Existence of Gibbs State for Continuous Gas with Many-Body Interaction

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Abstract

A continuous infinite system of point particles interacting via finite-range many-body potentials of superstable type is considered in the framework of classical statistical mechanics. We prove that for any temperature and chemical activity there exists at least one Gibbs state.

1 Introduction

One of the basic problems of equilibrium statistical mechanics is the construction of Gibbs states for continuous particle systems with many-body interactions. In the pioneering works by W. Greenberg [7] and H. Moral [13] the problem was analyzed via Kirkwood-Salsburg equations(KSE). For sufficiently small activity parameter z they proved existence of unique solution of KSE, but with rather unnatural assumptions on the potentials which, in fact, take place only for finite range and positive interactions. In [17] the convergence of the Brydges-Federbush type cluster expansion is proved for dilute continuous systems with n-body ($n \leq M$) interaction. The proof requires a stable potential satisfying an integrability condition and exponential decay of the many-body potentials at large distances. In the following paper [15] the authors consider the system of hard-core spheres interacting via infinite group of many body potentials (for all n) which are bounded and integrable. They prove the convergence of the Mayer series for the pressure in thermodynamic limit and establish the region of analyticity in the activity z. In the recent work by V.Belitsky and E.A.Pechersky [3] the problem of existence and uniqueness of Gibbs state in $\mathbb{R}^d$ with finite group of n-body interactions was investigated using the technique of Dobrushin's type [4], [5].

In this work we give a simple proof of the existence of Gibbs state with infinite group of many body potentials. We establish some kind of modified Ruelle's bound for finite volume correlation functions. It gives a possibility to prove existence of at least one Gibbs measure in thermodynamic limit. We consider these results as some further development in solving the problem.

In the next section we define the system and formulate main results. Section 3 is devoted to the proof of these results. The basic technical lemma is outlined in the Appendix.

2 Correlation functions

2.1 Configuration space

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. By $\mathcal{O}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all open and Borel sets, respectively. $\mathcal{O}_c(\mathbb{R}^d)$, $\mathcal{B}_c(\mathbb{R}^d)$ denote the systems of all sets in $\mathcal{O}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$, respectively, which are bounded.

The set of positions $\{x_i\}_{i\in\mathbb{N}}$ of identical particles is considered to be a locally finite subset in $\mathbb{R}^d$ and the set of all such subsets creates the configuration space:

$$\Gamma = \Gamma_{\mathbb{R}^d} := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\},$$

where $|A|$ denotes the cardinality of the set A. The symbol $|\cdot|$ may also represent the Lebesgue measure of the set, but the meaning will always be clear from the context. For any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ we denote by γ_Λ the projection of γ on Λ and the corresponding configuration space by Γ_Λ. We also need to define the space of finite configurations Γ_0:

$$\Gamma_0 = \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping $N_\Lambda : \Gamma \to \mathbb{N}_0$ of the form

$$N_\Lambda(\eta) := |\eta \cap \Lambda|.$$

The Borel σ-algebra $\mathfrak{B}(\Gamma)$ is equal to $\sigma(N_\Lambda \,|\Lambda \in \mathfrak{B}_c(\mathbb{R}^d))$ and additionally one may introduce the following filtration

$$\mathfrak{B}_\Lambda(\Gamma) := \sigma(N_{\Lambda'} \,|\Lambda' \in \mathcal{B}_c(\mathbb{R}^d), \ \Lambda' \subset \Lambda),$$

see [10], [11] for details.

By $\mathfrak{B}(\cdot)$ we denote the corresponding σ-algebras on Γ_Λ and Γ_0. For a given intensity measure $\sigma = zdx$ $(z > 0)$ on $\mathcal{B}(\mathbb{R}^d)$ and any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \middle| \, x_k \neq x_l \text{ if } k \neq l \right\}$$

and hence as a measure $\sigma^{(n)}$ on $\Gamma^{(n)}$ through the map

$$sym_n : \widetilde{(\mathbb{R}^d)^n} \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\} \in \Gamma^{(n)},$$

c.f. [8]. For simplicity we will write $(x)_n$ instead of $\{x_1, ..., x_n\} \in \Gamma^{(n)}$.

Define the Lebesgue-Poisson measure λ_σ on $\mathfrak{B}(\Gamma_0)$ by the formula:

$$\lambda_\sigma := \sum_{n \geq 0} \frac{1}{n!} \sigma^{(n)}.$$

The restriction of λ_σ to $\mathfrak{B}(\Gamma_\Lambda)$ we also denote by λ_σ. For a more detailed structure of the configuration spaces Γ, Γ_0, Γ_Λ see [1].

2.2 Interactions and Hamiltonians

We consider a general type of many-body interaction specified by a family of k-body potentials $V_k : \mathbb{R}^{dk} \to \mathbb{R}$, $k \geq 2$. About the potentials $\{V_k\}_{k \geq 2}$ we will assume:

A1. Finite range. There exists a constant $R > 0$, such that for any $k \geq 2$

$$V_k(x_1, ..., x_k) \equiv 0, \text{ if } \operatorname{diam}\{x_1, ..., x_k\} > R.$$

A2. Continuity.

$$V_k \in C(\widetilde{(\mathbb{R}^d)^k}), \ k \geq 2.$$

A3. Symmetry. For any $k \geq 2$, any $(x_1, ..., x_k) \in (\mathbb{R}^d)^k$, and any permutation π of numbers $\{1, \dots, k\}$

$$V_k(x_1, ..., x_k) = V_k(x_{\pi(1)}, ..., x_{\pi(k)}).$$

A4. Translation invariance. For any $k \geq 2$, any $(x_1, ..., x_k) \in (\mathbb{R}^d)^k$, and any $a \in \mathbb{R}^d$

$$V_k(x_1, ..., x_k) = V_k(x_1 + a, ..., x_k + a).$$

We are able now to introduce the Hamiltonian $U^V : \Gamma_0 \to \mathbb{R}\cup\{\infty\}$, which corresponds to the family of potentials $V := \{V_k\}_{k\geq 2}$ and which is defined by

$$U^V(\eta) = \sum_{k\geq 2} \sum_{\{x_1,...,x_k\}\subset\eta} V_k(x_1,...,x_k),\ \ \eta\in\Gamma_0,\ \ |\eta|\geq 2.$$

For the fixed family of potentials V we will write for short $U = U^V$ and for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ we will sometimes write $U_\Lambda(\eta)$ instead of $U(\eta)$.

A5. Strong Superstability. For any $k \geq 2$ the potential V_k can be represented as

$$V_k = V_k^+ + V_k^{(st)},$$

where V_k^+ is a nonnegative function such that for any $(x_1,\ldots,x_k) \in (\mathbb{R}^d)^k \backslash \widetilde{(\mathbb{R}^d)^k}$

$$V_k^+(x_1,\ldots,x_k) = +\infty,$$

and $V_k^{(st)}$ is stable, i.e. there exists a constant $B \geq 0$ such that for any configuration $\eta \in \Gamma_0$ holds

$$U^{V^{(st)}}(\eta) \geq -B|\eta|.$$

Let $\lambda \in \mathbb{R}_+$ be arbitrary. For each $r \in \mathbb{Z}^d$ we define an elementary cube

$$\Delta(r) = \{x \in \mathbb{R}^d \mid \lambda(r^i - 1/2) \leq x^i < \lambda(r^i + 1/2)\}.$$

These cubes form a partition of $\mathbb{R}^d$, which we denote by $\bar{\Delta}_\lambda$. We will sometimes write Δ instead of $\Delta(r)$, if a cube Δ is considered to be arbitrary and there is no reason to emphasize that it is centered at the concrete point $r \in \mathbb{Z}^d$. By $\mathcal{J}_\lambda(\mathbb{R}^d)$ we denote all finite unions of cubes of the form $\Delta(r)$ (such sets are used in the construction of the Jordan measure).

Let $k \geq 2$ and $N \in \mathbb{N}$ be arbitrary. For any $X_N = \cup_{j=1}^N \Delta_j \in \mathcal{J}_\lambda(\mathbb{R}^d)$ we define

$$I_k^{k_1,\ldots,k_N|\bar{k}}(\Delta_1,...,\Delta_N) :=$$

$$= \sum^{*}_{\substack{\Delta'_j\subset X_N^c,\\ 1\leq j\leq\bar{k}}} \sup_{\substack{(x)^i_{k_i}\subset\Delta_i,\,1\leq i\leq N,\\ y_1\in\Delta'_1,\ldots,y_{\bar{k}}\in\Delta'_{\bar{k}}}} |V_k^{(st),-}(x_1^1,\ldots,x_{k_N}^N, y_1,...,y_{\bar{k}})|, \tag{1}$$

and

$$v_k^{k_1,\dots,k_N}\Big(\Delta_1,\dots,\Delta_N\Big) := \inf_{(x)^i_{k_i}\subset\Delta_i,\,1\le i\le N,} V_k^+(x_1^1,\dots,x_{k_N}^N), \qquad (2)$$

where $\bar{k}\geq 1$ (equal to zero in (2)), $k_i\in\mathbb{N}_0$, $(x)^i_{k_i}=\{x_1^i,\dots,x^i_{k_i}\}$, $1\le i\le N$ such that $k_1+\dots+k_N\geq 1$ and $k_1+\dots+k_N+\bar{k}=k$. $V_k^{(st),-}$ denotes the negative part of $V_k^{(st)}$, and the symbol $\sum^*$ means that the sum extends only over different cubes, i.e. $\Delta_i'\neq\Delta_j'$, $i\neq j$, $1\le i,\,j\le\bar{k}$.

A6. Attraction-Repulsion relation. There exists $\lambda=\lambda_0>0$, such that for any $N\in\mathbb{N}$ and any $X_N=\cup_{j=1}^N\Delta_j\in\mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ (we omit dependence on the cubes in the notations of (1) and (2)) the following holds

- for an arbitrary $\Delta\in\bar{\Delta}_{\lambda_0}$ and any $k\geq 2$

$$V_k(x_1,\dots,x_k)\geq 0, \quad \{x_1,\dots x_k\}\subset\Delta$$

-

$$v_k^{k_1,\dots,k_N}\geq 8(\bar{I}^{(N)}_{k;\,k_1,\dots,k_N}+B),$$

and

$$\bar{I}^{(N)}=\sum_{k\geq 2}\bar{I}^{(N)}_k<\infty, \quad N\in\mathbb{N} \qquad (3)$$

where

$$\bar{I}^{(N)}_{k;\,k_1,\dots,k_N}=\sum_{l\geq 1} l I_{k+l}^{k_1,\dots,k_N|\,l}, \quad \bar{I}^{(N)}_k=\sum_{l\geq 1} l I_{k+l}^{1,\dots,1|k+l-N},$$

$k\geq N+1$, $k_1+\dots+k_N=k$.

In the sequel we write $\bar{\Delta}$ instead of $\bar{\Delta}_{\lambda_0}$.

Remark 2.1 *By the definition, $V_k^{st,-}$ describes attractive part of k-body interaction. Therefore, $I_k^{k_1,\dots,k_N|\bar{k}}(\Delta_1,\dots,\Delta_N)$ describes only attractive part of k-body interaction of fixed particles in cubes $\Delta_1,\dots,\Delta_N$ with "dilute configuration", i.e. no more then one particle is located in any cube Δ from $X_N^c=\mathbb{R}^d\setminus X_N$, $X_N=\cup_{j=1}^N\Delta_j$. Then, condition (3) means that the energy of*

k-body interaction decreases sufficiently fast with k. From the definition (2) and assumption **A6**, *it is clear that at least one cube from $\Delta_1, ..., \Delta_N$ contains more than one particle, and so $v_k^{k_1,\ldots,k_N}$ should be greater than contributions of all $k+l$-body attractive energies of interaction ($l \in N$) for sufficiently small λ.*

For a given $\bar{\gamma} \in \Gamma$ define the interaction energy between $\eta \in \Gamma_\Lambda$, $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $\bar{\gamma}_{\Lambda^c} = \bar{\gamma} \cap \Lambda^c$, $\Lambda^c = \mathbb{R}^d \setminus \Lambda$ as

$$W_\Lambda(\eta \mid \bar{\gamma}) = \sum_{k\geq 2} \sum_{\substack{m+n=k \\ m,n\geq 1}} \sum_{\substack{\{x_1,\ldots,x_m\}\subset\eta \\ \{y_1,\ldots,y_n\}\subset\bar{\gamma}_{\Lambda^c}}} V_k(x_1,\ldots,x_m,y_1,\ldots,y_n).$$

Define

$$U_\Lambda(\eta|\bar{\gamma}) = U_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma}).$$

A7. The order of interaction. For any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, $\eta \in \Gamma_\Lambda$ and $\bar{\gamma} \in \Gamma$ the interaction energy $W_\Lambda(\eta \mid \bar{\gamma})$ does not become $-\infty$ and the partition function

$$Z_\Lambda(\bar{\gamma}) = \int_{\Gamma_\Lambda} \exp\{-U_\Lambda(\eta|\ \bar{\gamma})\}\lambda_\sigma(d\eta) < \infty.$$

2.3 Gibbs specification and correlation functions.

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and let $\bar{\gamma} \in \Gamma$. The finite volume Gibbs state with boundary configuration $\bar{\gamma}$ for U, $z > 0$ and $\beta > 0$ is

$$\mu_\Lambda(d\eta|\ \bar{\gamma}) = \frac{\exp\{-\beta U_\Lambda(\eta|\ \bar{\gamma})\}}{Z_\Lambda(\bar{\gamma})}\lambda_{z\sigma}(d\eta).$$

Under assumption **A7**, the finite volume Gibbs state is well defined. When $\bar{\gamma} = \emptyset$, let $\mu_\Lambda(d\eta|\emptyset) \equiv \mu_\Lambda(d\eta)$.

The corresponding finite-volume correlation functions for boundary configuration $\bar{\gamma} \in \Gamma$ have the following form

$$\rho^\Lambda(\eta \mid \bar{\gamma}) = \frac{1}{Z_\Lambda(\bar{\gamma})} \int_{\Gamma_\Lambda} e^{-\beta U(\eta\cup\gamma|\bar{\gamma})}\lambda_\sigma(d\gamma), \quad \eta \in \Gamma_\Lambda. \tag{4}$$

Let $\{\pi_\Lambda\}$ denote the specification associated with z, β and the Hamiltonian U (see [14]), which is defined on Γ by

$$\pi_\Lambda(A|\ \bar{\gamma}) = \int_{A'} \mu_\Lambda(d\eta|\ \bar{\gamma}),$$

where $A' = \{\eta \in \Gamma_\Lambda : \ \eta \cup (\bar{\gamma}_{\Lambda^c}) \in A\}$, $A \in \mathfrak{B}(\Gamma)$.

A probability measure μ on Γ is called a Gibbs state for U, β and z if

$$\mu(\pi_\Lambda(A|\ \bar{\gamma})) = \mu(A)$$

for every $A \in \mathfrak{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

This relation is the well known (DLR)-equation (Dobrushin-Lanford-Ruelle equation), see [6] for more details. The class of all Gibbs states which correspond to the specifications $\{\pi_\Lambda\}_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)}$ we denote by $\mathcal{G}(V, z, \beta)$.

2.4 Main results.

Theorem 2.1 *Suppose that the interaction family V satisfies the assumptions* **A1-A7**. *Then, for any $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ and any $\beta, z \geq 0$ there exists a constant $\xi = \xi(\beta, z)$ (independent of Λ) such that the finite volume correlation function $\rho^\Lambda(\eta) = \rho^\Lambda(\eta \mid \emptyset)$ satisfies the following inequality*

$$\rho^\Lambda(\eta) \leq \xi^{|\eta|} e^{-\frac{1}{2}U^+(\eta)}, \ \eta \in \Gamma_\Lambda. \tag{5}$$

Remark 2.2 *The estimate (5) without exponent factor at the right-hand side is the well-known Ruelle bound [18]. We call (5) a generalized Ruelle bound. For 2-body interaction it was obtained in [1], [16].*

As a consequence of Theorem 2.1 the following theorem is fulfilled.

Theorem 2.2 *Let the interaction family V satisfy* **A1-A7**. *Then for any $z \geq 0$ and $\beta \geq 0$*

$$\mathcal{G}(V, z, \beta) \neq \emptyset.$$

Proof. Existence of the corresponding Gibbs state follows from the arguments which are based on the following observation. Let $\psi \in L^1(\mathbb{R}^d)$ be any positive function such that $\psi(x) \leq 1$, $x \in \mathbb{R}^d$, and let $\alpha(t)$, $t \in \mathbb{R}_+$ be any continuous decreasing function with the following conditions:

(1) $\alpha_0 := \lim_{t \to 0+} \alpha(t) = +\infty$;

(2) $\alpha_+ := \lim_{t \to +\infty} \alpha(t) \geq 1$;

Define,

$$\Gamma_\infty = \left\{ \gamma \in \Gamma \;\middle|\; \sum_{\{x,y\} \subset \gamma} \psi(x)\alpha(|x-y|)\psi(y) < \infty \right\}$$

and

$$E_\psi^\alpha(\gamma) = \sum_{\{x,y\} \subset \gamma} \psi(x)\alpha(|x-y|)\psi(y), \;\; \gamma \in \Gamma_\infty.$$

As shown in [9], for any $0 < D < \infty$ the set

$$\left\{ \gamma \in \Gamma \mid |E_\psi^\alpha(\gamma)| \leq D \right\}$$

is precompact in Γ, which is Polish space.

In this paper we consider α as any continuous decreasing function such that

$$\alpha(|x-y|) \leq e^{\frac{1}{2}V_2^+(x,y)}.$$

Obviously, chosen in such a way, this function satisfies the conditions above. Using the properties of the so-called K-transform (see [8]) and the Theorem 2.1, for any $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ we have

$$\int_\Gamma E_\psi^\alpha(\gamma) d\mu_\Lambda(\gamma) = \int_{\mathbb{R}^{2d}} \psi(x)\alpha(|x-y|)\psi(y)\rho_\Lambda^{(2)}(\{x,y\}) dx dy < C,$$

where $C \in \mathbb{R}_+$ is some constant.

Therefore, by Prokhorov theorem the family of measures

$$\{\mu_\Lambda \mid \Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)\}$$

is precompact, which implies the existence of at least one limit measure μ when $\Lambda \nearrow \mathbb{R}^d$. We will prove that corresponding limit measure is Gibbsian. Let μ_{Λ_n}, $n \geq 1$, where $\Lambda_n \nearrow \mathbb{R}^d$, $n \to \infty$ be the sequence which converges (in the sense of the Prokhorov theorem) to the measure μ, and let ρ^{Λ_n}, ρ be the corresponding correlation functions. It is well-known (see [6]) that probability measure μ on Γ is Gibbs, iff μ fulfills the *Georgii-Nguyen-Zessin* equation (GNZ), i.e. for all positive, $\mathcal{B}(\mathbb{R}^d) \times \mathfrak{B}(\Gamma)$ measurable functions H the following holds

$$\int_\Gamma \sum_{x\in\gamma} H(x,\gamma)\mu(d\gamma) = \int_\Gamma \int_{\mathbb{R}^d} H(x, \gamma \cup \{x\}) e^{-\beta W(\{x\}|\gamma)} \sigma(dx)\mu(d\gamma). \tag{6}$$

Moreover, using Mecke formula (see [6]), one can show that (6) holds for any measure μ_{Λ_n}, $n \geq 1$.

Let $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. The σ-algebra $\mathfrak{B}(\Gamma)$ is generated by sets of the form $A \cap \tilde{A}$ with $A \in \mathfrak{B}_\Lambda(\Gamma)$, $\tilde{A} \in \mathfrak{B}_{\mathbb{R}^d\setminus\Lambda}(\Gamma)$ and every measure on Γ is uniquely determined by its values on these sets.

Let us prove (6) for the function $H(x,\gamma) = 1\!\!1_\Lambda(x) 1\!\!1_A(\gamma) 1\!\!1_{\tilde{A}}(\gamma)$. Let $n \in \mathbb{N}$ be arbitrary. Using the properties of the K-transform (see [8]) we have

$$\int_{\Gamma_{\Lambda_n}} \sum_{x\in\gamma} 1\!\!1_\Lambda(x) 1\!\!1_A(\gamma) 1\!\!1_{\tilde{A}}(\gamma) \mu_{\Lambda_n}(d\gamma) \leq \int_{\Gamma_{\Lambda_n}} \sum_{x\in\gamma} 1\!\!1_\Lambda(x) \mu_{\Lambda_n}(d\gamma) =$$

$$= \int_\Lambda \rho^{\Lambda_n}(x)\sigma(dx) \leq z\xi|\Lambda|. \tag{7}$$

The right hand side of (6) for the measure μ_{Λ_n} is bounded by

$$\int_{\mathbb{R}^d} 1\!\!1_\Lambda(x) \int_{\Gamma_{\Lambda_n}} e^{-\beta W(\{x\}|\gamma)} \mu_{\Lambda_n}(d\gamma)\sigma(dx) =$$

$$= \int_{\mathbb{R}^d} 1\!\!1_\Lambda(x) \rho^{\Lambda_n}(x)\sigma(dx) \leq z\xi|\Lambda|, \tag{8}$$

where we have used the definition of the correlation function and Fubini theorem. Hence, there exists some subsequence $\{\mu_{\Lambda_{n_k}}\}_{k\geq 1}$ which ensures the fulfillment of (6) for the limit measure μ. The proof for the general positive function H follows from the fact that any positive measurable function can be approximated by the simple functions.

3 The proof of Theorem 2.1.

The proof is based on the expansion of the Lebesgue-Poisson integral for the correlation functions (4) into the series over some kind of dense configurations (see [16] and definition (3.4) therein).

3.1 Cluster expansion in densities of configurations.

The main idea of the construction consists in the use of the fact that if two or more particles are in one elementary cube $\Delta \in \bar{\Delta}$ then Gibbs factor $\exp[-\beta V_2(x_i, x_j)] \sim \exp[-\beta b]$, where

$$b = \inf_{\Delta \in \bar{\Delta}} \quad \inf_{x_1, x_2 \in \Delta} V_2^+(x_1, x_2) \tag{9}$$

and $b \to \infty$, when $\lambda \to 0$. The configurations with this property will be called *dense* configurations, as opposed to *dilute* configurations, in which no more than one particle is situated in any cube. The main technical idea consists in separation of the dilute parts of configurations from the dense parts. In order to do this we define an indicator function for the configuration γ_Λ, $\Lambda \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$ in the cube Δ:

$$\chi_n^\Delta(\gamma_\Lambda) \;=\; \chi_n^\Delta(\gamma_\Delta) \;=\begin{cases} 1, & \text{for } |\gamma_\Delta| = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then the indicator for *dilute* configurations is defined as

$$\chi_-^\Delta(\gamma_\Delta) \;=\; \chi_0^\Delta(\gamma_\Delta) + \chi_1^\Delta(\gamma_\Delta)$$

and for *dense* configurations as

$$\chi_+^\Delta(\gamma_\Delta) \;=\; \sum_{n \geq 2} \chi_n^\Delta(\gamma_\Delta).$$

To obtain decomposition we use the following partition of the unity:

$$1 \;=\; \prod_{\Delta \subset \Lambda} \left[\chi_-^\Delta(\gamma_\Delta) + \chi_+^\Delta(\gamma_\Delta)\right] \;=\; \sum_{\omega} \prod_{\Delta \subset \Lambda} \chi_{\omega(\Delta)}^\Delta(\gamma_\Delta), \tag{10}$$

where ω is the map from $\bar{\Delta} \cap \Lambda := \{\Delta \in \bar{\Delta} \,:\, \Delta \subset \Lambda\}$ into the set $\{+,\, -\}$, such that $\omega(\Delta) = +$ or $-$ for any $\Delta \in \bar{\Delta} \cap \Lambda$. Inserting (10) into (4) for $\bar{\gamma} = \emptyset$, we get

$$\rho^{\Lambda}(\eta) \;=\; \frac{1}{Z_{\Lambda}} \sum_{\omega} \int_{\Gamma_{\Lambda}} \prod_{\Delta \subset \Lambda} \chi^{\Delta}_{\omega(\Delta)}(\gamma_{\Delta}) e^{-\beta U(\eta \cup \gamma)} \lambda_{\sigma}(d\gamma), \tag{11}$$

where $Z_{\Lambda} = Z_{\Lambda}(\emptyset)$. Now we define the set

$$X = \bigcup_{\Delta \subset \Lambda \,:\, \omega(\Delta) = +} \Delta.$$

Then the sum over ω can be rewritten as the sum over all possible sets X in Λ. Namely,

$$\rho^{\Lambda}(\eta) \;=\; \frac{1}{Z_{\Lambda}} \sum_{\emptyset \subseteq X \subseteq \Lambda} \int_{\Gamma_{\Lambda}} \widetilde{\chi}^{X}_{+}(\gamma) \widetilde{\chi}^{X^c}_{-}(\gamma) e^{-\beta U(\eta \cup \gamma)} \lambda_{\sigma}(d\gamma),$$

where

$$\widetilde{\chi}^{X}_{\pm}(\gamma) \;=\; \prod_{\Delta \subset X} \chi^{\Delta}_{\pm}(\gamma_{\Delta})$$

For any $X \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$, $X \subseteq \Lambda$ define graph $G_R(X)$ with vertices in the centers of all elementary cubes $\Delta \subset X$ and lines $l(\Delta, \Delta')$ iff $dist(\Delta, \Delta') \leq R$. The number of lines depends on graph $G_R(X)$.

Definition 3.1 *The set X is called R-connected if the corresponding graph $G_R(X)$ is connected in ordinary way.*

R-connected set X is denoted by X^R. Then, every set X can be represented as some fixed partition

$$\{X\}^R_n := \{X^R_1, \ldots, X^R_n \mid \mathrm{dist}(X^R_i, X^R_j) > R, \quad \text{for} \quad i \neq j\},$$

and so the sum over all possible X in Λ can be rewritten as the sum over all possible sets $\{X\}^R_n$ (for $n = 0$, $X = \emptyset$). Furthermore, we replace the sum over all such sets by the sum over $X^R_1, \ldots, X^R_n$ independently, and remove the conditions $dist(X^R_i, X^R_j) > R$ by introducing the *hard-core* potential

$$\chi_R^{cor}(X)_n = \begin{cases} 0, & \text{there exist } X_i^R,\, X_j^R,\ i \neq j,\ dist(X_i^R,\, X_j^R) \leq R, \\ 1, & \text{otherwise.} \end{cases}$$

Then we get

$$\rho^\Lambda(\eta) = \frac{1}{Z_\Lambda} \sum_{n \geq 0} \frac{1}{n!} \sum_{X_1^R \subseteq \Lambda} \cdots \sum_{X_n^R \subseteq \Lambda} \chi_R^{cor}(X)_n \times$$

$$\times \int_{\Gamma_\Lambda} \widetilde{\chi}_+^X(\gamma) \widetilde{\chi}_-^{X^c}(\gamma) e^{-\beta U(\eta \cup \gamma)} \lambda_\sigma(d\gamma). \tag{12}$$

In the sequel, having in mind only R-connected components of X, we drop index R in the notation X_i^R, and summation $\sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda}$, for simplicity, will be denoted by $\sum_{(X)_n}$. Now, the last step in arranging our decomposition is as follows. Define the set

$$X_0 = \bigcup_{\Delta \subset \Lambda \,:\, dist(\Delta,\, \eta) \leq R} \Delta.$$

This set is fixed for fixed variable of the correlation function $\rho^\Lambda(\eta)$. Now, for every $n \geq 0$ we split the sum over $(X)_n$ into two sums. The first one is over those X_j, which do not intersect the region X_0 and the second one over those which intersect X_0. To distinguish the sets X_j which do not intersect and do intersect X_0, the latter sets are denoted by Y_j. There are $n!/k!(n-k)!$ possibilities when any k sets X_j do not intersect X_0 and $(n-k)$ sets Y_j intersect X_0. So the final expansion is the following:

$$\rho^\Lambda(\eta) = \frac{1}{Z_\Lambda} \sum_{n \geq 0} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{(X)_k} \sum_{(Y)_{n-k}} \chi_R^{cor}((X)_k, (Y)_{n-k}) \times$$

$$\times \int_{\Gamma_\Lambda} \lambda_\sigma(d\gamma) \widetilde{\chi}_+^X(\gamma) \widetilde{\chi}_-^{X^c}(\gamma) e^{-\beta U(\eta \cup \gamma)}, \tag{13}$$

where

$$X = \widetilde{X}_k \cup \widetilde{Y}_{n-k} := \left[\bigcup_{i=1}^{k} X_i \right] \bigcup \left[\bigcup_{j=1}^{n-k} Y_j \right].$$

3.2 The main estimates.

As the first step, let us split the exponent in (13) into four parts: the part which corresponds to the positive part of the energy of the configuration η , the interactions of the particles inside the region $X_0 \cup \widetilde{Y}_{n-k}$, inside $\Lambda \setminus (X_0 \cup \widetilde{Y}_{n-k})$ and interactions between them. Note that interaction between $X_0 \cup \widetilde{Y}_{n-k}$ and $\widetilde{X}_k$ is zero due to the finite range of potential. Therefore, considering $\gamma \in \Gamma_\Lambda \ : \ \gamma \cap \eta = \emptyset$ we get

$$e^{-\beta U(\eta \cup \gamma)} = e^{-\beta U^+(\eta)} E_1 E_2 E_0,$$

where

$$E_1(X_0 \cup \widetilde{Y}_{n-k}) = e^{-\beta U^{st}(\eta)} \prod_{l=1}^{n-k} e^{\beta W(\eta \,|\, \gamma_{Y_l}) - \frac{1}{2}\beta U^+(\gamma_{Y_l}) - \beta U^{st}(\gamma_{Y_l})},$$

$$E_2(X_0 \cup \widetilde{Y}_{n-k} \mid (X_0 \cup X)^c) = e^{-\beta W(\eta \,|\, \gamma_{X_0 \setminus \tilde{Y}_{n-k}})} \prod_{l=1}^{n-k} e^{-\beta[\frac{1}{2}U^+(\gamma_{Y_l}) + W(\gamma_{Y_l} \,|\, \gamma_{X^c})]},$$

and

$$E_0(\widetilde{Y}^c_{n-k}) = e^{-\beta U(\gamma_{\Lambda \setminus \tilde{Y}_{n-k}})}.$$

Lemma 3.1

$$E_1 \leq e^{\beta B|\eta|} \prod_{l=1}^{n-k} \prod_{\Delta \subset Y_l} e^{\beta B|\gamma_\Delta| - \frac{1}{2}\beta U^+(\gamma_\Delta)}. \tag{14}$$

Proof. Using **A5** we have

$$U^{(st)}(\eta \cup \gamma_{\widetilde{Y}_{n-k}}) \geq -B\Big(|\eta| + \sum_{l=1}^{n-k} \sum_{\Delta \subset Y_l} |\gamma_\Delta|\Big)$$

and

$$W^+(\eta \mid \gamma_{\widetilde{Y}_{n-k}}) \geq 0, \quad U^+(\gamma_{Y_l}) \geq \sum_{\Delta \subset Y_l} U^+(\gamma_\Delta). \quad \blacksquare$$

Lemma 3.2 *For any* $\gamma \in \Gamma$ *and* $\bar{\gamma} \in \bar{\Gamma}_{X^c}$, $X \in \mathcal{J}_{\lambda_0}(\mathbb{R}^d)$, $X \subseteq \Lambda$

$$\frac{1}{4}U^+(\gamma_X) + W(\gamma_X \mid \bar{\gamma}) \geq -\bar{I}|\gamma_X|, \tag{15}$$

where $\bar{I} := \bar{I}^{(1)}$ *(see (3)), and*

$$\bar{\Gamma}_{X^c} = \{\gamma \in \Gamma_{X^c} \mid \ |\gamma \cap \Delta| \leq 1, \text{ for all } \Delta \subset X^c\}$$

Proof. See Appendix. ■

Let us define

$$\partial\eta = \bigcup_{\Delta\,:\,\eta\cap\Delta\neq\emptyset} \Delta.$$

Now using the property of infinite divisibility of measure λ_σ and estimate (15) we can calculate the part of integral in (13)

$$e^{-\frac{1}{2}\beta U^+(\eta)} \int_{\Gamma_{\widetilde{Y}_{n-k}}} \widetilde{\chi}_+^{\widetilde{Y}_{n-k}}(\gamma) E_1 E_2 \lambda_\sigma(d\gamma) \leq$$

$$\leq e^{-\frac{1}{4}\beta U^+(\eta)+\beta|\eta|\bar{I}} \int_{\Gamma_{\widetilde{Y}_{n-k}}} \widetilde{\chi}_+^{\widetilde{Y}_{n-k}}(\gamma) e^{-\beta W(\eta \mid \gamma_{(X_0\cap\partial\eta)\setminus\widetilde{Y}_{n-k}})} E_1 \times$$

$$\times \prod_{l=1}^{n-k} e^{-\beta[\frac{1}{2}U^+(\gamma_{Y_l})+W(\gamma_{Y_l} \mid \gamma_{X^c})]} \lambda_\sigma(d\gamma). \tag{16}$$

Assumption **A6**, estimate (14) and trivial inequality

$$U^+(\eta) \geq \sum_{\Delta\subset\partial\eta} U^+(\eta_\Delta)$$

gives us the bound for the integral (16)

$$e^{\beta|\eta|(\bar{I}+B)+\beta\sum_{\Delta\subset\partial\eta}\bar{I}|\eta_\Delta|} \prod_{l=1}^{n-k} \prod_{\Delta\subset Y_l} I_\Delta,$$

where

$$I_\Delta = \int_{\Gamma_\Delta} \chi_+^\Delta(\gamma_\Delta) e^{-\beta\frac{1}{2}U^+(\gamma_\Delta)+\beta(B+\bar{I})|\gamma_\Delta|} \lambda_\sigma(d\gamma). \tag{17}$$

Focusing only on the 2-body positive part of interaction and taking into account the definition (9) we can estimate the last integral by

$$I_\Delta \le \varepsilon_1 = \frac{1}{2} z^2 \lambda_0^{2d} e^{-\beta(\frac{1}{2}b - 2\bar{I} - 2B)} \exp\{z \lambda_0^d e^{-\beta(\frac{3}{2}b - \bar{I} - B)}\}, \tag{18}$$

which is finite due to **A6**.

Now taking the maximum of E_0 in variable $\widetilde{Y}_{n-k}$ (we denote this maximum by $\bar{Y}_{n-k}$) and using elementary estimate

$$\chi_R^{cor}((X)_k, (Y)_{n-k}) \le \chi_R^{cor}(X)_k \tag{19}$$

we can estimate the sum over $(Y)_{n-k}$ by the following lemma:

Lemma 3.3 *(e.g. [12])*

$$\sum_{Y \cap X_0 \neq \emptyset} \varepsilon_1^{\frac{|Y|}{\lambda^d}} \le |\eta| c(d) \left(\frac{R}{\lambda}\right)^d \frac{\varepsilon}{1-\varepsilon} = |\eta| K, \tag{20}$$

where $c(d)$ is a constant which depends only on d and $\varepsilon = 4c(d) \left(\frac{R}{\lambda}\right)^d \varepsilon_1$.

For the *proof* in our case see [16].

The last step is as follows. The expansion like (12) can be constructed for partition function Z_{Λ_1} with $\Lambda_1 \subset \Lambda$. Denote it by

$$Z_{\Lambda_1} = \sum_{k \ge 0} \frac{1}{k!} Z_{\Lambda_1}^{(k)}. \tag{21}$$

Taking into account all previous estimates we get

$$\rho^\Lambda(\eta) \le \frac{1}{Z_\Lambda} e^{-\frac{1}{2}\beta U^+(\eta_\Lambda) + \beta(2\bar{I}+B)|\eta|} \sum_{n \ge 0} \sum_{k=0}^{n} \frac{(|\eta| K)^{n-k}}{k!(n-k)!} Z^k_{\Lambda \setminus \bar{Y}_{n-k}} =$$

$$= \frac{1}{Z_\Lambda} e^{-\frac{1}{2}\beta U^+(\eta_\Lambda) + \beta(2\bar{I}+B)|\eta|} \sum_{k \ge 0} \frac{1}{k!} \sum_{l \ge 0} \frac{(|\eta| K)^l}{l!} Z^k_{\Lambda \setminus \bar{Y}_l} =$$

$$= e^{-\frac{1}{2}\beta U^+(\eta_\Lambda) + \beta(2\bar{I}+B)|\eta|} \sum_{l \ge 0} \frac{(|\eta| K)^l}{l!} \frac{Z_{\Lambda \setminus \bar{Y}_l}}{Z_\Lambda}. \tag{22}$$

The fact that $Z_{\Lambda_1} \le Z_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$ gives the inequality

$$\rho^\Lambda(\eta) \le e^{-\frac{1}{2}\beta U^+(\eta)} e^{|\eta|(\beta(2\bar{I}+B)+K)}. \quad \blacksquare$$

A Proof of the lemma 3.2

Let $X = \cup_{j=1}^{N} \Delta_j$. Consider the configuration γ with $|\gamma_X| = m, |\gamma_{\Delta_1}| = m_1, ..., |\gamma_{\Delta_N}| = m_N$, $m_j \geq 1$ for $j = 1, ..., N$ and $m_1 + \cdots + m_N = m$. Let in the k-body interaction be involved $\bar{k} \geq 1$ particles from the dilute configuration $\bar{\gamma}_{X^c} \in \bar{\Gamma}_{X^c}$ and, correspondingly, q_1 particles of γ_X from Δ_1, which are situated in the points $x_1^{(1)}, ..., x_{q_1}^{(1)} \in \Delta_1$, ..., q_N particles $x_1^{(N)}, \ldots, x_{q_N}^{(N)}$ from Δ_N. It is clear that $q_1 + \cdots + q_N + \bar{k} = k$ and $0 \leq q_i \leq m_i$, $\bar{k} \geq 1$. Then the interaction energy between m particles of the configuration γ_X and $\bar{k}$ particles of dilute configuration $\bar{\gamma}_{X^c}$ can be written in the following form:

$$W_k(\gamma_X \mid \bar{\gamma}_{X^c}) = \sum_{\substack{0 \leq q_i \leq m_i, \bar{k} \geq 1 \\ q_1 + \cdots + k_N + \bar{k} = k}} \sum_{\{x_1^{(1)}, \ldots, x_{q_1}^{(1)}\} \in \gamma_{\Delta_1}} \cdots \sum_{\{x_1^{(N)}, \ldots, x_{q_N}^{(N)}\} \in \gamma_{\Delta_N}} \times$$

$$\times \sum_{\{y_1, \ldots, y_{\bar{k}}\} \in \bar{\gamma}_{X^c}} V_k(x_1^{(1)}, \ldots, x_{q_1}^{(1)}, \ldots, x_1^{(N)}, \ldots, x_{q_N}^{(N)}, y_1, \ldots, y_{\bar{k}}).$$

Then taking into account (1) we obtain

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \leq \sum_{\substack{0 \leq q_i \leq m_i, \bar{k} \geq 1 \\ q_1 + \cdots + q_N + \bar{k} = k}} \prod_{i=1}^{N} C_{m_i}^{q_i} \, I_k^{q_1, \ldots, q_N | \bar{k}}(\Delta_1, \ldots, \Delta_N), \tag{23}$$

where $C_m^k = m!/k!(m-k)!$. Let in the sequence $q_1, \ldots, q_N$ be nonzero correspondingly $q_{l_i} = k_{l_i}$ particles from Δ_{l_i}, $i = 1, \ldots, M$ involved in k-body interaction. Changing in (23) to the summation over $k_{l_1}, \ldots, k_{l_M}$:

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \leq \tag{24}$$

$$\leq \sum_{M=1}^{\min\{N, k-1\}} \sum_{1 \leq l_1 < l_2 < \cdots < l_M \leq N} \sum_{\substack{1 \leq k_{l_i} \leq m_{l_i}, \bar{k} \geq 1 \\ k_{l_1} + \cdots + k_{l_M} + \bar{k} = k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \, I_k^{k_{l_1}, \ldots, k_{l_M} | \bar{k}}(\Delta_{l_1}, \ldots, \Delta_{l_M})$$

Let among the cubes $\Delta_1, \ldots, \Delta_N$ be N_1 cubes with only one point of γ inside. Without loss of generality, we suppose that $m_j = 1$, $j = N - N_1 + 1, \ldots, N$. Split the summation over $1 \leq l_1 < l_2 < \cdots < l_M \leq N$ into the summation over $1 \le l_1 < l_2 < \cdots < l_S \le N - N_1$ over cubes $\Delta_1, \ldots, \Delta_{N-N_1}$ and the summation over $1 \le l'_1 < l'_2 < \cdots < l'_{S'} \le N_1$ over cubes $\Delta'_1, \ldots, \Delta'_{N_1}$. It is clear that $S + S' = M$ and S can take integer values from 0 to M. Therefore, we get additionally $M+1$ sums over S. Every value of $1 \le l'_1 < \cdots < l'_{S'} \le N_1$ corresponds to the dilute configuration. Hence, using the definition (1) we can apply the following formula:

$$\sum_{1 \le l'_1 < l'_2 < \cdots < l'_{S'} \le N_1} I_k^{k_{l_1}, \ldots, k_{l_S}, 1, \ldots, 1 | \bar{k}} (\Delta_{l_1}, \ldots, \Delta_{l_S}, \Delta_{l'_1}, \ldots, \Delta_{l'_{S'}}) \le$$

$$\le I_k^{k_{l_1}, \ldots, k_{l_S} | \bar{k} + S'} (\Delta_{l_1}, \ldots, \Delta_{l_S}),$$

yielding

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \le$$

$$\le \sum_{M=1}^{\min\{N-N_1, k-1\}} \sum_{1 \le l_1 < l_2 < \cdots < l_M \le N - N_1} \sum_{l=0}^{\min\{N_1, k-M-1\}} \sum_{\substack{1 \le k_{l_i} \le m_{l_i}, \bar{k} \ge 1 \\ k_{l_1} + \cdots + k_{l_M} + \bar{k} + l = k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times I_k^{k_{l_1}, \ldots, k_{l_M} | \bar{k} + l} (\Delta_{l_1}, \ldots, \Delta_{l_M}) + N_* \sum_{l'_1 = 1}^{\min\{N_1, k-1\}} I_k^{1 | k-1} (\Delta'_{l'_1}),$$

where $N_* = \min\{N, k-1\}$. Collecting the terms with $M = 1$, $k_{l_1} = 1$ in the first sum and the last sum, and selecting also the terms with $k_{l_1} = k_{l_2} = \cdots = k_{l_M} = 1$, summing up all inequalities in $k \ge 2$ and taking into account that $N_* \le k - 1$, we get

$$-W_k(\gamma_X \mid \bar{\gamma}_{X^c}) \le \bar{I} |\gamma_X| + W_1 + W_2,$$

where

$$W_1 = \sum_{M=2}^{N-N_1} \sum_{1 \le l_1 < l_2 < \cdots < l_M \le N - N_1} \prod_{i=1}^{M} C_{m_{l_i}}^{1} \sum_{k \ge M+1} (k - M) I_k^{1, \ldots, 1 | k-M} (\Delta_{l_1}, \ldots, \Delta_{l_M})$$

$$W_2 = \sum_{M=1}^{N-N_1} \sum_{1 \le l_1 < l_2 < \cdots < l_M \le N - N_1} \sum_{k \ge M+1} \sum_{\substack{1 \le k_{l_i} \le m_{l_i} \\ k_{l_1} + \cdots + k_{l_M} = k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times \sum_{l\geq 1} l I_{k+l}^{k_{l_1},\ldots,k_{l_M}|l}(\Delta_{l_1},\ldots,\Delta_{l_M}).$$

Using the same arguments, one can get almost the same inequality for the positive part of energy:

$$U^+(\gamma_X) \geq U_0,$$

where

$$U_0 = \sum_{M=1}^{N-N_1} \sum_{1\leq l_1<l_2<\cdots<l_M\leq N-N_1} \sum_{k\geq M+1} \sum_{\substack{1\leq k_{l_i}\leq m_{l_i}\\ k_{l_1}+\cdots+k_{l_M}=k}} \prod_{i=1}^{M} C_{m_{l_i}}^{k_{l_i}} \times$$

$$\times v_k^{k_{l_1},\ldots,k_{l_M}}(\Delta_{l_1},\ldots,\Delta_{l_M}).$$

Now it is clear from the assumptions **A6** that

$$\frac{1}{8}U_0 \geq W_1, \quad \text{and} \quad \frac{1}{8}U_0 \geq W_2,$$

which gives (15). ■

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