VERIFICATION THEOREMS FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS VIA A GENERALIZED FUKUSHIMA - DIRICHLET DECOMPOSITION

Fausto Gozzi and Francesco Russo

Università di Roma "La Sapienza"
Dipartimento di Matematica per le Decisioni
Via del Castro Laurenziano n.9, Roma, Italy

Université Paris 13
Institut Galilée, Mathématiques
99, av. JB Clément, F-99430 Villetaneuse, France

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Abstract. This paper is devoted to present a method of proving verification theorems for stochastic optimal control of finite dimensional diffusion processes without control in the diffusion term. The value function is assumed to be continuous in time and one time continuously differentiable in the space variable ($C^{0,1}$) instead of one time continuously differentiable in time and twice in space ($C^{1,2}$), like in the classical results. The results are obtained via a pathwise Fukushima-Dirichlet decomposition obtained in the stochastic calculus via regularization. As side effect of this paper we continue the development of that "pathwise" stochastic calculus. A comparison with other verification techniques is performed, in particular with the results in the viscosity solution setting.

1 Introduction

In this paper we want to present a method to get verification theorems for stochastic optimal control problems in finite dimension based on a generalized Fukushima-Dirichlet decomposition proved in Section 3. Since this Fukushima - Dirichlet decomposition holds for functions $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ that are C^0 in time and C^1 in space ($C^{0,1}$ in symbols), our verification theorem has the advantage of requiring less regularity of the value function V than the classical ones which need C^1 regularity in time and C^2 in space of V ($C^{1,2}$ in symbols), see e.g. [22, p. 140 ,163, 172]. It is also possible to prove a verification theorem in the case when V is only continuous (see e.g. [38], [57, Section 5.2]) in the framework of viscosity solutions: however such result applied to our cases is weaker than ours, as it requires more assumption on the candidate optimal strategy (see on this section 9 where also a comparison with others nonsmooth verification theorems is performed).

We can say that, on one side, we want to present a method to prove verification theorems for a class of stochastic optimal control problems while, on the other side, we want to illustrate a Fukushima-Dirichlet decomposition that fits well with the problem of finding optimal feedback controls in the stochastic framework, but it has its own interest at the level of stochastic calculus.

1.1 The Fukushima - Dirichlet decomposition

One classical tool in proving verification theorems in stochastic control theory is the Itô formula which generally allows to expand a $C^{1,2}$ function (say u) of a diffusion process (say X). Roughly speaking the Itô formula in fact gives a decomposition of $u(t,X)$ in a martingale part, say M (which is thrown away taking expectation in the case of deterministic data and expected cost) plus an absolutely continuous process, say A. Then, in case of deterministic data and expected cost, one uses the fact that u is a classical solution of a partial differential equation (PDE), which is in fact the Hamilton-Jacobi-Bellman (HJB) equation, to represent A in term of the Hamiltonian function. If one wants to repeat the above arguments when u is not $C^{1,2}$, a natural way is to try to extend the decomposition $u(\cdot,X) = M + A$ and the representation of A via the HJB PDE. This is what we perform in this paper in the case when $u \in C^{0,1}$ using a point of view that can be also applied to problems with stochastic data and pathwise cost (so the HJB becomes a stochastic PDE, see e.g. [13, 39, 40, 45]).

We propose in fact an extension of the classical Fukushima-Dirichlet decomposition. That decomposition is inspired by the theory of Dirichlet forms. A classical monography concerning this theory is [25] where one can find classical references on the subject. In Theorem 5.2.2 of [25], given a "good" symmetric Markov process $(X_t^x)_{t\geq 0}$ and a function belonging to some suitable space (Dirichlet space), it is possible to write

$$u(X_t) = u(x) + M_t^u + A_t^u \qquad (1)$$

where M is a local martingale and $(A_t^u)_{t\geq 0}$ is a zero quadratic variation process, for quasi-everywhere x, i.e. for x belonging to a zero capacity set. For instance, if $X = B$ is a classical Brownian motion in $\mathbb{R}^n$ then the Dirichlet space is $H^1(\mathbb{R}^n)$ and $M^u = \int_0^t \nabla u(B_u) dB_u$. We call (1) a Fukushima-Dirichlet decomposition. Our point of view is pathwise: as in [23], a process Y, as $u(X)$, which is the sum of a local martingale and a zero quadratic variation process, even without any link to Dirichlet forms, is called a Dirichlet process.

The papers [50] and [24] reinterpret A in a pathwise way as the covariation process $[\nabla u(B), B]$ transforming (1) in a true Itô's formula; the first work considers $u \in C^1(\mathbb{R}^n)$ and it extends the framework to reversible continuous semimartingales; the second work is connected with Brownian motion and $u \in H^1(\mathbb{R}^n)$. The literature on Itô's formula for non-smooth functions of semimartingales or diffusion processes has known a lot of development in the recent years, see for instance [43] for non-degenerate Brownian martingales, [20, 21] for non-degenerate 1-dimensional diffusions with bounded measurable drift or [18] in the jump case.

In our applications, however, the fact of identifying the remainder process A^u as a covariation is not so important since the goal is to give the representation of it via the data of the HJB PDE. So we come back to the spirit of the Fukushima-Dirichlet decomposition. Besides our "pathwise" approach to Dirichlet processes, the true novelty of this approach is the time-inhomogeneous version of the decomposition; this is in particular motivated by non-autonomous problems in control theory.

We will be able in particular to decompose $u(t, X_t)$ when $u \in C^{0,1}$ and X is a semimartingale. This will provide a *generalized Fukushima-Dirichlet* decomposition: the resulting process will be called a *weak Dirichlet process* and it will consist in the sum of a local martingale and a process which is "orthogonal" in the sense of the covariation to the space of local martingales. This decomposition, in another context, appears also in [17]. There is of course an approach to the theory of "time-dependent Dirichlet forms" developed for instance by [44, 53, 56] but it is not suitable fur our purposes.

1.2 The statement of our verification theorem in a model case

To clarify our results we describe briefly and informally below the framework and the statement of the verification theorem that we are going to prove in a model case. The precise statement and proof are given in Section 5; then in Sections 7, 8 applications to more specific (and somehow more difficult) classes of problems is given. We decided this structure since it is difficult to provide a single general result: we can say that we introduce a technique, based on the Fukushima-Dirichlet decomposition and on its representation, that can be adapted with some work to different settings each time with a different adaptation. First we take a given stochastic basis $\left(\Omega, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\right)$ that satisfies the so-called usual conditions, a finite dimensional Hilbert space $A = \mathbb{R}^n$ (the state space), a finite dimensional Hilbert space $E = \mathbb{R}^m$ (the noise space), a set $U \subseteq \mathbb{R}^k$ (the control space). We fix then a terminal time $T \in [0, +\infty]$ (the horizon of the problem, that can be finite or infinite but is fixed), an initial time and state $(t, x) \in [0, T] \times A$ (which will vary as usual in the dynamic programming approach). The state equation is (define $\mathcal{T}_t = [t, T] \cap \mathbb{R}$)

$$\begin{aligned} dy(s) &= [F_0(s, y(s)) + F_1(s, y(s), z(s))]\, ds + B(s, y(s))\, dW(s), \quad s \in \mathcal{T}_t. \\ y(t) &= x \end{aligned} \tag{2}$$

where the following holds (for a matrix B by $||B||$ we mean $\sum_{i,j} |b_{ij}|$ and, given E, F finite deimensional spaces, by $\mathcal{L}(E, F)$ we mean the set of linear operators $E \to F$).

Hypothesis 1.1 *1. $z : \mathcal{T}_t \times \Omega \to U$ (the control process) is measurable, locally integrable in t for a.e. $\omega \in \Omega$, adapted to the filtration $(\mathcal{F}_s)_{s\geq 0}$;*

2. $F_0 : \mathcal{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $F_1 : \mathcal{T}_0 \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $B : \mathcal{T}_0 \times \mathbb{R}^n \to \mathcal{L}(E, \mathbb{R}^n)$ are continuous.

3. *There exists* $C > 0$ *such that* $\forall s \in \mathcal{T}_0$, $\forall x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} \left|\langle F_0(s, x_1) - F_0(t, x_2), x_1 - x_2\rangle\right| + \|B(t, x_1) - B(t, x_2)\|^2 &\leq C|x_1 - x_2|^2, \\ \left|\langle F_0(t, x), x\rangle\right| + \|B(t, x)\| &\leq C\left[1 + |x|^2\right], \end{aligned}$$

and, there exists a constant K *such that,* $\forall t \in \mathcal{T}_0$, $\forall x \in \mathbb{R}^n$, $\forall z \in U$,

$$\begin{aligned} |F_1(t, x_1, z) - F_1(t, x_2, z)| &\leq K|x_1 - x_2| \\ |\langle F_1(t, x, z), x\rangle| &\leq K(1 + |x| + |z|). \end{aligned}$$

4. *For every* $x \in \mathbb{R}^n$

$$\int_0^T \left[|F_0(t, x)| + \|B(t, x)\|^2\right] dt < +\infty$$

We call $\mathcal{Z}_{ad}(t)$ the set of admissible control strategies when the initial time and state is (t, x) defined as

$$\mathcal{Z}_{ad}(t) = \left\{ \begin{array}{c} z : \mathcal{T}_t \times \Omega \to U, \quad z \text{ measurable, adapted to } (\mathcal{F}_s)_{s \geq 0}, \\ \text{locally integrable in } t \text{ for a.e. } \omega \in \Omega \end{array} \right\}$$

and call $y(s; t, x, z)$ the state process associated with a given $z \in \mathcal{Z}_{ad}(t)$; this is the strong solution of the equation (2) which exists and it is unique for any given z thanks to Theorem 1.2 in [37, p.2].

Consider now the case $T < +\infty$. We try to minimize the functional

$$J(t, x; z) = \mathbb{E}\left\{\int_t^T l(s, y(s; t, x, z), z(s))\, ds + \phi(y(T; t, x, z))\right\} \tag{3}$$

over all control processes $z \in \mathcal{Z}_{ad}(t)$.

On the coefficients we assume the following.

Hypothesis 1.2 $l : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *are continuous and such that* $J(t, x; z)$ *is well defined for each* $z \in \mathcal{Z}_{ad}(t)$ *(i.e.* $s \to l(s, y(s), z(s))$ *is semiintegrable in* $[t, T]$*).*

Remark 1.3 *The requirement that* $J(t, x; z)$ *is well defined for each* $z \in \mathcal{Z}_{ad}(t)$ *is assumed explicitely to cover a more general class of problems. This is obvious under additional assumptions, e.g. that* l *and* ϕ *be bounded below. However some interesting problem arising in economics contains functions* l *like* $-\ln z$ *and in this case the semiintegrability of* $s \to -\ln z(s)$ *for each* $z \in \mathcal{Z}_{ad}(t)$ *follows by ad hoc arguments. These are problems with state constraints that are not explicitely treated in this paper but we want to set our framework so to be able to treat such class of problems.* ■

The value function is defined as

$$V(t,x) = \inf_{z \in \mathcal{Z}_{ad}(t)} J(t,x;z), \tag{4}$$

and a control $z^* \in \mathcal{Z}_{ad}(t)$ is optimal at (t,x) if $V(t,x) = J(t,x;z^*)$. The current value Hamiltonian is defined, for $(t,x,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U$ as

$$H_{CV}(t,x,p;z) = \langle F_0(t,x), p\rangle + \langle F_1(t,x,z), p\rangle + l(t,x,z)$$

and the (minimum value) Hamiltonian as

$$H(t,x,p) = \inf_{z \in U} H_{CV}(t,x,p;z).$$

Since the first term of $H_{CV}(t,x,p;z)$ does not depend on the control z we will usually define

$$H_{CV}^0(t,x,p;z) = \langle F_1(t,x,z), p\rangle + l(t,x,z)$$

and

$$H^0(t,x,p) = \inf_{z \in U} H_{CV}^0(t,x,p;z) \tag{5}$$

so we can write

$$H_{CV}(t,x,p;z) =: \langle F_0(t,x), p\rangle + H_{CV}^0(t,x,p;z) \tag{6}$$

and

$$H(t,x,p) =: \langle F_0(t,x), p\rangle + H^0(t,x,p). \tag{7}$$

When $T < +\infty$ the Hamilton-Jacobi-Bellman (HJB) equation for the value function is a semilinear parabolic PDE ($(t,x) \in [0,T] \times \mathbb{R}^n$)

$$\begin{aligned} -\partial_t v(t,x) = \frac{1}{2}\mathrm{Tr}\left[B^*(t,x)\, \partial_{xx} v(t,x) B(t,x)\right] + \langle F_0(t,x), \partial_x v(t,x)\rangle \\ + H^0(t,x,\partial_x v(t,x)), \end{aligned} \tag{8}$$

with the final condition

$$v(T,x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{9}$$

To ensure finiteness and continuity of the Hamiltonian we need also to add the following assumption.

Hypothesis 1.4 *The functions l, F_1 and the set U are such that the value function is always $< +\infty$ and the Hamiltonian $H^0(t,x,p)$ is well defined, finite and continuous for every $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.*

Remark 1.5 *The above Hypothesis 1.4 is satisfied e.g. when l, ϕ are continuous and U is compact. Another possibility is to take U unbounded, F_1 sublinear, $l(t,x,z) = g(x) + h(z)$ with g continuous and bounded, h continuous and such that*

$$|h(z)|/|z| \longrightarrow +\infty \qquad \text{as } |z| \to +\infty;$$

this case was studied e.g in [7, 6, 30, 31, 33] even in the infinite dimensional case (see on this Sections 7, 8). ∎

The statement of the classical verification theorem for this model problem when $T < +\infty$ is the following, see Definition 5.1 for the definition of strict solution of equation (8)-(9).

Theorem 1.6 *Assume that Hypotheses 1.1, 1.2, 1.4 hold true. If v is a polinomially growing classical solution of the HJB equation (8)-(9) on $[0,T] \times \mathbb{R}^n$ and $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$, then:*

(i) $v \leq V$ *on* $[0,T] \times \mathbb{R}^n$*;*

(ii) fix $(t,x) \in [0,T] \times \mathbb{R}^n$; if $z \in \mathcal{Z}_{ad}(t)$ is such that, calling $y(s) = y(s;t,x,z)$

$$H^0(s, y(s), \partial_x v(s, y(s))) = H^0_{CV}(s, y(s), \partial_x v(s, y(s)), z(s)), \quad \mathbb{P}\text{-}a.s.$$

for a.e. $s \in [t,T]$, then z is optimal at (t,x) and $v(t,x) = V(t,x)$ for every $s \in [t,T]$.

This theorem states a sufficient optimality condition and its proof is based on Itô's formula, see e.g. [57, p.268]. In the classical context also necessary conditions and existence of optimal feedback can be proved, see again [57, p.268].

The statement of our Verification Theorem in the model problem described above is very similar. We give it in the case when the following nondegeneracy hypothesis hold

$$B^{-1}(t,x) F_1(t,x,z) \text{ is bounded on } [0,T] \times \mathbb{R}^n \times U \tag{10}$$

leaving the discussion for the general case to Section 5 (strong solution of the HJB equation (8)-(9) are defined in Definition 5.2).

Theorem 1.7 *Assume that Hypotheses 1.1, 1.2, 1.4 and (10) hold true. If v is a polinomially growing strong solution of the HJB equation (8)-(9) and $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$, then (i) and (ii) of Theorem 5.5 above hold.*

In fact we will go further:

- proving a result for the case when (10) does not hold;
- showing that in our setting we can obtain a necessary condition and the existence of optimal feedback controls on the same line of the classical results, see Section 6.

Moreover in a paper in preparation we will show that also some cases where the drift is a distribution can be treated and also cases where the solution of HJB enjoys weaker regularity.

When $T = +\infty$, F_0, F_1 and B do not depend on time, we consider the problem of maximizing

$$J(t,x;z) = \mathbb{E}\left\{ \int_t^{+\infty} e^{-\lambda t} l_1(y(s;t,x,z), z(s))\, ds \right\}$$

where $l(t,x,z) = e^{-\lambda t} l_1(x,z)$, with $\lambda > 0$, l_1 continuous and bounded. In this case the value function V is defined as in (4) but its dependence on t becomes trivial as $V(t,x) = e^{\lambda t} V(0,x)$ for every t and x. Then we set $t = 0$ and call $V_0(x) = V(0,x)$. The HJB equation for $V_0(x)$ becomes an elliptic PDE ($x \in \mathbb{R}^n$)

$$\lambda v(x) = \frac{1}{2} \mathrm{Tr}\left[B^*(x)\, \partial_{xx} v(x) B(x)\right] + \langle F_0(t,x), \partial_x v(t,x) \rangle + H^1(x, \partial_x v(t,x)), \quad (11)$$

where

$$H^1(x,p) = \inf_{z \in U} \{\langle F_1(x,z), p \rangle + l_1(x,z)\} =: \inf_{z \in U} H^1_{CV}(x,p;z). \quad (12)$$

In this elliptic case the statement of our verification theorem is the following (for the definition of strong solution of the HJB equation (11) see Definition 5.8).

Theorem 1.8 *Assume that Hypotheses 1.1 and (10) hold. Assume also that l_1 is continuous and bounded and that Hypothesis 1.4 hold true for the Hamiltonian H^1. If v is a bounded strong solution of the HJB equation (11) and $v \in C^1(\mathbb{R}^n)$ then $v \leq V_0$. Moreover a sufficient condition for optimality holds in term of v: fix $x \in \mathbb{R}^n$; if $z \in \mathcal{Z}_{ad}(0)$ is such that, calling $y(s) = y(s;0,x,z)$*

$$H^1(y(s), \partial_x v(s, y(s))) = H^1_{CV}(y(s), \partial_x v(s, y(s)), z(s)), \quad \text{for a.e. } s \geq 0, \quad \mathbb{P}-a.s.$$

then z is optimal at $(0,x)$ and $v(x) = V_0(x)$.

1.3 Other Verification Theorems for stochastic optimal control problems

We recall that in the literature on stochastic optimal control other verification theorems for not $C^{1,2}$ functions were proved. Roughly speaking we can say that each of them (and the technique of proof, too) is strictly connected with the concept of weak solution of the HJB equation that is considered. We have in fact the following.

- *(Strong solutions).* In [30, 31, 33] (see also [8, 9], [14, 15], [29]) in a setting similar to ours but in infinite dimension, a verification theorem is proved assuming that there exists a strong solution v of HJB which is C^1 in space. In fact the method outlined here goes in the same direction, but improves and generalizes in the finite dimensional case the ideas contained there. Such improvement is not a straightforward one, as we need to use completely different tools to get our results and we get more powerful theorems. We refer to Section 9 for explanations on this point.

- *(Viscosity solutions).* In [38, 57] a verification technique for viscosity solutions is introduced and studied. Such technique adapts to the case when v is only continuous and so is very general and applicable to cases when the control enters in the diffusion coefficients B. However in the case of our interest, i.e. when v is C^1 in space, such technique gives weaker results as it requires more assumptions on the coefficients F_0, F_1, B and on the candidate optimal strategy; see Section 9 for explanations.

- *(Mild solutions).* In [26, Theorem 7.2] a verification theorem is given when the HJB equation admits mild solutions which are C^1 in space (see Definition 7.13 or Remark 8.3) and when the solution can be represented using the solution of a suitable backward SDE. The results available with this technique are limited to infinite dimensional cases where mild solutions exist and Girsanov theorem can be applied in a suitable way. Such restrictions prevent the use of this technique (at least to the present state of the art) e.g. in the cases described in Sections 7 and 8, see Section 9 for explanations.

As we said above our paper wants to go deeper in the technique of verification theorems for the case of strong solutions giving examples of application that cannot be covered with other techniques or with the previous results of [30, 31, 33]. It is clear that our approach can be used in a larger variety of cases as such papers deals with very specific cases; in Sections 7 and 8 we choose two representative cases where the results obtained appear to be new and better than what can be proved with other techniques. In particular in Section 8 we provide an explicit example where the value function is C^1 but cannot be C^2 in space.

Finally we want to stress the fact that the Fukushima-Dirichlet decomposition we get below is in fact stronger than what we need to prove the Verification Theorem. In fact for such purpose it would be enough to prove a Dynkin-type formula (roughly speaking an Itô formula after expectation) which is in general easier. We decided to state and prove the Fukushima-Dirichlet decomposition below since it can help to deal with stochastic control problems where the criterion to minimize does not contain expectation (e.g. pathwise optimality and optimality in probability). In such context the HJB equation becomes a stochastic PDE and to get a verification theorem, Dynkin-type formulae cannot be used. See e.g. [13, 39, 40, 45] on this subject.

To sum up we think that the interest of our verification results is the fact that

- we obtain them using only the fact that the solution v of HJB belongs to $C^{0,1}$ and not necessarily to $C^{1,2}$ obtaining better results than other techniques.
- they can be applied also to problems with pathwise optimality and optimality in probability.

Remark 1.9 *As a first step of our work, we consider here the case of a stochastic optimal control problem in finite dimension with no state constraints. We are aware of the fact that in many cases, the HJB equation associated with the control problem admits a $C^{1,2}$ solution, so that our method does not give a real advantage in this case. However*

- *in some cases, mainly degenerate ones (see Sections 7 and 8) it is known that the solution of HJB is C^0 (or C^α with Hölder exponent $\alpha > 0$) in time and C^1 in space but $C^{1,2}$ regularity is not known at this stage. In particular we show an explicit example where the value function is C^1 but cannot be C^2 in space;*

- *we think that such technique could be extended also in cases where the solution of HJB is not $C^{0,1}$ and a weaker decomposition hold; we do not treat this case here leaving it for further investigation;*

- *we consider problems without state constraints; again we think that some state constraints problems can be treated with our approach but we do not handle them here: we provide an example with exit time which usually presents similar difficulties;*

- *we think that our method can be extended to the infinite dimensional case, where $C^{1,2}$ regularity results for the solution of HJB equation are much less known while C^1 regularity can be found in a variety of cases (see e.g. [7, 6, 30, 31, 33, 8, 9]). We do not perform this extension here leaving it for further work.*

Remark 1.10 *Similar ideas as in this work have been expressed with a different formalism in [10]. There the authors have implemented a generalized Itô formula in the Krylov spirit for proving an existence and uniqueness result for a generalized solution of Bellman equation for controlled processes, in a non-degeneracy situation.*

1.4 Plan of the paper

The plan of the paper is the following: after some preliminaries in Section 2, Section 3 is devoted to basic facts about the Fukushima-Dirichlet decomposition and its generalization which holds for instance for $C^{0,1}$ functions of semimartingales; in Section 4 we describe how to apply the decomposition to get representation of the solution of linear PDE's, a basic step to perform our application to control problem. Section 5 contains the proof of the verification Theorems 5.5 and 1.8 for the model problem described above in Subsection 1.2. Sections 7 and 8 contain applications of our technique to more specific classes of problems where other techniques are more difficult to use. The first is a case of exit time problem where the HJB equation is nondegenerate but $C^{1,2}$ regularity is not known to hold due to the lack of regularity of the coefficients; the second is case where the HJB equation is degenerate. Finally in Section 9 we compare our result with others verification techniques.

2 Preliminaries

2.1 Notation

Throughout this paper we will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a given stochastic basis (where $\mathcal{F}$ stands for a given filtration $(\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions). Given a finite dimensional real Hilbert space E, W will denote a cylindrical Brownian motion with values in E and adapted to $(\mathcal{F}_s)_{s\geq 0}$. Given $0 \leq t \leq T \leq +\infty$ and setting $\mathcal{T}_t = [t,T] \cap \mathbb{R}$ the symbol $\mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$, will denote the space of all continuous processes adapted to the filtration $\mathcal{F}$ with values in E. This is a Fréchet space if endowed with the topology of the uniform convergence in probability (*u.c.p.* from now on).

To be more precise this means that, given a sequence $(X^n) \subseteq \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ and $X \in \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; E)$ we have

$$X^n \to X$$

if and only if for every $\varepsilon > 0$, $t_1 \in \mathcal{T}_t$

$$\lim_{n \to +\infty} \sup_{s \in [t,t_1]} \mathbb{P}\left(|X_s^n - X_s|_E > \varepsilon\right) = 0.$$

Given a random time $\tau \geq t$ and a process $(X_s)_{s \in \mathcal{T}_t}$, we denote by X^τ the stopped process defined by $X_s^\tau = X_{s \wedge \tau}$. The space of all processes in $[t, T]$, adapted to $\mathcal{F}$ and square integrable with values in E is denoted by $L_{\mathcal{F}}^2(t, T; E)$. S^n will denote the space of all symmetric matrices of dimension n.

Let $k \in \mathbb{N}$. As usual $C^k(\mathbb{R}^n)$ is the space of all functions : $\mathbb{R}^n \to \mathbb{R}$ that are continuous together with their derivatives up to the order k. This is a Fréchet space equipped with the seminorms

$$\sup_{x \in K} |u(x)|_{\mathbb{R}} + \sup_{x \in K} |\partial_x u(x)|_{\mathbb{R}^n} + \sup_{x \in K} |\partial_{xx} u(x)|_{\mathbb{R}^{n \times n}}$$

for every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by C^k when no confusion may arise. The symbol $C_b^k(\mathbb{R}^n)$ will denote the Banach space of all continuous and bounded functions : $\mathbb{R}^n \to \mathbb{R}$ endowed with the usual sup norm. Passing to parabolic spaces we denote by $C^0(\mathcal{T}_t \times \mathbb{R}^n)$ the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s, x) \mapsto u(s, x)$$

that are continuous. This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x) \in [t,t_1] \times K} |u(s, x)|_{\mathbb{R}}$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$). Moreover we will denote by $C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ (respectively $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$), the space of all functions

$$u : \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}, \qquad (s, x) \mapsto u(s, x)$$

that are continuous together with their derivatives $\partial_t u$, $\partial_x u$, $\partial_{xx} u$ (respectively $\partial_x u$). This space is a Fréchet space equipped with the seminorms

$$\sup_{(s,x) \in [t,t_1] \times K} |u(s, x)|_{\mathbb{R}} + \sup_{(s,x) \in [t,t_1] \times K} |\partial_s u(s, x)|_{\mathbb{R}^n}$$
$$+ \sup_{(s,x) \in [t,t_1] \times K} |\partial_x u(s, x)|_{\mathbb{R}^n} + \sup_{(s,x) \in [t,t_1] \times K} |\partial_{xx} u(s, x)|_{\mathbb{R}^{n \times n}}$$

(respectively

$$\sup_{(s,x) \in [t,t_1] \times K} |u(s, x)|_{\mathbb{R}} + \sup_{(s,x) \in [t,t_1] \times K} |\partial_x u(s, x)|_{\mathbb{R}^n} \,)$$

for every $t_1 > 0$ and every compact set $K \subset\subset \mathbb{R}^n$. This space will be denoted simply by $C^{1,2}$ (respectively $C^{0,1}$) when no confusion may arise.

Similarly, for $\alpha, \beta \in [0,1]$ one defines $C^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$ (or simply $C^{\alpha,1+\beta}$) as the subspace of $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ of functions $u : \mathcal{T}_t \times \mathbb{R}^n \mapsto \mathbb{R}$ such that are $u(\cdot, x)$ is $\alpha-$Hölder continuous and $\partial_x u(s, \cdot)$ is $\beta-$Hölder continuous (with the agreement that 0-Hölder continuity means just continuity).

Similarly to $C_b^k(\mathbb{R}^n)$ we define the Banach spaces $C_b^0(\mathcal{T}_t \times \mathbb{R}^n)$ $C_b^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$, $C_b^{\alpha,1+\beta}(\mathcal{T}_t \times \mathbb{R}^n)$, $C_b^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$.

2.2 The calculus via regularization

We will follow here a framework of generalized calculus introduced by Russo and Vallois [48, 49, 50, 51] essentially in the one-dimensional case with some results [18, 49] in the multidimensional case. For simplicity we will remain in the framework of continuous processes. Throughout this subsection we fix, as above, $0 \leq t \leq T \leq +\infty$ and set $\mathcal{T}_t = [t, T] \cap \mathbb{R}$. We first recall some one dimensional consideration. For two processes $(X_s)_{s \in \mathcal{T}_t}$, $(Y_s)_{s \in \mathcal{T}_t}$, we define the forward integral and the covariation as follows ($s \in \mathcal{T}_t$)

$$\int_t^s X_r d^- Y_r = \lim_{\varepsilon \to 0} \int_t^s X_r \frac{Y_{r+\varepsilon} - Y_r}{\varepsilon} dr \tag{13}$$

$$[X, Y]_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^s (X_{r+\varepsilon} - X_r)(Y_{r+\varepsilon} - Y_r)\, dr \tag{14}$$

if those quantities exist in the sense of *u.c.p.* This ensures that the forward integral defined in (13) and the covariation process defined in (14) are continuous processes. It can be seen that the covariation is a bilinear and symmetric operator.

If $\tau \geq t$ is a random (non necessarily) stopping time, the following equality holds:

$$[X, Y]^\tau = [X^\tau, Y^\tau]. \tag{15}$$

If $(X^1, ..., X^n)$ is a vector of continuous processes we say that it has all its mutual brackets if $[X^i, X^j]$ exist for any $1 \leq i, j \leq n$. If $X^1, ..., X^n$ have all their mutual brackets then by polarisation (i.e. writing a bilinear form as a sum/difference of quadratic forms) we know that $[X^i, X^j]$ $(1 \leq i, j \leq n)$ are locally bounded variation processes.

If $[X, X]$ exists, then X is said to be a finite quadratic variation process; $[X, X]$ is called the quadratic variation of X. If $[X, X] = 0$ then X is said to be a zero quadratic variation process. A bounded variation process is a zero quadratic variation process. If S^1, S^2 are $(\mathcal{F}_s)$-semimartingales then $[S^1, S^2]$ coincides with the classical bracket $\langle S^1, S^2 \rangle$. If H is a continuous $(\mathcal{F}_s)$-predictable process then $\int_t^s H_r d^- S_r$ is the classical Itô integral $\int_t^s H_r dS_r$.

Remark 2.1 *Let X (respectively A) be a finite (respectively zero) quadratic variation process. Then (X, A) has all its mutual brackets and $[X, A] = 0$.* ■

We recall now an easy extension of stability results, see [19, th. 2.9] that will be used in subsection 3.2.

Proposition 2.2 *Let* $V = \left(V^1,...V^m\right)$ *(respectively* $X = \left(X^1,...X^n\right)$*) be a vector of continuous processes on* $\mathcal{T}_t$ *with bounded variation processes (respectively having all its mutual brackets). Let* $f, g \in C_{\text{loc}}^{\frac{1}{2}+\gamma,1}\left(\mathbb{R}^m \times \mathbb{R}^n\right)$ *(*$\gamma > 0$*). Then* $\forall s \in [t,T]$

$$[f(V,X), g(V,X)]_s = \sum_{i,j=1}^{n} \int_t^s \partial_{x_i} f(V,X)\, \partial_{x_j} g(V,X)\, d\left[X^i, Y^j\right]_r .$$

Proof. We give a sketch of the proof as the arguments are similar to the ones used in [19, th. 2.9]. By localisation $C_{\text{loc}}^{\frac{1}{2}+\gamma,1}$ can be replaced by $C^{\frac{1}{2}+\gamma,1}$. The case where f and g do not depend on V was treated for instance in [49, 51]. Since the bracket is a bilinear form, using polarisation techniques we can take $g = f$. For simplicity we set here $m = n = 1$. For given $\varepsilon > 0$ we write, when $r \in \mathcal{T}_t$

$$f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_r) = J_1(r,\varepsilon) + J_2(r,\varepsilon)$$

where

$$\begin{aligned} J_1(r,\varepsilon) &= f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_{r+\varepsilon}) \\ J_2(r,\varepsilon) &= f(V_r, X_{r+\varepsilon}) - f(V_r, X_r). \end{aligned}$$

Therefore, for $s \in \mathcal{T}_t$

$$\frac{1}{\varepsilon}\int_t^s [f(V_{r+\varepsilon}, X_{r+\varepsilon}) - f(V_r, X_r)]^2 dr \le \frac{2}{\varepsilon}\int_t^s J_1^2(r,\varepsilon)\, dr + \frac{2}{\varepsilon}\int_t^s J_2^2(r,\varepsilon)\, dr.$$

Now

$$\frac{2}{\varepsilon}\int_t^s J_1^2(r,\varepsilon)\, dr \le c^2(s,f,V)\frac{2}{\varepsilon}\int_t^s \left(|V_{r+\varepsilon} - V_r|^{\frac{1}{2}+\gamma}\right)^2 dr$$

where $c(s,f,V)$ is a (random) Hölder constant of f such that

$$|f(v_1,x) - f(v_2,x)| \le c(s,f,V)\,|v_1 - v_2|^{\frac{1}{2}+\gamma},$$

$$\forall v_1, v_2 \in \left[\inf_{r\in[t,s]} V_r,\ \sup_{r\in[t,s]} V_r\right], \quad \forall x \in \left[\inf_{r\in[t,s]} X_r,\ \sup_{r\in[t,s]} X_r\right].$$

Then we get

$$\frac{2}{\varepsilon}\int_t^s J_1^2(r,\varepsilon)\, dr \le c^2(s,f,V)\frac{2}{\varepsilon}\int_t^s |V_{r+\varepsilon} - V_r|^{1+2\gamma} dr.$$

Since V is a bounded variation process, this term converges to zero in probability.

On the other hand

$$J_2(r,\varepsilon) = \partial_x f(V_r, X_r)(X_{r+\varepsilon} - X_r) + J_3(r,\varepsilon)$$

where $J_3(r,\varepsilon)$ converges *u.c.p.* to zero, as in [19, th. 2.9]. Therefore, similarly as in [49] we have

$$\frac{2}{\varepsilon}\int_0^t J_2^2(s,\varepsilon)\, ds \to \sum_{i,j=1}^{n}\int_0^t (\partial_x f(V_s, X_s))^2\, d[X,X]_s$$

and so the claim. ∎

For our purposes we need to express integrals and covariation in a multidimensional setting, in the spirit of [18].

If $X = \left(X^1, ..., X^n\right)^*$ is a vector of continuous processes in $\mathcal{T}_t$, Y is a $m \times n$ matrix of continuous processes in $\mathcal{T}_t$, $\left(Y^{i,j}\right)_{1\le i\le m, 1\le j\le n}$ then the symbol $\int_t^s Y d^- X$ denotes, whenever it exists, the *u.c.p.* limit of $\int_t^s Y_r \frac{X_{r+\varepsilon}-X_r}{\varepsilon} dr$ where the product is intended in the matrix sense. Similarly, il A is a $n \times d$ matrix $\left(A^{j,k}\right)_{1\le j\le n, 1\le k\le d}$ then $[Y, A]_s$ is the $m \times d$ real matrix constituted by the following *u.c.p.* limit (if it exists)

$$\frac{1}{\varepsilon}\int_t^s \left(Y_{r+\varepsilon} - Y_r\right)\left(A_{r+\varepsilon} - A_r\right) dr.$$

Clearly the matrix operation cannot be commutative in general. Let now A, X, Y, C be real matrix valued processes which are successively compatible for the matrix product. We define

$$\int_t^s A_r d\left[X, Y\right]_r C_r = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^s A_r \left(X_{r+\varepsilon} - X_r\right)\left(Y_{r+\varepsilon} - Y_r\right) C_r dr$$

where the limit is intended in the *u.c.p.* sense. The previous stability transformations (Proposition 2.2 above) can be extended to the case of vector valued functions. This is pointed out in next remark.

Remark 2.3 *Let $f \in C^{\frac{1}{2}+\gamma,1}_{\mathrm{loc}}\left(\mathcal{T}_t \times \mathbb{R}^n; \mathbb{R}^p\right)$, $g \in C^{\frac{1}{2}+\gamma,1}_{\mathrm{loc}}\left(\mathcal{T}_t \times \mathbb{R}^m; \mathbb{R}^q\right)$, $X = \left(X^1, ..., X^n\right)^*$, $Y = \left(Y^1, ..., Y^m\right)^*$ such that (X, Y) has all its mutual brackets. Let V^1, V^2 be bounded variation processes. Then*

$$\left[f\left(V^1, X\right), g\left(V^2, Y\right)\right]_s = \int_t^s \partial_x f\left(V^1, X\right) d\left[X, Y^*\right] \partial_x g\left(V^2, Y\right)^*$$

for every $s \in \mathcal{T}_t$. ∎

One refined result is the vector Itô formula whose proof follows similarly as in [49, 19], where the involved stochastic integrals were scalar.

Proposition 2.4 *Let $f \in C^{1,2}\left(\mathcal{T}_t \times \mathbb{R}^n\right)$. Let $X = \left(X^1, ..., X^n\right)^*$, having all its mutual brackets, V be a bounded variation process indexed by $\mathcal{T}_t$. Then*

$$\begin{aligned} f\left(V_s, X_s\right) &= f\left(V_t, X_t\right) + \int_t^s \partial_x f\left(V_r, X_r\right) d^- X_r \\ &\quad + \int_t^s \partial_v f\left(V_r, X_r\right) dV_r \\ &\quad + \frac{1}{2}\int_t^s \partial_{xx} f\left(V_r, X_r\right) d\left[X, X^*\right]. \end{aligned}$$

Remark 2.5 *From the above statement it follows that, in particular, the integral* $\int_t^s \partial_x f(V_r, X_r)\, d^- X_r$ *automatically exists.* ■

Remark 2.6 *Let* $W = (W^1, ..., W^n)^*$ *be an* $(\mathcal{F}_s)$*-Brownian motion. Then* $[W, W^*]_s = (\delta_{i,j})\, s$. ■

3 Fukushima-Dirichlet decomposition

3.1 Definitions and remarks

Throughout all this section we fix, as above, $0 \leq t \leq T \leq +\infty$ and set $\mathcal{T}_t = [t, T] \cap \mathbb{R}$.

Definition 3.1 *D is called an* $(\mathcal{F}_s)$*-Dirichlet process in* $\mathcal{T}_t$ *if it is* $(\mathcal{F}_s)$*-adapted and can be written as*

$$D = M + A \tag{16}$$

where

(i) M *is an* $(\mathcal{F}_s)$*-martingale,*

(ii) A *is a zero quadratic variation process such that (for convenience)* $A_0 = 0$.

A vector $D = (D^1, ..., D^n)$ *is said to be Dirichlet if it has all its mutual brackets and every* D^i *is Dirichlet.*

Remark 3.2 *An* $(\mathcal{F}_s)$*-semimartingale is an* $(\mathcal{F}_s)$*-Dirichlet process.* ■

Remark 3.3 *The decomposition (16) is unique, see for instance [52].* ■

The concept of Dirichlet process can be weakened for our purposes. This class of processes appears in [16] implicitely. It is useful because it includes $C^{0,1}$ functions of semimartingales (see Proposition 3.9) which are not Dirichlet processes but they preserve a sort of orthogonal decomposition. This fact will be used in the applications to optimal control to treat a wider class of problems.

Definition 3.4 *A process D is called* $(\mathcal{F}_s)$*-* ***weak Dirichlet*** *(continuous) process in* $\mathcal{T}_t$ *if it can be written as*

$$D = M + A \tag{17}$$

where

(i) M *is an* $(\mathcal{F}_s)$*- continuous local martingale,*

(ii) A *is a process such that* $[A, N] = 0$ *for every* $(\mathcal{F}_s)$*- local martingale* N*. (For convenience* $A_0 = 0$*)*

A will be said ***weak zero energy*** *process.*

Remark 3.5 *The decomposition (17) is unique. In fact, let*

$$D = M^1 + A^1 = M^2 + A^2$$

where M^1, M^2, A^1, A^2 fulfill properties (i) and (ii) of Definition 3.4. Then we have $M + A = 0$ where $M = M^1 - M^2$ and $A = A^1 - A^2$ is such that $[A, N] = 0$ for every $(\mathcal{F}_s)$- local martingale N.

It is now enough to evaluate the covariation of both members with M to get $[M, M] = 0$. Since $A_0 = 0$ then $M_0 = 0$ and consequently $M \equiv 0$.

Remark 3.6 *[17] provides an example of weak Dirichlet process coming from convolutions of local martingales. If for every $s > 0$, $(G(s, \cdot))$ is a continuous random field such that $(G(s, \cdot))$ is $(\mathcal{F}_r)$-predictable and M is an $(\mathcal{F}_r)$ local martingale, then $X_s = \int_t^s G(s, r) dM_r$ defines a weak Dirichlet process.*

Definition 3.7 *A vector $D = \left(D^1, ..., D^n\right)$ is said to be a $(\mathcal{F}_s)$-* ***weak Dirichlet*** *(continuous) process if every D^i is $(\mathcal{F}_s)$- weak Dirichlet (continuous) process. A vector $A = \left(A^1, ..., A^n\right)$ is said to be a $(\mathcal{F}_s)$-* ***weak zero energy*** *process if every A^i is $(\mathcal{F}_s)$- weak zero energy process.*

The aim now is to study what happens to a Dirichlet process after a $C^{0,1}$ type transformation. It is well known, see for instance [52], that a C^1 function of a finite quadratic variation process (respectively Dirichlet process) is a finite quadratic variation (respectively Dirichlet) process. Here, motivated by applications to optimal control, see the introduction, subsection 1.1, we look at the (possibly) inhomogeneous case showing two different results: the first (Proposition 3.8) that give stability for Dirichlet processes in the inhomogeneous case for functions in $C_{\text{loc}}^{\frac{1}{2}+\gamma,1}(\mathcal{T}_t \times \mathbb{R}^n)$, the second (Proposition 3.9) that states that a function $C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ of a semimartingale is a weak Dirichlet process.

The second result comes from the need of treating optimal control where the state process is a semimartingale that solves a classical SDE's, which is the case we treat in this paper. The first result comes from the idea of investigating (in a work in preparation) the case of SDE's with distributional drift whose solutions are Dirichlet processes but not semimartingales.

3.2 The decomposition for $C^{\frac{1}{2}+\gamma,1}$ functions

Let us start with the stability result for Dirichlet processes taking $(D_s)_{s\in\mathcal{T}_t}$ be an $(\mathcal{F}_s)$- Dirichlet process with decomposition (16).

Proposition 3.8 *For every $u \in C^{\frac{1}{2}+\gamma,1}(\mathcal{T}_t \times \mathbb{R}^n)$ we have, for $s \in \mathcal{T}_t$*

$$u(s, D_s) = u(t, D_t) + \int_t^s \partial_x u(r, D_r)\, dM_r + \int_t^s \partial_x u(r, D_r)\, d^- A_r + \mathcal{A}_t^D(u)_s$$

where $\mathcal{A}_t^D : C^{\frac{1}{2}+\gamma,1}(\mathcal{T}_t \times \mathbb{R}^n) \to \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; \mathbb{R}^n)$ is a linear map having the following properties:

a) $\mathcal{A}_t^D$ *is continuous;*

b) if $u \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ *then, for* $s \in \mathcal{T}_t$

$$\mathcal{A}_t^D(u)_s = \int_t^s \partial_s u(r, D_r)\, dr + \int_t^s \partial_{xx} u(r, D_r)\, d[M, M]_r$$

c) if $u \in C^{\frac{1}{2}+\gamma,1}(\mathcal{T}_t \times \mathbb{R}^n)$ *then* $t \to \mathcal{B}_t^D(u)_s := \int_t^s \partial_x u(r, D_r)\, d^- A_r + \mathcal{A}_t^D(u)_s$ *is a zero quadratic variation process.*

Proof. We set $t = 0$ for simplicity and so use the notation $\mathcal{A}_0^D$, $\mathcal{B}_0^D$ and $\mathcal{T}_0$. Property a) follows simply by writing

$$\mathcal{A}_t^D(u)_s = u(s, D_s) - u(0, D_0) - \int_0^s \partial_x u(r, D_r)\, dM_r - \int_0^s \partial_x u(r, D_r)\, d^- A_r$$

and observing that the process defined on the right hand side has the required continuity property.

Property b) follows from Proposition 2.4 applied reversely. Indeed, given $u \in C^{1,2}(\mathcal{T}_0 \times \mathbb{R}^n)$, Proposition 2.4 can be applied. In particular

$$\int_0^s \partial_x u(r, D_r)\, d^- D_r$$

exists; this implies that also

$$\int_0^s \partial_x u(r, D_r)\, d^- A_r$$

exists since

$$\int_0^s \partial_x u(r, D_r)\, d^- M_r = \int_0^s \partial_x u(r, D_r)\, dM_r$$

is the classical Itô integral.

Property c) follows if we can show that the process

$$\mathcal{B}_0^D(u)_s := \int_0^s \partial_x u(r, D_r)\, d^- A_r + \mathcal{A}_0^D(u)_s = u(s, D_s) - u(0, D_0) - \int_0^s \partial_x u(r, D_r)\, dM_r$$

is a zero quadratic variation process. We operate with the bilinearity of the covariation process and we evaluate

(i) $[u(\cdot, D), u(\cdot, D)]$

(ii) $\left[u(\cdot, D), \int_0^\cdot \partial_x u(r, D_r)\, dM_r\right]$

(iii) $\left[\int_0^\cdot \partial_x u(r, D_r)\, dM_r, \int_0^\cdot \partial_x u(r, D_r)\, dM_r\right]$

as follows.

(i) We apply Proposition 2.2 to get that

$$[u(\cdot, D), u(\cdot, D)]_s = \int_0^s \partial_x u(s, D_r)\, d[D, D^*]_r\, \partial_x u(r, D_r)^*.$$

(ii) Setting $N_t = \int_0^s \partial_x u(r, D_r)\, dM_r$, Remark 2.1 implies that (N, D) has all its mutual brackets; therefore again Proposition 2.2 implies that

$$[u(\cdot, D), N^*]_s = \int_0^s \partial_x u(r, D_r)\, d[D, N^*]_r.$$

On the other hand, by Remark 2.1

$$[D, N^*]_t = [M, N^*]_s = \int_0^s \partial_x u(r, D_r)\, d[M, M^*]_r\, \partial_x u(r, D_r)^*.$$

(iii) The fact that the covariation of semimartingales coincides with the classical covariation (see [49]) gives

$$\begin{aligned} &\left[\int_0^\cdot \partial_x u(r, D_r)\, dM_r, \int_0^\cdot \partial_x u(r, D_r)\, dM_r\right]_s \\ = &\int_0^s \partial_x u(r, D_r)\, d[M, M^*]_r\, \partial_x u(r, D_r)^*. \end{aligned}$$

Finally by Remark 2.1 and the decomposition we get that

$$[D, D^*] = [M, M^*].$$

The bilinearity of the covariation allows now to conclude. ■

3.3 The decomposition for $C^{0,1}$ functions

We now go on with a result concerning weak Dirichlet processes. Suppose $(D_s)_{s\in\mathcal{T}_t}$ to be an $(\mathcal{F}_s)$- Dirichlet process with decomposition (16) where A is a zero quadratic variation process. Given a $C^{0,1}$ function u of D, we cannot expect that $Z = u(\cdot, D)$ is a Dirichlet process. However one can hope that it is at least a weak Dirichlet process. At this stage, this result is not known; however in the following Proposition we show that the result is true if D is an $(\mathcal{F}_s)$- semimartingale.

Proposition 3.9 *Suppose $(D_s)_{s\in\mathcal{T}_t}$ to be an $(\mathcal{F}_s)$- semimartingale. For every $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ we have, for $s \in \mathcal{T}_t$,*

$$u(s, D_s) = u(t, D_t) + \int_t^s \partial_x u(r, D_r)\, dM_r + \int_t^s \partial_x u(r, D_r)\, dA_r + \mathcal{A}_t^D(u)_s \quad (18)$$

where $\mathcal{A}_t^D : C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n) \to \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; \mathbb{R}^n)$ is a linear map having the following properties:

a) $\mathcal{A}_t^D$ *is continuous;*

b) if $u \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ *then*

$$\mathcal{A}_t^D(u)_s = \int_t^s \partial_s u(r, D_r)\, dr + \int_t^s \partial_{xx} u(r, D_r)\, d[M, M]_r$$

c) if $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ *then* $s \to \mathcal{B}_t^D(u)_s := \int_t^s \partial_x u(r, D_r)\, dA_r + \mathcal{A}_t^D(u)_s$ *is a weak zero energy process.*

Remark 3.10 *Given a bounded stopping time* τ *with values in* $\mathcal{T}_t$*, it is easy to see that decomposition (18) still holds for the stopped process* D^τ*. In fact given a* $(\mathcal{F}_s)$*-martingale* N*, and an* $(\mathcal{F}_s)$*- weak zero energy process* A*, we have* $[N, A^\tau] = [N, A]^\tau = 0$.

Proof. Without restriction of generality, we can suppose $t = 0$ and $T < +\infty$, so $\mathcal{T}_t = [0, T]$.

The only point to check is c), as a) and b) can be proved exactly as in the proof of Proposition 3.8. Since A is a bounded variation process, to prove c) we only have to check that

$$\left[u(\cdot, D) - \int_0^{\cdot} \partial_x u(r, D_r)\, dM_r, N\right] = 0$$

for every one dimensional $(\mathcal{F}_r)$-martingale N. Since the covariation of semimartingales coincides with the classical covariation

$$\left[\int_0^{\cdot} \partial_x u(r, D_r)\, dM_r, N\right] = \int_0^{\cdot} \partial_x u(r, D_r)\, d[M, N]_r\,,$$

it remains to check that, for every $s \in [0, T]$,

$$[u(\cdot, D), N]_s = \int_0^s \partial_x u(r, D_r)\, d[M, N]_r$$

For this, we have to evaluate the *u.c.p.* limit of

$$\int_0^s [u(r+\varepsilon, D_{r+\varepsilon}) - u(r, D_r)]\, \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr$$

in probability. This can be written as the sum of two terms:

$$I_1(s, \varepsilon) = \int_0^s (u(r+\varepsilon, D_{r+\varepsilon}) - u(r+\varepsilon, D_r))\, \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr$$

$$I_2(s, \varepsilon) = \int_0^s (u(r+\varepsilon, D_r) - u(r, D_r))\, \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr.$$

First we prove that $I_1(s, \varepsilon)$ goes to $\int_0^t \partial_x u(r, D_r)\, d[M, N]_r$. In fact

$$\begin{aligned} I_1(s, \varepsilon) &= \int_0^s (u(r+\varepsilon, D_{r+\varepsilon}) - u(r+\varepsilon, D_r))\, \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr \\ &= \int_0^s \partial_x u(r+\varepsilon, D_r)\, (D_{r+\varepsilon} - D_r)\, \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_1(s, \varepsilon) \qquad (19) \end{aligned}$$

where $R_1(s,\varepsilon) \to 0$ *u.c.p.* as $\varepsilon \to 0$. Indeed

$$\begin{aligned} R_1(s,\varepsilon) &= \int_0^s \left[\int_0^1 \left[\partial_x u\left(r+\varepsilon, D_r + \lambda\left(D_{r+\varepsilon} - D_r\right)\right) - \partial_x u\left(r+\varepsilon, D_r\right)\right] d\lambda \right. \\ &\quad \left. \frac{N_{r+\varepsilon} - N_r}{\varepsilon}\left(D_{r+\varepsilon} - D_r\right)\right] dr \end{aligned}$$

and the claim follows by the continuity of $\partial_x u$ and from the estimate

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^T \left(N_{r+\varepsilon} - N_r\right)\left(D_{r+\varepsilon} - D_r\right) dr \\ \le\ &\left[\frac{1}{\varepsilon}\int_0^T \left(N_{r+\varepsilon} - N_r\right)^2 dr \cdot \frac{1}{\varepsilon}\int_0^T \left(D_{r+\varepsilon} - D_r\right)^2 dr\right]^{\frac{1}{2}} \stackrel{\varepsilon\to 0}{\longrightarrow} \left([N]\cdot[D]\right)^{\frac{1}{2}}. \quad (20) \end{aligned}$$

On the other hand the first term in (19) can be rewritten as

$$\int_0^s \partial_x u\left(r, D_r\right)\left(D_{r+\varepsilon} - D_r\right) \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr + R_2(s,\varepsilon) \qquad (21)$$

where $R_2(s,\varepsilon) \to 0$ *u.c.p.* arguing as for $R_1(s,\varepsilon)$. The integral in (21) goes then *u.c.p.* to $\int_0^s \partial_x u\left(r, D_r\right) d\left[M,N\right]_r$, since by (20) the measures $\frac{(N_{r+\varepsilon}-N_r)(D_{r+\varepsilon}-D_r)}{\varepsilon} dr$ weakly converge to $d\left[N,D\right]$ as $\varepsilon \to 0$.

It remains to show that $I_2(s,\varepsilon) \to 0$ *u.c.p.* for every $s \in [0,T]$ as $\varepsilon \to 0$. By using suitable localization theorems (e.g. as usually done for instance in [47], section IV.1), it is enough to suppose u to be with compact support and N to be a square integrable martingale. Then we evaluate

$$\mathbb{E}\left(\sup_{0\le s\le T} |I_2(s,\varepsilon)|^2\right) \qquad (22)$$

Now, we have, exchanging integrals

$$\begin{aligned} I_2(s,\varepsilon) &= \int_0^s [u(r+\varepsilon, D_r) - u(r, D_r)] \frac{N_{r+\varepsilon} - N_r}{\varepsilon} dr \\ &= \int_0^s [u(r+\varepsilon, D_r) - u(r, D_r)] dr \int_r^{r+\varepsilon} \frac{1}{\varepsilon} dN_\lambda \\ &= \int_0^s \frac{dN_\lambda}{\varepsilon} \int_{(\lambda-\varepsilon)\vee 0}^{\lambda} [u(r+\varepsilon, D_r) - u(r, D_r)] dr. \end{aligned}$$

Doob inequality implies that (22) is smaller than

$$4\mathbb{E}\left\{\int_0^T d[N]_\lambda \left(\frac{1}{\varepsilon}\int_{(\lambda-\varepsilon)\vee 0}^{\lambda} [u(r+\varepsilon, D_r) - u(r, D_r)] dr\right)^2\right\}$$

The fact that u is uniformly continuous on compact sets and the Lebesgue dominated convergence theorem, imply the result. ∎

A significant bracket evaluation, in the spirit of Proposition 2.2, but for $C^{0,1}-$ functions of semimartingales is the following. For simplicity we formulate the one-dimensional case, even if it extends to the multidimensional case.

Corollary 3.11 *Let S be a $(\mathcal{F}_s)-$ semimartingale in $\mathcal{T}_t$, $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R})$. Then*

$$[u(\cdot, S), S]_t = \int_0^t \partial_x u(s, S_s) d[S]_s$$

Proof. Let $S = M + V$ be the decomposition of S with M being a local martingale and V a finite variation process with $V_0 = 0$. Then

$$u(t, S_t) = u(0, S_0) + \int_0^t \partial_x u(s, S_s) dM_s + \tilde{A}_t,$$

where $\tilde{A}$ is a weak zero energy process. In particular, a classical localization argument shows that $[\tilde{A}, M] = 0$. On the other hand, obviously $[\tilde{A}, V] = 0$; consequently, by linearity and since the covariation of local martingales is the classical convolution, the result follows. ■

Now, to develop applications to optimal control, we apply previous decomposition to semimartingales that are solutions of classical SDE's. Consider a classical Itô process in $\mathcal{T}_t$

$$S_s = x + \int_t^s H_r dW_r + \int_t^s K_r dr$$

where W is a n-dimensional Brownian motion, K an $(\mathcal{F}_s)$-adapted process, H an $(\mathcal{F}_s)$-predictable process.

Corollary 3.12 *For every $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ we have*

$$u(s, S_s) = u(t, S_t) + \int_t^s \partial_x u(r, S_r) H_r dW_r + \int_t^s \partial_x u(r, S_r) K_r dr + \mathcal{A}^S_{1,t}(u)_s \quad (23)$$

where $\mathcal{A}^S_{1,t} : C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n) \to \mathcal{C}_{\mathcal{F}}(\mathcal{T}_t \times \Omega; \mathbb{R}^n)$ is a linear map having the following properties:

a) $\mathcal{A}^S_1$ is continuous;

b) if $u \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$ then

$$\mathcal{A}^S_{1,t}(u)_s = \int_t^s \partial_s u(r, S_r) dr + \int_t^s \frac{1}{2} \mathrm{Tr}\left[H_r^* \partial_{xx} u(r, S_r) H_r\right] dr$$

c) if $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ then $\mathcal{B}^D_{1,t}(u)_s := \int_t^s \partial_x u(r, S_r) K_r dr + \mathcal{A}^S_{1,t}(u)_s$ is a weak zero energy process; if $u \in C^{\frac{1}{2}+\gamma,1}_{\mathrm{loc}}(\mathcal{T}_t \times \mathbb{R}^n)$ for some $\gamma > 0$, then $\mathcal{B}^D_{1,t}(u)_s$ is a zero quadratic variation process.

Proof. It follows directly from the Propositions 3.8 and 3.9 setting $D = S$, $\mathcal{A}^S_{1,t}(u) = \mathcal{A}^S_t(u)$ so

$$M_s = \int_t^s H_r dW_r$$

$$A_s = \int_t^s K_r dr$$

and from straightforward calculations. ∎

Remark 3.13 *Let $b_1 : \Omega \times \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1 : \Omega \times \mathcal{T}_t \times \mathbb{R}^n \to \mathbb{R}^{m\times n}$ be progressively measurable, continuous in s, x and satisfying, for every $s \in \mathcal{T}_t$ and $x_1, x_2, x \in \mathbb{R}^n$,*

$$\begin{aligned} |\langle b_1(s,x_1) - b_1(s,x_2), x_1 - x_2\rangle| + ||\sigma_1(s,x_1) - \sigma_1(s,x_2)||^2 &\le K_s |x_1 - x_2|^2, \\ |\langle b_1(s,x), x\rangle| + ||\sigma_1(s,x)||^2 &\le K_t\left[1 + |x|^2\right] \quad (24) \\ \int_t^T \left[|b_1(s,x)| + ||\sigma_1(s,x)||^2\right] ds &< +\infty, \end{aligned}$$

for a suitable a.s. integrable process K_t. Let (S_s) be the diffusion process that is the unique solution of the SDE

$$dS_s = b_1(s, S_s)\, ds + \sigma_1(s, S_s)\, dW_s; \qquad S_t = x. \tag{25}$$

It exists and it is unique in the strong sense thanks to Theorem 1.2 [37, p.2]. For every $u \in C^{0,1}(\mathcal{T}_t \times \mathbb{R}^n)$ the above decomposition (23) can be written as

$$u(s, S_s) = u(t, S_t) + \int_t^s \partial_x u(r, S_r)\, \sigma_1(r, S_r)\, dW_r + \int_t^s \partial_x u(r, S_r)\, b_1(r, S_r)\, dr + \mathcal{A}^S_{1,t}(u)_s \tag{26}$$

with

$$\mathcal{A}^S_{1,t}(u)_s = \int_t^s \left(\partial_s u(r, S_r) + \frac{1}{2} Tr\ \left[\sigma_1(r, S_r)^* \partial_{xx} u(r, S_r)\, \sigma_1(r, S_r)\right]\right) dr \tag{27}$$

when $u \in C^{1,2}(\mathcal{T}_t \times \mathbb{R}^n)$.

Note moreover that the decomposition (26) - (27) still holds if we do not assume anything about b_1 and σ_1 excepted for the fact that S is a weak solution of (25), the processes $H_s = b_1(s, S_s), K_s = \sigma_1(s, S_s)$ are adapted and $\int_0^T H_s^2 ds + \int_0^T |K_s| ds < +\infty$, a.s. ∎

4 Representation of the operators $\mathcal{A}$ and $\mathcal{B}$ when u solves a suitable PDE.

Here we want to develop the connection between suitable (deterministic) linear differential operators and our (stochastic) operators $\mathcal{A}$ and $\mathcal{B}$ introduced in the previous section. This connection is well known and obvious when u is the $C^{1,2}$ solution of a second order PDE. Our aim is to extend the validity of such representation when u is only a $C^{0,1}$ solution (in a suitable sense that we will define below). This will be used as a key tool in the applications to optimal control.

4.1 Strong solutions of parabolic PDE's

Let $0 < T < +\infty$, consider two continuous functions

$$b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

and the linear parabolic operator

$$\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq C^0([0,T] \times \mathbb{R}^n) \longrightarrow C^0([0,T] \times \mathbb{R}^n)$$

$$D(\mathcal{L}_0) = C^{1,2}([0,T] \times \mathbb{R}^n)$$

$$\mathcal{L}_0 u(t,x) = \partial_t u(t,x) + \langle b(t,x), \partial_x u(t,x) \rangle + \frac{1}{2} \text{Tr } [\sigma^*(t,x) \partial_{xx} u(t,x) \sigma(t,x)].$$

Defining

$$L_0(t) u(t,x) = \langle b(t,x), \partial_x u(t,x) \rangle + \frac{1}{2} \text{Tr } [\sigma^*(t,x) \partial_{xx} u(t,x) \sigma(t,x)],$$

we can write

$$\mathcal{L}_0 u(t,x) = \partial_t u(t,x) + L_0(t) u(t,x).$$

Recall that an operator $\mathcal{M} : D(\mathcal{M}) \subseteq F \to G$ (F, G suitable Fréchet spaces) is closable if, given any sequence $(u_n)_{n \in \mathbb{N}} \subseteq D(\mathcal{M})$, we have

$$\left.\begin{array}{ll} u_n \longrightarrow 0, & \text{in } F \\ \mathcal{M} u_n \longrightarrow \eta & \text{in } G \end{array}\right\} \Longrightarrow \eta = 0$$

When $\mathcal{L}_0$ is closable we denote by $\mathcal{L}$ its closure and recall that

$$u \in D(\mathcal{L}) \Longleftrightarrow \exists (u_n) \subset C^{1,2}([0,T] \times \mathbb{R}^n) : \left\{ \begin{array}{l} u_n \longrightarrow u \\ \mathcal{L}_0 u_n \longrightarrow \mathcal{L} u \end{array} \right. \text{ in } C^0([0,T] \times \mathbb{R}^n)$$

Now, given $\phi \in C^0(\mathbb{R}^n)$ and $h \in C^0([0,T] \times \mathbb{R}^n)$ we consider the inhomogenous backward parabolic problem

$$\left\{ \begin{array}{l} \partial_t u(t,x) + \langle b(t,x), \partial_x u(t,x) \rangle + \frac{1}{2} \text{Tr } [\sigma^*(t,x) \partial_{xx} u(t,x) \sigma(t,x)] = h(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T], \qquad x \in \mathbb{R}^n \\ u(T,x) = \phi(x), \qquad\qquad x \in \mathbb{R}^n \end{array} \right. \tag{28}$$

that can rewritten as

$$\partial_t u(t,x) + L_0(t) u(t,x) = h(t,x); \qquad u(T,x) = \phi(x)$$

or

$$\mathcal{L}_0 u(s,x) = h(s,x); \qquad u(T,x) = \phi(x)$$

Definition 4.1 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strict solution to the backward Cauchy problem (28) if $u \in D(\mathcal{L}_0)$ and (28) holds.*

Definition 4.2 *We say that* $u \in C^0([0,T] \times \mathbb{R}^n)$ *is a strong solution to the backward Cauchy problem (28) if there exists a sequence* $(u_n) \subset D(\mathcal{L}_0)$ *and two sequences* $(\phi_n) \subseteq C^0(\mathbb{R}^n)$, $(h_n) \subseteq C^0([0,T] \times \mathbb{R}^n)$, *such that*

1. *For every* $n \in \mathbb{N}$ u_n *is a strict solution of the problem*

$$\mathcal{L}_0 u_n(t,x) = h_n(t,x); \qquad u_n(T,x) = \phi_n(x)$$

2. *The following limits hold*

$$\begin{array}{l} u_n \longrightarrow u \text{ in } C^0([0,T] \times \mathbb{R}^n) \\ h_n \longrightarrow h \text{ in } C^0([0,T] \times \mathbb{R}^n) \\ \phi_n \longrightarrow \phi \text{ in } C^0(\mathbb{R}^n) \end{array}$$

4.2 The representation result

We state first a technical lemma whose proof is elementary.

Lemma 4.3 *Let* $f_n, f : [0,T] \to \mathbb{R}$, $n \in \mathbb{N} \cup \{+\infty\}$, *continuous such that* $f_n \to f$ *uniformly. For a fixed constant* $K > 0$, *we define*

$$\tau_n = \inf \{t \in [0,T] : |f_n(t)| \geq K\}$$

with the convention that $\inf \emptyset = T$. *Then*

$$\lim_{n \to +\infty} f_n^{\tau_n} = f^\tau$$

uniformly, where f^τ *(respectively* $f_n^{\tau_n}$*) is the stopped function defined by* $f^\tau(t) = f^\tau(\tau \wedge t)$ *(respectively* $f^{\tau_n}(t) = f^{\tau_n}(\tau_n \wedge t)$*).*

¿From the above lemma we get the following Proposition.

Proposition 4.4 *The set* $\mathcal{M}_{loc}$ *of all* $(\mathcal{F}_t)-$ *continuous local martigales is a closed subset of* $\mathcal{C}_{\mathcal{F}}([0,T] \times \Omega; \mathbb{R})$ *endowed with the u.c.p. topology.*

Proof. Let $(M_n(t), \quad t \in [0,T])_{n \in \mathbb{N}}$ be a sequence of local continuous martingales converging *u.c.p.* to a continuous martingale M. For $K > 0$ we define the following stopping times:

$$\tau^n = \inf \{t \in [0,T] : |M_n(t)| \geq K\}$$

$$\tau = \inf \{t \in [0,T] : |M(t)| \geq K\}$$

with the convention that $\inf \emptyset = T$.

Lemma 4.3 implies that

$$M_n^{\tau^n} \longrightarrow M^\tau \qquad u.c.p. \tag{29}$$

In order to conclude, it suffices to show that M^τ is a square integrable martingale. Using (29) above and Lebesgue dominated convergence theorem, we have

$$M_n(\tau^n) \longrightarrow M(\tau) \qquad \text{in } L^2(\Omega)$$

or equivalently

$$M_n^{\tau^n}(T) \longrightarrow M^\tau(T) \qquad \text{in } L^2(\Omega).$$

The fact that M^τ is a square integrable martingale follows from Proposition 5.23, Ch. 1 of [36]. ∎

We are now able to state a useful representation result. Below we fix $t \in [0,T]$ and, since now $T < +\infty$, we have $\mathcal{T}_t = [t,T]$.

Theorem 4.5 *Let*

$$b_1 : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

be a continuous progressively measurable field (continuous in (s,x)) and

$$b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \text{resp.}\sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$ be a strong solution of the Cauchy problem (28).

Fix $t \in [0,T]$, $x \in \mathbb{R}^n$ and let (S_s) be a solution to the SDE

$$dS_s = b_1(s,S_s)\,ds + \sigma(s,S_s)\,dW_s; \qquad S_t = x.$$

Then

$$\mathcal{B}_{1_t}^S(u)_s = \int_t^s h(r,S_r)\,dr + \int_t^s \langle \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr. \tag{30}$$

so that

$$\mathcal{A}_{1,t}^S(u)_s = \int_t^s h(r,S_r)\,dr - \int_t^s \langle \partial_x u(r,S_r), b(r,S_r)\rangle\,dr.$$

provided that the following assumption be verified for every $s \in \mathcal{T}_t$

$$\lim_{n\to+\infty} \int_t^s \langle \partial_x u_n(r,S_r) - \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr = 0, \qquad u.c.p. \tag{31}$$

Proof. We set $t = 0$ for simplicity. Let $u_n \longrightarrow u$ in $C^0([0,T] \times \mathbb{R}^n)$ be a sequence such that $\mathcal{L}_0 u_n = h_n \longrightarrow h$ in $C^0([0,T] \times \mathbb{R}^n)$. By Itô's formula

$$\begin{aligned} u_n(s,S_s) &= u_n(0,S_0) + \int_0^s \mathcal{L}_0 u_n(r,S_r)\,dr + \int_0^s \langle \partial_x u_n(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr \\ &\quad + \int_0^s \partial_x u_n(r,S_r)\,\sigma(r,S_r)\,dW_r. \end{aligned}$$

¿From (31) we conclude that

$$\begin{aligned} M_s^n &= u_n(s,S_s) - u_n(0,S_0) - \int_0^s \mathcal{L}_0 u_n(r,S_r)\,dr \\ &\quad - \int_0^s \langle \partial_x u_n(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr \end{aligned}$$

converges *u.c.p.* to

$$\begin{aligned} M_s &= u(s,S_s) - u(0,S_0) - \int_0^s h(r,S_r)\,dr \\ &\quad - \int_0^s \langle \partial_x u(r,S_r), b_1(r,S_r) - b(r,S_r)\rangle\,dr. \end{aligned}$$

Using Proposition 4.4 above we get that M is an $(\mathcal{F}_s)$-local martingale and the result follows by Corollary 3.12 by identification of the weak zero energy processes. ■

The above result depends on the extra assumption (31) which is essential but not easy to check. We give first a special (but useful) case where it holds and then an improvement for the nondegenerate case. We have the following.

Remark 4.6 *If*

$$\lim_{n\to+\infty} \partial_x u_n = \partial_x u \qquad \text{in } C^0([0,T]\times\mathbb{R}^n)$$

then Assumption (31) is verified. This means that the result of Proposition 4.4 above applies if we know that u is a strong solution in a more restrictive sense, i.e. substituting the point 2 of Definition 4.2 with

$$\begin{aligned} &u_n \longrightarrow u \text{ in } C^0([0,T]\times\mathbb{R}^n) \\ &\partial_x u_n \longrightarrow \partial_x u \text{ in } C^0([0,T]\times\mathbb{R}^n) \\ &h_n \longrightarrow h \text{ in } C^0([0,T]\times\mathbb{R}^n) \\ &\phi_n \longrightarrow \phi \text{ in } C^0(\mathbb{R}^n). \end{aligned}$$

This is a particular case of our setting and it is the one used e.g in [30, 31, 33] to get the verification result. We can say that in these works a result like Theorem 4.5 is proved under the assumption that u is a strong solution in this more restrictive sense. It is worth to note that in such simplified setting the proof of Theorem 4.5 follows simply by using standard convergence arguments. In particular there one does not need to use the Fukushima-Dirichlet decomposition presented in Section 3. So, from the metodological point of view there is a serious difference with the result of Theorem 4.5, see Section 9 for comments. ■

A more significant achievement concerns the nondegenerate case. It is illustrated in the following corollary.

Corollary 4.7 *We make the same assumption of Theorem 4.5 except (31) which we replace by the assumption that*

$$\sigma^{-1}(b_1 - b) \text{ is bounded.} \tag{32}$$

where σ^{-1} stands for the pseudo-inverse of σ. Then the same conclusion of Theorem 4.5 holds.

Proof. Setting $t = 0$ for simplicity we write

$$\beta_s = W_s + \int_0^s \left(\sigma^{-1}(b_1 - b)\right)(r, S_r)\, dr$$

and applying Girsanov Theorem, there is a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $(\beta_s)_{s\in[0,T]}$ is an $(\mathcal{F}_s)$- standard Brownian motion.

So, under $\mathbb{Q}$ the process (S_s) fulfills equation

$$dS_s = b(s, S_s)\, ds + \sigma(s, S_s)\, dW_s; \qquad S_0 = x.$$

Under $\mathbb{Q}$, Assumption (31) is trivially verified since $b = b_1$ and so we have (the tilde stands for operators under the new probability $\mathbb{Q}$)

$$\tilde{\mathcal{B}}^S_{1,0}(u)_s = \int_0^s h(r, S_r)\, dr,$$

$$\mathcal{A}^S_{1,0}(u)_s = \int_0^s h(r, S_r)\, dr - \int_0^s \langle \partial_x u(r, S_r), b(r, S_r)\rangle\, dr,$$

and

$$\begin{aligned} u(s, S_s) &= u(0, S_0) + \int_0^s \partial_x u(r, S_r)\, \sigma(r, S_r)\, d\beta_r \\ &\quad + \int_0^s \langle \partial_x u_n(r, S_r), b(r, S_r)\rangle\, dr + \mathcal{A}^S_{1,0}(u)_s. \end{aligned}$$

Expressing β in terms of W we obtain the result. ∎

4.3 Some useful consequences

The previous result has some important consequences.

Remark 4.8 *From Remark 3.10 it follows that the conclusion of the above Theorem 4.5 and Corollary 4.7 holds also if we stop the processes $\mathcal{A}$ and $\mathcal{B}$ with a stopping time $t \le \tau \le T$. More precisely we have*

$$\mathcal{B}^S_{1,t}(u)_{s\wedge\tau} = \int_t^{s\wedge\tau} h(r, S_r)\, dr - \int_t^{s\wedge\tau} \langle \partial_x u(r, S_r), b_1(r, S_r) - b(r, S_r)\rangle\, dr$$

so that

$$\mathcal{A}^S_{1,t}(u)_{s\wedge\tau} = \int_t^{s\wedge\tau} h(r, S_r)\, dr - \int_t^{s\wedge\tau} \langle \partial_x u(r, S_r), b(r, S_r)\rangle\, dr.$$

This fact will be useful in Section 7. ∎

Remark 4.9 *The results of the Theorem 4.5 and of Corollary 4.7 above still hold true with suitable modifications if we assume that, instead of having* $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$*:*

(i) the strong solution u *belongs to* $C^0([0,T] \times \mathbb{R}^n) \cap C^{0,1}([\varepsilon,T] \times \mathbb{R}^n)$ *for every small* $\varepsilon > 0$*;*

(ii) for some $\beta \in (0,1)$ *the map* $(t,x) \to t^\beta \partial_x u(t,x)$ *belong to* $C^{0,1}([0,T] \times \mathbb{R}^n)$*.*

The proof of Theorem 4.5 in this case is a straightforward generalization of the one presented above: we do not give it here to avoid technicalities. In fact in proving the verification theorem in Section 7 we will deal with initial data only continuous so with solutions u *satisfying (i) and (ii) above and possibly not* $C^{0,1}$*. This difficulty will be faced directly in the proof of verification theorem 7.19 by approximating the initial data with* C^1 *ones, using (30) and passing to the limit.*

Similarly also the results of Theorem 5.5, can be generalized to this case. ■

Remark 4.10 *From the above Theorem 4.5 it follows that the process* $\mathcal{B}^S_{1,t}(u)_s$ *is in fact a semimartingale (and also absolutely continuous). So we can say that, if* $u \in C^{0,1}([0,T] \times \mathbb{R}^n)$ *is a strong solution of the Cauchy problem (28) then in the decomposition of Proposition 3.9 the weak zero energy process is in fact a finite variation process. In particular we can say that in this case, the result of Proposition 3.9 can be seen as a stability results of semimartingales through inhomogeneous* $C^{0,1}$ *transformations that are strong solutions of the Cauchy problem (28) for suitable data* $\phi \in C^0(\mathbb{R}^n)$ *and* $h \in C^0([0,T] \times \mathbb{R}^n)$*.* ■

4.4 The elliptic case

We devote the last part of this subsection to apply the same setting above to elliptic problems. Consider the inhomogenous elliptic problem

$$\lambda u(x) + L_0 u(x) + h(x) = 0, \qquad \forall x \in \mathbb{R}^n. \tag{33}$$

where $D(L_0) = C^2(\mathbb{R}^n)$ and

$$L_0 u(x) = \langle b(x), \partial_x u(s,x) \rangle + \frac{1}{2} \text{Tr} \, [\sigma^*(x) \, \partial_{xx} u(x) \, \sigma(x)]$$

Definition 4.11 *We say that* u *is a strict solution to the elliptic problem (33) if* $u \in D(L_0)$ *and (33) holds.*

Definition 4.12 *We say that* u *is a strong solution to the elliptic problem (33) if there exists a sequence* $(u_n) \subseteq D(L_0)$ *and a sequence* $(h_n) \subseteq C^0(\mathbb{R}^n)$*, such that*

1. For every $n \in \mathbb{N}$ u_n *is a strict solution of the problem*

$$\lambda u_n(x) - L_0 u_n(x) = h_n(x), \qquad \forall x \in \mathbb{R}^n;$$

2. The following limits hold

$$u_n \longrightarrow u \text{ in } C^0(\mathbb{R}^n)$$
$$h_n \longrightarrow h \text{ in } C^0(\mathbb{R}^n)$$

Note that if L is the closure of L_0 in $C^0(\mathbb{R}^n)$ then a strong solution u, by construction, belongs to $D(L)$. We now exploit the above setting to show that, for functions $u \in C^1(\mathbb{R}^n)$ that are strong solutions of the Cauchy problem (33) the following results holds; we omit the proof as it is completely similar and even simpler to the one of Theorem 4.5 for the parabolic case.

Theorem 4.13 *Let*

$$b_1 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n,$$

be a continuous progressively measurable process (continuous in x) and

$$b : \mathbb{R}^n \to \mathbb{R}^n, \qquad \text{resp.} \sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

be continuous functions. Let S_s be a solution to the SDE

$$dS_s = b_1(S_s)\,ds + \sigma(S_s)\,dW_s; \qquad S_0 = x.$$

Let $u \in C^1(\mathbb{R}^n)$ be a strong solution of the Cauchy problem (33). Assume that

$$\lim_{n \to +\infty} \int_0^s \langle \partial_x u_n(S_r) - \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle\, dr = 0, \qquad u.c.p. \tag{34}$$

or that (32) holds. Then for every $t \geq 0$ we have

$$\mathcal{B}^S_{1,0}(u)_s = \int_0^s h(S_r)\,dr + \int_0^s \langle \partial_x u(S_r), b_1(S_r) - b(S_r) \rangle\, dr$$

and

$$\mathcal{A}^S_{1,0}(u)_s = \int_0^s h(S_r)\,dr - \int_0^s \langle \partial_x u(S_r), b(S_r) \rangle\, dr.$$

5 The proof of the verification theorem in our model case

In this section we give the proof of the verification theorem 5.5 for the model problem described in the introduction. Then in Sections 7 and 8 we will consider two families of problems to which our technique apply.

Consider first the parabolic case fixing the horizon $T < +\infty$ and let us recall the optimal control problem (2) - (3) described in the introduction (Subsection 1.2). Under Hypotheses 1.1 1.2 and 1.4 we seek to minimize the cost (depending on the initial data $(t,x) \in [0,T] \times \mathbb{R}^n$)

$$J(t,x;z) = \mathbb{E}\left\{ \int_t^T l(s, y(s;t,x,z), z(s))\,ds + \phi(y(T;t,x,z)) \right\}$$

where $z \in \mathcal{Z}_{ad}(t)$ and $y(s;t,x,z)$ is the unique strong solution of the state equation

$$\begin{cases} dy(s) = [F_0(s,y(s)) + F_1(s,y(s),z(s))]\,ds + B(s,y(s))\,dW(s), & s \in \mathcal{T}_t, \\ y(t) = x. \end{cases}$$

We define the operator

$$\mathcal{L}_0 : D(\mathcal{L}_0) \subseteq C^0([0,T] \times \mathbb{R}^n) \longrightarrow C^0([0,T] \times \mathbb{R}^n)$$

$$D(\mathcal{L}_0) = C^{1,2}([0,T] \times \mathbb{R}^n)$$

$$\mathcal{L}_0 u(t,x) = \partial_t u(t,x) + \langle F_0(t,x), \partial_x u(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\ [B^*(t,x)\,\partial_{xx}u(t,x)\,B(t,x)].$$

The HJB equation associated with the problem (2) - (3) can then be written as

$$\mathcal{L}_0 u(t,x) + H^0(t,x,\partial_x u(t,x)) = 0, \quad u(T,x) = \phi(x) \tag{35}$$

where H^0 is given in (5). Let us give the following definitions.

Definition 5.1 *We say that $u \in C^0([0,T] \times \mathbb{R}^n)$ is a strict solution of the HJB equation (35) if $u \in D(\mathcal{L}_0)$ and (35) holds on $[0,T] \times \mathbb{R}^n$.*

Definition 5.2 *A function $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$ is a strong solution of the HJB equation (35) if, setting $h_0(t,x) = -H^0(t,x,\partial_x v(t,x))$, v is a strong solution of the backward Cauchy problem*

$$\mathcal{L}_0 v(t,x) = h_0(t,x); \qquad v(T,x) = \phi(x)$$

in the sense of Definition 4.2.

Assume the following.

Hypothesis 5.3 *There exists a function $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$ which is a strong solution of the HJB equation in the sense of Definition 5.2 and is polynomially growing with its space derivative in the variable x. Moreover:*

(i) either we can choose the approximating sequence (v_n) of Definition 5.2 so that for every $0 \le t \le s \le T$ and for every admissible control $z \in \mathcal{Z}_{ad}(t)$

$$\lim_{n\to+\infty} \int_t^s \langle \partial_x v_n(r,S_r) - \partial_x v(r,S_r), F_1(r,y(r),z(r))\rangle\,dr = 0, \qquad u.c.p.$$

(ii) or the function

$$(t,x,z) \to B^{-1}(t,x)\,F_1(t,x,z),$$

where B^{-1} stands for the pseudo-inverse of B, is well defined and bounded on $[0,T] \times \mathbb{R}^n \times U$.

Remark 5.4 *All the results below still hold true with suitable modifications if we assume that:*

- *the strong solution* v *belongs to* $v \in C^0([0,T] \times \mathbb{R}^n) \cap C^{0,1}([\varepsilon, T] \times \mathbb{R}^n)$ *for every small* $\varepsilon > 0$*;*
- *for some* $\beta \in (0,1)$ *the map* $(t,x) \to t^\beta \partial_x v(t,x)$ *belongs to* $C^0([0,T] \times \mathbb{R}^n)$*.*

The proof of Theorem 5.5 in this case is a straightforward generalization of the one presented here: we do not give it here to avoid technicalities since here we deal with a model problem. In Section 7 we will deal with such difficulty.

We give now here the precise statement of Theorem 1.7.

Theorem 5.5 *Assume that Hypotheses 1.1, 1.2, 1.4 and 5.3 hold. Let* H^0_{CV}*,* H^0 *be as in (5)-(6). Let* $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$ *be a strong solution of (35) and fix* $(t,x) \in [0,T] \times \mathbb{R}^n$ *which is polynomially growing with its space derivative in the variable* x*. Then*

(i) $v \leq V$ *on* $[0,T] \times \mathbb{R}^n$*.*

(ii) If z *is an admissible control at* (t,x) *that satisfies (setting* $y(s) = y(s;t,x,z)$*)*

$$H^0(s, y(s), \partial_x v(s, y(s))) = H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s)), \qquad (36)$$

for a.e. $s \in [t,T]$*,* $\mathbb{P}-$*almost surely. Then* z *is optimal at* (t,x) *and* $v(t,x) = V(t,x)$*.*

The proof of this theorem follows by the following fundamental identity that we state as a lemma.

Lemma 5.6 *Assume that Hypotheses 1.1, 1.2, 1.4 and 5.3 hold. Let* $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$ *be a strong solution of (35). Then, for every* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $z \in \mathcal{Z}_{ad}(t)$ *such that* $J(t,x;z) < +\infty$ *the following identity holds*

$$J(t,x;z) = v(t,x)$$

$$+\mathbb{E}\int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s))\right] ds \qquad (37)$$

where $y(s) \stackrel{def}{=} y(s;t,x,z)$ *is the solution of (2) associated to the control* z*.*

Proof. For the sake of completeness we first show how the proof goes in the case when $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and is a strict solution of (35). First we observe that the boundedness of the datum l implies that also v is bounded and that for every $(t,x) \in [0,T] \times \mathbb{R}^n$, $z \in \mathcal{Z}_{ad}(t)$ the functional $J(t,x;z)$ is finite. We now use Itô's formula applied to the function $v(t,x)$ by obtaining, for every $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\begin{aligned} v(T, y(T)) &= v(t, y(t)) + \int_t^T \langle \partial_x v(s, y(s)), B(s, y(s))\, dW(s) \rangle \\ &+ \int_t^T \langle F_0(s, y(s)), \partial_x v(s, y(s)) \rangle\, ds \\ &+ \int_t^T \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s)) \rangle\, ds \\ &+ \int_t^T \partial_s v(s, y(s))\, ds \\ &+ \frac{1}{2} \int_t^T Tr\left[B^*(s, y(s))\, \partial_{xx} v(s, y(s))\, B(s, y(s))\right] ds \end{aligned} \tag{38}$$

(which is exactly the decomposition of Remark 3.13 in the smooth case) so, by taking expectation of both sides (finite since we have the polinomial growth assumption) we get the so-called Dynkin formula

$$\begin{aligned} \mathbb{E} v(T, y(T)) &= \mathbb{E} v(t, y(t)) + \mathbb{E} \int_t^T \mathcal{L}_0 v(s, y(s))\, ds \\ &+ \mathbb{E} \int_t^T \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s)) \rangle\, ds. \end{aligned}$$

Now by (35) we get for every $s \geq 0$

$$\mathcal{L}_0 v(s, y(s)) = -H^0(s, y(s), \partial_x v(s, y(s)))$$

which yields

$$\mathbb{E} v(T, y(T)) = \mathbb{E} v(t, y(t))$$

$$+\mathbb{E} \int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s)) \rangle\right] ds.$$

Now observe that the last line, by its definition, is not smaller than $\mathbb{E} \int_t^T -l(s, y(s), z(s))\, ds$. This implies that this integral is finite or $-\infty$ since we know, by the polynomial growth of v, that the terms in the first line are finite. If it is $-\infty$ we have nothing to do. If not, we add and subtract $\mathbb{E} \int_t^T -l(s, y(s), z(s))\, ds$, use that $v(T, y(T)) = \phi(y(T))$ and that $y(t) = x$ to get

$$J(t, x; z) = v(t, x)$$

$$\begin{aligned} +\mathbb{E} \int_t^T &\left[-H^0(s, y(s), \partial_x v(s, y(s)))\right. \\ &\left. + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s)) \rangle + l(s, y(s), z(s))\right] ds \end{aligned}$$

which is the claim (37).

We now show how the proof goes when $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$. The difference is due to the fact that the term

$$I := \int_t^T \partial_s v(s, y(s))\, ds + \frac{1}{2} \int_t^T Tr\left[B^*(s, y(s))\, \partial_{xx} v(s, y(s))\, B(s, y(s))\right] ds$$

in the third and fourth line of equation (38) is not well defined now. However, thanks to the decomposition of Remark 3.13 for $\sigma_1 = B$, $b_1 = F_0 + F_1$, we can write, since $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$,

$$\begin{aligned} & v(T, y(T)) \\ = \;& v(t, y(t)) + \int_t^T \langle \partial_x v(s, y(s)), B(s, y(s))\, dW(s)\rangle \\ & + \int_t^T \langle F_0(s, y(s)), \partial_x v(s, y(s))\rangle\, ds \\ & + \int_t^T \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\, ds \\ & + \mathcal{A}_{1,t}^{y(\cdot;t,x,z)}(v)_T \end{aligned} \tag{39}$$

so that the bad term I has now been substituted by

$$I = \mathcal{A}_{1,t}^{y(\cdot;t,x,z)}(v)_T.$$

Moreover by Hypothesis 5.3 we know that v is a strong solution (in the sense of Definition 4.2) of

$$\mathcal{L}_0 v(t,x) = h_0(t,x); \qquad v(T,x) = \phi(x)$$

where we have set $h_0(t,x) = -H^0(t, x, \partial_x v(t,x))$. Then applying now Theorem 4.5 for the operator $\mathcal{L}_0$ (so now $b = F_0$ and $b_1 = F_0 + F_1$) we get that

$$\begin{aligned} & \mathcal{A}_{1,t}^{y(\cdot;t,x,z)}(v)_T \\ = \;& \int_t^T h_0(s, y(s))\, ds - \int_t^T \langle F_0(s, y(s)), \partial_x v(s, y(s))\rangle\, ds \\ = \;& \int_t^T -H^0(s, y(s), \partial_x v(s, y(s)))\, ds - \int_t^T \langle F_0(s, y(s)), \partial_x v(s, y(s))\rangle\, ds \end{aligned}$$

which yields, substituting in the decomposition (39),

$$\begin{aligned} & v(T, y(T)) \\ = \;& v(t, y(t)) + \int_t^T \langle \partial_x v(s, y(s)), B(s, y(s))\, dW(s)\rangle \\ & + \int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\right] ds. \end{aligned}$$

Now, taking expectation, adding and subtracting $\int_t^T l(s, y(s), z(s))\, ds$ (which is a.s. finite using the same argument for the smooth case), using that $v(T, y(T)) = \phi(y(T))$ and that $y(t) = x$ we get

$$\mathbb{E}\left[\int_t^T l(s, y(s), z(s))\, ds + \phi(y(T))\right] = v(t, x)$$

$$+\mathbb{E}\int_t^T \left[-H^0(s, y(s), \partial_x v(s, y(s))) + H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s))\right] ds$$

and the claim again follows. ■

Proof of Theorem 5.5. By the definition of H^0 and H^0_{CV}, for every $(t, x) \in [0, T] \times \mathbb{R}^n$, $z \in \mathcal{Z}_{ad}(t)$ and for a.e. $s \in \mathcal{T}_t$, the following inequality holds $\mathbb{P}$-almost surely (setting $y(s) = y(s; t, x, z)$)

$$-H^0(s, y(s), \partial_x v(s, y(s))) + H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s)) \geq 0, \qquad (40)$$

and then by the fundamental identity (37) it follows $v(t, x) \leq J(t, x; z)$ for every admissible z. This gives $v \leq V$ and so part (i) of the Theorem.

Now consider an admissible control z such that, for a.e. $s \in [t, T]$, $\mathbb{P} - a.s.$

$$H^0(s, y(s), \partial_x v(s, y(s))) = H^0_{CV}(s, y(s), \partial_x v(s, y(s)); z(s)). \qquad (41)$$

By the fundamental identity (37) we have

$$v(t, x) = J(t, x; z)$$

which implies optimality and $v(t, x) = V(t, x)$. ■

Remark 5.7 *To get the metodological novelty in the above proof it is crucial to note the following. Since we know that u is the limit of classical solutions, the first idea to approach the above proof (and that has been used e.g. in the papers [30],[31],[33]), is probably to prove the fundamental identity for the approximating solutions u_n and then pass to the limit for $n \to \infty$. However if one tries to do this one needs to use the uniform convergence of the space derivatives $\partial_x u_n$ to $\partial_x u$. This is not needed with our method, see also on this Remark 4.6.* ■

Let us go now to the proof of Verification Theorem 1.8 for the infinite horizon case. Set $T = +\infty$, assume that F_0, F_1 and B do not depend on time and that $l(t, x, z) = e^{\lambda t} l_1(x, z)$ $(\lambda > 0)$, $\phi = 0$; then we can set $t = 0$ (since the dependence of the value function V on t becomes trivial in this case) and the HJB equations for $V(x)$ becomes an elliptic PDE. In this case we set

$$L_0 u(x) = \langle F_0(x), \partial_x u(x)\rangle + \frac{1}{2}\mathrm{Tr}\ [B^*(x)\, \partial_{xx} u(x)\, B(x)]$$

and we rewrite the HJB equation (11) as

$$\lambda u(x) + L_0 u(x) + H^1(x, \partial_x u(x)) = 0, \quad x \in \mathbb{R}^n. \qquad (42)$$

where H^1 is defined in (12). Let us give the following definition.

Definition 5.8 *A function $v \in C^1(\mathbb{R}^n)$ is a strong solution of the HJB equation (42) if, setting $h_0(x) = -H^1(x, \partial_x v(x))$, v is a strong solution of the linear problem*

$$\lambda v(x) + L_0 v(x) = h_0(x);$$

in the sense of Definition 4.12.

Assume the following.

Hypothesis 5.9 *There exists a function $v \in C^1(\mathbb{R}^n)$ which is a strong solution of the HJB equation (42) in the sense of Definition 5.8. Moreover:*

(i) either we can choose the approximating sequence (v_n) of Definition 5.8 so that for every $s \geq 0$ and for every admissible control $z \in \mathcal{Z}_{ad}(0)$

$$\lim_{n\to+\infty} \int_0^s \langle \partial_x v_n(S_r) - \partial_x v(S_r), F_1(y(r), z(r))\rangle\, dr = 0, \qquad u.c.p.$$

(ii) or the function

$$(x, z) \to B^{-1}(x) F_1(x, z)$$

where B^{-1} stands for the pseudo-inverse of B, is well defined and bounded on $\mathbb{R}^n \times U$.

The precise statement of the verification theorem is the following

Theorem 5.10 *Assume that Hypotheses 1.1, and 5.9 hold. Assume also that l_1 is continuous and bounded and that Hypothesis 1.4 hold true for the function V_0 and the Hamiltonian H^1. If v is a bounded strong solution of the HJB equation (42) and $v \in C^1(\mathbb{R}^n)$ then $v \leq V_0$ on $\mathbb{R}^n$. Morerover if z is an admissible control at $(0, x)$ that satisfies (setting $y(s) = y(s; 0, x, z)$)*

$$H^1(y(s), \partial_x v(y(s))) = H^1_{CV}(y(s), \partial_x v(y(s)); z(s))$$

for a.e. $s \in [0, +\infty)$, $\mathbb{P}-$almost surely, then z is optimal and $v(x) = V_0(x)$.

Proof. The proof goes exactly along the same lines as the finite horizon case. First one observes that the boundedness of the datum l_1 implies that also V_0 is bounded and that for every $x \in \mathbb{R}^n$, $z \in \mathcal{Z}_{ad}(0)$ the functional $J(0, x; z)$ is finite. Then one proves the following fundamental identity

$$\begin{aligned} & J(0, x; z) \\ = \; & v(x) + \mathbb{E}\int_0^{+\infty} e^{-\lambda s}\left[-H^1(y(s), \partial_x v(y(s))) + H^1_{CV}(y(s), \partial_x v(y(s)); z(s))\right] ds. \end{aligned}$$

As in the finite horizon case we use the decomposition of Remark 3.13 for $\sigma_1 = B$ and $b_1 = F_0 + F_1$, but taking $t = 0$ and replacing $v(t, x)$ by $e^{-\lambda t} v(x)$. Observe that, if $v \in C^2(\mathbb{R}^n)$ then the function $w(t, x) = e^{-\lambda t} v(x)$ solves the equation

$$w_t(t, x) + L_0 w(t, x) = -e^{-\lambda t} H^1(x, \partial_x v(x))$$

Then since $v \in C^1(\mathbb{R}^n)$, we get for every $T_1 > 0$

$$
\begin{aligned}
& e^{-\lambda T_1} v\left(y\left(T_1\right)\right) \\
= \; & v(y(0))+\int_0^{T_1} e^{-\lambda s}\left\langle\partial_x v(y(s)), B(y(s)) d W(s)\right\rangle \\
& +\int_0^{T_1} e^{-\lambda s}\left\langle F_0(y(s)), \partial_x v(y(s))\right\rangle d s \\
& +\int_0^{T_1} e^{-\lambda s}\left\langle F_1(y(s), z(s)), \partial_x v(y(s))\right\rangle d s \\
& +\mathcal{A}_{1,0}^{y(\cdot ; 0, x, z)}\left(e^{\lambda \cdot} v\right)_{T_1}
\end{aligned}
\tag{43}
$$

Then applying now Theorem 4.13 for the operator L_0 (so now $b = F_0$ and $b_1 = F_0 + F_1$) we get that

$$
\begin{aligned}
& \mathcal{A}_{1,0}^{y(\cdot ; 0, x, z)}\left(e^{\lambda \cdot} v\right)_{T_1} \\
= \; & \int_0^{T_1} e^{-\lambda s} h_0(y(s)) d s-\int_0^{T_1} e^{-\lambda s}\left\langle F_0(y(s)), \partial_x v(y(s))\right\rangle d s \\
= \; & -\int_0^{T_1} e^{-\lambda s} H^1\left(y(s), \partial_x v(s, y(s))\right) d s-\int_0^{T_1} e^{-\lambda s}\left\langle F_0(y(s)), \partial_x v(y(s))\right\rangle d s
\end{aligned}
$$

which yields, substituting in the decomposition (43),

$$
\begin{aligned}
& e^{-\lambda T_1} v\left(y\left(T_1\right)\right) \\
= \; & v(y(0))+\int_0^{T_1} e^{-\lambda s}\left\langle\partial_x v(y(s)), B(y(s)) d W(s)\right\rangle \\
& +\int_0^{T_1} e^{-\lambda s}\left[-H^1\left(s, y(s), \partial_x v(s, y(s))\right)+\left\langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\right\rangle\right] d s .
\end{aligned}
$$

Now, taking expectation, adding and subtracting $\int_0^{T_1} e^{-\lambda s} l_1(y(s), z(s))\, ds$ (which is a.s. finite arguing as in the proof for the parabolic case) and using that $y(0) = x$ we get

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^{T_1} e^{-\lambda s} l_1(y(s), z(s)) d s+e^{-\lambda T_1} v\left(y\left(T_1\right)\right)\right] \\
= \; & v(x)+\mathbb{E} \int_0^{T_1} e^{-\lambda s}\left[-H^1\left(y(s), \partial_x v(y(s))\right)+H_{CV}^1\left(y(s), \partial_x v(y(s)) ; z(s)\right)\right] d s
\end{aligned}
$$

and the claim again follows taking the limit for $T_1 \to +\infty$ and using the boundedness of v. The rest of the proof is exactly the same as in the finite horizon case ∎

Remark 5.11 *It must be noted that the assumption on the boundedness of l_1 and v is made for simplicity of exposition. What is really needed in this infinite horizon case is that J is well defined for each $z \in \mathcal{Z}_{ad}(0)$ and that* $\lim_{T_1 \to +\infty} e^{-\lambda T_1} v(y(T_1)) =$

0 *for every* $z \in \mathcal{Z}_{ad}(0)$. *This allows to pass to the limit in the last formula. In some cases this can be checked directly, in other cases much weaker conditions can be imposed (e.g. sublinearity of* l_1 *and estimates on the solution* y *as* $T_1 \to +\infty$*).* ■

6 Necessary Conditions and Optimal feedback controls

Here we want simply to show that under additional assumptions (mainly about existence/uniqueness of the maximum of the Hamiltonian and of the solution of the closed loop equation) we can get necessary conditions and optimal feedback controls. This is a consequence of the verification Theorems 5.5 and 5.10 which has some importance for applications. For the sake of brevity we give the results only for the finite horizon case observing that completely analogous results hold true for the infinite horizon case. We start by making a remark about the so-called "weak formulation" of a stochastic control problem.

Remark 6.1 *The setting introduced in Section 1.2 corresponds to the so-called "strong formulation" of a stochastic control problem. For certain purposes (namely the existence of optimal feedbacks) it is convenient to consider the "weak formulation", letting the stochastic basis vary. In such formulation one considers as the set* $\overline{\mathcal{Z}}_{ad}(t)$ *of admissible controls as the set of 5-tuples* $(\Omega, \mathcal{F}, \mathbb{P}, W, z)$ *such that*

- $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete filtered probability space with the filtration* $\mathcal{F}$ *satisfying the usual conditions,*
- W *is an* E *valued* m*-dimensional Brownian motion on* $[t, T]$*,*
- *the process* z *is, measurable,* $(\mathcal{F}_s)$*-adapted and a.s. locally integrable.*

We will use the notation $(\Omega, \mathcal{F}, \mathbb{P}, W, z) \in \overline{\mathcal{Z}}_{ad}(t)$ *and we will call this control strategies weakly admissible (weakly optimal when they are optimal). When no ambiguity arises we will leave aside the probability space and the white noise (regarding it as fixed) and simply consider* $z \in \mathcal{Z}_{ad}(t)$. *We will specify when the weak concept of admissible control is needed. One can see e.g. the books [57, p.64], [22, p.141, 160] for comments on these formulations.* ■

We start recalling a case when the sufficient condition of the verification Theorem 5.5 becomes also necessary.

Proposition 6.2 *Assume that the hypotheses of Theorem 5.5 hold true. We also assume that the value function* V *is a strong solution of the HJB equation. Then an admissible control* $z \in \mathcal{Z}(t)$ *is optimal at* $(t, x) \in [0, T] \times \mathbb{R}^n$ *if and only if (36) holds with* V *in place of* v.

Proof. The fundamental identity in this case gives

$$J(t,x;z) = V(t,x)$$

$$+\mathbb{E}\int_t^T \left[H^0\left(s,y(s),\partial_x V(s,y(s))\right) - H^0_{CV}\left(s,y(s),\partial_x V(s,y(s));z(s)\right)\right] ds$$

and this immediately gives the claim. ∎

Remark 6.3 *One case where the above Proposition 6.2 can be applied is when:*

- *it is known that V is the unique viscosity solution of the HJB equation;*
- *it is known that strong solutions are also viscosity solutions.*

This happens e.g. in the example of Section 8. ∎

We now pass to the existence of feedbacks. First we recall the definition of (weakly) admissible feedback map.

Definition 6.4 *A measurable map $G : [0,T] \times \mathbb{R}^n \longmapsto U$ is a (weakly) admissible feedback map if for every $(t,x) \in [0,T] \times \mathbb{R}^n$ the closed loop equation ($s \in [t,T]$)*

$$\begin{aligned} y(s) &= x + \int_t^s F_1\left(r,y(r),G(r,y(r))\right) dr \\ &\quad + \int_t^s F_0\left(r,y(r)\right) dr + \int_t^s B\left(r,y(r)\right) dW(r), \end{aligned} \tag{44}$$

admits a (weak) solution $y_{G,t,x}(\cdot)$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$ and the control strategy $z_{G,t,x}(s) = G(s,y_{G,t,x}(s))$ is (weakly) admissible.

If we have an admissible feedback map G then the control strategy $z_{G,t,x}(s) = G(s,y_{G,t,x}(s))$ is by definition (weakly) admissible and so, given G, we have, for every $(t,x) \in [0,T] \times \mathbb{R}^n$ an admissible (weak) couple $(z_{G,t,x}(\cdot), y_{G,t,x}(\cdot))$. Our goal is to find an admissible feedback map G such that, for every $(t,x) \in [0,T] \times \mathbb{R}^n$, the strategy $z_{G,t,x}$ is (weakly) optimal. Such G will be called a (weakly) optimal feedback map.

Proposition 6.5 *Assume that the Hypotheses of Theorem 5.5 hold true. Assume also that:*

(i) the maximum of the Hamiltonian (5) exists and it is possible to define a measurable map

$$\begin{aligned} G_0 &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longmapsto U \\ G_0(t,x,p) &\in \arg\max_z H_{CV}(t,x,p;z); \end{aligned}$$

(ii) the map $G(t,x) = G_0(t,x,\partial_x v(t,x))$ is a (weakly) admissible feedback map.

Then for every $(t,x) \in [0,T] \times \mathbb{R}^n$, $z_{G,t,x}$ is a (weakly) optimal control, $y_{G,t,x}$ is the corresponding (weakly) optimal state and $v = V$.

Finally if $\arg\max_{z \in U_0} H_{CV}(t,x,v_x(t,x);z)$ is always a singleton and the closed loop equation (44) admits a unique (weak) solution, then the optimal control is (weakly) unique.

Proof. This is an obvious consequence of the verification Theorem 5.5. ∎

Remark 6.6 *We observe that if we have existence of the (weak) solution of the closed loop equation (44) for every strong solution of the HJB equation (35), then we automatically have that the strong solution is unique.* ∎

Remark 6.7 *We finally observe that Propositions 6.2 and 6.5 holds also in the infinite horizon case with obvious changes. We will use the infinite horizon version of Proposition 6.5 in the study of the one dimensional example in Section 8.2.* ∎

7 Application 1: a class of exit time control problems with non degenerate diffusion

Here we show that our technique for proving verification theorems works for a family of stochastic optimal control problems with exit time and nondegenerate diffusion where the solutions of the associated HJB equation are not known to be $C^{1,2}$.

7.1 The problem

Let first $\mathbb{R}^n$ be the state space, $\mathbb{R}^n$ be the space of noises and U (a given subset of a Polish space) be the control space. Let T be a fixed finite horizon, $(\Omega, \mathcal{F}, \mathbb{P})$ be a given stochastic basis (where $\mathcal{F}$ stands for a given filtration $(\mathcal{F}_s)_{s\in[0,T]}$), W be a cylindrical Brownian motion with values in $\mathbb{R}^n$ adapted to $(\mathcal{F}_s)_{s\in[0,T]}$. Consider a stochastic controlled system in $\mathbb{R}^n$ with fixed finite horizon T and initial time $t \in [0,T)$ governed by the state equation

$$\begin{aligned} dy(s) &= [F_0(s,y(s)) + F_1(s,y(s),z(s))]\,ds + B(y(s))\,dW(s), \qquad s \in [t,T] \\ y(t) &= x. \end{aligned} \tag{45}$$

We consider an open bounded domain $\mathcal{O} \subseteq \mathbb{R}^n$ with uniformly C^2 boundary (see e.g. [41, pp.2-3] for the definition). The initial datum x belongs to $\mathcal{O}$ and we call $\tau_{\mathcal{O}}$ the first exit time of the process y from this open set, i.e.

$$\tau_{\mathcal{O}}(\omega) = \inf\{s > t : y(s;\omega) \in \mathcal{O}^c\}$$

Moreover $F_0 : [0,T] \times \bar{\mathcal{O}} \times U \to \mathbb{R}^n$, $F_1 : [0,T] \times \bar{\mathcal{O}} \times U \to \mathbb{R}^n$, $B : [0,T] \times \bar{\mathcal{O}} \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$, ($\bar{\mathcal{O}}$ stands for the closure of the set $\mathcal{O}$) satisfy

Hypothesis 7.1 *F_0 and F_1 are continuous and bounded.*

Hypothesis 7.2 *B is time independent, uniformly continuous and bounded and nondegenerate i.e. there exists $\lambda_0 > 0$ such that*

$$\lambda_0^{-1} \|\xi\|_X \geq \sum_{i,j=1}^{d} \left[B(x) B^T(x)\right]_{i,j} \xi_i \xi_j \geq \lambda_0 \|\xi\|_X \qquad \forall (t,x) \in [0,T] \times \bar{\mathcal{O}}, \qquad \forall \xi \in X$$

Remark 7.3 *First note that the above Hypothesis 7.2 implies that B is invertible with bounded inverse, so we fall in the framework of Corollary 4.7. Moreover it is possible to extend our results to the case when B is time dependent, provided suitable regularity conditions are satisfied. Such conditions are needed to apply the results of semigroup theory in the subsection below and substantially require that the domain of the operator*

$$L(t) u = \frac{1}{2} \mathrm{Tr}\left[B^*(t,x) B(t,x) \partial_{xx} u\right]$$

is constant and that a Hölder continuity in time hold, see on this [41, 3, 4, Section 3.1.2]. We leave aside these assumptions to keep a more simplified setting where, anyway, $C^{1,2}$ regularity does not hold in general. ■

About the control strategy $z : [t,T] \times \Omega \to U$ we assume that it belongs to a given set $\mathcal{Z}_{ad}(t)$ of stochastic processes defined on $[t,T] \times \Omega$ with values in a fixed Polish space U. More precisely we will assume the following.

Hypothesis 7.4 *The control space U is a Polish space, while $\mathcal{Z}_{ad}(t)$ is the space of all measurable processes $z : [t,T] \times \Omega \to U$ adapted to the filtration $\mathcal{F}$.*

Remark 7.5 *The above setting corresponds to the so-called "strong formulation" of a stochastic control problem. To look at the existence of optimal feedbacks it is sometime convenient to consider the "weak formulation", letting the stochastic basis to vary, see Remark 6.1. The concept of admissible control is needed in this case to find optimal feedbacks.* ■

The following Proposition can be proved in the same way as in [54], even if the drift is random. In fact Theorem 6.4.3 therein makes use of Girsanov transformation to eliminate the drift; the same can be done here. Then we can apply Corollary 6.4.4 and Theorem 7.2.1 of [54].

Proposition 7.6 *Assume that Hypotheses 7.1, 7.2 hold. Then, for all $z \in \mathcal{Z}_{ad}(t)$, equation (45) has a weak solution*

$$y(\cdot; t, x, z) \in M_W^1(t,T;X).$$

This solution is unique in the sense of probability law.

Remark 7.7 *In the case of dimension 1, no continuity on the diffusion coefficients is required, see [54] Exercise 7.3.3 at page 192.* ■

We now consider the following stochastic optimal control problem with exit time. Minimize the cost functional

$$J(t,x;z) = \mathbb{E}\left[\int_t^{\tau_{\mathcal{O}}\wedge T} l\left(s, y\left(s;t,x,z\right), z\left(s\right)\right) ds \right. \qquad (46)$$

$$\left. + I_{\{\tau_{\mathcal{O}}<T\}}\psi\left(\tau_{\mathcal{O}}, y\left(\tau_{\mathcal{O}};t,x,z\right)\right) + I_{\{\tau_{\mathcal{O}}\geq T\}}\phi\left(y\left(T;t,x,z\right)\right)\right]$$

over all controls $z \in \mathcal{Z}_{ad}(t)$. Here $y(\cdot;t,x,z)$ is the solution of the equation (45) and we assume that l, ψ, ϕ satisfy

Hypothesis 7.8 $l \in C_b\left([0,T]\times\bar{\mathcal{O}}\right)$, $\phi \in C_b\left(\bar{\mathcal{O}}\right)$ $\psi \in C_b^{\frac{1+\beta}{2},1+\beta}\left([0,T]\times\partial\mathcal{O}\right)$ *(for some* $\beta > 0$*),* $\psi(T,x) = \phi(x)$ *on* $\partial\mathcal{O}$.

Remark 7.9 *The above hypothesis is needed to apply the results of semigroup theory in the subsection below. They allow to prove generation of analytic semigroup for the operator*

$$Lu = \frac{1}{2}\mathrm{Tr}\left[B^*(x)\,B(x)\,\partial_{xx}u\right]$$

(with 0 Dirichlet boundary conditions) and to get well posedness of associated boundary value problem in $\mathcal{O}$ *(see [41, Ch.5]).* ■

The value function of this problem is defined as

$$V(t,x) = \inf\left\{J(t,x;z) \ : \ z \in \mathcal{Z}_{ad}(t)\right\} \qquad (47)$$

and a control $z^* \in \mathcal{Z}_{ad}(t)$ and such that $V(t,x) = J(t,x;z^*)$ is said to be *optimal* with respect to the initial time and state (t,x). The corresponding HJB equation is

$$\begin{cases} v(t,x) + \frac{1}{2}\mathrm{Tr}\left[B(x)\,\partial_{xx}v(t,x)B^*(x)\right] = \langle F_0(t,x), \partial_x v(t,x)\rangle + H\left(t,x,\partial_x v(t,x)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad t\in[0,T], \quad x\in\mathcal{O}, \\ v(T,x) = \phi(x), \qquad x\in\mathcal{O}, \\ v(t,x) = \psi(t,x), \qquad t\in[0,T], s\in\partial\mathcal{O}, \end{cases} \qquad (48)$$

where

$$H(t,x,p) = \inf_{z\in U} H_{CV}(t,x,p;z), \qquad (49)$$

with

$$H_{CV}(t,x,p;z) = \langle F_1(t,x,z), p\rangle + l(t,x,z),$$

being the *current value Hamiltonian.*

Proposition 7.10 *Under Hypotheses 7.1, 7.2, 7.4, 7.8, the Hamiltonian* $H(t,x,p)$ *defined in (49) is continuous in* $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$*. Moreover there exists a constant* $C>0$ *such that*

$$\begin{aligned} |H(t,x,p)-H(t,x,q)| &\leq C|p-q| \\ |H(t,x,p)| &\leq C(1+|p|) \end{aligned} \tag{50}$$

Proof. It is enough to apply the definition of the Hamiltonian and use the continuity and the boundedness of F_1 and l. ∎

7.2 Strong solutions of the HJB equation

The HJB equation (48) above is a semilinear parabolic equation with continuous coefficients. Since the second order term is nondegenerate (Hypothesis 7.2) one expects interior regularity results for the solution even if the boundary data are merely continuous, as it is under our assumptions. In particular the following result, taken from [12], Theorems 9.1 and 9.2, apply.

Theorem 7.11 *Under Hypotheses 7.1, 7.2, 7.4, 7.8, there exists a viscosity solution* u *of (48) and* u *belong to the space* $C_{loc}^{\frac{1+\alpha}{2},1+\alpha}([0,T)\times\mathcal{O})\cap C([0,T]\times\bar{\mathcal{O}})$ *for some* $\alpha\in(0,1)$.

We note also that it is not known if the solution is classical in the interior. This is known, using theorems of [5], if the data are supposed to be Hölder continuous in (t,x), which we do not assume here. Moreover even in the linear case, if the coefficients are only continuous, solutions are only $W^{2,p}$, so they are also $C^{1,\alpha}$. An example in this direction is in [28, Ch.4].

This means that we are exactly in the case where it makes sense to apply our technique to prove verification theorems: no classical solution but solutions (in a generalized sense) with at least $C^{0,1}$ regularity. Once this is clear, we need to check if our solutions (that exists at least in the viscosity sense thanks to the above Theorem 7.11) are also strong solutions in the sense of Definition 5.2 (suitably modified to take care of the boundary datum ψ). Such a result is not available in the literature in this form but it can be easily deduced using the results on analytic semigroups contained e.g. in [41]. We explain here below how this can be done in the case when $\psi=0$ (the general case can be treated with the ideas explained in [41, Section 5.1.2]). In this case the operator

$$\begin{aligned} L &: D(L)\subseteq C_b(\bar{\mathcal{O}})\to C_b(\bar{\mathcal{O}}) \\ D(L) &= \left\{\eta\in\cap_{p\geq1}W_{loc}^{2,p}(\mathcal{O}): \quad \eta, L\eta\in C_b(\bar{\mathcal{O}}), \quad \eta|_{\partial\mathcal{O}}=0\right\} \\ (L\eta)(x) &= \frac{1}{2}\mathrm{Tr}\left[B(x)\,\partial_{xx}\eta(x)B^*(x)\right] \end{aligned}$$

generates an analytic semigroup $\{e^{tL}, t\geq 0\}$, [41, p.97, Corollary 3.1.21]. Moreover, given any initial datum $\phi\in C_b(\bar{\mathcal{O}})$ and any function H continuous in $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$

and satisfying (50) we can apply a modification of the argument used to prove Proposition 7.3.4 in [41, p.281] to get existence and uniqueness of a solution u of the integral equation

$$\begin{aligned} u(t,x) &= \left(e^{(T-t)L}\phi\right)(x) \\ &\quad + \int_t^T \left(e^{(T-s)L}\left[\langle F_0(s,\cdot), \partial_x u(s,\cdot)\rangle + H(s,\cdot,\partial_x u(s,\cdot))\right]\right)(x)\, ds \end{aligned} \tag{51}$$

which can be considered as an integral form of the PDE (48) when $\psi = 0$ (in the general case one needs to lift the function ψ into the equation obtaining an extra term in (51), see on this [41, Remark 5.1.14, p.195]) written using the variation of constants formula. More precisely we have the following definitions and results.

Definition 7.12 *We say that a function w belongs to the space $\Sigma^{1,\alpha}\left([0,T]\times\bar{\mathcal{O}}\right)$ if $w \in C_b\left([0,T]\times\bar{\mathcal{O}}\right)$, w is Fréchet differentiable in $x\in\mathcal{O}$, $\partial_x w \in C_b\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$ for every $\varepsilon \in (0,T)$, and*

$$\sup_{(t,x)\in[0,T)\times\bar{\mathcal{O}}} \left|(T-t)^{\alpha}\,\partial_x w(t,x)\right| < +\infty.$$

Definition 7.13 *A function $u \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ that satisfies the integral equation (51) for every $(t,x)\in[0,T]\times\bar{\mathcal{O}}$ is called a* mild solution *of the HJB equation (48).*

Theorem 7.14 *Assume that B and F_0 belong to $C_b\left(\bar{\mathcal{O}}\right)$. Moreover let H be continuous in $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}^n$ and satisfies (50). Then there exists a unique mild solution $u \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ of the HJB equation (48).*

Proof. The proof is a standard application of the contraction mapping principle, see [41, Proposition 7.3.4] and also [7, 30]. ∎

Now we want to show that such a mild solution is a strong solution in the sense that it can be seen as the limit of smooth solutions.

We rewrite here the definition of strong solution for our case since it is slightly weaker than the one used in Section 5 due to the possible singularity of the spatial gradient at $t = 0$, see Remarks 4.9 and 5.4.

Definition 7.15 *A function $u \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$ is a strong solution of the equation (48) if it satisfies the boundary and final conditions*

$$\begin{aligned} &u(T,x) = \phi(x), && x\in\mathcal{O}, \\ &u(t,x) = \psi(t,x), && t\in[0,T], s\in\partial\mathcal{O}, \end{aligned}$$

and if, for every $\varepsilon > 0$, it is a strong solution (in the sense of Definition 4.2) of the linear parabolic problem

$$\begin{aligned} w_t + Lw &= h, \qquad t\in[0,T], \qquad x\in\mathcal{O}, \\ w(T-\varepsilon,x) &= u(T-\varepsilon,x), \qquad x\in\mathcal{O}, \end{aligned}$$

where $h = \left[\langle F_0(\cdot,\cdot), \partial_x u(\cdot,\cdot)\rangle + H(\cdot,\cdot,\partial_x u(\cdot,\cdot))\right]$, i.e. if, for every $\varepsilon > 0$ there exist sequences u_n^{ε}, h_n^{ε}, ϕ_n^{ε} such that

1. *for every n* u_n^ε *is a strict solution (i.e. it belong to* $C^{1,2}\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$ *and satisfy the equalities on* $[0,T-\varepsilon]\times\bar{\mathcal{O}})$ *of the approximating problem*

$$v_t + Lv = h_n^\varepsilon, \quad v(T-\varepsilon,x) = \phi_n^\varepsilon(x)$$

2. u_n^ε *converges to* u *uniformly in* $[0,T-\varepsilon]\times\bar{\mathcal{O}}$*;*

3. h_n^ε *converges to* h *uniformly in* $[0,T-\varepsilon]\times\bar{\mathcal{O}}$*;*

4. ϕ_n^ε *converges to* $u(T-\varepsilon,\cdot)$ *uniformly in* $\bar{\mathcal{O}}$*;*

Given the above Definition 7.15 we can apply directly Proposition 4.1.8 (see also Theorem 5.1.11) of [41] to get the following.

Theorem 7.16 *The mild solution of (48) is also strong.*

Proof. It is enough to apply Proposition 4.1.8 and Theorem 5.1.11 of [41] to the nonhomogenous linear parabolic problem

$$v_t + Lv = h, \quad v(T,x) = u(T-\varepsilon,x)$$

where we set

$$h(t,x) = \langle F_0(t,x), \partial_x u(t,x)\rangle + H(t,x,\partial_x u(t,x)).$$

In fact for every $\varepsilon > 0$ such function h belongs to the space $C\left([0,T-\varepsilon]; C\left(\bar{\mathcal{O}}\right)\right) = C\left([0,T-\varepsilon]\times\bar{\mathcal{O}}\right)$. ∎

Remark 7.17 *In the case when* ψ *is not* 0 *the above results still hold true using the same techniques shown in [41], Theorem 5.1.16, and 5.1.17. We do not do it here for simplicity of exposition.*

Remark 7.18 *If the coefficients are only Borel measurable and not necessarily continuous then is well known, see for instance [1], that the semigroup has a density with respect to the Lebesgue measure and that it fulfills the classical Aronson estimates. The methodology of Fukushima - Dirichlet decomposition can be extended to this case. We do not develop it here since in this work, for practical reasons we only use the topology of continuous functions.*

7.3 The verification theorem

Now we prove a verification theorem for the optimal control problem above (45) - (46). The proof is a modification of the proof given in Section 5. The main differences are due to

1. a singularity of the first derivative and so different definition of strong solution;

2. constraints on the set $\mathcal{O}$ and so presence of boundary data in x (stopping time).

The statement of the verification theorem in this case is the following

Theorem 7.19 *Assume that Hypotheses 7.1, 7.2, 7.4, 7.8, hold. Let* $v \in \Sigma^{1,\frac{1}{2}}\left([0,T] \times \bar{\mathcal{O}}\right)$ *be a strong solution of (48). Then*

(i) $v \leq V$ *on* $[0,T] \times \bar{\mathcal{O}}$.

(ii) If z *is an admissible control at* $(t,x) \in [0,T] \times \mathcal{O}$ *that satisfies (setting* $y(s) = y(s;t,x,z)$*), for a.e.* $s \in [t,T]$,

$$H\left(s, y(s), \partial_x v(s, y(s))\right) = H_{CV}\left(s, y(s), \partial_x v(s, y(s)); z(s)\right),$$

then z *is optimal at* (t,x) *and* $v(t,x) = V(t,x)$.

The proof of this theorem follows as usual by the following fundamental identity that we state as a lemma.

Lemma 7.20 *Assume that Hypotheses 7.1, 7.2, 7.4, 7.8, hold. Let* $v \in \Sigma^{1,\frac{1}{2}}\left([0,T] \times \bar{\mathcal{O}}\right)$ *be a strong solution of (48). Then, for every* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ *and* $z \in \mathcal{Z}_{ad}(t)$ *the following identity holds*

$$J(t,x;z) = v(t,x)$$

$$+\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T}\left[-H\left(s, y(s), \partial_x v(s, y(s))\right) + H_{CV}\left(s, y(s), \partial_x v(s, y(s)); z(s)\right)\right] ds$$

where $y(s) \stackrel{def}{=} y(s;t,x,z)$ *is the solution of (45) associated with the control* z.

Proof. Since $v \in \Sigma^{1,\frac{1}{2}}\left([0,T] \times \bar{\mathcal{O}}\right)$ then for every $\varepsilon > 0$ we have $v \in C^{0,1}\left([0,T-\varepsilon] \times \bar{\mathcal{O}}\right)$ and v is a strong solution of the problem

$$\begin{aligned} w_t + Lw &= h, \qquad t \in [0,T], \qquad x \in \mathcal{O}, \\ w(T-\varepsilon, x) &= v(T-\varepsilon, x), \qquad x \in \mathcal{O}, \end{aligned}$$

where $h = \left[\langle F_0(\cdot,\cdot), \partial_x v(\cdot,\cdot)\rangle + H(\cdot,\cdot,\partial_x v(\cdot,\cdot))\right]$, (see Definition 7.15). Moreover, thanks to the decomposition of Remark 3.13 for $\sigma_1 = B$ and $b_1 = F_0 + F_1$, we can write,

$$\begin{aligned} v\left(\tau_{\mathcal{O}} \wedge (T-\varepsilon), y\left(\tau_{\mathcal{O}} \wedge (T-\varepsilon)\right)\right) &= v(t, y(t)) \\ &+ \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle \partial_x v(s, y(s)), B(y(s))\, dW(s)\rangle \\ &+ \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle F_0(s, y(s)), \partial_x v(s, y(s))\rangle\, ds \\ &+ \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle F_1(s, y(s), z(s)), \partial_x v(s, y(s))\rangle\, ds \\ &+ \mathcal{A}_{1,t}^{y(\cdot;t,x,z)}(v)_{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \end{aligned} \qquad (52)$$

Then applying now Theorem 4.5 and Remark 4.8 for the operator $\mathcal{L}_0 = \partial_t + L$ (so now $b = 0$ and $b_1 = F_0 + F_1$), differently from what we used to find (52) we get that

$$\begin{aligned} & \mathcal{A}_{1,t}^{y(\cdot;t,x,z)}(v)_{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \\ = & \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} h(s,y(s))\,ds \\ = & \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} H(s,y(s),\partial_x v(s,y(s)))\,ds \\ & -\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle F_0(s,y(s)),\partial_x v(s,y(s))\rangle\,ds \end{aligned}$$

which yields, substituting in the decomposition (52),

$$\begin{aligned} & v(\tau_{\mathcal{O}}\wedge(T-\varepsilon),y(\tau_{\mathcal{O}}\wedge(T-\varepsilon))) = v(t,y(t)) \\ & +\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle \partial_x v(s,y(s)),B(y(s))\,dW(s)\rangle \\ & +\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} H(s,y(s),\partial_x v(s,y(s)))\,ds \\ & +\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle F_1(s,y(s),z(s)),\partial_x v(s,y(s))\rangle\,ds. \end{aligned}$$

Adding and subtracting $\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds$ (which is always finite since l is bounded) and using that $y(t) = x$ we get

$$\begin{aligned} & \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds + v(\tau_{\mathcal{O}}\wedge(T-\varepsilon),y(\tau_{\mathcal{O}}\wedge(T-\varepsilon))) \\ = & \; v(t,x) + \int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \langle \partial_x v(s,y(s)),B(s,y(s))\,dW(s)\rangle \\ & +\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \left[-H(s,y(s),\partial_x v(s,y(s))) + H_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right] ds. \end{aligned}$$

Taking the expectation we arrive at the following formula

$$\begin{aligned} & \mathbb{E}\left[\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l(s,y(s),z(s))\,ds + v(\tau_{\mathcal{O}}\wedge(T-\varepsilon),y(\tau_{\mathcal{O}}\wedge(T-\varepsilon)))\right] \\ = & \; v(t,x) \\ & +\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} \left[-H(s,y(s),\partial_x v(s,y(s))) + H_{CV}(s,y(s),\partial_x v(s,y(s));z(s))\right] ds. \end{aligned}$$

Let now $\varepsilon \to 0^+$. By continuity of v (and since by definition the strong solution satisfies the boundary and final condition) we get,

$$\begin{aligned} & \mathbb{E}v(\tau_{\mathcal{O}}\wedge(T-\varepsilon),y(\tau_{\mathcal{O}}\wedge(T-\varepsilon))) \\ \to & \; \mathbb{E}\left[I_{\{\tau_{\mathcal{O}}\geq T\}}\phi(y(T)) + I_{\{\tau_{\mathcal{O}}<T\}}\psi(\tau_{\mathcal{O}},y(\tau_{\mathcal{O}};t,x,z))\right]. \end{aligned}$$

Moreover, by the boundedness of l we get

$$\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)} l\left(s,y\left(s\right),z\left(s\right)\right)ds \to \mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T} l\left(s,y\left(s\right),z\left(s\right)\right)ds.$$

Finally we recall that, by the definition of the Hamiltonian (49) for a suitable $C_0 > 0$ we have

$$\begin{aligned} 0 &\leq -H\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) + H_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right) \\ &\leq C_0\left(1+\left|\partial_x v\left(s,y\left(s\right)\right)\right|\right), \end{aligned}$$

so that, for every $s \in [0,T)$, thanks to the fact that $v \in \Sigma^{1,\frac{1}{2}}\left([0,T]\times\bar{\mathcal{O}}\right)$

$$\begin{aligned} &\mathbb{E}\left[-H\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) + H_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\right]ds \\ \leq\ & C_0\left(1+\mathbb{E}\left|\partial_x v\left(s,y\left(s\right)\right)\right|\right) \\ \leq\ & C_1\left(T-s\right)^{-\frac{1}{2}} \end{aligned}$$

which allows us to apply the dominated convergence theorem obtaining

$$\begin{aligned} &\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge(T-\varepsilon)}\left[-H^0\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) + H^0_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\right]ds \\ \to\ & \mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T}\left[-H^0\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) + H^0_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\right]ds. \end{aligned}$$

All the above yields then

$$\begin{aligned} &\mathbb{E}\left[\int_t^{\tau_{\mathcal{O}}\wedge T} l\left(s,y\left(s\right),z\left(s\right)\right)ds + \phi\left(y\left(T\right)\right)\right] \\ =\ & v\left(t,x\right) \\ &+\mathbb{E}\int_t^{\tau_{\mathcal{O}}\wedge T}\left[-H\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) + H_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right)\right]ds. \end{aligned}$$

and so the claim follows. ■

Proof of Theorem 7.19. By the definition of H and H_{CV}, for every $z \in \mathcal{Z}_{ad}\left(t\right)$, $x \in \mathcal{O}$ and for a.e. $s \in \mathcal{T}_t$, the following inequality holds $\mathbb{P}$-almost surely (setting $y\left(s\right) = y\left(s;t,x,z\right)$)

$$H\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right)\right) - H_{CV}\left(s,y\left(s\right),\partial_x v\left(s,y\left(s\right)\right);z\left(s\right)\right) \geq 0. \qquad (53)$$

The rest of the proof follows exactly the one of Theorem 5.5. ■

Remark 7.21 *Arguing as in Subsection 6 it is possible to prove a necessary condition and the existence of optimal feedbacks.* ■

8 Application 2: a class of problems with degenerate diffusion

Here we consider a class of problems where the second order operator in the HJB equation is degenerate.

8.1 The problem

Consider the problem (2) - (3) introduced in Subsection 1.2, but with the following additional assumption.

Hypothesis 8.1 *The matrix BB^* is degenerate in the sense that there exists at least one point $x \in \mathbb{R}^n$ such that BB^* is not invertible.*

This assumption implies that the HJB equation (35) and (42) are degenerate (parabolic or elliptic). For such equations a general regularity theory like the one available in the uniformly parabolic or elliptic case is not known. So it is difficult to obtain existence of $C^{1,2}$ (C^2) solutions in this case. There is more hope to obtain solutions with regularity $C^{0,1}$ (C^1)or at least continuous with existence of the space derivatives in the weak sense (this case is not covered in the present paper but is the subject of our current research). This means that, on one hand, this kind of problems is a source of possible sharp applications of our verification Theorems 5.5, 1.8. On the other hand in this case it is difficult to have Hypothesis 5.3-(ii) to be satisfied. So one needs to have convergence of the derivatives in the sense of Hypothesis 5.3-(i), which may be not easy to check. We will give an example when this can be done in Subsection 8.2.

Remark 8.2 *A possible methodology to prove existence of $C^{0,1}$ solutions of the parabolic degenerate equation (35) is the following (consider the case where all data are autonomous). Take the degenerate elliptic operator*

$$L_1\eta\left(x\right) = \left\langle F_0\left(x\right), \partial_x \eta\left(x\right)\right\rangle + \frac{1}{2} Tr\ \left[B^*\left(x\right)\partial_{xx}\eta\left(x\right)B\left(x\right)\right],$$

defined in $C^2\left(\mathbb{R}^n\right)$. We can write the HJB equation (35) as

$$v_t\left(t,x\right) + L_1 v\left(t,x\right) + H^0\left(x, \partial_x v\left(t,x\right)\right) = 0,$$

$$v\left(T,x\right) = \phi\left(x\right).$$

At this point, if the operator L_1 is "sufficiently good", e.g. if it generates a smoothing semigroup (see the papers [2, 11, 27, 32] on this) one can try to apply some perturbation arguments finding existence of a strong solutions. Also Remark 5.4 apply. ∎

The next subsection below is devoted to present one dimensional example in the finite horizon case.

8.2 A one-dimensional example

We present here a family of examples of stochastic optimal control problems with finite horizon (that can be extended to the infinite horizon case), which come from optimal investment models. For this reason the problems are taken with the sup, so also Hamiltonians are different from the rest of the paper. The proofs are only sketched for brevity.

An advertising model

This example comes from advertising models where the state variable y is the goodwill (i.e. the number of people that are aware of the product) the control variable z is the investment in advertising and one maximizes the profit from selling the product on a given finite horizon; see on this e.g. [35] and the references therein.

The state space is $\mathbb{R}$, the control space is $U = \mathbb{R}^+$. The noise space is $\mathbb{R}$ and W is a standard 1-dimensional Brownian motion. Given an initial point $x \in \mathbb{R}$ and parameters $\alpha, \beta > 0$, the state equation is

$$\begin{cases} dy\,(s) = [-\alpha y\,(s) + z\,(s)]\,ds + \beta y\,(s)\,dW\,(s) \\ y\,(t) = x. \end{cases} \tag{54}$$

Given $\rho > 0$, $c : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, increasing and convex and such that

$$\lim_{z \to +\infty} \frac{c\,(z)}{z} = +\infty, \tag{55}$$

$h : \mathbb{R} \to \mathbb{R}$, continuous and strictly increasing, we maximize the profit functional

$$J(t, x; z) = \mathbb{E}\left\{ \int_t^T -c\,(z\,(s))\,ds + h\,(y\,(T)) \right\}$$

over all controls $z \in \mathcal{Z}_{ad}\,(t)$ where

$$\begin{aligned} \mathcal{Z}_{ad}(t) \;&=\; \{z : \mathcal{T}_t \times \Omega \to U, \quad \text{measurable, a.s. locally integrable} \\ &\qquad \text{and adapted to } (\mathcal{F}_s)_{s \geq 0}\}. \end{aligned}$$

Here $y\,(\cdot) = y\,(\cdot; x, z) = y\,(\cdot; t, x, z)$ is the strong solution of the equation (54) on $[0, +\infty)$. Here Hypotheses 1.1, 1.2 and 1.4 are clearly satisfied. The value function is

$$V(t, x) = \sup\{J(t, x; z) \;:\; z \in \mathcal{Z}_{ad}\,(0)\}$$

and the corresponding Hamilton-Jacobi-Bellman equation (HJB from now on) reads as follows

$$\begin{cases} \partial_t v\,(t, x) + \frac{1}{2}\beta^2 x^2 \partial_{xx} v(t, x) - \langle \alpha x, \partial_x v\,(t, x) \rangle + H^0\,(\partial_x v\,(t, x)) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T], \qquad x \in \mathbb{R} \\ v\,(T, x) = h\,(x), \qquad x \in \mathbb{R} \end{cases} \tag{56}$$

where the Hamiltonian H^0 is given by

$$H^0\,(p) = \sup_{z \geq 0} \{zp - c\,(z)\} = c^*\,(p)$$

(where c^* is the Legendre transform of c) and it is always finite and continuous thanks to (55).

Remark 8.3 *We stress the fact that, via the approach described in Remark 8.2, we can prove, eventually restricting the assumptions on data, that there exists a strong solution of the HJB equation (56). In fact, the operator*

$$L_1\eta(x) = \langle \alpha x, \partial_x \eta(x)\rangle + \frac{1}{2}\beta^2 x^2 \partial_{xx}\eta(x),$$

generates an analytic semigroup (see e.g. [32] and the references therein) in the space C^0 with domain C^2 with suitable weights. This allows us, using a perturbation method similar to the one of [31], to show, under some restriction on the data, the existence of a mild solution of HJB equation (56), i.e. a solution of the integral equation

$$v(t,\cdot) = e^{(T-t)L_1}h + \int_t^T e^{(T-s)L_1} H^0(\partial_x v(s,\cdot))\, ds$$

belonging to the space $C^{0,1}$ with suitable weights. Such mild solution turns out to be strong in the sense of Definition 5.2 easily: by simply taking suitable approximations of the initial datum h and of $H^0(\partial_x v(s,\cdot))$, as it is done again in [31] or in [41], one finds solutions u_n of approximating problems that converges to u. It is not however trivial to see if for such approximating sequence we also have the convergence required in Hypothesis 5.3-(i). Below we see that it is true in a simple case. ■

Choosing data in a suitable way we can write explicitely a strong solution which satisfies our assumptions but is not C^2 in space. Let $c(z) = z^{1+\eta}$ with $\eta \in (0,1)$, and $h(x) = |x|^{1+\eta}\, sgn(x)$. Then the Hamiltonian is

$$H^0(p) = \eta\left(\frac{[p]^+}{1+\eta}\right)^{1+\frac{1}{\eta}}$$

with the maximum point

$$z = \left(\frac{[p]^+}{1+\eta}\right)^{\frac{1}{\eta}}. \tag{57}$$

If we look for a solution of the form $w(t,x) = f(t)\cdot|x|^{1+\eta}\, sgn(x)$ we see that the function

$$v(t,x) = \begin{cases} a(t)\,|x|^{1+\eta}\, sgn(x) & \text{for } x>0 \\ 0 & \text{for } x=0 \\ b(t)\,|x|^{1+\eta}\, sgn(x) & \text{for } x<0 \end{cases}$$

satisfies for every $x \neq 0$ the HJB equation (56) above if $a > 0$ and $b < 0$ are, respectively, the solution of the Cauchy problem

$$a'(t) = -\left[\frac{1}{2}\beta^2\eta(1+\eta) - \alpha(1+\eta)\right]a(t) - \eta a(t)^{1+\frac{1}{\eta}}; \qquad a(T) = 1,$$

respectively

$$b'(t) = \left[\frac{1}{2}\beta^2\eta(1+\eta) - \alpha(1+\eta)\right] b(t); \qquad b(T) = -1;$$

Now the second Cauchy problem admits a unique global solution on $[0,T]$, which is always strictly negative. The first does the same under suitable conditions on the data (e.g. that $\frac{1}{2}\beta^2\eta < \alpha$): assume from now on that this is the case. We can easily see that $v \in C^{0,1}([0,T]\times\mathbb{R})$. Moreover we may prove, by approximating v by $v_n = v\cdot\theta_n$ (where θ_n is a suitable cut-off function such that $\theta_n = 0$ for $|x| \le \frac{1}{n}$ and $\theta_n = 1$ for $|x| \ge \frac{2}{n}$) that v is a strong solution of (56) and that $\partial_x v_n \to \partial_x v$ uniformly on compact sets. This fact says that we have a strong solution to which standard verification do not apply but which falls into our cases. In particular not only Theorem 1.8 but also Proposition 6.5 applies. Indeed the maximum of the Hamiltonian is reached in the unique point z given by (57), so we can set, with the notation of Proposition 6.5

$$G_0(t,x,p) = G_0(p) = \left(\frac{[p]^+}{1+\eta}\right)^{\frac{1}{\eta}}.$$

Recalling that

$$\partial_x v(t,x) = \begin{cases} a(t)(1+\eta)|x|^\eta & \text{for } x>0 \\ 0 & \text{for } x=0 \\ -b(t)(1+\eta)|x|^\eta & \text{for } x<0 \end{cases}$$

we have

$$G(t,x) = G_0(\partial_x v(t,x)) = \begin{cases} a(t)^{\frac{1}{\eta}}|x| & \text{for } x>0 \\ 0 & \text{for } x=0 \\ [-b(t)]^{\frac{1}{\eta}}|x| & \text{for } x<0 \end{cases}$$

It is easy to check that G is an admissible feedback map (along with Definition 6.4). In fact, setting, for $x>0$

$$z(s) = a(s)^{\frac{1}{\eta}}|y(s)|,$$

the closed loop equation is

$$\begin{cases} dy(s) = \left[-\alpha y(s) + a(s)^{\frac{1}{\eta}}|y(s)|\right]dt + \beta y(s)\,dW(s) \\ y(t) = x. \end{cases}$$

This equation always features existence and uniqueness of a strong solution $y_{G,t,x}$ which is a.s. strictly positive and so $z_{G,t,x}$ is admissible and $z_{G,t,x}(s) = G(s, y_{G,t,x}(s))$ for $s\in[t,T]$. The same argument can be applied to the cases $x<0$ and $x=0$. This means that, from Proposition 6.5, for every fixed $(t,x)\in[0,T]\times\mathbb{R}$ the couple $(z_{G,t,x}, y_{G,t,x})$ (again with the notations of Proposition 6.5) is optimal. Moreover the optimal control (state) is unique. In particular, when $x>0$ ($x<0$) then we have a unique optimal couple where the optimal state is a.s. positive (negative). When $x=0$ the zero strategy is optimal. These facts cannot be immediately deduced using other verification theorems. Of course one could argue with ad hoc arguments applied to this problem but this is another story. We observe that this also implies that v is the value function.

Remark 8.4 *The procedure outlined in this example may be applied to other kind of problems with bilinear state equation and current cost with growth* $1+\eta$ *(*$\eta \in (0,1)$*), eventually with infinite horizon. We think e.g. to optimal investment models where adjustment cost are not quadratic but of growth* $1+\eta$ *(see e.g. [55]). Also one could treat similar models when no explicit solution can be found but it is possible to prove that the there exists a strong* $C^{0,1}$ *solution of the HJB equation.*

Finally we mention two possible extensions of the problem introduced above that are interesting for economic applications (namely growth theory, optimal investment, optimal portfolio models): the case when state constraints arise and the case when the space gradient of the value function presents some singularities. Take for example the advertising example with $\eta < 0$*. These cases do not fall in our assumptions but we think that our procedure can be extended to cover them in a future work.* ∎

9 Comparison with other verification results

In this section we compare our results with other known verification techniques for non smooth solution of the HJB equation. A discussion about this matter has already been done in Section 1.2. Here we give a more detailed analysis.

9.1 The classical result

We start by recalling that the classical verification theorems for stochastic optimal control problems (see e.g. [22, p.163]) adapted to our case are perfectly equal to our Theorems 5.5, 1.8 excepted for the following facts:

- they assume, in the finite horizon case, $v \in C^{1,2}([0,T] \times \mathbb{R}^n)$ instead of $v \in C^{0,1}([0,T] \times \mathbb{R}^n)$;
- they assume, in the infinite horizon case, $v \in C^2(\mathbb{R}^n)$ instead of $v \in C^1(\mathbb{R}^n)$.

In Section 7 we have seen an example where v is surely $C^{0,1}([0,T] \times \mathbb{R}^n)$ but it is not clear if it is also $C^{1,2}([0,T] \times \mathbb{R}^n)$. Moreover, in Section 8 we have seen two examples where v is surely $C^1(\mathbb{R}^n)$ and not $C^2(\mathbb{R}^n)$.

9.2 The approximation result

By approximation result we mean the verification theorem, proved e.g. in [30] in a special infinite dimensional case, that is proved using the approach that we have briefly recalled in Remark 4.6. The statement of such result is completely similar to our Theorems 5.5 or 1.8. The only difference is that to prove this theorem one needs to know that the function v is a strong solution of HJB equation in a more restrictive sense (we may call them "stronger" solutions) pointed out in Remark 4.6: it is required that also the space derivative of the approximating sequence converges uniformly on compact sets, i.e. $\partial_x u_n \to \partial_x u$. In our case in place of this we have a weaker requirement, namely that, or Hypothesis 5.3-(i), or 5.3-(ii) is satisfied. This means that for proving this kind of results we need to check a more restrictive condition more that may not be satisfied. For example:

- in Application 1 the fact that the solution is strong, follows directly from theorems on semigroups stated in [41]. The convergence of the space derivative in such cases is not trivial at all and may be not true.
- in Application 2, in the first one dimensional example, as pointed out in Remark 8.3, it is straightforward to prove that u is a strong solution in the sense of Definition 5.2 while it is not trivial to check that is a "stronger" solution as defined Remark 4.6. In the special examples of Subsection 8.2 this is true.

We finally point out that, to avoid the requirement of the convergence of the derivative on compact sets we need to use a completely different (and much more complex) approach based on the Fukushima-Dirichlet decomposition and on the representation result Theorem 4.5.

9.3 The viscosity solution result

When the HJB equation admits a viscosity solution it is possible to prove a verification theorem which is very general since it deals with only continuous solutions of HJB, and with the case when also the diffusion coefficient is controlled, but which is less useful when we know that we have strong solutions. The theorem, in the model case studied in this paper, is the following (see e.g. [34, 38, 57] for the proof and precise definition of superdifferentials $D^{1,2,+}_{t+,x} v(s, y^*(s))$ used here).

Theorem 9.1 *Consider the problem (2) - (3) with the following assumptions. The data F_0, F_1, B, l, ϕ are all uniformly continuous. Moreover they are Lipschitz continuous in the variable x uniformly with respect to the other variables. Moreover, for a suitable constant $M > 0$*

$$|F_0(t,0)|, |F_1(t,0,z)|, |B(t,0)|, |l(t,0,z)|, |\phi(0)| \leq M.$$

Finally the control space U is a Polish space and the set of admissible controls $\mathcal{Z}_{ad}(t)$ is given by all measurable and adapted processes on $[t,T]$ with values in U.

Let $v \in C^0([0,T] \times \mathbb{R}^n)$ be a viscosity solution of the HJB equation (8). Then:

- *$v(t,x) \leq V(t,x)$, for any $(t,x) \in [0,T) \times \mathbb{R}^n$.*
- *Let $(y^*(\cdot), z^*(\cdot))$ be a given admissible pair for the problem starting at (t,x). Suppose that there exists $(p^*, q^*, Q^*) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(t,T;S^n)$ such that for a.e.$s \in [t,T]$,*

$$(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s)), \ \mathbb{P} - a.s.$$

and

$$\begin{aligned} & p^*(s) - \frac{1}{2}\mathrm{Tr}\left[B^*\left(s, x^*(s)\right) Q^*\left(s\right) B\left(s, x^*(s)\right)\right] - \left\langle F_0\left(s, x^*(s)\right), q^*\left(s\right)\right\rangle \\ = \ & H^0_{CV}(s, x^*(s), q^*(s); z^*(s)) \\ = \ & H^0(s, x^*(s), q^*(s)), \qquad \mathbb{P} - a.s. \end{aligned}$$

then $(y^*(\cdot), z^*(\cdot))$ *is an optimal pair for the problem starting at* (t, x).

In the case we know that there exists a strong solution $v \in C^{0,1}$ then to apply the above theorem we need to know that v is also a viscosity solution. This is not difficult as the concept of viscosity solution is more general in a wide range of cases (this is true since classical solutions are always viscosity solutions and limits of viscosity solutions is still a viscosity solution, see e.g. [22]); if so then we know that it must be

$$q^*\left(s\right) = \partial_x v\left(s, y^*\left(s\right)\right).$$

The above sufficient condition states then that:

if there exists $(p^*, q^*, Q^*) \in L^2_{\mathcal{F}}(t, T; \mathbb{R}) \times L^2_{\mathcal{F}}(t, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(t, T; S^n)$ such that for a.e.$s \in [t, T]$,

$$(p^*(s), q^*\left(s\right), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s)), \ \mathbb{P} - a.s.$$

(so $q^*\left(s\right) = \partial_x v\left(s, y^*\left(s\right)\right)$) and

$$\begin{aligned} & p^*(s) - \frac{1}{2}\mathrm{Tr}\left[B^*\left(s, x^*(s)\right) Q^*\left(s\right) B\left(s, x^*(s)\right)\right] - \left\langle F_0\left(s, x^*(s)\right), \partial_x v\left(s, y^*\left(s\right)\right)\right\rangle \\ = \ & H^0_{CV}\left(s, x^*(s), \partial_x v\left(s, y^*\left(s\right)\right); z^*(s)\right) \\ = \ & H^0\left(s, x^*(s), \partial_x v\left(s, y^*\left(s\right)\right)\right), \ \mathbb{P} - a.s. \end{aligned}$$

then $(y^*(\cdot), z^*(\cdot))$ is an optimal pair for the problem starting at (t, x).

Our sufficient conditions instead states that:

if for a.e.$s \in [t, T]$, and

$$H^0_{CV}\left(s, x^*(s), \partial_x v\left(s, y^*\left(s\right)\right); z^*(s)\right) = H^0\left(s, x^*(s), \partial_x v\left(s, y^*\left(s\right)\right)\right), \ \mathbb{P} - a.s.$$

then $(y^*(\cdot), z^*(\cdot))$ is an optimal pair for the problem starting at (t, x).

The main difference is that in the Theorem 9.1 above we need to know about the existence of a couple of processes p^* and Q^* (that play the role of time derivative and second space derivative) which are in fact not needed in our statement. This is not a trivial difficulty in general. In particular our statement shows that, when strong solutions exist, one can get sharper sufficient conditions. Similar and even stronger considerations can be done for the necessary conditions and the existence of optimal feedbacks. In particular the analogous of the necessary condition stated by Proposition 6.2 in the viscosity setting says only that for every possible $(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s))$, $\mathbb{P} - a.s.$ (so $q^*\left(s\right) = \partial_x v\left(s, y^*\left(s\right)\right)$) we have

$$\begin{aligned} & p^*(t) - \frac{1}{2}\mathrm{Tr}\left[B^*\left(t, x^*(t)\right) Q^*\left(t\right) B\left(t, x^*(t)\right)\right] - \left\langle F_0\left(t, x^*(t)\right), \partial_x v\left(t, y^*\left(t\right)\right)\right\rangle \\ \leq \ & H^0_{CV}\left(t, x^*(t), \partial_x v\left(t, y^*\left(t\right)\right); z^*(t)\right) \end{aligned}$$

and this *do not* implie a priori that

$$H^0_{CV}\left(t, x^*(t), \partial_x v\left(t, y^*\left(t\right)\right); z^*\left(t\right)\right) = H^0\left(t, x^*(t), \partial_x v\left(t, y^*\left(t\right)\right)\right), \ \mathbb{P}-a.s$$

since it may happen that

$$\begin{aligned} & p^*(t) - \frac{1}{2}\mathrm{Tr}\left[B^*\left(t, x^*(t)\right) Q^*\left(t\right) B\left(t, x^*(t)\right)\right] - \left\langle F_0\left(t, x^*(t)\right), \partial_x v\left(t, y^*\left(t\right)\right)\right\rangle \\ < \ & H^0\left(t, x^*(t), \partial_x v\left(t, y^*\left(t\right)\right)\right) \end{aligned}$$

for all possible $(p^*(s), q^*(s), Q^*(s)) \in D^{1,2,+}_{t+,x} v(s, y^*(s))$, $\mathbb{P}-a.s.$. So in this case we can say that the difference between the two cases is wider.

Remark 9.2 *We must also note another point. The assumptions made in Theorem 9.1 about the data are much more restricitve than our assumptions made for the model problem of Section 7. We think that Theorem 9.1 could be extended to more general situations but this extension is not trivial and not known at this stage.*

Of course the importance of the verification theorem for viscosity solutions is that it covers the cases when the control enters also in the diffusion coefficients and when the viscosity solutions has no further regularity. So it is in a sense natural that such result is more restrictive than ours when applied to cases when the control enters only in the drift and a strong solution exists. ■

9.4 The backward equations result

This is a verification technique introduced for instance in [46] and [26] that is used when it is possible to find a mild solution to the HJB equation (in the sense recalled in Subsection 7.2) and when it can be represented via the solution of a forward - backward system, see on this [26, 42]. The most recent and complete result is given in [26] in the infinite dimensional case and we take the setting from it adapting the statement to our problem. Similar results in other context are given e.g. in [46] and references therein.

Consider then the problem (2) - (3) with the following assumptions.

1. The state space is $X = \mathbb{R}^n$ while the control space is U, a bounded subset of of $\mathbb{R}^m$.

2. The data F_0, B are continuously differentiable with bounded first derivative in the variable x uniformly with respect to t.

3. $F_1\left(t, x, z\right) = B\left(t, x\right) R\left(t, x\right) z$ where $R : [0, T] \times X \to L\left(U, X\right)$ is continuously differentiable in x, bounded with its space derivative.

4. l is continuous and polynomially growing.

5. ϕ is continuously differentiable with bounded first derivatives.

6. Defining the modified current value Hamiltonian $K_{CV}(t,x,q;z) = l(t,x,z) + \langle q,z\rangle_{\mathbb{R}^m}$ we assume that $K_{MAX} = \sup_{z\in U} K_{CV}(t,x,q;z)$ is measurable, continuously differentiable in (x,p) with bounded derivative with respect to p and with polynomially growing derivative with respect to x.

7. There exists a unique maximum point of the modified current value Hamiltonian K_{CV} given by $\Gamma(t,x,p)$ where Γ is continuous.

The result is the following.

Theorem 9.3 *Assume Hypotheses 1-7 above. Fix $(t,x)\in[0,T]\times X$. For all admissible control strategies z starting at (t,x) we have $J(t,x;z)\geq v(t,x)$ and the equality holds if and only if the following feedback law is verified by z and $y(\cdot;t,x,z)$:*

$$z(s) = \Gamma(s,y(s),R(s,y(s))^*G(s,y(s))^*\partial_x v(s,y(s))), \quad \mathbb{P}-a.s.\ for\ a.e.\ s\in[t,T]. \tag{58}$$

Finally there exists at least an admissible control strategy z starting at (t,x) for which (58) holds. In such a system the closed loop equation:

$$\begin{cases} dy(s) &= F_0(s,y(s)) + B(s,y(s))R(s,y(s))\Gamma_1(s,y(s))\,ds \\ &\quad +B(s,y(s))\,dW_s, \qquad s\in[t,T], \\ y(t) &= x\in X, \end{cases} \tag{59}$$

where we set $\Gamma_1(s,y(s)) = \Gamma(s,y(s),R(s,y(s))^*B(s,y(s))^*\nabla_x v(s,y(s)))$, admits a solution $\bar{y}$ and if $\bar{z}(s) = \Gamma(s,\bar{y}(s),R(s,\bar{y}(s))^*G(s,\bar{y}(s))^*\nabla_x v(s,\bar{y}(s)))$ then the couple $(\bar{z},\bar{y})$ is optimal for the control problem.

We point out the following.

- The dependence on the control in the state equation is linear and through the same operator B that drives the noise. This is not a casual restriction but it seems to be a limitation due to the technique used in the proof (mainly due to the use of Girsanov transform). It is not clear how to deal with more general cases as pointed out in [26, Section 7]. So, at present stage, we cannot apply this technique to the cases described in Section 7. Moreover the control set is bounded and this means that we cannot apply Theorem 9.3 to the two examples of Section 8.2.

- The regularity assumptions on the coefficients are quite strong and in any case quite less general than our assumptions. In particular continuous differentiability is always required also in the Hamiltonian (the assumption 6 is substantially a regularity assumption on H).

- The results quoted above is proven for an infinite dimensional case where mild solutions exist and Girsanov theorem can be applied in a suitable way. Here we give a finite dimensional version of it. Of course the infinite dimensional case imposes certain restrictions and probably some technical assumptions may be removed. However some assumptions that strongly limits the applicability of the result (linearity in the control, smoothness of the coefficients) seem to constitute a structural limit of the technique used there.

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