Stochastic integrals and stochastic differential equations with respect to compensated Poisson random measures in infinite dimensional Hilbert spaces

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Contents

Introduction

The purpose of this paper is to give a complete proof of the existence of a mild solution of a stochastic differential equation with respect to a compensated Poisson random measure by a fixpoint argument in the spirit of [DaPrZa 96]. This will be done within the following framework.
Let $(H, \langle\ ,\ \rangle)$ be an infinite dimensional, separable Hilbert space, $(U, \mathcal{B}, \nu)$ a σ-finite measure space and $(\Omega, \mathcal{F}, P)$ a complete probability space with filtration $\mathcal{F}_t$, $t \geq 0$ such that $\mathcal{F}_0$ contains all P-nullset of $\mathcal{F}$. Consider the following stochastic differential equation in H on the intervall $[0, T]$, $T > 0$:

$$(1) \qquad \begin{cases} dX(t) &= [AX(t) + F(X(t))]\,dt + B(X(t), y)\; q(dt, dy) \\ X(0) &= \xi \end{cases}$$

where

- $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of linear, bounded operators on H,
- $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable,
- $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable,
- $q(dt, dy) := \Pi(dt, dy) - \lambda(dt) \otimes \nu(dy)$, is a compensated Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ where Π is a Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ with intensity measure $\lambda(ds) \otimes \nu(dy)$,
- ξ is an H-valued, $\mathcal{F}_0$-measurable random variable.

A mild solution of equation (1) is an H-valued predictable process such that

$$\begin{aligned}X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\,ds \\ + \int_0^{t+}\int_U S(t-s)B(X(s),y)\,q(ds,dy) \quad P\text{-a.s.}\end{aligned}$$

for all $t \in [0,T]$.

The organization of this paper is as follows.

In Chapter 1 we present the definition of that type of stochastic integral with respect to a compensated Poisson random measure which we use in this paper. For this end, in Section 1 and 2 we first repeat the notions of Poisson random measures and Poisson point processes where we refer to the book [IkWa 81].
In Section 3, the construction of the stochastic integral of Hilbert space valued predictable processes with respect to a compensated Poisson random measure with intensity measure $\lambda(ds) \otimes \nu(dy)$ will be done by an isometric formula in the style of the definition of the stochastic integral with respect to the Wiener process in [DaPrZa 92] or square integrable martingales in [Me 82]. For real valued processes this can be found in [BeLi 82]. Independently, this definition was done in [Rue 2003].
Denote by $\mathcal{E}$ the space of elementary processes where an H-valued process $\Phi(t) : \Omega \times U \to H$, $t \in [0,T]$, on $(\Omega \times U, \mathcal{F} \otimes \mathcal{B}, P \otimes \nu)$ is said to be *elementary* if there exist $0 = t_0 < t_1 < \cdots < t_k = T$ and for $m \in \{0, \ldots, k-1\}$ exist $B_1^m, \ldots, B_{I(m)}^m \in \Gamma_p$, $I(m) \in \mathbb{N}$, pairwise disjoint, such that

$$\Phi = \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]t_m,t_{m+1}]\times B_i^m}$$

where $x_i^m \in H$ and $F_i^m \in \mathcal{F}_{t_m}$, $1 \le i \le I(m)$, $0 \le m \le k-1$.
Define

$$\begin{aligned}&\mathrm{Int}(\Phi)(t,\omega) \\ &:= \int_0^{t+}\int_U \Phi(s,y)\,q(ds,dy)(\omega) := \int_0^T\int_U 1_{]0,t]}(s)\Phi(s,y)\,q(ds,dy)(\omega) \\ &:= \sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m}(\omega)(q(\omega)(t_{m+1}\wedge t, B_i^m) - q(\omega)(t_m \wedge t, B_i^m)),\end{aligned}$$

$t \in [0,T]$ and $\omega \in \Omega$.
Then, if $\Phi \in \mathcal{E}$, $\mathrm{Int}(\Phi) \in \mathcal{M}_T^2(H)$ which denotes the space of all square inte-

grable H-valued martingales and we obtain the following isometric formula

$$\|\mathrm{Int}(\Phi)\|^2_{\mathcal{M}^2_T} := sup_{t\in[0,T]} E[\| \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)\|^2]$$
$$= E[\int_0^T \int_U \|\Phi(s,y)\|^2\, \nu(dy)\, ds] =: \|\Phi\|_T,$$

i.e. Int: $(\mathcal{E}, \| \, \|_T) \to (\mathcal{M}^2_T(H), \| \, \|_{\mathcal{M}^2_T})$ is an isometric transformation and can therefore be extended to the space $\bar{\mathcal{E}}^{\| \, \|_T}$. $\bar{\mathcal{E}}^{\| \, \|_T}$ can be characterized by

$$\mathcal{N}^2_q(T,U,H) = L^2([0,T] \times \Omega \times U, P_T(U), P \otimes \lambda \otimes \nu; H).$$

The main emphazis is on the Chapter 2 where we prove the existence of the mild solution

$$X(\xi) \in \mathcal{H}^2(T,H) := \{Y(t), t \in [0,T] \mid Y \text{ is an } H\text{-predictable process s.t.} \\ \|Y\|_{\mathcal{H}^2} := \sup_{t\in[0,T]} E[\|Y(t)\|^2] < \infty\}$$

of problem (1) and the continuity of the mapping $X : L^2(\Omega, \mathcal{F}_0, P, H) \to \mathcal{H}^2(T,H)$.
A mild solution of the stochastic differential equation (1) is defined implicitly by $X(\xi) = \mathcal{F}(\xi, X(\xi))$, where $\mathcal{F} : L^2(\Omega, \mathcal{F}_0, P, H) \times \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is given by

$$\mathcal{F}(\xi, Y)(t) = S(t)\xi + \int_0^t S(t-s)F(Y(s))\, ds \\ + \int_0^{t+} \int_U S(t-s)B(Y(s),y)\, q(ds,dy), \quad t \in [0,T].$$

To obtain the existence of the solution, first, we have to show that $\mathcal{F}(\xi, Y)$ is well defined for all $\xi \in L^2(\Omega, \mathcal{F}_0, P, H)$ and $Y \in \mathcal{H}^2(T,H)$ and is an element of $\mathcal{H}^2(T,H)$. In particular, this includes the proof of the existence of a predictable version of the stochastic integral denoted by

$$\int_0^{t-} \int_U S(t-s)B(Y(s),y)\, q(ds,dy), \quad t \in [0,T].$$

Secondly, to apply a fixpoint argument, we have to prove that $\mathcal{F}$ is a contraction in the second variable.

In a future paper the differential dependence of the mild solution on the initial data will be examined and it will be proved that

$$X : L^2(\Omega, \mathcal{F}_0, P, H) \to \mathcal{H}^2(T,H)$$

is Gâteaux differentiable.

Chapter 1

The Stochastic Integral with Respect to Poisson Point Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(U, \mathcal{B})$ a measurable space.

1.1 Poisson random measures

Let $\mathbb{M}$ be the space of non-negative (possibly infinte) integral-valued measures on $(U, \mathcal{B})$ and

$$\mathcal{B}_{\mathbb{M}} := \sigma(\mathbb{M} \to \mathbb{Z}_+ \cup \{+\infty\}, \mu \mapsto \mu(B) \,|\, B \in \mathcal{B})$$

Definition 1.1 (Poisson random measure). A random variable $\Pi : (\Omega, \mathcal{F}) \to (\mathbb{M}, \mathcal{B}_{\mathbb{M}})$ is called *Poisson random measure* on $(U, \mathcal{B})$ if the following conditions hold:

(i) For all $B \in \mathcal{B}$: $\Pi(B) : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ is Poisson distributed with parameter $E(\Pi(B))$, i.e.:

$$P(\Pi(B) = n) = exp\big(-E(\Pi(B))\big)(E(\Pi(B)))^n/n!, \ n \in \mathbb{N} \cup \{0\}$$

If $E(\Pi(B)) = +\infty$ then $\Pi(B) = +\infty$ P-a.s.

(ii) If $B_1, \dots, B_m \in \mathcal{B}$ are pairwise disjoint then $\Pi(B_1), \dots, \Pi(B_m)$ are independent.

Remark 1.2. If Π is a Poisson random measure then the mapping $\Omega \to \mathbb{Z}_+ \cup \{+\infty\}$, $\omega \mapsto \Pi(\omega)(B)$, $B \in \mathcal{B}$, is $\mathcal{F}$-measurable since the mapping $\Omega \to \mathbb{M}$, $\omega \mapsto \Pi(\omega)$ is $\mathcal{F}/\mathcal{B}_{\mathbb{M}}$-measurable by Definition 1.1 and since the mapping $\mathbb{M} \to \mathbb{Z}_+ \cup \{+\infty\}$, $\mu \mapsto \mu(B)$ is $\mathcal{B}_{\mathbb{M}}$-measurable by the definition of $\mathcal{B}_{\mathbb{M}}$.

Lemma 1.3. *Let $m \in \mathbb{N}$ and μ and ν be two probability measures on $[0,\infty[^m$. If for all $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$*

$$\int_{[0,\infty[^m} e^{-\langle \alpha, x\rangle}\,\mu(dx) = \int_{[0,\infty[^m} e^{-\sum_{j=1}^m \alpha_j x_j}\,\mu(d(x_1,\dots,x_m))$$
$$= \int_{[0,\infty[^m} e^{-\sum_{j=1}^m \alpha_j x_j}\,\nu(d(x_1,\dots,x_m)) = \int_{[0,\infty[^m} e^{-\langle \alpha, x\rangle}\,\nu(dx).$$

then $\mu = \nu$.

Proof. Denote by $\mathcal{H}$ the space of all $\mathcal{B}(\mathbb{R}_+^m)$-measurable functions $f : \mathbb{R}_+^m \to \mathbb{R}$ such that $\int_{\mathbb{R}_+^m} f\,d\mu = \int_{\mathbb{R}_+^m} f\,d\nu$. Then $\mathcal{H}$ is a monotone vector space. Moreover define

$$\mathcal{A} := \{\mathbb{R}_+^m \to \mathbb{R}, x \mapsto \exp(-\sum_{j=1}^m \alpha_j x_j)\,|\,\alpha_j \in \mathbb{Q}_+, 1 \le j \le m\}.$$

Then $\mathcal{A}$ is a class of bounded, measurable functions, which is closed under multiplication and which is a subset of $\mathcal{H}$ by assumption. By the monoton class theorem it follows that $\sigma(\mathcal{A})_b \subset \mathcal{H}$.
Moreover, $\mathcal{A} \subset \{f : \mathbb{R}_+^m \to \mathbb{R}\,|\, f \text{ is } \mathcal{B}(\mathbb{R}_+^m)\text{-measurable}\}$ is countable and separates the points of $\mathbb{R}_+^m$. Thus, we obtain that $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}_+^m)$ and $\mathcal{B}(\mathbb{R}_+^m)_b \subset \mathcal{H}$. In particular, we get for $A \in \mathcal{B}(\mathbb{R}_+^m)$ that $\mu(A) = \nu(A)$. □

Lemma 1.4. *Let X be a Poissonian random variable on $(\Omega, \mathcal{F}, P)$ with parameter $c > 0$, i.e. $X : \Omega \to \mathbb{Z}_+ \cup \{+\infty\}$ such that for all $n \in \mathbb{N} \cup \{0\}$: $P(X = n) = c^n \frac{exp(-c)}{n!}$. Then*

$$E(e^{\alpha X}) = \int_0^\infty e^{\alpha x}\,P \circ X^{-1}(dx) = \sum_{n=0}^\infty e^{n\alpha} e^{-c}\frac{c^n}{n!} = exp(c\,(e^\alpha - 1))\,\forall \alpha \in \mathbb{R}$$

Theorem 1.5. *Given a σ-finite measure ν on $(U, \mathcal{B})$ there exists a Poisson random measure Π on $(U, \mathcal{B})$ with $E(\Pi(B)) = \nu(B)$ for all $B \in \mathcal{B}$. ν is then called the* mean measure *or* intensity measure *of the Poisson random measure Π.*

Proof. [IkWa 81,Theorem 8.1, p.42]

Step 1. $\nu(U) < \infty$
Let N be a Poissonian random variable with parameter $c := \nu(U)$.
Moreover let $\xi_1, \xi_2, \dots$ be independent U-valued random variables with distribution $\frac{1}{c}\,\nu$, also independent of N.
Define $\Pi := \sum_{k=1}^{N} \delta_{\xi_k}$.

Claim 1. Let $B \in \mathcal{B}$.Then $\Pi(B)$ is Poisson distributed with parameter $\nu(B)$.

Let $s \leq 0$, then

$$\begin{aligned}
&E(e^{s\Pi(B)})\\
&= E\big[\exp(s\sum_{k=1}^{N}\delta_{\xi_k}(B))\big] \quad \text{, if } N = 0 \text{ then } \sum_{k=1}^{N}\delta_{\xi_k}(B) = 0\\
&= E\big[\sum_{n=0}^{\infty}\exp(s\sum_{k=1}^{n}1_B(\xi_k))1_{\{N=n\}}\big]\\
&= \sum_{n=0}^{\infty}E\big[\prod_{k=1}^{n}\exp(s1_B(\xi_k))1_{\{N=n\}}\big]\\
&= \sum_{n=0}^{\infty}E\big[\prod_{k=1}^{n}\exp(s1_B(\xi_k))\big]P(N=n)\\
&= \sum_{n=0}^{\infty}\big(E\big[\exp(s1_B(\xi_1))\big]\big)^n e^{-c}\frac{c^n}{n!}\\
&= \exp\big(c\,(E[\exp(s1_B(\xi_1))] - 1)\big)\\
&= \exp\big(c\,P(\xi_1 \in B)e^s + c\,P(\xi_1 \in B^c) - c)\big)\\
&= \exp\big(c\,\frac{\nu(B)}{c}e^s + c\,(1 - \frac{\nu(B)}{c}) - c\big)\\
&= \exp\big(\nu(B)(e^s - 1)\big)
\end{aligned}$$

By Lemma 1.4 and Lemma 1.3 the assertion follows.

Claim 2. Let $B_1, \dots, B_m \in \mathcal{B}$ pairwise disjoint. Then $\Pi(B_1), \dots, \Pi(B_m)$ are independent.

Let $s_1, \dots, s_m \in \mathbb{R}_-$, then:

$$\int_{[0,\infty[^m}\exp(\sum_{j=1}^{m}s_jx_j)\,P\circ(\Pi(B_1),\dots,\Pi(B_m))^{-1}d(x_1,\dots,x_m)$$

$$
\begin{aligned}
&= E\big[\exp(\sum_{j=1}^{m} s_j \Pi(B_j))\big] \\
&= E\big[\sum_{n=0}^{\infty} \exp(\sum_{j=1}^{m} s_j \sum_{k=1}^{n} 1_{B_j}(\xi_k)) 1_{\{N=n\}}\big] \\
&= \sum_{n=0}^{\infty} E\big[\prod_{k=1}^{n} \exp(\sum_{j=1}^{m} s_j 1_{B_j}(\xi_k))\big] e^{-c}\frac{c^n}{n!} \\
&= \sum_{n=0}^{\infty} \big(E\big[\exp(\sum_{j=1}^{m} s_j 1_{B_j}(\xi_1))\big]\big)^n e^{-c}\frac{c^n}{n!} \\
&= \exp\big(c\,\big(E\big[\exp(\sum_{j=1}^{m} s_j 1_{B_j}(\xi_1))\big] - 1\big)\big) \\
&= \exp\Big(c\,\big(E\big[\ 1_{\{\xi_1 \in \bigcup_{j=1}^{m} B_j\}} \exp(\sum_{j=1}^{m} s_j 1_{B_j}(\xi_1)) \\
&\qquad\qquad + 1_{\{\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c\}} \exp(\sum_{j=1}^{m} s_j 1_{B_j}(\xi_1))\big] - 1\big)\Big) \\
&= \exp\Big(c\big(E\big[\sum_{j=1}^{m} 1_{\{\xi_1 \in B_j\}} e^{s_j} + 1_{\{\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c\}}\big] - 1\big)\Big) \\
&= \exp\Big(c\big(\sum_{j=1}^{m} P(\xi_1 \in B_j) e^{s_j} + P(\xi_1 \in (\bigcup_{j=1}^{m} B_j)^c) - 1\big)\Big) \\
&= \exp\Big(c\big(\sum_{j=1}^{m} \frac{\nu(B_j)}{c} e^{s_j} + (1 - \sum_{j=1}^{m} \frac{\nu(B_j)}{c}) - 1\big)\Big) \\
&= \exp\Big(\sum_{j=1}^{m} \nu(B_j)(e^{s_j} - 1)\big) = \prod_{j=1}^{m} \exp(\nu(B_j)(e^{s_j} - 1)\Big) \\
&= \prod_{j=1}^{m} \int_0^{\infty} \exp(s_j x_j)\, P \circ \Pi(B_j)^{-1}(dx_j) \\
&= \int_{[0,\infty[^m} \exp(\sum_{j=1}^{m} s_j x_j)\, P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1} d(x_1, \ldots, x_m)
\end{aligned}
$$

Hence, by Proposition 1.3, we can conclude that

$$
P \circ (\Pi(B_1), \ldots, \Pi(B_m))^{-1} = P \circ \Pi(B_1)^{-1} \otimes \cdots \otimes P \circ \Pi(B_m)^{-1}
$$

which implies the required independence.

Step 2. ν is σ-finite
There exist $U_i \in \mathcal{B}$, $i \in \mathbb{N}$, pairwise disjoint such that $\nu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $U = \bigcup_{i=1}^{\infty} U_i$. Set $\nu_i := \nu(\cdot \cap U_i)$, $i \in \mathbb{N}$.
For $i \in \mathbb{N}$ let N_i be a Poissonian random variable with parameter $c_i := \nu(U_i)$ and $\xi_1^i, \xi_2^i, \ldots$ independent U_i-valued random variables with distribution $\frac{1}{c_i}\nu_i$, also independent of N_i. Moreover the families of random variables $\{N_i, \xi_1^i, \xi_2^i, \ldots\}_{i\in\mathbb{N}}$ are independent.
Let Π_i be the Poisson random measure on U_i associated with N_i and $\xi_1^i, \xi_2^i, \ldots$ with intensity measure ν_i as defined in Step 1.
Define $\Pi := \sum_{i=1}^{\infty} \Pi_i := \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{\xi_k^i}$. Then one has for $B \in \mathcal{B}$ that

$$\begin{aligned}\Pi(B) &= \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{\xi_k^i}(B) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_B(\xi_k^i) = \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} 1_{B\cap U_i}(\xi_k^i) \\ &= \sum_{i=1}^{\infty} \Pi_i(B \cap U_i)\end{aligned}$$

Claim 1. Let $B \in \mathcal{B}$ with $E[\Pi(B)] < \infty$ then

$$\begin{aligned}\nu(B) &= \sum_{i=1}^{\infty} \nu(B \cap U_i) = \sum_{i=1}^{\infty} E[\Pi_i(B \cap U_i)] \text{ , by Step1, Claim1} \\ &= E[\Pi(B)] < \infty.\end{aligned}$$

Then $\Pi(B)$ is Poisson distributed with parameter $\nu(B)$.

Let $s \le 0$, then:

$$\begin{aligned}E[e^{s\Pi(B)}] &= \lim_{m\to\infty} E\big[\exp(s \sum_{i=1}^{m} \Pi_i(B \cap U_i))\big] = \lim_{m\to\infty} \prod_{i=1}^{m} E\big[\exp(s\, \Pi_i(B \cap U_i))\big], \\ &\quad \text{since the families of random variables } \{N_i, \xi_1^i, \xi_2^i, \ldots\}_{i\in\mathbb{N}} \text{ are independent,} \\ &= \lim_{m\to\infty} \prod_{i=1}^{m} \exp(\nu(B \cap U_i)(e^s - 1)) \quad \text{, by Step 1} \\ &= \exp(\nu(B)(e^s - 1))\end{aligned}$$

By Lemma 1.4 and Lemma 1.3 the assertion follows.

Claim 2. Let $B \in \mathcal{B}$ with $\nu(B) = E[\Pi(B)] = +\infty$. Then $\Pi(B) = +\infty$ P-a.s..

$$P(\Pi(B) = +\infty) = P(\bigcap_{m\in\mathbb{N}} \bigcup_{i\ge m} \{\Pi_i(B \cap U_i) > 0\})$$

Since

$$
\begin{aligned}
P(\bigcap_{i\geq m}\{\Pi_i(B\cap U_i)>0\}^c) &= P(\bigcap_{i\geq m}\{\Pi_i(B\cap U_i)=0\})\\
= \lim_{n\to\infty} P(\bigcap_{i=m}^{m+n}\{\Pi_i(B\cap U_i)=0\}) &= \lim_{n\to\infty}\prod_{i=m}^{m+n} e^{-\nu(B\cap U_i)}\\
= \lim_{n\to\infty}\exp(-\sum_{i=m}^{m+n}\nu(B\cap U_i)) &= 0
\end{aligned}
$$

it follows that $P(\bigcup_{i\geq m}\{\Pi_i(B\cap U_i)>0\})=1$ for all $m\in\mathbb{N}$ and therefore $P(\Pi(B)=+\infty)=1$.

Claim 3. Let $B_1,\dots,B_m\in\mathcal{B}$ pairwise disjoint. Then $\Pi(B_1),\dots,\Pi(B_m)$ are independent.

If $E[\Pi(B_j)]<\infty$ for all $j\in\{1,\dots,m\}$ then one gets for all $s_1,\dots,s_m\in\mathbb{R}_-$ that

$$
\begin{aligned}
E\big[\exp(\sum_{j=1}^{m}s_j\Pi(B_j))\big] &= E\big[\exp(\sum_{i=1}^{\infty}\sum_{j=1}^{m}s_j\Pi_i(B_j\cap U_i))\big]\\
&= \lim_{n\to\infty}E\big[\exp(\sum_{i=1}^{n}\sum_{j=1}^{m}s_j\Pi_i(B_j\cap U_i))\big]\\
&= \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{j=1}^{m}E\big[\exp(s_j\Pi_i(B_j\cap U_i))\big]\\
&= \lim_{n\to\infty}\prod_{i=1}^{n}\prod_{j=1}^{m}\exp\big(\nu(B_j\cap U_i)(e^{s_j}-1)\big)\\
&= \prod_{j=1}^{m}\exp\big(\nu(B_j)(e^{s_j}-1)\big)
\end{aligned}
$$

If there exists $i\in\{1,\dots,m\}$ with $E[\Pi(B_i)]=\infty$, then, by Step 2, Claim 2, $\Pi(B_i)=\infty$ P-a.s. Let $\{i_1,\dots,i_n\}\subset\{1,\dots,m\}$, then the independence of $\Pi(B_{i_1}),\dots,\Pi(B_{i_n})$ follows from the case $E[\Pi(B_j)]<\infty$ for all $j\in\{1,\dots,m\}$ and the above statement.

$\square$

1.2 Point processes and Poisson point processes

Definition 1.6 (Point function on U). A *point function p on U* is a mapping $p : D_p \subset (0,\infty) \to U$ where the domain D_p is a countable subset of $(0,\infty)$.
p defines a measure $N_p(dt, dy)$ on $((0,\infty) \times U, \mathcal{B}((0,\infty)) \otimes \mathcal{B})$ in the following way:
Define $\bar{p} : (0,\infty) \to (0,\infty) \times U$, $t \mapsto (t, p(t))$ and denote by c the counting measure on $(D_p, \mathcal{P}(D_p))$, i.e. $c(A) := |A|$ for all $A \in \mathcal{P}(D_p)$.
For $\bar{B} \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}$ define

$$N_p(\bar{B}) := c(\bar{p}^{-1}(\bar{B})).$$

Then, in particular, we have for all $A \in \mathcal{B}((0,\infty))$ and $B \in \mathcal{B}$

$$N_p(A \times B) := \#\{t \in D_p | t \in A, p(t) \in B\}.$$

Notation: $N_p(t, B) := N_p(]0,t] \times B)$, $t \geq 0$, $B \in \mathcal{B}$

Let $\mathcal{P}_U$ be the space of all point functions on U and

$$\mathcal{B}_{\mathcal{P}_U} := \sigma(\mathcal{P}_U \to \mathbb{Z}_+ \cup \{+\infty\}, p \mapsto N_p(]0,t] \times B) \,|\, t > 0, B \in \mathcal{B})$$

Definition 1.7 (Point process). (i) A *point process on U* is a random variable $p : (\Omega, \mathcal{F}) \to (\mathcal{P}_U, \mathcal{B}_{\mathcal{P}_U})$.

(ii) A point process p is called *stationary* if for every $t > 0$ p and $\theta_t p$ have the same probability law, where $\theta_t p$ is defined by $D_{\theta_t p} := \{s \in (0,\infty) \,|\, s + t \in D_p\}$ and $(\theta_t p)(s) := p(s+t)$.

(iii) A point process is called *Poisson point process* if there exists a Poisson random measure Π on $(0,\infty) \times U$ such that there exists $N \in \mathcal{F}$, $P(N) = 0$, such that for all $\omega \in N^c$ and for all $\bar{B} \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}$: $N_{p(w)}(\bar{B}) = \Pi(\omega)(\bar{B})$.

(iv) A point process p is called *σ-finite* if there exist $U_i \in \mathcal{B}$, $i \in \mathbb{N}$, $U_i \uparrow U$, $i \to \infty$, and $E[N_p(t, U_i)] < \infty$ for all $t > 0$ and $i \in \mathbb{N}$.

The statement of the following proposition about stationary Poisson point processes can be found in [IkWa 81, I.9 Point processes and Poisson point processes, p.43]

Proposition 1.8. *Let p be a σ-finite Poisson point process. Then p is stationary if and only if there exists a σ-finite measure ν on $(U, \mathcal{B})$ such that*

$$E[N_p(dt, dy)] = \lambda(dt) \otimes \nu(dy)$$

where λ denotes the Lebesgue-measure on $(0, \infty)$. ν is called characteristic *measure of p.*

Theorem 1.9. *Given a σ-finite measure ν on $(U, \mathcal{B})$ there exists a stationary Poisson point process on U with characteristic measure ν.*

Proof. Let Π be a Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ with intensity measure $\lambda \otimes \nu$ where λ denotes the Lebesgue-measure on$((0, \infty), \mathcal{B}((0, \infty)))$. Remember the construction of Π in the proof of Theorem 1.5:
There exist U_i, $i \in \mathbb{N}$, pairwise disjoint sucht that $U = \bigcup_{i \in \mathbb{N}} U_i$ and $c_i := \nu(U_i) < \infty$. For $i \in \mathbb{N}$ let

- N_i be a Poissonian random variable with parameter c_i,
- $\xi_k^i = (t_k^i, x_k^i)$, $k \in \mathbb{N}$, i.i.d. $]i-1, i] \times U_i$-valued random variables with distribution $\lambda \otimes (\frac{1}{c_i}\nu(\cdot \cap U_i)$, also independent of N_i.

Moreover the families of random variables $\{N_i, \xi_1^i, \xi_2^i, \dots\}$, $i \in \mathbb{N}$, are independent.
Then

$$\Pi := \sum_{i=1}^{\infty} \Pi_i := \sum_{i=1}^{\infty} \sum_{k=1}^{N_i} \delta_{(t_k^i, x_k^i)}$$

is a Poisson random measure on $((0, \infty) \times U, \mathcal{B}((0, \infty)) \otimes \mathcal{B})$ with intensity measure $\lambda \otimes \nu$ and for $\bar{B} \in \mathcal{B}((0, \infty)) \otimes \mathcal{B}$ holds

$$\Pi(\bar{B}) = \sum_{i=1}^{\infty} \Pi_i(\bar{B} \cap (]i-1, i] \times U_i)) \tag{1.1}$$

.
Then there exists a P-nullset $N \in \mathcal{F}$ such that for all $\omega \in N^c$:
$\Pi(\omega)(\{t\} \times U) = 1$ or 0 for all $t > 0$, since

$$P(\bigcup_{t>0} \{\Pi(\{t\} \times U) > 1\}) = P(\bigcup_{i=1}^{\infty} \bigcup_{t \in]i-1, i]} \{\Pi(\{t\} \times U) > 1\})$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\infty} P\Big(\bigcup_{t\in]i-1,i]} \{\Pi(\{t\}\times U_i) > 1\}\Big) \\
&\leq \sum_{i=1}^{\infty} P\Big(\bigcup_{n\neq m} \bigcup_{t\in]i-1,i]} \{\delta_{\xi^i_n}(\{t\}\times U_i) = 1\} \cap \{\delta_{\xi^i_m}(\{t\}\times U_i) = 1\}\Big) \\
&\leq \sum_{i=1}^{\infty} \sum_{n\neq m} P\Big(\bigcup_{t\in]i-1,i]} \{t^i_n = t^i_m = t\}\Big) \\
&= \sum_{i=1}^{\infty} \sum_{n\neq m} \lambda\otimes\lambda(\{(t,t)\,|\,t\in]i-1,i]\}) \\
&= 0
\end{aligned}
$$

If $\omega \in N^c$ and $t \in]i-1,i]$, then

$$
\begin{aligned}
&\Pi(\omega(\{t\}\times U)) = 1 \\
\iff &\sum_{k=1}^{N_i(\omega)} \delta_{(t^i_k(\omega),x^i_k(\omega))}(\{t\}\times U_i) = \Pi_i(\omega)(\{t\}\times U_i) \\
&= \Pi(\omega)(\{t\}\times U) \text{ , by equation (1.1),} \\
&= 1 \\
\iff &\exists!\, k \in \{1,\dots,N_i(\omega)\} \text{ such that } t = t^i_k(\omega)
\end{aligned}
$$

In this case we set

$$
p(\omega)(t) := x^i_k(\omega) \text{ and } D_{p(\omega)} := \{t\in(0,\infty)\,|\,\Pi(\omega)(\{t\}\times U) \neq 0\}
$$

If $\omega \in N$ then define $p_0 \in \mathcal{P}_U$ by $D_p := \{t_0\} \subset (0,\infty)$ and $p_0(t_0) = x_0 \in U$ and set $p(\omega) = p_0$.

Claim 1. $N_{p(\omega)} = \Pi(\omega)$ for all $\omega \in N^c$.
Let $\omega \in N^c$, $A \in \mathcal{B}((0,\infty))$ and $B \in \mathcal{B}$ then:

$$
\begin{aligned}
&\Pi(\omega)(A\times B) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{N_i(\omega)} \delta_{(t^i_k,x^i_k)(\omega)}(A\cap]i-1,i] \times B\cap U_i) \\
&= \sum_{i=1}^{\infty} \#\{s\in]i-1,i]\,|\, s\in A, \exists k\in\{1,\dots,N_i(\omega)\} \text{ such that } s = t^i_k(\omega) \\
&\qquad\qquad \text{and } x^i_k(\omega) \in B\cap U_i\} \\
&= \sum_{i=1}^{\infty} \#\{s\in]i-1,i]\,|\, s\in A, \exists!\, k\in\{1,\dots,N_i(\omega)\} \text{ such that } s = t^i_k(\omega) \\
&\qquad\qquad \text{and } x^i_k(\omega) \in B\cap U_i\},
\end{aligned}
$$

$$\begin{aligned} &\quad \text{since } \Pi(\omega)(\{s\} \times U) \in \{0,1\} \text{ for all } s \in [0,\infty[,\\ &= \#\{s \in D_{p(\omega)} \,|\, s \in A, p(\omega)(s) \in B\},\\ &\quad \text{by the definition of } p,\\ &= N_{p(\omega)}(A \times B) \end{aligned}$$

Claim 2. For all $\bar{B} \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}$ the mapping $N_p(\bar{B})$ is $\mathcal{F}$-measurable and $E[N_p(dt,dx)] = \lambda(dt) \otimes \nu(dx)$.
Since $N_p(\bar{B}) = \Pi(\bar{B})$ P-a.s. the measurability is obvious by Remark 1.2 and the completness of $(\Omega, ,P)$.Now $E[N_p(\bar{B})]$ is well defined and we obtain that $E[N_p(\bar{B})] = E[\Pi(\bar{B})] = \lambda \otimes \nu(\bar{B})$, since Π is a Poisson random measure with intensity measure $\lambda(dt) \otimes \nu(dx)$.

Claim 3. $p : \Omega \to \mathcal{P}_U$ is $\mathcal{F}/\mathcal{B}_{\mathcal{P}_U}$-measurable.

$$\begin{aligned} \mathcal{B}_{\mathcal{P}_U} &= \sigma(\mathcal{P}_U \to \mathbb{Z}_+ \cup \{+\infty\}, p \mapsto N_p(]0,t] \times B) \,|\, t > 0, B \in \mathcal{B})\\ &= \sigma(\{p \in \mathcal{P}_U \,|\, N(t,B) = m\} \,|\, t > 0, B \in \mathcal{B}, m \in \mathbb{Z}_+) \end{aligned}$$

and for $t > 0$, $B \in \mathcal{B}$, $m \in \mathbb{Z}_+$ one gets by Claim 2 that

$$\{p \in \{N_\cdot(t,B) = m\}\} = \{N_p(t,B) = m\} \in \mathcal{F}.$$

By Claim 1 - 3 it follows that p is a Poisson point process with characteristic measure ν. By Proposition 1.8 p is stationary. □

1.3 Stochastic integrals with respect to Poisson point processes

Let $\mathcal{F}_t$, $t \geq 0$, be a filtration on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$.

Definition 1.10. A point process p is called $(\mathcal{F}_t)$-*adapted* if for every $t > 0$ and $B \in \mathcal{B}$ $N_p(t,B)$ is $\mathcal{F}_t$-measurable.

For an arbitrary point process p define the following set
$\Gamma_p := \{B \in \mathcal{B} \,|\, E[N_p(t,B)] < \infty \text{ for all } t > 0\}$.

Definition 1.11. An $(\mathcal{F}_t)$-adapted point process p on U is said to be of *class (QL)* (quasi-left-continuous) with respect to $\mathcal{F}_t$, $t \geq 0$, if it is σ-finite and there exists for all $B \in \mathcal{B}$ a process $\hat{N}_p(t,B)$, $t \geq 0$, such that

(i) for $B \in \Gamma_p$ $t \mapsto \hat{N}_p(t,B)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process,

(ii) for all $t \geq 0$ and P-a.e. $\omega \in \Omega$: $\hat{N}_p(\omega)(t,\cdot)$ is a σ-finite measure on $(U,\mathcal{B})$,

(iii) for $B \in \Gamma_p$ $q(t,B) := N_p(t,B) - \hat{N}_p(t,B)$, $t \geq 0$, is an $(\mathcal{F}_t)$-martingale

$\hat{N}_p$ is called the *compensator* of the point process p and q the *compensated Poisson random measure* of p.

Definition 1.12. A point process p is called an $(\mathcal{F}_t)$-*Poisson point process* if it is an $(\mathcal{F}_t)$-adapted, σ-finite Poisson point process such that $\{N_p(]t,t+h] \times B) \,|\, h > 0, B \in \mathcal{B}\}$ is independent of $\mathcal{F}_t$ for all $t \geq 0$.

Remark 1.13. Let p be a σ-finite Poisson point process on U. Then there exists a filtration $\mathcal{F}_t$, $t \geq 0$, on $(\Omega,\mathcal{F},P)$ such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$ and p is an $(\mathcal{F}_t)$-Poisson point process.

Proof. Define $\mathcal{N} := \{N \in \mathcal{F} \,|\, P(N) = 0\}$ and for $t \geq 0$

$$\mathcal{F}_t := \sigma(N_p(t,B) \,|\, B \in \mathcal{B}) \cup \mathcal{N}.$$

Then p is an $(\mathcal{F}_t)$-adapted, σ-finite Poisson point process.
Moreover $\sigma(N_p(t,B) \,|\, B \in \mathcal{B}) \cup \mathcal{N} = \sigma(\Pi(]0,t] \times B) \,|\, B \in \mathcal{B}) \cup \mathcal{N}$ is independent of $\sigma(\Pi(]t,t+h] \times B) \,|\, h > 0, B \in \mathcal{B}) \cup \mathcal{N}$ by Definition 1.1 (ii) since $]0,t] \times B$ and $]t,t+h] \times \tilde{B}$ are disjoint for all $h > 0$ and $B, \tilde{B} \in \mathcal{B}$. Since

$$\begin{aligned}&\sigma(\Pi(]t,t+h] \times B) \,|\, h > 0, B \in \mathcal{B}) \cup \mathcal{N}\\ =\,&\sigma(N_p(]t,t+h] \times B) \,|\, h > 0, B \in \mathcal{B}) \cup \mathcal{N}\end{aligned}$$

the assertion follows. □

For the rest of this section fix a σ-finite measure ν on $(U,\mathcal{B})$ and a stationary $(\mathcal{F}_t)$-Poisson point process p on U with characteristic measure ν.

Proposition 1.14. *p is of class (QL) with compensator $\hat{N}_p(t,B) = t\nu(B)$, $t \geq 0$, $B \in \mathcal{B}$.*

Proof. Set for $t \geq 0$ and $B \in \mathcal{B}$: $\hat{N}_p(t,B) := t\nu(B)$.
Then condition (i) and (ii) of Definition 1.11 are fulfilled. Moreover, for $B \in \Gamma_p$ $q(t,B) := N_p(t,B) - \hat{N}_p(t,B)$, $t \geq 0$, is $(\mathcal{F}_t)$-adapted. It remains to

check that for $B \in \Gamma_p$ $q(t, B)$, $t \geq 0$, has the martingale property.
For this end let $0 \leq s < t < \infty$ and $F_s \in \mathcal{F}_s$, then

$$\begin{aligned} &E[q(t,B)1_{F_s}] = E[(N_p(t,B) - \hat{N}_p(t,B))1_{F_s}] \\ =& E[N_p(t,B)1_{F_s}] - t\nu(B)P(F_s) \\ =& E[(N_p(t,B) - N_p(s,B))1_{F_s}] + E[N_p(s,B)1_{F_s}] - t\nu(B)P(F_s) \\ =& E[N_p(t,B) - N_p(s,B)]P(F_s) + E[N_p(s,B)1_{F_s}] - (t-s)\nu(B)P(F_s) \\ &- s\nu(B)P(F_s) \\ =& E[(N_p(s,B)1_{F_s}] - s\nu(B)P(F_s) \\ =& E[(N_p(s,B) - \hat{N}_p(s,B))1_{F_s}] \\ =& E[q(s,B)1_{F_s}] \end{aligned}$$

□

Remark 1.15. If $t \in [0, \infty[$ and

$$B \in \Gamma_p = \{B \in \mathcal{B} \,|\, E[N_p(t,B)] < \infty \text{ for all } t > 0\} = \{B \in \mathcal{B} \,|\, \nu(B) < \infty\}$$

then $q(t, B) \in \mathbb{R}$ P-a.s. since $q(t, B) = N_p(t, B) - t\nu(B)$ where $N_p(t, B) < \infty$ P-a.s. as $E[N_p(t, B)] < \infty$.
If $0 \leq s \leq t < \infty$ and $B \in \Gamma_p$ then

$$\begin{aligned} q(t,B) - q(s,B) &= N_p(t,B) - N_p(s,B) - (t-s)\nu(B) \\ &= N_p(]s,t] \times B) - (t-s)\nu(B) \quad P\text{-a.s.} \end{aligned}$$

Notation: In the following we will use the following notation:
$q(]s, t] \times B) := N_p(]s, t] \times B) - (t - s)\nu(B)$, $0 \leq s \leq t < \infty$, $B \in \mathcal{B}$.

Proposition 1.16. *For $A \in \Gamma_p$ $(q(t, A),\ t \geq 0)$ is an element of $\mathcal{M}^2$ and we have for $A_1, A_2 \in \Gamma_p$ that*

$$\langle q(\cdot, A_1), q(\cdot, A_2)\rangle(t) = \hat{N}_p(t, A_1 \cap A_2),\ t \geq 0.$$

In particular, this means that for all $A \in \Gamma_p$ the following holds:
$M(t) := q(t, A)^2 - \hat{N}_p(t, A)$, $t \geq 0$, is an $(\mathcal{F}_t)_{t\geq 0}$-martingale and in this case:
$E[M(t)] = E[M(0)] = 0$ for all $t \geq 0$.

Proof. [Ikeda, Watanabe, Theorem 3.1, p.60; Lemma 3.1, p.60] □

Step 1. Definition of the stochastic integral for elementary processes
Let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space and fix $T > 0$.
The class $\mathcal{E}$ of all elementary processes is determined by the following definition

Definition 1.17. An H-valued process $\Phi(t) : \Omega \times U \to H$, $t \in [0,T]$, on $(\Omega \times U, \mathcal{F} \otimes \mathcal{B}, P \otimes \nu)$ is said to be *elementary* if there exist $0 = t_0 < t_1 < \cdots < t_k = T$, $k \in \mathbb{N}$, and for $m \in \{0, \dots, k-1\}$ exist $B_1^m, \dots, B_{I(m)}^m \in \Gamma_p$, pairwise disjoint, $I(m) \in \mathbb{N}$, such that

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]t_m,t_{m+1}] \times B_i^m}$$

where $x_i^m \in H$ and $F_i^m \in \mathcal{F}_{t_m}$, $1 \le i \le I(m)$, $0 \le m \le k-1$.

For $\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]tm,t_{m+1}] \times B_i^m} \in \mathcal{E}$ define the stochastic integral process by

$$\begin{aligned}
&\operatorname{Int}(\Phi)(t,\omega) \\
:= &\int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy)(\omega) := \int_0^T \int_U 1_{]0,t]}(s) \Phi(s,y)\, q(ds,dy)(\omega) \\
:= &\sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m}(\omega)(q(\omega)(t_{m+1} \wedge t, B_i^m) - q(\omega)(t_m \wedge t, B_i^m)),
\end{aligned}$$

$t \in [0,T]$ and $\omega \in \Omega$.

Proposition 1.18.
If $\Phi \in \mathcal{E}$ then $\left(\int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy), t \in [0,T]\right) \in \mathcal{M}_T^2(H)$ and

$$\begin{aligned}
&\|\operatorname{Int}(\Phi)\|_{\mathcal{M}_T^2}^2 := \sup\nolimits_{t \in [0,T]} E[\| \int_0^{t+} \int_U \Phi(s,y)\, q(ds,dy) \|^2] \\
= &E[\int_0^T \int_U \|\Phi(s,y)\|^2\, \nu(dy)\, ds] =: \|\Phi\|_T
\end{aligned}$$

Proof.

Claim 1. $\operatorname{Int}(\Phi)$ is $(\mathcal{F}_t)$-adapted.

Let $t \in [0,T]$ then:

$$\begin{aligned}
&\operatorname{Int}(\Phi)(t) \\
= &\sum_{\substack{m \in \{0,\dots,k-1\} \\ t_m \le t}} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} (N_p(t_{m+1} \wedge t, B_i^m) - N_p(t_m, B_i^m) - (t_{m+1} \wedge t - t_m) \\
&\nu(B_i^m))
\end{aligned}$$

which is $\mathcal{F}_t$-measurable since p is $(\mathcal{F}_t)$-adapted.

Claim 2. For all $t \in [0,T]$:

$$E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big] = E\big[\int_0^t \int_U \|\Phi(s,y)\|^2\, \nu(dy)ds\big] < \infty :$$

$$\begin{aligned}
&E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big] \\
&= E\big[\|\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} q(]t_m \wedge t, t_{m+1}\wedge t] \times B_i^m)\|^2\big] \\
&= E\big[\sum_{\substack{m=0\\ t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \|x_i^m 1_{F_i^m} q(]t_m \wedge t, t_{m+1}\wedge t] \times B_i^m)\|^2 \\
&\quad + 2 \sum_{\substack{0\le m<n\le k-1\\ t_n\le t}} \sum_{\substack{(i,j)\in\{1,\dots,I(m)\}\\ \times\{1,\dots,I(n)\}}} \langle x_i^m \Delta_i^m, x_j^n \Delta_j^n\rangle\big]
\end{aligned}$$

where $\Delta_h^l := q(]t_l \wedge t, t_{l+1}\wedge t] \times A_h^B)$, $0 \le l \le k-1$, $1 \le h \le I(l)$.

1.: For $m \in \{0,\dots,k-1\}$, $t_m \le t$, $i \in \{1,\dots,I(m)\}$ holds:

$$E\big[\|x_i^m 1_{F_i^m} q(]t_m \wedge t, t_{m+1}\wedge t] \times B_i^m)\|^2\big] \le E\big[\|x_i^m \Delta_i^m\|^2\big] < \infty :$$

For this purpose let $0 \le s \le t \le T$ and $B \in \Gamma_p$, then:

$$\begin{aligned}
E\big[q(]s,t] \times B)^2\big] &= E\big[(q(t,B) - q(s,B))^2\big] \\
&= E\big[\underbrace{q(t,B)^2}_{(a)} - 2\underbrace{q(t,B)q(s,B)}_{(b)} + q(s,B)^2\big]
\end{aligned}$$

(a) By Proposition 1.16 and Proposition 1.14 it follows that

$$E[q(t,B)^2] = E[\hat{N}_p(t,B)] = t\nu(B) < \infty.$$

(b) Since $|q(]s,t] \times B)|$ and $|q(s,B)|$ are independent we get that

$$\begin{aligned}
E\big[|q(t,B)q(s,B)|\big] &\le E\big[|q(]s,t] \times B)q(s,B)|\big] + E\big[q(s,B)^2\big] \\
&= E\big[|q(]s,t] \times B)|\big]E\big[|q(s,B)|\big] + E\big[q(s,B)^2\big] \\
&< \infty.
\end{aligned}$$

From (a) and (b) it follows that $E\big[q(]s,t] \times B)^2\big] < \infty$. Moreover we obtain that

$$
\begin{aligned}
(1.2) \qquad & E\big[q(]s,t]\times B)^2\big] \\
&= E\big[q(t,B)^2\big] - 2E\big[q(t,B)q(s,B)\big] + E\big[q(s,B)^2\big] \\
&= E\big[q(t,B)^2\big] - 2E\big[q(]s,t]\times B)q(s,B)\big] - E\big[q(s,B)^2\big] \\
&= t\nu(B) - 2E\big[q(]s,t]\times B)\big]E\big[q(s,B)\big] - s\nu(B) \\
&= (t-s)\nu(B), \quad \text{as } E\big[q(s,B)\big] = E[N_p(]0,s]\times B] - s\nu(B) = 0
\end{aligned}
$$

2.: For $m,n \in \{0,\dots,k-1\}$, $m<n$, $t_n \le t$, $i \in \{1,\dots,I(m)\}$, $j \in \{1,\dots,I(n)\}$ holds:

$$E\big[|\langle x_i^m 1_{F_i^m}\Delta_i^m, x_j^n 1_{F_j^n}\Delta_j^n\rangle|\big] \le E\big[|\langle x_i^m\Delta_i^m, x_j^n\rangle||\Delta_j^n|\big] < \infty :$$

Since $m<n$ and $t_m < t_n \le t$ we get that

$$]t_m\wedge t, t_{m+1}\wedge t]\cap]t_n\wedge t, t_{n+1}\wedge t] =]t_m, t_{m+1}]\cap]t_n, t_{n+1}\wedge t] = \emptyset$$

therefore $|\Delta_j^n|$ and $\langle x_i^m, x_j^n\rangle|\Delta_i^m|$ are independent and we obtain that

$$E\big[|\langle x_i^m\Delta_i^m, x_j^n\rangle||\Delta_j^n|\big] = E\big[|\langle x_i^m\Delta_i^m, x_j^n\rangle|\big]E\big[|\Delta_j^n|\big] < \infty.$$

3.: For $m,n \in \{0,\dots,k-1\}$, $m<n$, $t_n \le t$, $i \in \{1,\dots,I(m)\}$, $j \in \{1,\dots,I(n)\}$ holds:

$$
\begin{aligned}
& E\big[\langle x_i^m 1_{F_i^m}\Delta_i^m, x_j^n 1_{F_j^n}\Delta_j^n\rangle\big] \\
&= E\big[\langle x_i^m 1_{F_i^m}\Delta_i^m, x_j^n 1_{F_j^n}\rangle\Delta_j^n\big] \\
&= E\big[\langle x_i^m 1_{F_i^m}\Delta_i^m, x_j^n 1_{F_j^n}\rangle\big]E[\Delta_j^n] \\
&= 0 \ \text{, since } E[\Delta_j^n] = 0.
\end{aligned}
$$

By **1.-3.** one gets for all $t \in [0,T]$ that

$$
\begin{aligned}
& E\big[\|\mathrm{Int}(\Phi)(t)\|^2\big] \\
&= E\Big[\Big\|\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} q(]t_m\wedge t, t_{m+1}\wedge t]\times B_i^m)\Big\|^2\Big] \\
&= E\Big[\sum_{\substack{m=0\\ t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \|x_i^m 1_{F_i^m} q(]t_m\wedge t, t_{m+1}\wedge t]\times B_i^m)\|^2 \\
&\qquad + 2\sum_{\substack{0\le m<n\le k-1\\ t_n\le t}}\ \sum_{\substack{(i,j)\in\{1,\dots,I(m)\}\\ \times\{1,\dots,I(n)\}}} \langle x_i^m\Delta_i^m, x_j^n\Delta_j^n\rangle\Big] \\
&= \sum_{\substack{m=0\\ t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} E\big[\|x_i^m 1_{F_i^m} q(]t_m\wedge t, t_{m+1}\wedge t]\times B_i^m)\|^2\big]
\end{aligned}
$$

$$= \sum_{\substack{m=0\\ t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \|x_i^m\|^2 P(F_i^m) E\big[q(]t_m\wedge t, t_{m+1}\wedge t]\times B_i^m)^2\big],$$

since $F_i^m \in \mathcal{F}_{t_m}$ and $q(]t_m, t_{m+1}\wedge t]\times B_i^m)$ is independent of $\mathcal{F}_{t_m}$,

$$= \sum_{\substack{m=0\\ t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \|x_i^m\|^2 P(F_i^m)(t_{m+1}\wedge t - t_m\wedge t)\nu(B_i^m),$$

by equation (1.2),

$$= E\Big[\int_0^t \int_U \|\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]t_m,t_{m+1}]\times B_i^m}\|^2\,\nu(dy)ds\Big]$$

$$= E\Big[\int_0^t \int_U \|\Phi(s,y)\|^2\,\nu(dy)ds\Big]$$

Claim 3. $\mathrm{Int}(\Phi)(t)$, $t\in[0,T]$, is an $(\mathcal{F}_t)$-martingale.
Let $0\le s<t\le T$ and $F_s\in\mathcal{F}_s$ then:

$$\int_{F_s}\int_0^{t+}\int_U \Phi(r,y)\,q(dr,dy)dP$$

$$= \int_{F_s}\sum_{m=0}^{k-1}\sum_{i=1}^{I(m)} x_i^m 1_{F_i^m}(q(t_{m+1}\wedge t, B_i^m) - q(t_m\wedge t, B_i^m))\,dP$$

$$= \sum_{\substack{m=0\\ t_m\le s}}^{k-1}\sum_{i=1}^{I(m)} \int_{F_s} x_i^m 1_{F_i^m}(q(t_{m+1}\wedge t, B_i^m) - q(t_m\wedge s, B_i^m))\,dP$$

$$+ \sum_{\substack{m=0\\ s<t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \int_{F_s} x_i^m 1_{F_i^m}(q(t_{m+1}\wedge t, B_i^m) - q(t_m, B_i^m))\,dP$$

$$+ \sum_{\substack{m=0\\ s<t<t_m}}^{k-1}\sum_{i=1}^{I(m)} \int_{F_s} x_i^m 1_{F_i^m}\underbrace{(q(t, B_i^m) - q(t, B_i^m))}_{=0}\,dP$$

$$= \sum_{\substack{m=0\\ t_m\le s}}^{k-1}\sum_{i=1}^{I(m)} \int_{F_s} x_i^m 1_{F_i^m}\big(E\big[q(t_{m+1}\wedge t, B_i^m)|\mathcal{F}_s\big] - q(t_m\wedge s, B_i^m)\big)\,dP$$

$$+ \sum_{\substack{m=0\\ s<t_m\le t}}^{k-1}\sum_{i=1}^{I(m)} \int_{F_s} x_i^m 1_{F_i^m}\underbrace{\big(E\big[q(t_{m+1}\wedge t, B_i^m)|\mathcal{F}_{t_m}\big] - q(t_m, B_i^m)\big)}_{=0,\text{ since } q(\cdot,B_i^m)\text{ is an }(\mathcal{F}_t)\text{-martingale}}\,dP$$

$$+\sum_{\substack{m=0\\ s<t<t_m}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} x_i^m 1_{F_i^m}\underbrace{(q(s,B_i^m)-q(s,B_i^m))}_{=0}\,dP$$

$$=\sum_{\substack{m=0\\ t_m\le s}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} x_i^m 1_{F_i^m}(q(t_{m+1}\wedge s,B_i^m)-q(t_m\wedge s,B_i^m))\,dP,$$

since $q(t_{m+1}\wedge\cdot,B_i^m)$ is an $(\mathcal{F}_t)$-martingale

$$+\sum_{\substack{m=0\\ s<t_m\le t}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} x_i^m 1_{F_i^m}\underbrace{(q(t_{m+1}\wedge s,B_i^m)-q(t_m\wedge s,B_i^m))}_{=0}\,dP$$

$$+\sum_{\substack{m=0\\ s<t<t_m}}^{k-1}\sum_{i=1}^{I(m)}\int_{F_s} x_i^m 1_{F_i^m}(q(t_{m+1}\wedge s,B_i^m)-q(t_m\wedge s,B_i^m))\,dP$$

$$=\int_{F_s}\int_0^{s+}\int_U \Phi(r,y)\,q(dr,dy)dP$$

□

In this way one has found the semi norm $\|\,\|_T$ on $\mathcal{E}$ such that $\mathrm{Int}:(\mathcal{E},\|\,\|_T)\to(\mathcal{M}_T^2(H),\|\,\|_{\mathcal{M}_T^2})$ is an isometric transformation. To get a norm on $\mathcal{E}$ one has to consider equivalence classes of elementary processes with respect to $\|\,\|_T$. For simplicity, the space of equivalence classes will be denoted by $\mathcal{E}$, too.
Since $\mathcal{E}$ is dense in the absract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ w.r.t. $\|\,\|_T$ it is clear that there is a unique isometric extension of Int to $\bar{\mathcal{E}}$.

Step 2. Characterization of $\bar{\mathcal{E}}$
Define the predictable σ-field on $[0,T]\times\Omega\times U$ by

$$\begin{aligned}
&\mathcal{P}_T(U)\\
&:=\sigma(g:[0,T]\times\Omega\times U\to H\,|\,g\text{ is }(\underbrace{\mathcal{F}_t\times\mathcal{B}}_{\tilde{\mathcal{F}}_t})-\text{adapted and left-continuous})\\
&=\sigma(\{]s,t]\times\tilde{F}_s\,|\,0\le s\le t\le T,\tilde{F}_s\in\tilde{\mathcal{F}}_s\}\cup\{\{0\}\times\tilde{F}_0\,|\,\tilde{F}_0\in\tilde{\mathcal{F}}_0\})\\
&=\sigma(\{]s,t]\times F_s\times B\,|\,0\le s\le t\le T,F_s\in\mathcal{F}_s,B\in\mathcal{B}\}\\
&\qquad\cup\{\{0\}\times F_0\times B\,|\,F_0\in\mathcal{F}_0\times\mathcal{B}\})
\end{aligned}$$

At this point, for the sake of completness, also define the predictable σ-field on $[0,T]\times\Omega$ by

$$\mathcal{P}_T:=\sigma(g:[0,T]\times\Omega\to\mathbb{R},|\,g\text{ is }(\mathcal{F}_t)\text{-adapted and left-continuous})$$

$$= \sigma(\underbrace{\{]s,t] \times F_s \,|\, 0 \le s \le t \le T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \,|\, F_0 \in \mathcal{F}_0\}}_{:=\mathcal{A}})$$

Let $\tilde{H}$ be an arbitrary Hilbert space. If $Y : [0,T] \times \Omega \to \tilde{H}$ is $\mathcal{P}_T/\mathcal{B}(\tilde{H})$-measurable it is called ($\tilde{H}$-)predictable.

Remark 1.19. (i) If $B \in \mathcal{B}([0,T])$ then $B \times \Omega \times U \in \mathcal{P}_T(U)$.

(ii) If $A \in \mathcal{P}_T$ and $B \in \mathcal{B}$ then $A \times B \in \mathcal{P}_T(U)$.

Proof. (i)

$$\begin{aligned} B \times \Omega \times U &\in \mathcal{B}([0,T]) \otimes \{\Omega, \emptyset\} \otimes \{U, \emptyset\} \\ &= \sigma(\{]s,t] \times \Omega \times U \,|\, 0 \le s \le t \le T\} \cup \{[0,T] \times \Omega \times U\}) \\ &\subset \mathcal{P}_T(U) \end{aligned}$$

(ii)

$$\begin{aligned} A \times B &\in \mathcal{P}_T \otimes \{B, \emptyset\} = \sigma(\{A \times B \,|\, A \in \mathcal{A}\} \cup \{[0,T] \times \Omega \times B\}) \\ &\subset \mathcal{P}_T(U) \end{aligned}$$

□

Furthermore, for the next proposition we need the following lemma:

Lemma 1.20. *Let E be a metric space with metric d and let $f : \Omega \to E$ be strongly measurable, i.e. it is Borel measurable and $f(\Omega) \subset E$ is separable. Then there exists a sequence f_n, $n \in \mathbb{N}$, of simple E-valued functions (i.e. f_n is $\mathcal{F}/\mathcal{B}(E)$-measurable and takes only a finite number of values) such that for arbitrary $\omega \in \Omega$ the sequence $d(f_n(\omega), f(\omega))$, $n \in \mathbb{N}$, is monotonely decreasing to zero.*

Proof. [DaPrZa 92, Lemma 1.1, p.16] □

Proposition 1.21. *If Φ is an $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable process and*

$$E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \nu(dy)ds] < \infty$$

then there exists a sequence of elementary processes Φ_n, $n \in \mathbb{N}$, such that $\|\Phi - \Phi_n\|_T \to 0$ as $n \to \infty$.

Proof. There exist $U_n \in \mathcal{B}$, $n \in \mathbb{N}$, with $\nu(U_n) < \infty$ such that $U_n \uparrow U$ as $n \to \infty$. Then $1_{U_n}\Phi : [0,T]\times\Omega\times U_n \to H$ is $\mathcal{P}_T(U)\cap([0,T]\times\Omega\times U_n)/\mathcal{B}(H)$-measurable.
Moreover

$$\begin{aligned}
(1.3) \qquad & \mathcal{P}_T(U) \cap ([0,T] \times \Omega \times U_n) \\
&= \sigma(\,\{]s,t] \times F_s \times B \,|\, 0 \le s \le t \le T, F_s \in \mathcal{F}_s, B \in \mathcal{B} \cap U_n\} \\
&\qquad \cup \{\{0\} \times F_0 \times B \,|\, F_0 \in \mathcal{F}_0, B \in \mathcal{B} \cap U_n\}) \\
&=: \mathcal{P}_T(U_n) \ :
\end{aligned}$$

Therefore one gets that $1_{U_n}\Phi \,:\, [0,T] \times \Omega \times U_n \to H$ is $\mathcal{P}_T(U_n)/\mathcal{B}(H)$-measurable. Then there exists a sequence Φ_k^n, $k \in \mathbb{N}$, of simple random variables of the following form

$$\Phi_k^n = \sum_{m=1}^{M_k} x_m^k 1_{A_m^k},\ x_m^k \in H,\ A_m^k \in \mathcal{P}_T(U_n),\ 1 \le m \le M_k,\ k \in \mathbb{N},$$

such that $\|1_{U_n}\Phi - \Phi_k^n\| \downarrow 0$ as $k \to \infty$ by Lemma 1.20. Since

$$\begin{aligned}
\|1_{U_n}\Phi - \Phi_k^n\| \le \|1_{U_n}\Phi\| + \|\Phi_1^n\| \le \|1_{U_n}\Phi\| + \sum_{m=1}^{M_1} \|x_m^1\| 1_{A_m^1} \\
\in L^2([0,T] \times \Omega \times U_n, \mathcal{P}_T(U_n), \lambda \otimes P \otimes \nu)
\end{aligned}$$

one gets by Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\|1_{U_n}(\Phi - \Phi_k^n)\|_T^2 &= E[\int_0^T \int_U \|1_{U_n}(\Phi - \Phi_k^n)\|^2 \, d\nu \, d\lambda] \\
&= E[\int_0^T \int_{U_n} \|1_{U_n}\Phi - \Phi_k^n\|^2 \, d\nu \, d\lambda] \to 0 \text{ as } k \to \infty
\end{aligned}$$

Choose for $n \in \mathbb{N}$ $k(n) \in \mathbb{N}$ such that $\|1_{U_n}(\Phi - \Phi_{k(n)}^n)\|_T < \frac{1}{n}$, then

$$\|\Phi - 1_{U_n}\Phi_{k(n)}^n\|_T \le \|\Phi - 1_{U_n}\Phi\|_T + \|1_{U_n}(\Phi - \Phi_{k(n)}^n)\|_T$$

where the first summand converges to 0 by Lebesgue's dominated convergence theorem and the second summand is smaller than $\frac{1}{n}$.
Thus the assertion of the Proposition is reduced to the case $\Phi = x1_A$ where $x \in H$ and $A \in \mathcal{P}_T(U_n)$ for some $n \in \mathbb{N}$. Then there is a sequence of elemntary processes Φ_k, $k \in \mathbb{N}$, such that $\|\Phi - \Phi_k\|_T \to 0$ as $k \to \infty$:

To get this result it is sufficient to prove that for any $\varepsilon > 0$ there is a finite sum $\Lambda = \bigcup_{i=1}^N A_i$ of predictable rectangles

$$\begin{aligned} A_i \in \mathcal{A}_n :=& \{]s,t] \times F_s \times B \,|\, 0 \le s \le t \le T, F_s \in \mathcal{F}_s, B \in \mathcal{B} \cap U_n\} \\ & \cup \{\{0\} \times F_0 \times B \,|\, F_0 \in \mathcal{F}_0, B \in \mathcal{B} \cap U_n\},\ 1 \le i \le N, \end{aligned}$$

such that $P \otimes \lambda \otimes \nu(A \triangle \Lambda) \le \varepsilon$, since then one obtains that $\sum_{i=1}^N x 1_{A_i}$ is an elementary process, as $x1_{A_i}$, $1 \le i \le N$, are elementary processes and $\mathcal{E}$ is a linear space, and

$$\begin{aligned} \|x1_A - \sum_{i=1}^N x1_{A_i}\|_T &= \big(E[\int_0^T \int_U \|x(1_A - \sum_{k=1}^N 1_{A_i})\|^2 \, d\nu \, d\lambda]\big)^{\frac{1}{2}} \\ &\le \|x\| P \otimes \lambda \otimes \nu(A \triangle \Lambda) \le \|x\| \varepsilon \end{aligned}$$

Hence define $\mathcal{K} := \{\bigcup_{i \in I} A_i \,|\, |I| < \infty, A_i \in \mathcal{A}_n, i \in I\}$ then $\mathcal{K}$ is stable under finite intersections. Now let $\mathcal{G}$ be the family of all $A \in \mathcal{P}_T(U_n)$ which can be approximated by elements of $\mathcal{K}$ in the above sense. Then $\mathcal{G}$ is a Dynkin system and therefore $\mathcal{P}_T(U_n) = \sigma(\mathcal{K}) = \mathcal{D}(\mathcal{K}) \subset \mathcal{G}$ as $\mathcal{K} \subset \mathcal{G}$. $\square$

Define

$$\begin{aligned} \mathcal{N}_q^2(T,U,H) := \{&\Phi : [0,T] \times \Omega \times U \to H \,|\, \Phi \text{ is } \mathcal{P}_T(U)/\mathcal{B}(H)\text{-measurable} \\ &\text{and } \|\Phi\|_T := \big(E[\int_0^T \int_U \|\Phi(s,y)\|^2 \, \nu(dy) \, ds]\big)^{\frac{1}{2}} < \infty\} \end{aligned}$$

Then $\mathcal{E} \subset \mathcal{N}_q^2(T,U,H)$ and

$$\mathcal{N}_q^2(T,U,H) = L^2([0,T] \times \Omega \times U, P_T(U), P \otimes \lambda \otimes \nu, H)$$

is complete since $(H, \| \, \|)$ is complete. Therefore $\bar{\mathcal{E}} \subset \mathcal{N}_q^2(T,U,H)$ and by the previous proposition it follows that $\bar{\mathcal{E}} \supset \mathcal{N}_q^2(T,U,H)$. So finally one gets that $\bar{\mathcal{E}} = \mathcal{N}_q^2(T,U,H)$

1.4 Properties of the stochastic integral

Proposition 1.22. *Assume that $\Phi \in \mathcal{N}_q^2(T,U,H)$ and $u \in [0,T]$. Then $1_{]0,u]}\Phi \in \mathcal{N}_q^2(T,U,H)$ and for all $t \in [0,T]$*

$$\int_0^{t+} \int_U 1_{]0,u]} \Phi(s,y) \, q(ds,dy) = \int_0^{(t \wedge u)+} \int_U \Phi(s,y) \, q(ds,dy) \quad P\text{-a.s.}.$$

Proof.

Step 1. Let Φ be an elementary process, i.e.

$$\Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]t_m, t_{m+1}] \times A_i^m} \in \mathcal{E}$$

Then

$$1_{]u,T]} \Phi = \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} 1_{]t_m \vee u, t_{m+1} \vee u] \times A_i^m}$$

is an elementary process since $F_i^m \in \mathcal{F}_{t_m \vee u}$. Concerning the integral of $1_{]0,u]}\Phi$ one obtains that

$$\begin{aligned}
&\int_0^{t+} \int_U 1_{]0,u]}(s)\Phi(s)\, q(ds, dy) \\
&= \int_0^{t+} \int_U \Phi\, q(ds, dy) - \int_0^{t+} \int_U 1_{]u,T]}(s)\Phi\, q(ds, dy) \\
&= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} \big(q(t_{m+1} \wedge t, A_i^m) - q(t_m \wedge t, A_i^m) - q((t_{m+1} \vee u) \wedge t, A_i^m) \\
&\qquad + q((t_m \vee u) \wedge t, A_i^m)\big) \\
&= \sum_{m=0}^{k-1} \sum_{i=1}^{I(m)} x_i^m 1_{F_i^m} (q(t_{m+1} \wedge u \wedge t, A_i^m) - q(t_m \wedge u \wedge t, A_i^m)) \\
&= \int_0^{(t\wedge u)+} \int_U \Phi(s)\, q(ds, dy)
\end{aligned}$$

Step 2. Let now $\Phi \in \mathcal{N}_q^2(T, U, H)$. Then there exists a sequence of elementary processes Φ_n, $n \in \mathbb{N}$, such that $\|\Phi_n - \Phi\|_T \to 0$ as $n \to \infty$. Then it is clear that $\|1_{]0,u]}\Phi_n - 1_{]0,u]}\Phi\|_T \to 0$ as $n \to \infty$. By the defintion of the stochastic integral it follows that for all $t \in [0, T]$

$$\begin{aligned}
&E\Big[\Big\| \int_0^{(t\wedge u)+} \int_U \Phi_n(s, y)\, q(ds, dy) - \int_0^{(t\wedge u)+} \int_U \Phi(s, y)\, q(ds, dy) \Big\|^2\Big] \\
&+ E\Big[\Big\| \int_0^{t+} \int_U 1_{]0,u]}(s)\Phi_n(s, y)\, q(ds, dy) - \int_0^{t+} \int_U 1_{]0,u]}(s)\Phi(s, y)\, q(ds, dy) \Big\|^2\Big] \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

which implies that for all $t \in [0, T]$ there exists a subsequence $n_k(t)$, $k \in \mathbb{N}$, such that

$$\int_0^{(t\wedge u)+} \int_U \Phi_{n_k(t)}(s, y)\, q(ds, dy) \underset{k\to\infty}{\longrightarrow} \int_0^{(t\wedge u)+} \int_U \Phi(s, y)\, q(ds, dy) \; P-\text{a.s}$$

$$\int_0^{t+}\int_U 1_{]0,u]}(s)\Phi_{n_k(t)}(s,y)\,q(ds,dy) \underset{k\to\infty}{\longrightarrow} \int_0^{t+}\int_U 1_{]0,u]}(s)\Phi(s,y)\,q(ds,dy)\ P-\text{a.s.}.$$

Then by Step 1 the assertion follows.

□

Chapter 2

Existence of the Mild Solution

As in the previous chapter let $(H, \langle\ ,\ \rangle)$ be a separable Hilbert space, $(U, \mathcal{B}, \nu)$ a σ-finite measure space and $(\Omega, \mathcal{F}, P)$ a complete probability space with filtration $\mathcal{F}_t$, $t \geq 0$, such that $\mathcal{F}_0$ contains all P-nullsets of $\mathcal{F}$.
We fix a stationary $(\mathcal{F}_t)$-Poisson point process on U with characteristic measure ν. Moreover let $T > 0$ and consider the following type of stochastic differential equations in H

$$\begin{cases} dX(t) &= [AX(t) + F(X(t))]\, dt + B(X(t), y)\, q(dt, dy) \\ X(0) &= \xi \end{cases} \tag{2.1}$$

where

- $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of linear, bounded operators on H,
- $F : H \to H$ is $\mathcal{B}(H)/\mathcal{B}(H)$-measurable,
- $B : H \times U \to H$ is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable,
- $q(t, B)$, $t \geq 0$, $B \in \Gamma_p$, is the compensated Poisson random measure of p,
- ξ is an H-valued, $\mathcal{F}_0$-measurable random variable.

Remark 2.1. If we call $M_T := \sup_{t \in [0,T]} \|S(t)\|_{L(H)}$ then $M_T < \infty$.

Proof. For example by [Pa 83, Theorem 2.2, p.4] there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\|_{L(H)} \leq M e^{\omega t} \quad \text{for all } t \geq 0$$

□

Definition 2.2 (Mild solution). An H-valued predictable process $X(t)$, $t \in [0,T]$, is called a mild solution of equation (2.1) if

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))\,ds + \int_0^{t+} \int_U S(t-s)B(X(s),y)\,q(ds,dy) \quad P\text{-a.s.}$$

for all $t \in [0,T]$. In particular the appearing integrals have to be well defined.

To get the existence of a mild solution on $[0,T]$ we make the following assumptions

Hypothesis H.0

- $F : H \to H$ is Lipschitz-continuous, i.e. that there exists a constant $C > 0$ such that

$$\|F(x) - F(y)\| \leq C\|x - y\| \quad \text{for all } x, y \in H,$$

- there exists a square integrable mapping $K : [0,T] \to [0,\infty[$ such that

$$\int_U \|S(t)(B(x,y) - B(z,y))\|^2\,\nu(dy) \leq K^2(t)\|x - y\|^2$$

$$\int_U \|S(t)B(x,y)\|^2\,\nu(dy) \leq K(t)(1 + \|x\|)$$

Now we introduce the space where we want to find the mild solution of the above problem. We define

$$\mathcal{H}^2(T,H) := \{Y(t), t \in [0,T] \mid Y \text{ is an } H\text{-predictable process such that } \sup_{t\in[0,T]} E[\|Y(t)\|^2] < \infty\}$$

and for $Y \in \mathcal{H}^2(T,H)$

$$\|Y\|_{\mathcal{H}^2} := \sup_{t\in[0,T]} \big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}$$

Then $(\mathcal{H}^2(T,H), \|\ \|_{\mathcal{H}^2})$ is a Banach space.
For technical reasons we also consider the norms $\|\ \|_{2,\lambda,T}$, $\lambda \geq 0$, on $\mathcal{H}^2(T,H)$ given by

$$\|Y\|_{2,\lambda,T} := \sup_{t\in[0,T]} e^{-\lambda t}\big(E[\|Y(t)\|^2]\big)^{\frac{1}{2}}$$

Then $\| \|_{\mathcal{H}^2} = \| \|_{2,0,T}$ and all norms $\| \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent.
For simplicity we use the following notations

$$\mathcal{H}^2(T,H) := (\mathcal{H}^2(T,H), \| \|_{\mathcal{H}^2})$$

and

$$\mathcal{H}^{2,\lambda}(T,H) := (\mathcal{H}^2(T,H), \| \|_{2,\lambda,T}),\ \lambda > 0.$$

Theorem 2.3. *Assume that the coefficients A, F and B fullfill the conditions of Hypothesis H.0 then for every initial condition $\xi \in L^2(\Omega, \mathcal{F}_0, P, H) =: L_0^2$ there exists a unique mild solution $X(\xi)(t)$, $t \in [0,T]$, of equation (2.1).*
In addition we even obtain that the mapping

$$X : L_0^2 \to \mathcal{H}^2(T,H)$$

is Lipschitz continuous.

For the proof of the theorem we need the following lemmas.

Lemma 2.4. *If $Y : [0,T] \times \Omega \times U \to H$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable then the mapping*

$$[0,T] \times \Omega \times U \to H,\ (s,\omega,y) \mapsto 1_{]0,t]}(s)S(t-s)Y(s,\omega,y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$.

Step 1. Consider the case that Y is a simple process given by

$$Y = \sum_{k=1}^{n} x_k 1_{A_k}$$

where $x_k \in H$, $1 \leq k \leq n$, and $A_k \in \mathcal{P}_T(U)$, $1 \leq k \leq n$, is a disjoint covering of $[0,T] \times \Omega \times U$. Then we obtain that

$$\begin{aligned} \tilde{Y} : [0,T] \times \Omega \times U &\to H \\ (s,\omega,y) &\mapsto 1_{]0,t]}(s)S(t-s)Y(s,\omega,y) \\ &= 1_{]0,t]}(s) \sum_{k=1}^{n} S(t-s)x_k 1_{A_k}(s,\omega,y) \end{aligned}$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable since for $B \in \mathcal{B}(H)$ we get that

$$\tilde{Y}^{-1}(B) = \bigcup_{k=1}^{n} \big(\{s \in [0,T] \mid 1_{]0,t]}(s)S(t-s)x_k \in B\} \times \Omega \times U\big) \cap A_k$$

where $\{s \in [0,T] \mid 1_{]0,t]}(s)S(t-s)x_k \in B\} \in \mathcal{B}([0,T])$ by the strong continuity of the semigroup $S(t)$, $t \in [0,T]$. By Lemma 1.19 (i) we can conclude that $\tilde{Y}^{-1}(B) \in \mathcal{P}_T(U)$.

Step 2. Let Y be an arbitrary $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable process.
Then there exists a sequence Y_n, $n \in \mathbb{N}$, of simple $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable random variables such that $Y_n \to Y$ pointwisely a $n \to \infty$. Since $S(t) \in L(H)$ for all $t \in [0,T]$ the assertion follows.

□

Lemma 2.5. *Let Φ be a process on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ with values in a Banach space E. If Φ is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and stochastically continuous then there exists a predictable version of Φ.*
In particular, if $\Phi(t) \in L^2(\Omega, \mathcal{F}_t, P, E)$ and $\Phi : [0,T] \to L^2(\Omega, \mathcal{F}, P, E)$ is continuous then there exists a predictable version of Φ.

Proof. [DaPrZa 92, Proposition 3.6 (ii), p.76] □

Proof of Theorem 2.3. Let $t \in [0,T]$, $\xi \in L_0^2$ and $Y \in \mathcal{H}^2(T,H)$ and define

$$\begin{aligned}\mathcal{F}(\xi, Y)(t) := S(t)\xi &+ \int_0^t S(t-s)F(X(s))\, ds \\ &+ \int_0^{t+} S(t-s)B(X(s),y)\, q(ds,dy)\end{aligned}$$

Then a mild solution of problem (2.1) with initial condition $\xi \in L_0^2$ is by Definition 2.2 an H-predictable process such that $\mathcal{F}(\xi, X(\xi))(t) = X(\xi)(t)$ P-a.s. for all $t \in [0,T]$. Thus we have to search for an implicit function $X : L_0^2 \to \mathcal{H}^2(T,H)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$ in $\mathcal{H}^2(T,H)$.
For this reason we prove that $\mathcal{F}$ as a mapping from $L_0^2 \times \mathcal{H}^2(T,H)$ to $\mathcal{H}^2(T,H)$ is well defined and we show that there exists $\lambda \geq 0$ such that

$$\mathcal{F} : L_0^2 \times \mathcal{H}^{2,\lambda}(T,H) \to \mathcal{H}^{2,\lambda}(T,H)$$

is a contraction in the second variable, i.e. that there exists $L_{T,\lambda} < 1$ such that for all $\xi \in L_0^2$ and $Y, \tilde{Y} \in \mathcal{H}^{2,\lambda}(T,H)$

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\xi, \tilde{Y})\|_{2,\lambda,T} \leq L_{T,\lambda} \|Y - \tilde{Y}\|_{2,\lambda,T}.$$

Then the existence and uniqueness of the mild solution $X(\xi) \in \mathcal{H}^{2,\lambda}(T,H)$ of (2.1) with initial condition $\xi \in L_0^2$ follows by Banach's fixpoint theorem. Since the norms $\| \; \|_{2,\lambda,T}$, $\lambda \geq 0$, are equivalent we consider $X(\xi)$ as an element of $\mathcal{H}^2(T,H)$ and get the existence of the imlicit function $X : L_0^2 \to \mathcal{H}^2(T,H)$ such that $\mathcal{F}(\xi, X(\xi)) = X(\xi)$.

Step 1. The mapping $\mathcal{F} : L_0^2 \times \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$ is well defined.

Let $\xi \in L_0^2$ and $Y \in \mathcal{H}^2(T,H)$ then, by [FrKn 2002], $(S(t)\xi)_{t\in[0,T]} \in \mathcal{H}^2(T,H)$, $1_{]0,t]}(\cdot)S(t-\cdot)F(Y(\cdot))$ is P-a.s. Bochner integrable on $[0,T]$ and the process

$$\Big(\int_0^t S(t-s)F(Y(s))\,ds\Big)_{t\in[0,T]}$$

is an element of $\mathcal{H}^2(T,H)$.
Therefore it remains to prove that:
$(1_{]0,t]}(\cdot)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$ for all $t \in [0,T]$ and that there is a version of

$$\Big(\int_0^t \int_U S(t-s)B(X(s),y)\; q(ds,dy)\Big)_{t\in[0,T]}$$

which is an element of $\mathcal{H}^2(T,H)$.

Claim 1. If $Y \in \mathcal{H}^2(T,H)$ then:
$\Phi := (1_{]0,t]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$ for all $\in [0,T]$.

Let $t \in [0,T]$. First, we prove that the mapping

$$[0,T] \times \Omega \times U \to H,\; (s,\omega,y) \mapsto 1_{]0,t]}(s)S(t-s)B(Y(s,\omega),y)$$

is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable. By Lemma 2.4 we have to check if the mapping $(s,\omega,y) \mapsto B(Y(s,\omega),y)$ is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable.
The mapping $F : [0,T] \times \Omega \times U \to H \times U$, $(s,\omega,y) \mapsto (Y(s,\omega),y)$ is $\mathcal{P}_T(U)/\mathcal{B}(H) \otimes \mathcal{B}$-measurable since for $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}$ we have that

$$F^{-1}(A \times B) = \underbrace{Y^{-1}(A)}_{\in \mathcal{P}_T} \times B \in \mathcal{P}_T(U) \text{ by Lemma 1.19 (ii).}$$

Moreover B is $\mathcal{B}(H) \otimes \mathcal{B}/\mathcal{B}(H)$-measurable by assumption.
With respect to the norm $\| \; \|_T$ of Φ we obtain

$$\|\Phi\|_T^2 = E\Big[\int_0^t \int_U \|1_{]0,t]}(s)S(t-s)B(Y(s),y)\|^2\,\nu(dy)\,ds\Big]$$

$$\leq E\Big[\int_0^t K(t-s)(1+\|Y(s)\|)\,ds\Big]$$
$$\leq (1+\|Y\|_{\mathcal{H}^2})\int_0^T K(s)\,ds$$
$$< \infty$$

Claim 2. If $Y \in \mathcal{H}^2(T,H)$ then there is a predictable version of

$$(Z(t))_{t\in[0,T]} := \Big(\int_0^{t+}\int_U S(t-s)B(Y(s),y)\ q(ds,dy)\Big)_{t\in[0,T]}$$

which is an element of $\mathcal{H}^2(T,H)$.

Since $(1_{]0,t]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,T]} \in \mathcal{N}_q^2(T,U,H)$ for all $t \in [0,T]$ we get by the isometric formula that

$$\sup_{t\in[0,T]} E\Big[\|\int_0^{t+}\int_U S(t-s)B(Y(s),y)\,q(ds,dy)\|^2\Big]$$
$$= \sup_{t\in[0,T]} E\Big[\int_0^t\int_U \|S(t-s)B(Y(s),y)\|^2\,\nu(dy)\,ds\Big]$$
$$\leq \big(1+\|Y\|_{\mathcal{H}^2}\big)\int_0^T K(s)\,ds$$
$$<\infty$$

To prove the existence of the predictable version we will use Lemma 2.5. For this purpose we will show that the process Z is adapted to $\mathcal{F}_t$, $t\in[0,T]$, and continuous as a mapping from $[0,T]$ to $L^2(\Omega,\mathcal{F},P,H)$.
Let $\alpha > 1$ and define for $t \in [0,T]$

$$Z^\alpha(t) := \int_0^{(\frac{t}{\alpha})+}\int_U S(t-s)B(Y(s),y)\,q(ds,dy)$$
$$= \int_0^{(\frac{t}{\alpha})+}\int_U S(t-\alpha s)S((\alpha-1)s)B(Y(s),y)\,q(ds,dy)$$

where we used semigroup property.
Set $\Phi^\alpha(s,y) := S((\alpha-1)s)B(Y(s),y)$ then one can show analogously to the proof of the $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurability of the mapping
$(s,\omega,y) \mapsto 1_{]0,t]}(s)S(t-s)B(Y(s,\omega),y)$ that Φ^α is $\mathcal{P}_T(U)/\mathcal{B}(H)$-measurable.
Moreover

$$E\Big[\int_0^t\int_U \|S((\alpha-1)s)B(Y(s),y)\|^2\,\nu(dy)\,ds\Big]$$

$$
\begin{aligned}
&\le (1+\|Y\|_{\mathcal{H}^2}) \int_0^T K((\alpha-1)s)\,ds \\
&= (1+\|Y\|_{\mathcal{H}^2}) \frac{1}{\alpha-1} \int_0^T K(s)\,ds \\
&< \infty
\end{aligned}
$$

Therefore we obtain that $\Phi^\alpha \in \mathcal{N}_q^2(T,U,H)$.
Now we show that the mapping $Z^\alpha : [0,T] \to L^2(\Omega,\mathcal{F},P,H)$ is continuous for all $\alpha > 1$. For this reason let $0 \le u \le t \le T$.

$$
\begin{aligned}
&\big(E\big[\| \int_0^{(\frac{t}{\alpha})+} \int_U S(t-\alpha s)\Phi^\alpha(s,y)\,q(ds,dy) - \int_0^{(\frac{u}{\alpha})+} \int_U S(u-\alpha s)\Phi^\alpha(s,y) \\
&\qquad q(ds,dy)\|^2\big]\big)^{\frac{1}{2}}, \\
&= \big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y) - 1_{]0,\frac{u}{\alpha}]}(s) S(u-\alpha s)\Phi^\alpha(s,y) \\
&\qquad q(ds,dy)\|^2\big]\big)^{\frac{1}{2}}
\end{aligned}
$$

by Proposition 1.22,

$$
\begin{aligned}
&= \big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{u}{\alpha}]}(s)(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y) \\
&\qquad + 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y)\,q(ds,dy)\|^2\big]\big)^{\frac{1}{2}} \\
&\le \big(E\big[\| \int_0^{T+} \int_U 1_{]0,\frac{u}{\alpha}]}(s)(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\,q(ds,dy)\|^2\big]\big)^{\frac{1}{2}} \\
&\quad + \big(E\big[\| \int_0^{T+} \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) S(t-\alpha s)\Phi^\alpha(s,y)\,q(ds,dy)\|^2\big]\big)^{\frac{1}{2}} \\
&= \big(E\big[\int_0^{\frac{u}{\alpha}} \int_U \|(S(t-\alpha s) - S(u-\alpha s))\Phi^\alpha(s,y)\|^2\,\nu(dy)\,ds\big]\big)^{\frac{1}{2}} \\
&\quad + \big(E\big[\int_0^T \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) \|S(t-\alpha s)\Phi^\alpha(s,y)\|^2\,\nu(dy)\,ds\big]\big)^{\frac{1}{2}},
\end{aligned}
$$

by the isometric formula.

(1.) The first summand converges to 0 as $u \uparrow t$ or $t \downarrow u$ by Lebesgue's dominated convergence theorem since the integrand converges pointwisely to 0 as $u \uparrow t$ or $t \downarrow u$ by the strong continuity of the semigroup and can be estimated independently of u and t by $4M_T^2\|\Phi^\alpha\|^2(s,y)$, $(s,y) \in [0,T] \times U$,

where $E\big[\int_0^T \int_U \|\Phi^\alpha(s,y)\|^2\,\nu(dy)\,ds\big] < \infty$.
(2.) The second summand can be estimated by

$$\big(E\big[\int_0^T \int_U 1_{]\frac{u}{\alpha},\frac{t}{\alpha}]}(s) M_T^2 \|\Phi^\alpha(s,y)\|^2\,\nu(dy)\,ds\big]\big)^{\frac{1}{2}} \to 0$$

and therefore converges to 0 by Lebesgue's dominated convergence theorem as $u \uparrow t$ or $t \downarrow u$.
To obtain the continuity of $Z : [0,T] \to L^2(\Omega, \mathcal{F}, P)$ we prove the uniform convergence of Z^{α_n}, $n \in \mathbb{N}$, to Z in $L^2(\Omega, \mathcal{F}, P, H)$ for an arbitrary sequence α_n, $n \in \mathbb{N}$, with $\alpha_n \downarrow 1$ as $n \to \infty$:

$$\begin{aligned}
&E\big[\|\int_0^{(\frac{t}{\alpha_n})+} \int_U S(t-\alpha_n s)\Phi^{\alpha_n}(s,y)\,q(ds,dy) - \int_0^{t+}\int_U S(t-s)B(Y(s),y)\\
&\qquad q(ds,dy)\|^2\big]\\
=&E\big[\|\int_0^{T+}\int_U 1_{]0,\frac{t}{\alpha_n}]}(s)S(t-s)B(Y(s),y) - 1_{]0,t]}(s)S(t-s)B(Y(s),y)\\
&\qquad q(ds,dy)\|^2\big]\\
=&E\big[\|\int_0^{T+}\int_U 1_{]\frac{t}{\alpha_n},t]}(s)S(t-s)B(Y(s),y)\,q(ds,dy)\|^2\big]\\
=&E\big[\int_{\frac{t}{\alpha_n}}^t \int_U \|S(t-s)B(Y(s),y)\|^2\,\nu(dy)\,ds\big]\\
\le&E\big[\int_{\frac{t}{\alpha_n}}^t K(t-s)(1+\|Y(s)\|)\,ds\big]\\
\le&\big(1+\|Y\|_{\mathcal{H}^2}\big)\big(t-\frac{t}{\alpha_n}\big)^{\frac{1}{2}}\big(\int_0^T K^2(s)\,ds\big)^{\frac{1}{2}}\\
\le&\big(1+\|Y\|_{\mathcal{H}^2}\big)\big(\frac{\alpha_n-1}{\alpha_n}T\big)^{\frac{1}{2}}\big(\int_0^T K^2(s)\,ds\big)^{\frac{1}{2}}
\end{aligned}$$

where $\dfrac{\alpha_n-1}{\alpha_n}T \to 0$ as $n \to \infty$.
Moreover we know for all $t \in [0,T]$ that

$$\big(\int_0^{u+}\int_U 1_{]0,u]}(s)S(t-s)B(Y(s),y)\,q(ds,dy)\big)_{u\in[0,t]} \in \mathcal{M}_t^2(H)$$

since $(1_{]0,u]}(s)S(t-s)B(Y(s),\cdot))_{s\in[0,t]} \in \mathcal{N}_q^2(t,U,H)$. That means in particular that the process

$$Z(t) = \int_0^{t+}\int_U 1_{]0,t]}(s)S(t-s)B(Y(s),y)\,q(ds,dy),\ t\in[0,T] \text{ is } (\mathcal{F}_t)\text{-adapted.}$$

Together with the continuity of Z in $L^2(\Omega, \mathcal{F}, P < H)$, by Lemma 2.5, this implies the existence of a predictable version of $Z(t)$, $t \in [0,T]$, denoted by

$$\Big(\int_0^{t-} \int_U S(t-s)B(Y(s),y)\, q(ds,dy)\Big)_{t\in[0,T]}.$$

Therefore we have finally proved that

$$\mathcal{F} : L_0^2 \times \mathcal{H}^2(T,H) \to \mathcal{H}^2(T,H)$$

Claim 3. There exists $\lambda \geq 0$ such that for all $\xi \in L_0^2$

$$\mathcal{F}(\xi,\cdot) : \mathcal{H}^{2,\lambda}(T,H) \to \mathcal{H}^{2,\lambda}(T,H)$$

is a contraction where the contraction constant $L_{T,\lambda} < 1$ does not depend on ξ.

Let $Y, \tilde{Y} \in \mathcal{H}^2(T,H)$, $\xi \in L_0^2$. Then we get for $\lambda \geq 0$ that

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\lambda t} \|\big(\mathcal{F}(\xi,Y) - \mathcal{F}(\xi,\tilde{Y})\big)(t)\|_{L^2}\\
&\leq \sup_{t\in[0,T]} e^{-\lambda t} \|\int_0^t S(t-s)[F(Y(s)) - F(\tilde{Y}(s))]\, ds\|_{L^2}\\
&\quad + \sup_{t\in[0,T]} e^{-\lambda t} \|\int_0^{t+} \int_U S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\, q(ds,dy)\|_{L^2}
\end{aligned}$$

The first summand can be estimated by

$$\underbrace{M_T C T^{\frac{1}{2}} \Big(\frac{1}{2\lambda}\Big)^{\frac{1}{2}}}_{\to 0 \text{ as } \lambda\to\infty} \|Y - \tilde{Y}\|_{2,\lambda,T},$$

for the proof see [FrKn 2002, Theorem 3.2., Step 3, p.81].
By the isometric formula we get the following estimation for the second summand:

$$\begin{aligned}
&E\big[\|\int_0^{t+} \int_U S(t-s)B(Y(s),y)\, q(ds,dy) - \int_0^{t+} \int_U S(t-s)B(\tilde{Y}(s),y)\, q(ds,dy)\|^2\big]\\
&= E\big[\int_0^t \int_U \|S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\|^2\, \nu(dy)\, ds\big]\\
&\leq E\big[\int_0^t K^2(t-s)\|Y(s) - \tilde{Y}(s)\|^2\, ds\big]
\end{aligned}$$

$$\leq \int_0^t e^{\lambda s} K^2(t-s)\, ds \|Y - \tilde{Y}\|^2_{2,\lambda,T}$$
$$= \|Y - \tilde{Y}\|^2_{2,\lambda,T} \underbrace{e^{-\lambda t} \int_0^T e^{-\lambda s} K^2(s)\, ds}_{\to 0 \text{ as } \lambda \to \infty}$$

Therefore we obtain that

$$\sup_{t\in[0,T]} e^{-\lambda t} \| \int_0^{t+} \int_U S(t-s)[B(Y(s),y) - B(\tilde{Y}(s),y)]\, q(ds,dy)\|_{L^2}$$
$$\leq \Big(\int_0^t e^{-\lambda s} K^2(s)\, ds \Big)^{\frac{1}{2}} \|Y - \tilde{Y}\|_{2,\lambda,T}$$

Thus we have finally proved that there exists $\lambda \geq 0$ such that there exists $L_{T,\lambda} < 1$ with

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\xi, \tilde{Y})\|_{2,\lambda,T} \leq L_{T,\lambda} \|Y - \tilde{Y}\|_{2,\lambda,T}$$

for all $\xi \in L_0^2$, $Y, \tilde{Y} \in \mathcal{H}^{2,\lambda}(T,H)$. Hence the existence of a unique implicit function

$$X : L_0^2 \to \mathcal{H}^2(T,H)$$
$$\xi \mapsto X(\xi) = \mathcal{F}(\xi, X(\xi))$$

is verified.

Claim 4. The mapping $X : L_0^2 \to \mathcal{H}^2(T,H)$ is Lipschitz continuous.

By Theorem A.1 (ii) and the equivalence of the norms $\| \ \|_{2,\lambda,T}$, $\lambda \geq 0$, we only have to check that the mappings

$$\mathcal{F}(\cdot, Y) : L_0^2 \to \mathcal{H}^2(T,H)$$

are Lipschitz continuous for all $Y \in \mathcal{H}^2(T,H)$ where the Lipschitz constant does not depend on Y.
But this assertion holds as for all $\xi, \zeta \in L_0^2$ and $Y \in \mathcal{H}^2(T,H)$

$$\|\mathcal{F}(\xi, Y) - \mathcal{F}(\zeta, Y)\|_{\mathcal{H}^2} = \|S(\cdot)(\xi - \zeta)\|_{\mathcal{H}^2} \leq M_T \|\xi - \zeta\|_{L^2}.$$

□

Appendix A

Continuity of Implicit Functions

We fix two Banach spaces $(E, \| \ \|)$ and $(\Lambda, \| \ \|_\Lambda)$.
Consider a mapping $G : \Lambda \times E \to E$ such that there exists an $\alpha \in [0, 1[$ such that

$$\|G(\lambda, x) - G(\lambda, y)\| \leq \alpha \|x - y\| \qquad \text{for all } \lambda \in \Lambda \text{ and all } x, y \in E$$

Then we get by Banach's fixpoint theorem that there exists exactly one mapping $\varphi : \Lambda \to E$ such that

$$\varphi(\lambda) = G(\lambda, \varphi(\lambda)) \text{ for all } \lambda \in \Lambda.$$

Theorem A.1 (Continuity of the implicit function). *(i) If we assume in addition that the mapping $\lambda \mapsto G(\lambda, x)$ is continuous from Λ to E for all $x \in E$ we get that $\varphi : \Lambda \to E$ is continuous.*

(ii) If the mappings $\lambda \mapsto G(\lambda, x)$ are not only continuous from Λ to E for all $x \in E$ but there even exists a $L \geq 0$ such that $\|G(\lambda, x) - G(\tilde{\lambda}, x)\|_E \leq L \|\lambda - \tilde{\lambda}\|_\Lambda$ for all $x \in E$ then the mapping $\varphi : \Lambda \to E$ is Lipschitz continuous.

Proof. [FrKn 2002, Theorem D.1, p.164] □

Bibliography

[BeLi 82] Bensoussan, A., Lions, J.-L.: Impulse control and quasivariational inequalities. Mathematical Methods of Information Science, 11. Gauthier-Villars, Paris (1982)

[Co 80] Cohn, D.L.: Measure theory. Boston: Birkhäuser 1980

[DaPrZa 92] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions. Cambridge: Cambridge University Press 1992

[DaPrZa 96] Da Prato, G., Zabczyk, J.: Ergodicity for infinite dimensional systems. Cambridge: Cambridge University Press 1996

[FrKn 2002] Frieler, K., Knoche, C.: Solutions of stochastic differential equations in infinite dimensional Hilbert spaces and their dependence on initial data. Diplomathesis, Bielefeld 2002

[IkWa 81] Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam-Oxford-New York: North-Holland Publishing Company 1981

[Me 82] Metivier, M.: Semimartingales. A course on stochastic processes.Berlin-New York: W. de Gruyter 1982

[Rü 2003] Rüdiger, B.: Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. Sonderforschungsbereich 611, Rheinischen Friedrich-Wilhelm-Universität Bonn, Preprint Nr.62, submitted to PTRF (2003)