_j^n 1_{F_j^n} \Delta_j^n \rangle|] \leq E[|\langle x_i^m \Delta_i^m, x_j^n \Delta_j^n \rangle|] < \infty : \]
Since \( m < n \) and \( t_m < t_n \leq t \) we get that
\[ |t_m \land t, t_{m+1} \land t[\cap]t_n \land t, t_{n+1} \land t| = |t_m, t_{m+1}[\cap]t_n, t_{n+1} \land t| = \emptyset \]
therefore \( \Delta_j^n \) and \( \langle x_i^m, x_j^n \rangle | \Delta_i^m \) are independent and we obtain that
\[ E[|\langle x_i^m \Delta_i^m, x_j^n \Delta_j^n \rangle|] = E[|\langle x_i^m \Delta_i^m, x_j^n \Delta_j^n \rangle|] E[|\Delta_j^n|] < \infty. \]

3.: For \( m, n \in \{0, \ldots, k - 1\}, m < n, t_n \leq t, i \in \{1, \ldots, I(m)\}, j \in \{1, \ldots, I(n)\} \) holds:
\[ E[|\langle x_i^m 1_{F_i^m} \Delta_i^m, x_j^n 1_{F_j^n} \Delta_j^n \rangle|] \]
\[ = E[|\langle x_i^m 1_{F_i^m} \Delta_i^m, x_j^n 1_{F_j^n} \Delta_j^n \rangle|] \]
\[ = E[|\langle x_i^m 1_{F_i^m} \Delta_i^m, x_j^n 1_{F_j^n} \Delta_j^n \rangle|] E[\Delta_j^n] \]
\[ = 0, \quad \text{since } E[\Delta_j^n] = 0. \]

By 1.-3. one gets for all \( t \in [0, T] \) that
\[ E[|I_{t}(\Phi)(t)|^2] \]
\[ = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} E[\|x_i^m 1_{F_i^m} q([t_m \land t, t_{m+1} \land t] \times B_i^m)\|^2] \]
\[ = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} E[\|x_i^m 1_{F_i^m} q([t_m \land t, t_{m+1} \land t] \times B_i^m)\|^2] + \sum_{0 \leq m < n \leq k-1 (i,j) \in \{1, \ldots, I(m)\} \times \{1, \ldots, I(n)\}} E[\langle x_i^m \Delta_i^m, x_j^n \Delta_j^n \rangle] \]
\[ = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} E[\|x_i^m 1_{F_i^m} q([t_m \land t, t_{m+1} \land t] \times B_i^m)\|^2] \]
Claim 3. Int(\(\Phi\))(t), \(t \in [0, T]\), is an \((\mathcal{F}_t)\)-martingale. Let \(0 \leq s < t \leq T\) and \(F_s \in \mathcal{F}_s\) then:

\[
\int_{F_s} \int_0^{t+} \int_U \Phi(r, y) q(dr, dy) dP
\]

\[
= \int_{F_s} \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} x_i^m 1_{F_i^m} \left(q(t_{m+1} \wedge t, B_{i}^{m}) - q(t_m \wedge t, B_{i}^{m})\right) dP
\]

\[
= \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \int_{F_s} x_i^m 1_{F_i^m} \left(q(t_{m+1} \wedge t, B_{i}^{m}) - q(t_m \wedge s, B_{i}^{m})\right) dP
\]

\[
+ \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \int_{F_s} x_i^m 1_{F_i^m} \left(q(t_{m+1} \wedge t, B_{i}^{m}) - q(t_m, B_{i}^{m})\right) dP
\]

\[
= \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \int_{F_s} x_i^m 1_{F_i^m} \left(E[q(t_{m+1} \wedge t, B_{i}^{m}) | \mathcal{F}_s] - q(t_m \wedge s, B_{i}^{m})\right) dP
\]

\[
+ \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \int_{F_s} x_i^m 1_{F_i^m} \left(E[q(t_{m+1} \wedge t, B_{i}^{m}) | \mathcal{F}_t] - q(t_m \wedge s, B_{i}^{m})\right) dP
\]

since \(F_i^m \in \mathcal{F}_{t_m}\) and \(q([t_m, t_{m+1} \wedge t]) \times B_{i}^{m}\) is independent of \(\mathcal{F}_{t_m}\),

\[
= \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \sum_{t_m \leq t} \|x_i^m\|^2 P(F_i^m) E[q([t_m \wedge t, t_{m+1} \wedge t] \times B_{i}^{m})^2],
\]

by equation (1.2),

\[
E \left[ \int_0^t \int_U \sum_{m=0}^{k-1} \sum_{i=1}^{l(m)} \|x_i^m\|^2 P(F_i^m) (t_{m+1} \wedge t - t_m \wedge t) \nu(B_{i}^{m}) \right]
\]

\[
= E \left[ \int_0^t \int_U \|\Phi(s, y)\|^2 \nu(dy) ds \right]
\]
\[ + \sum_{m=0}^{k-1} \sum_{s < t < t_m} \int_{F_s} x_i^m 1_{F_i^m} (q(s, B_i^m) - q(s, B_i^m)) \, dP \]

\[ = \sum_{m=0}^{k-1} \sum_{s < t < t_m} \int_{F_s} x_i^m 1_{F_i^m} (q(t_{m+1} \wedge s, B_i^m) - q(t_m \wedge s, B_i^m)) \, dP, \]

since \(q(t_{m+1} \wedge \cdot, B_i^m)\) is an \((\mathcal{F}_t)\)-martingale

\[ + \sum_{m=0}^{k-1} \sum_{s < t < t_m} \int_{F_s} x_i^m 1_{F_i^m} (q(t_{m+1} \wedge s, B_i^m) - q(t_m \wedge s, B_i^m)) \, dP \]

\[ + \sum_{m=0}^{k-1} \sum_{s < t < t_m} \int_{F_s} x_i^m 1_{F_i^m} (q(t_{m+1} \wedge s, B_i^m) - q(t_m \wedge s, B_i^m)) \, dP \]

\[ = \int_{F_s} \int_0^{s^+} \int_U \Phi(r, y) q(dr, dy) \, dP \]

\[\square\]

In this way one has found the semi norm \(\| \cdot \|_T\) on \(E\) such that \(\text{Int} : (\mathcal{E}, \| \cdot \|_T) \to (\mathcal{M}_T^2(H), \| \cdot \|_{\mathcal{M}_T^2})\) is an isometric transformation. To get a norm on \(E\) one has to consider equivalence classes of elementary processes with respect to \(\| \cdot \|_T\). For simplicity, the space of equivalence classes will be denoted by \(\tilde{E}\), too.

Since \(\mathcal{E}\) is dense in the abstract completion \(\tilde{\mathcal{E}}\) of \(\mathcal{E}\) w.r.t. \(\| \cdot \|_T\) it is clear that there is a unique isometric extension of \(\text{Int}\) to \(\tilde{\mathcal{E}}\).

**Step 2. Characterization of \(\tilde{\mathcal{E}}\)**

Define the predictable \(\sigma\)-field on \([0, T] \times \Omega \times U\) by

\[\mathcal{P}_T(U) := \sigma(g : [0, T] \times \Omega \times U \to H \mid g \text{ is } (\mathcal{F}_t \times \mathcal{B}) \text{ adapted and left-continuous})\]

\[= \sigma(\{[s, t] \times \tilde{F}_s \mid 0 \leq s \leq t \leq T, \tilde{F}_s \in \tilde{\mathcal{F}}_s\} \cup \{\{0\} \times \tilde{F}_0 \mid \tilde{F}_0 \in \tilde{\mathcal{F}}_0\})\]

\[= \sigma(\{[s, t] \times F_s \times B \mid 0 \leq s \leq t \leq T, F_s \in \mathcal{F}_s, B \in \mathcal{B}\}
\]

\[\cup \{\{0\} \times F_0 \times B \mid F_0 \in \mathcal{F}_0 \times \mathcal{B}\})\]

At this point, for the sake of completeness, also define the predictable \(\sigma\)-field on \([0, T] \times \Omega\) by

\[\mathcal{P}_T := \sigma(g : [0, T] \times \Omega \to \mathbb{R}, \mid g \text{ is } (\mathcal{F}_t)\text{-adapted and left-continuous})\]
\[
\sigma(\{(s,t) \times F_s | 0 \leq s \leq t \leq T, F_s \in \mathcal{F}_s\} \cup \{0 \times F_0 | F_0 \in \mathcal{F}_0\}) := A
\]

Let \( \tilde{H} \) be an arbitrary Hilbert space. If \( Y : [0,T] \times \Omega \to \tilde{H} \) is \( \mathcal{P}_T/\mathcal{B}(\tilde{H}) \)-measurable it is called \((\tilde{H})\)-predictable.

**Remark 1.19.**
(i) If \( B \in \mathcal{B}([0,T]) \) then \( B \times \Omega \times U \in \mathcal{P}_T(U) \).
(ii) If \( A \in \mathcal{P}_T \) and \( B \in \mathcal{B} \) then \( A \times B \in \mathcal{P}_T(U) \).

**Proof.** (i)
\[
B \times \Omega \times U \in \mathcal{B}([0,T]) \otimes \{\Omega, \emptyset\} \otimes \{U, \emptyset\}
= \sigma(\{(s,t) \times \Omega \times U | 0 \leq s \leq t \leq T\} \cup \{0 \times \Omega \times U\})
\subset \mathcal{P}_T(U)
\]
(ii)
\[
A \times B \in \mathcal{P}_T \otimes \{B, \emptyset\} = \sigma(\{A \times B | A \in \mathcal{A}\} \cup \{0 \times \Omega \times B\})
\subset \mathcal{P}_T(U)
\]

Furthermore, for the next proposition we need the following lemma:

**Lemma 1.20.** Let \( E \) be a metric space with metric \( d \) and let \( f : \Omega \to E \) be strongly measurable, i.e. it is Borel measurable and \( f(\Omega) \subset E \) is separable. Then there exists a sequence \( f_n, n \in \mathbb{N} \), of simple \( E \)-valued functions (i.e. \( f_n \) is \( \mathcal{F}/\mathcal{B}(E) \)-measurable and takes only a finite number of values) such that for arbitrary \( \omega \in \Omega \) the sequence \( d(f_n(\omega), f(\omega)) \), \( n \in \mathbb{N} \), is monotonely decreasing to zero.

**Proof.** [DaPrZa 92, Lemma 1.1, p.16]

**Proposition 1.21.** If \( \Phi \) is an \( \mathcal{P}_T(U)/\mathcal{B}(H) \)-measurable process and
\[
E[\int_0^T \int_U \|\Phi(s,y)\|^2 \nu(dy)ds] < \infty
\]
then there exists a sequence of elementary processes \( \Phi_n, n \in \mathbb{N} \), such that \( \|\Phi - \Phi_n\|_T \to 0 \) as \( n \to \infty \).
Proof. There exist \( U_n \in B, n \in \mathbb{N} \), with \( \nu(U_n) < \infty \) such that \( U_n \uparrow U \) as \( n \to \infty \). Then \( 1_{U_n} \Phi : [0, T] \times \Omega \times U_n \to H \) is \( \mathcal{P}_T(U) \cap ([0, T] \times \Omega \times U_n) / \mathcal{B}(H) \)-measurable.

Moreover

\[
\mathcal{P}_T(U) \cap ([0, T] \times \Omega \times U_n) = \sigma(\{s, t\} \times F_s \times B \mid 0 \leq s \leq t, F_s \in \mathcal{F}_s, B \in \mathcal{B} \cap U_n) \\
\cup \{0 \times F_0 \times B \mid F_0 \in \mathcal{F}_0, B \in \mathcal{B} \cap U_n\} \\
=: \mathcal{P}_T(U_n)
\]

Therefore one gets that \( 1_{U_n} \Phi : [0, T] \times \Omega \times U_n \to H \) is \( \mathcal{P}_T(U_n) / \mathcal{B}(H) \)-measurable. Then there exists a sequence \( \Phi_n^k, k \in \mathbb{N} \), of simple random variables of the following form

\[
\Phi_k^n = \sum_{m=1}^{M_k} x_m^k 1_{A_m^k}, \quad x_m^k \in H, \quad A_m^k \in \mathcal{P}_T(U_n), \quad 1 \leq m \leq M_k, \quad k \in \mathbb{N},
\]

such that \( \|1_{U_n} \Phi - \Phi_k^n\| \downarrow 0 \) as \( k \to \infty \) by Lemma 1.20. Since

\[
\|1_{U_n} \Phi - \Phi_k^n\| \leq \|1_{U_n} \Phi\| + \|\Phi_k^n\| \leq \|1_{U_n} \Phi\| + \sum_{m=1}^{M_k} \|x_m^k\| 1_{A_m^k}
\]

\[
\in L^2([0, T] \times \Omega \times U_n, \mathcal{P}_T(U_n), \lambda \otimes \mathcal{P} \otimes \nu)
\]

one gets by Lebesgue’s dominated convergence theorem that

\[
\|1_{U_n} (\Phi - \Phi_k^n)\|^2_T = E\left[\int_0^T \int_{U_n} \|1_{U_n} (\Phi - \Phi_k^n)\|^2 \, d\nu \, d\lambda\right]
\]

\[
= E\left[\int_0^T \int_{U_n} \|1_{U_n} \Phi - \Phi_k^n\|^2 \, d\nu \, d\lambda\right] \to 0 \quad \text{as} \quad k \to \infty
\]

Choose for \( n \in \mathbb{N} \) \( k(n) \in \mathbb{N} \) such that \( \|1_{U_n} (\Phi - \Phi_k^n)\|_T < \frac{1}{n} \), then

\[
\|\Phi - 1_{U_n} \Phi_{k(n)}\|_T \leq \|\Phi - 1_{U_n} \Phi\|_T + \|1_{U_n} (\Phi - \Phi_k^n)\|_T
\]

where the first summand converges to 0 by Lebesgue’s dominated convergence theorem and the second summand is smaller than \( \frac{1}{n} \).

Thus the assertion of the Proposition is reduced to the case \( \Phi = x1_A \) where \( x \in H \) and \( A \in \mathcal{P}_T(U_n) \) for some \( n \in \mathbb{N} \). Then there is a sequence of elementary processes \( \Phi_k, k \in \mathbb{N} \), such that \( \|\Phi - \Phi_k\|_T \to 0 \) as \( k \to \infty \):
To get this result it is sufficient to prove that for any $\varepsilon > 0$ there is a finite sum $\Lambda = \bigcup_{i=1}^{N} A_i$ of predictable rectangles

$$A_i \in A_n := \{[s, t] \times F_s \times B \mid 0 \leq s \leq t \leq T, F_s \in F_s, B \in B \cap U_n\}$$

such that $P \otimes \lambda \otimes \nu(A \triangle \Lambda) \leq \varepsilon$, since then one obtains that $\sum_{i=1}^{N} x1_{A_i}$ is an elementary process, as $x1_{A_i}, 1 \leq i \leq N,$ are elementary processes and $E$ is a linear space, and

$$\|x1_A - \sum_{i=1}^{N} x1_{A_i}\|_T = (E[\int_{0}^{T} \int_{U} \|x(1_A - \sum_{i=1}^{N} 1_{A_i})\|^2 d\nu d\lambda])^{\frac{1}{2}} \leq \|x\| \varepsilon$$

Hence define $K := \{\bigcup_{i \in I} A_i \mid |I| < \infty, A_i \in A_n, i \in I\}$ then $K$ is stable under finite intersections. Now let $\mathcal{G}$ be the family of all $A \in \mathcal{P}_T(U_n)$ which can be approximated by elements of $K$ in the above sense. Then $\mathcal{G}$ is a Dynkin system and therefore $\mathcal{P}_T(U_n) = \sigma(K) = D(K) \subset \mathcal{G}$ as $K \subset \mathcal{G}$. 

Define

$$\mathcal{N}_q^2(T, U, H) := \{\Phi : [0, T] \times \Omega \times U \to H \mid \Phi \text{ is } \mathcal{P}_T(U)/\mathcal{B}(H)\text{-measurable and } \|\Phi\|_T := (E[\int_{0}^{T} \int_{U} \|\Phi(s, y)\|^2 d\nu(dy) ds])^{\frac{1}{2}} < \infty\}$$

Then $\mathcal{E} \subset \mathcal{N}_q^2(T, U, H)$ and

$$\mathcal{N}_q^2(T, U, H) = L^2([0, T] \times \Omega \times U, \mathcal{P}_T(U), P \otimes \lambda \otimes \nu, H)$$

is complete since $(H, \|\|)$ is complete. Therefore $\mathcal{E} \subset \mathcal{N}_q^2(T, U, H)$ and by the previous proposition it follows that $\mathcal{E} \supset \mathcal{N}_q^2(T, U, H)$. So finally one gets that $\mathcal{E} = \mathcal{N}_q^2(T, U, H)$

### 1.4 Properties of the stochastic integral

**Proposition 1.22.** Assume that $\Phi \in \mathcal{N}_q^2(T, U, H)$ and $u \in [0, T]$. Then $1_{[0, u]}\Phi \in \mathcal{N}_q^2(T, U, H)$ and for all $t \in [0, T]$

$$\int_{0}^{t+} \int_{U} 1_{[0, u]}\Phi(s, y) q(ds, dy) = \int_{0}^{(t \wedge u)+} \int_{U} \Phi(s, y) q(ds, dy) \quad P\text{-a.s.}$$
Proof.

**Step 1.** Let \( \Phi \) be an elementary process, i.e.

\[
\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x^m_i 1_{F^m_i 1_{[t_m, t_{m+1}]}} A^m_i \in \mathcal{E}
\]

Then

\[
1_{[u,T]} \Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x^m_i 1_{F^m_i 1_{[t_m \vee u, t_{m+1} \vee u]}} A^m_i
\]

is an elementary process since \( F^m_i \in \mathcal{F}_{t_m \vee u} \). Concerning the integral of \( 1_{[0,u]} \Phi \) one obtains that

\[
\int_0^{t+} \int_U 1_{[0,u]}(s) \Phi(s) q(ds, dy)
= \int_0^{t+} \int_U \Phi q(ds, dy) - \int_0^{t+} \int_U 1_{[u,T]}(s) \Phi q(ds, dy)
= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x^m_i 1_{F^m_i} \big(q(t_{m+1} \wedge t, A^m_i) - q(t_m \wedge t, A^m_i) - q((t_{m+1} \vee u) \wedge t, A^m_i) + q((t_m \vee u) \wedge t, A^m_i)\big)
= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x^m_i 1_{F^m_i} (q(t_{m+1} \wedge u \wedge t, A^m_i) - q(t_m \wedge u \wedge t, A^m_i))
= \int_0^{(t \wedge u)+} \int_U \Phi(s) q(ds, dy)
\]

**Step 2.** Let now \( \Phi \in \mathcal{N}^2(T, U, H) \). Then there exists a sequence of elementary processes \( \Phi_n, \ n \in \mathbb{N} \), such that \( \|\Phi_n - \Phi\|_T \to 0 \) as \( n \to \infty \). Then it is clear that \( \|1_{[0,u]} \Phi_n - 1_{[0,u]} \Phi\|_T \to 0 \) as \( n \to \infty \). By the definition of the stochastic integral it follows that for all \( t \in [0, T] \)

\[
E\left[\|\int_0^{(t \wedge u)+} \int_U \Phi_n(s, y) q(ds, dy) - \int_0^{(t \wedge u)+} \Phi(s, y) q(ds, dy)\|^2\right] + E\left[\|\int_0^{t+} \int_U 1_{[0,u]}(s) \Phi_n(s, y) q(ds, dy) - \int_0^{t+} \int_U 1_{[0,u]}(s) \Phi(s, y) q(ds, dy)\|^2\right]
\to 0 \text{ as } n \to \infty
\]

which implies that for all \( t \in [0, T] \) there exists a subsequence \( n_k(t), \ k \in \mathbb{N} \), such that

\[
\int_0^{(t \wedge u)+} \int_U \Phi_{n_k(t)}(s, y) q(ds, dy) \xrightarrow{k \to \infty} \int_0^{(t \wedge u)+} \int_U \Phi(s, y) q(ds, dy) \ P - \text{a.s}
\]
\[ \int_0^{t+} \int_U 1_{[0,u]}(s) \Phi_{n_k(t)}(s, y) \, q(ds, dy) \xrightarrow{k \to \infty} \int_0^{t+} \int_U 1_{[0,u]}(s) \Phi(s, y) \, q(ds, dy) \quad P \text{ a.s.} \]

Then by Step 1 the assertion follows.
Chapter 2

Existence of the Mild Solution

As in the previous chapter let \((H, \langle , \rangle)\) be a separable Hilbert space, \((U, \mathcal{B}, \nu)\) a \(\sigma\)-finite measure space and \((\Omega, \mathcal{F}, P)\) a complete probability space with filtration \(\mathcal{F}_t, t \geq 0\), such that \(\mathcal{F}_0\) contains all \(P\)-nullsets of \(\mathcal{F}\).

We fix a stationary \((\mathcal{F}_t)\)-Poisson point process on \(U\) with characteristic measure \(\nu\). Moreover let \(T > 0\) and consider the following type of stochastic differential equations in \(H\)

\[
\begin{align*}
    dX(t) &= [AX(t) + F(X(t))] \, dt + B(X(t), y) \, q(dt, dy) \\
    X(0) &= \xi
\end{align*}
\]

(2.1)

where

- \(A : D(A) \subset H \to H\) is the infinitesimal generator of a \(C_0\)-semigroup \(S(t), t \geq 0\), of linear, bounded operators on \(H\),
- \(F : H \to H\) is \(\mathcal{B}(H)/\mathcal{B}(H)\)-measurable,
- \(B : H \times U \to H\) is \(\mathcal{B}(H) \otimes \mathcal{B}(H)\)-measurable,
- \(q(t, B), t \geq 0, B \in \Gamma_p\), is the compensated Poisson random measure of \(p\),
- \(\xi\) is an \(H\)-valued, \(\mathcal{F}_0\)-measurable random variable.

**Remark 2.1.** If we call \(M_T := \sup_{t \in [0,T]} \|S(t)\|_{L(H)}\) then \(M_T < \infty\).

**Proof.** For example by [Pa 83, Theorem 2.2, p.4] there exist constants \(\omega \geq 0\) and \(M \geq 1\) such that

\[
\|S(t)\|_{L(H)} \leq Me^{\omega t} \quad \text{for all } t \geq 0
\]
Definition 2.2 (Mild solution). An \( H \)-valued predictable process \( X(t) \), \( t \in [0, T] \), is called a mild solution of equation (2.1) if

\[
X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\,ds
+ \int_0^{t+} \int_U S(t-s)B(X(s), y)\,q(ds, dy) \quad \text{P-a.s.}
\]

for all \( t \in [0, T] \). In particular the appearing integrals have to be well defined.

To get the existence of a mild solution on \([0, T]\) we make the following assumptions

Hypothesis H.0

- \( F : H \to H \) is Lipschitz-continuous, i.e. that there exists a constant \( C > 0 \) such that
  \[
  \|F(x) - F(y)\| \leq C\|x - y\| \quad \text{for all } x, y \in H,
  \]
- there exists a square integrable mapping \( K : [0, T] \to [0, \infty[ \) such that
  \[
  \int_U \|S(t)(B(x, y) - B(z, y))\|^2 \nu(dy) \leq K^2(t)\|x - y\|^2
  \]
  \[
  \int_U \|S(t)B(x, y)\|^2 \nu(dy) \leq K(t)(1 + \|x\|)
  \]

Now we introduce the space where we want to find the mild solution of the above problem. We define

\[
\mathcal{H}^2(T, H) := \{ Y(t), t \in [0, T] \mid Y \text{ is an } H\text{-predictable process such that } \sup_{t \in [0, T]} E[\|Y(t)\|^2] < \infty \}
\]

and for \( Y \in \mathcal{H}^2(T, H) \)

\[
\|Y\|_{\mathcal{H}^2} := \sup_{t \in [0, T]} \left( E[\|Y(t)\|^2] \right)^{\frac{1}{2}}
\]

Then \((\mathcal{H}^2(T, H), \| \cdot \|_{\mathcal{H}^2})\) is a Banach space.

For technical reasons we also consider the norms \( \|Y\|_{2,\lambda,T}, \lambda \geq 0 \), on \( \mathcal{H}^2(T, H) \) given by

\[
\|Y\|_{2,\lambda,T} := \sup_{t \in [0, T]} e^{-\lambda t} \left( E[\|Y(t)\|^2] \right)^{\frac{1}{2}}
\]
Then \( \| \|_{\mathcal{H}^2} = \| \|_{2,0,T} \) and all norms \( \| \|_{2,\lambda,T}, \lambda \geq 0 \), are equivalent.

For simplicity we use the following notations

\[
\mathcal{H}^2(T, H) := (\mathcal{H}^2(T, H), \| \cdot \|_{\mathcal{H}^2})
\]

and

\[
\mathcal{H}^{2,\lambda}(T, H) := (\mathcal{H}^2(T, H), \| \cdot \|_{2,\lambda,T}), \lambda > 0.
\]

**Theorem 2.3.** Assume that the coefficients \( A, F \) and \( B \) fulfill the conditions of Hypothesis H.0 then for every initial condition \( \xi \in L^2(\Omega, \mathcal{F}_0, P, H) =: L^2_0 \) there exists a unique mild solution \( X(\xi)(t), t \in [0, T], \) of equation (2.1).

In addition we even obtain that the mapping

\[
X : L^2_0 \to \mathcal{H}^2(T, H)
\]

is Lipschitz continuous.

For the proof of the theorem we need the following lemmas.

**Lemma 2.4.** If \( Y : [0, T] \times \Omega \times U \to H \) is \( \mathcal{P}_T(U)/\mathcal{B}(H) \)-measurable then the mapping

\[
[0, T] \times \Omega \times U \to H, (s, \omega, y) \mapsto 1_{[0,t]}(s)S(t-s)Y(s, \omega, y)
\]

is \( \mathcal{P}_T(U)/\mathcal{B}(H) \)-measurable for all \( t \in [0, T] \).

**Proof.** Let \( t \in [0, T] \).

**Step 1.** Consider the case that \( Y \) is a simple process given by

\[
Y = \sum_{k=1}^n x_k 1_{A_k}
\]

where \( x_k \in H, 1 \leq k \leq n \), and \( A_k \in \mathcal{P}_T(U), 1 \leq k \leq n \), is a disjoint covering of \([0, T] \times \Omega \times U\). Then we obtain that

\[
\tilde{Y} : [0, T] \times \Omega \times U \to H
\]

\[
(s, \omega, y) \mapsto 1_{[0,t]}(s)S(t-s)Y(s, \omega, y)
\]

\[
= 1_{[0,t]}(s) \sum_{k=1}^n S(t-s)x_k 1_{A_k}(s, \omega, y)
\]
is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable since for $B \in \mathcal{B}(H)$ we get that

$$\tilde{Y}^{-1}(B) = \bigcup_{k=1}^{n} \left( \{ s \in [0, T] \mid 1_{[0, t]}(s)S(t-s)x_k \in B \} \times \Omega \times U \right) \cap A_k$$

where $\{ s \in [0, T] \mid 1_{[0, t]}(s)S(t-s)x_k \in B \} \in \mathcal{B}([0, T])$ by the strong continuity of the semigroup $S(t)$, $t \in [0, T]$. By Lemma 1.19 (i) we can conclude that $\tilde{Y}^{-1}(B) \in \mathcal{P}_T(U)$.

**Step 2.** Let $Y$ be an arbitrary $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable process. Then there exists a sequence $Y_n$, $n \in \mathbb{N}$, of simple $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable random variables such that $Y_n \to Y$ pointwisely a $n \to \infty$. Since $S(t) \in L(H)$ for all $t \in [0, T]$ the assertion follows.

Lemma 2.5. Let $\Phi$ be a process on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$ with values in a Banach space $E$. If $\Phi$ is adapted to $\mathcal{F}_t$, $t \in [0, T]$, and stochastically continuous then there exists a predictable version of $\Phi$.

In particular, if $\Phi(t) \in L^2(\Omega, \mathcal{F}_t, P, E)$ and $\Phi : [0, T] \to L^2(\Omega, \mathcal{F}, P, E)$ is continuous then there exists a predictable version of $\Phi$.

**Proof.** [DaPrZa 92, Proposition 3.6 (ii), p.76]

**Proof of Theorem 2.3.** Let $t \in [0, T]$, $\xi \in L_0^2$ and $Y \in \mathcal{H}^2(T, H)$ and define

$$\mathcal{F}(\xi, Y)(t) := S(t)\xi + \int_0^t S(t-s)F(X(s)) \, ds$$

$$\quad + \int_0^{t+} S(t-s)B(X(s), y) \, q(ds, dy)$$

Then a mild solution of problem (2.1) with initial condition $\xi \in L_0^2$ is by Definition 2.2 an $H$-predictable process such that $\mathcal{F}(\xi, X(\xi))(t) = X(\xi)(t)$ $P$-a.s. for all $t \in [0, T]$. Thus we have to search for an implicit function $X : L_0^2 \to \mathcal{H}^2(T, H)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$ in $\mathcal{H}^2(T, H)$.

For this reason we prove that $\mathcal{F}$ as a mapping from $L_0^2 \times \mathcal{H}^2(T, H)$ to $\mathcal{H}^2(T, H)$ is well defined and we show that there exists $\lambda \geq 0$ such that

$$\mathcal{F} : L_0^2 \times \mathcal{H}^{2,\lambda}(T, H) \to \mathcal{H}^{2,\lambda}(T, H)$$

is a contraction in the second variable, i.e. that there exists $L_{T,\lambda} < 1$ such that for all $\xi \in L_0^2$ and $Y, \tilde{Y} \in \mathcal{H}^{2,\lambda}(T, H)$

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\xi, \tilde{Y})\|_{2,\lambda, T} \leq L_{T,\lambda}\|Y - \tilde{Y}\|_{2,\lambda, T}.$$
Then the existence and uniqueness of the mild solution $X(\xi) \in \mathcal{H}^{2,\lambda}(T, H)$ of \eqref{2.1} with initial condition $\xi \in L_0^2$ follows by Banach’s fixpoint theorem. Since the norms $\| \cdot \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent we consider $X(\xi)$ as an element of $\mathcal{H}^2(T, H)$ and get the existence of the implicit function $X : L_0^2 \to \mathcal{H}^2(T, H)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$.

**Step 1.** The mapping $\mathcal{F} : L_0^2 \times \mathcal{H}^2(T, H) \to \mathcal{H}^2(T, H)$ is well defined.

Let $\xi \in L_0^2$ and $Y \in \mathcal{H}^2(T, H)$ then, by [FrKn 2002], $(S(t)\xi)_{t \in [0,T]} \in \mathcal{H}^2(T, H)$, $1_{[0,t]}(\cdot)S(t-\cdot)F(Y(\cdot))$ is $P$-a.s. Bochner integrable on $[0,T]$ and the process

$$\left( \int_0^t S(t-s)F(Y(s)) \, ds \right)_{t \in [0,T]}$$

is an element of $\mathcal{H}^2(T, H)$.

Therefore it remains to prove that:

$$(1_{[0,t]}(\cdot)S(t-s)B(Y(s), \cdot))_{s \in [0,T]} \in \mathcal{N}^2_q(T, U, H)$$

for all $t \in [0,T]$ and that there is a version of

$$\left( \int_0^t \int_U S(t-s)B(X(s), y) \, q(ds, dy) \right)_{t \in [0,T]}$$

which is an element of $\mathcal{H}^2(T, H)$.

**Claim 1.** If $Y \in \mathcal{H}^2(T, H)$ then:

$$\Phi := (1_{[0,t]}(s)S(t-s)B(Y(s), \cdot))_{s \in [0,T]} \in \mathcal{N}^2_q(T, U, H)$$

for all $t \in [0,T]$.

Let $t \in [0,T]$. First, we prove that the mapping

$$[0,T] \times \Omega \times U \to H, \ (s, \omega, y) \mapsto 1_{[0,t]}(s)S(t-s)B(Y(s, \omega), y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. By Lemma 2.4 we have to check if the mapping

$$(s, \omega, y) \mapsto B(Y(s, \omega), y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable.

The mapping $F : [0,T] \times \Omega \times U \to H \times U, \ (s, \omega, y) \mapsto (Y(s, \omega), y)$ is $\mathcal{P}_T(U)/\mathcal{B}(H) \otimes \mathcal{B}$-measurable since for $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}$ we have that

$$F^{-1}(A \times B) = Y^{-1}(A) \times B \in \mathcal{P}_T(U)$$

by Lemma 1.19 (ii).

Moreover $B$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable by assumption.

With respect to the norm $\| \|_T$ of $\Phi$ we obtain

$$\|\Phi\|_T^2 = E \left[ \int_0^t \int_U \|1_{[0,t]}(s)S(t-s)B(Y(s), y)\|^2 \nu(dy) \, ds \right]$$
\[ \leq E\left[ \int_0^t K(t-s)(1 + \|Y(s)\|) \, ds \right] \]
\[ \leq (1 + \|Y\|_{\mathcal{H}^2}) \int_0^T K(s) \, ds \]
\[ < \infty \]

**Claim 2.** If \( Y \in \mathcal{H}^2(T, H) \) then there is a predictable version of

\[ (Z(t))_{t \in [0, T]} := \left( \int_0^{t+} \int_U S(t-s)B(Y(s), y) \, q(ds, dy) \right)_{t \in [0, T]} \]

which is an element of \( \mathcal{H}^2(T, H) \).

Since \((1_{[0,t]}(s)S(t-s)B(Y(s), \cdot))_{s \in [0,T]} \in \mathcal{N}^2_q(T, U, H)\) for all \( t \in [0, T] \) we get by the isometric formula that

\[ \sup_{t \in [0,T]} E\left[ \| \int_0^{t+} \int_U S(t-s)B(Y(s), y) \, q(ds, dy) \|^2 \right] \]
\[ = \sup_{t \in [0,T]} E\left[ \int_0^t \| S(t-s)B(Y(s), y) \|^2 \, \nu(dy) \, ds \right] \]
\[ \leq (1 + \|Y\|_{\mathcal{H}^2}) \int_0^T K(s) \, ds \]
\[ < \infty \]

To prove the existence of the predictable version we will use Lemma 2.5. For this purpose we will show that the process \( Z \) is adapted to \( \mathcal{F}_t, t \in [0, T] \), and continuous as a mapping from \([0, T]\) to \( L^2(\Omega, \mathcal{F}, P, H) \).

Let \( \alpha > 1 \) and define for \( t \in [0, T] \)

\[ Z^\alpha(t) := \int_0^{\left(\frac{t}{\alpha}\right)+} \int_U S(t-s)B(Y(s), y) \, q(ds, dy) \]
\[ = \int_0^{\left(\frac{t}{\alpha}\right)+} \int_U S(t-\alpha s)S((\alpha - 1)s)B(Y(s), y) \, q(ds, dy) \]

where we used semigroup property.

Set \( \Phi^\alpha(s, y) := S((\alpha - 1)s)B(Y(s), y) \) then one can show analogously to the proof of the \( \mathcal{P}_T(U)/\mathcal{B}(H) \)-measurability of the mapping \((s, \omega, y) \mapsto 1_{[0,t]}(s)S(t-s)B(Y(s, \omega), y)\) that \( \Phi^\alpha \) is \( \mathcal{P}_T(U)/\mathcal{B}(H) \)-measurable. Moreover

\[ E\left[ \int_0^t \int_U \| S((\alpha - 1)s)B(Y(s), y) \|^2 \, \nu(dy) \, ds \right] \]
\[ \leq (1 + \|Y\|_{\mathcal{H}^2}) \int_0^T K((\alpha - 1)s) \, ds \]
\[ = (1 + \|Y\|_{\mathcal{H}^2}) \frac{1}{\alpha - 1} \int_0^T K(s) \, ds \]
\[ < \infty \]

Therefore we obtain that \( \Phi^\alpha \in \mathcal{N}^2_q(T, U, H) \).

Now we show that the mapping \( Z^\alpha : [0, T] \rightarrow L^2(\Omega, \mathcal{F}, P, H) \) is continuous for all \( \alpha > 1 \). For this reason let \( 0 \leq u \leq t \leq T \).

\[ \left( E\left[ \left\| \int_0^{T^+} \int_U S(t - \alpha s) \Phi^\alpha(s, y) q(ds, dy) - \int_0^{T^+} \int_U S(u - \alpha s) \Phi^\alpha(s, y) q(ds, dy) \right\|^2 \right]\right)^{\frac{1}{2}}, \]
\[ = \left( E\left[ \left\| \int_0^{T^+} \int_U \left[ 1_{[0, u]}(s) S(t - \alpha s) \Phi^\alpha(s, y) - 1_{[0, u]}(s) S(u - \alpha s) \Phi^\alpha(s, y) \right] q(ds, dy) \right\|^2 \right]\right)^{\frac{1}{2}} \]

by Proposition 1.22,

\[ \leq \left( E\left[ \left\| \int_0^{T^+} \int_U \left[ 1_{[0, u]}(s) S(t - \alpha s) - S(u - \alpha s) \right] \Phi^\alpha(s, y) q(ds, dy) \right\|^2 \right]\right)^{\frac{1}{2}} + \left( E\left[ \left\| \int_0^{T^+} \int_U \left[ 1_{[0, u]}(s) S(t - \alpha s) \Phi^\alpha(s, y) q(ds, dy) \right]\right\|^2 \right]\right)^{\frac{1}{2}} \]
\[ = \left( E\left[ \int_0^{T^+} \left\| (S(t - \alpha s) - S(u - \alpha s)) \Phi^\alpha(s, y) \right\|^2 \nu(dy) \, ds \right]\right)^{\frac{1}{2}} + \left( E\left[ \int_0^{T^+} \left\| (S(t - \alpha s) - S(u - \alpha s)) \Phi^\alpha(s, y) \right\|^2 \nu(dy) \, ds \right]\right)^{\frac{1}{2}}, \]

by the isometric formula.

(1.) The first summand converges to 0 as \( u \uparrow t \) or \( t \downarrow u \) by Lebesgue’s dominated convergence theorem since the integrand converges pointwisely to 0 as \( u \uparrow t \) or \( t \downarrow u \) by the strong continuity of the semigroup and can be estimated independently of \( u \) and \( t \) by \( 4M^2_B \| \Phi^\alpha \|^2(s, y), (s, y) \in [0, T] \times U \).
where \(E\left[\int_0^T \int_U \|\Phi^\alpha(s, y)\|^2 \nu(dy) \, ds\right] < \infty\).

(2.) The second summand can be estimated by

\[
\left( E\left[\int_0^T \int_U 1_{\left\{ \frac{s}{n} \leq t \right\}}(s) M_n^2 \|\Phi^\alpha(s, y)\|^2 \nu(dy) \, ds\right]\right)^{\frac{1}{2}} \\
\to 0
\]

and therefore converges to 0 by Lebesgue’s dominated convergence theorem as \(u \uparrow t\) or \(t \downarrow u\).

To obtain the continuity of \(Z : [0, T] \to L^2(\Omega, \mathcal{F}, P)\) we prove the uniform convergence of \(Z^{\alpha_n}, n \in \mathbb{N}\), to \(Z\) in \(L^2(\Omega, \mathcal{F}, P, H)\) for an arbitrary sequence \(\alpha_n, n \in \mathbb{N}\), with \(\alpha_n \downarrow 1\) as \(n \to \infty\):

\[
E\left[\left\|\int_0^T \int_U \left( S(t - \alpha_n s) \Phi^\alpha(s, y) q(ds, dy) - \int_0^t \int_U S(t - s) B(Y(s), y) q(ds, dy)\right)\right\|^2\right] \\
= E\left[\left\|\int_0^T \int_U 1_{[0, \frac{t}{\alpha_n}]}(s) S(t - s) B(Y(s), y) q(ds, dy)\right\|^2\right] \\
= E\left[\left\|\int_0^{t+} \int_U \|S(t - s) B(Y(s), y)\|^2 q(ds, dy)\right\|^2\right] \\
\leq E\left[\int_0^T K(t - s) (1 + \|Y(s)\|) ds\right] \\
\leq (1 + \|Y\|_{L^2}) \left( t - \frac{t}{\alpha_n}\right)^{\frac{1}{2}} \left( \int_0^T K^2(s) ds\right)^{\frac{1}{2}} \\
\leq (1 + \|Y\|_{L^2}) \left( \frac{\alpha_n - 1}{\alpha_n} T\right)^{\frac{1}{2}} \left( \int_0^T K^2(s) ds\right)^{\frac{1}{2}}
\]

where \(\frac{\alpha_n - 1}{\alpha_n} T \to 0\) as \(n \to \infty\).

Moreover we know for all \(t \in [0, T]\) that

\[
\left( \int_0^{t+} \int_U 1_{[0, u]}(s) S(t - s) B(Y(s), y) q(ds, dy)\right)_{u \in [0, t]} \in \mathcal{M}_t^2(H)
\]

since \((1_{[0, u]}(s) S(t - s) B(Y(s), \cdot))_{s \in [0, t]} \in \mathcal{N}_t^2(t, U, H)\). That means in particular that the process

\[Z(t) = \int_0^{t+} \int_U 1_{[0, t]}(s) S(t - s) B(Y(s), y) q(ds, dy), \ t \in [0, T]\]

is \((\mathcal{F}_t)\)-adapted.
Together with the continuity of $Z$ in $L^2(\Omega, \mathcal{F}, P < H)$, by Lemma 2.5, this implies the existence of a predictable version of $Z(t)$, $t \in [0, T]$, denoted by

$$
\left( \int_0^t \int_U S(t-s)B(Y(s), y) q(ds, dy) \right)_{t \in [0, T]}.
$$

Therefore we have finally proved that

$$
\mathcal{F} : L^2_0 \times \mathcal{H}^2(T, H) \rightarrow \mathcal{H}^2(T, H)
$$

**Claim 3.** There exists $\lambda \geq 0$ such that for all $\xi \in L^2_0$

$$
\mathcal{F}(\xi, \cdot) : \mathcal{H}^{2,\lambda}(T, H) \rightarrow \mathcal{H}^{2,\lambda}(T, H)
$$

is a contraction where the contraction constant $L_{T,\lambda} < 1$ does not depend on $\xi$.

Let $Y, \tilde{Y} \in \mathcal{H}^2(T, H)$, $\xi \in L^2_0$. Then we get for $\lambda \geq 0$ that

$$
\sup_{t \in [0, T]} e^{-\lambda t} \| (\mathcal{F}(\xi, Y) - \mathcal{F}(\xi, \tilde{Y})(t) \|_{L^2}
$$

$$
\leq \sup_{t \in [0, T]} e^{-\lambda t} \int_0^t S(t-s)[F(Y(s)) - F(\tilde{Y}(s))] ds \|_{L^2}
$$

$$
+ \sup_{t \in [0, T]} e^{-\lambda t} \int_0^{t+} \int_U S(t-s)[B(Y(s), y) - B(\tilde{Y}(s), y)] q(ds, dy) \|_{L^2}
$$

The first summand can be estimated by

$$
M_T C T^{1/4} \left( \frac{1}{2\lambda} \right)^{1/2} \| Y - \tilde{Y} \|_{2,\lambda, T},
$$

for the proof see [FrKn 2002, Theorem 3.2., Step 3, p.81].

By the isometric formula we get the following estimation for the second summand:

$$
E[\| \int_0^{t+} \int_U S(t-s)B(Y(s), y) q(ds, dy) - \int_0^{t+} \int_U S(t-s)B(\tilde{Y}(s), y) q(ds, dy) \|^2]
$$

$$
= E\left[ \int_0^1 \int_U \| S(t-s)[B(Y(s), y) - B(\tilde{Y}(s), y)] \|^2 \nu(dy) ds \right]
$$

$$
\leq E\left[ \int_0^1 K^2(t-s) \| Y(s) - \tilde{Y}(s) \|^2 ds \right]
$$
36

\[ \leq \int_0^t e^{\lambda s} K^2(t - s) \, ds \|Y - \tilde{Y}\|_{2,\lambda,T}^2 \]

\[ = \|Y - \tilde{Y}\|_{2,\lambda,T}^2 e^{-\lambda T} \int_0^T e^{-\lambda s} K^2(s) \, ds \]

Therefore we obtain that

\[ \lim_{\lambda \to \infty} \sup_{t \in [0,T]} e^{-\lambda t} \| \int_0^{t^+} \int_U S(t - s) [B(Y(s), y) - B(\tilde{Y}(s), y)] q(ds, dy) \|_{L^2} \leq (\int_0^t e^{-\lambda s} K^2(s) \, ds)^{\frac{1}{2}} \|Y - \tilde{Y}\|_{2,\lambda,T} \]

Thus we have finally proved that there exists \( \lambda \geq 0 \) such that there exists \( L_{T,\lambda} < 1 \) with

\[ \| \mathcal{F}(\xi, Y) - \mathcal{F}(\zeta, \tilde{Y}) \|_{2,\lambda,T} \leq L_{T,\lambda} \| Y - \tilde{Y} \|_{2,\lambda,T} \]

for all \( \xi \in L_0^2 \), \( Y, \tilde{Y} \in \mathcal{H}^{2,\lambda}(T, H) \). Hence the existence of a unique implicit function

\[ X : L_0^2 \to \mathcal{H}^2(T, H) \]

\[ \xi \mapsto X(\xi) = \mathcal{F}(\xi, X(\xi)) \]

is verified.

**Claim 4.** The mapping \( X : L_0^2 \to \mathcal{H}^2(T, H) \) is Lipschitz continuous.

By Theorem A.1 (ii) and the equivalence of the norms \( \| \cdot \|_{2,\lambda,T}, \lambda \geq 0 \), we only have to check that the mappings

\[ \mathcal{F}(\cdot, Y) : L_0^2 \to \mathcal{H}^2(T, H) \]

are Lipschitz continuous for all \( Y \in \mathcal{H}^2(T, H) \) where the Lipschitz constant does not depend on \( Y \).

But this assertion holds as for all \( \xi, \zeta \in L_0^2 \) and \( Y \in \mathcal{H}^2(T, H) \)

\[ \| \mathcal{F}(\xi, Y) - \mathcal{F}(\zeta, Y) \|_{\mathcal{H}^2} = \| S(\cdot)(\xi - \zeta) \|_{\mathcal{H}^2} \leq M_T \| \xi - \zeta \|_{L^2}. \]

\( \square \)
Appendix A

Continuity of Implicit Functions

We fix two Banach spaces \((E, \|\|)\) and \((\Lambda, \|\|_{\Lambda})\).
Consider a mapping \(G : \Lambda \times E \rightarrow E\) such that there exists an \(\alpha \in [0, 1]\) such that
\[
\|G(\lambda, x) - G(\lambda, y)\| \leq \alpha \|x - y\|
\]
for all \(\lambda \in \Lambda\) and all \(x, y \in E\).

Then we get by Banach’s fixpoint theorem that there exists exactly one mapping \(\varphi : \Lambda \rightarrow E\) such that
\[
\varphi(\lambda) = G(\lambda, \varphi(\lambda)) \text{ for all } \lambda \in \Lambda.
\]

**Theorem A.1 (Continuity of the implicit function).**

(i) If we assume in addition that the mapping \(\lambda \mapsto G(\lambda, x)\) is continuous from \(\Lambda\) to \(E\) for all \(x \in E\) we get that \(\varphi : \Lambda \rightarrow E\) is continuous.

(ii) If the mappings \(\lambda \mapsto G(\lambda, x)\) are not only continuous from \(\Lambda\) to \(E\) for all \(x \in E\) but there even exists a \(L \geq 0\) such that
\[
\|G(\lambda, x) - G(\tilde{\lambda}, x)\|_{E} \leq L\|\lambda - \tilde{\lambda}\|_{\Lambda}\text{ for all } x \in E
\]
then the mapping \(\varphi : \Lambda \rightarrow E\) is Lipschitz continuous.

**Proof.** [FrKn 2002, Theorem D.1, p.164]
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