Traveling avalanche waves in spatially discrete bistable reaction–diffusion systems

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Abstract

Infinitely extended two-dimensional reaction–diffusion lattices composed of bistable cells are considered. A class of particular stable stationary solutions, called pattern solutions, is introduced and examples are given. Pattern solutions persist at high diffusion coefficients whereas all other stable stationary solutions, with the exception of the constant solutions, disappear one after the other when the diffusion constant is increased. Furthermore, the new concept of avalanche wave is introduced, where upon a sufficiently large perturbation, a pattern solution is transformed progressively into a constant solution or into another stable stationary solution that exists at a given diffusion constant. These waves exist even for (odd-) symmetrical nonlinearities of the individual cells, whereas it is well known that in this case other waves, as e.g. kinks, do not propagate. The existence of certain classes of avalanche waves is discussed theoretically and the theoretical results are confirmed numerically.

Key words: spatially discrete systems, nonlinear lattices, bistable nonlinearity, reaction-diffusion systems, traveling waves, pattern solution.

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1 Introduction

Reaction–diffusion equations constitute good models for many systems in physics [8, 23, 25, 28, 30, 37], chemistry[38, 60], biology[3, 4, 6, 13, 21, 27, 32, 34, 35, 39, 56, 63, 64] and engineering [5, 15, 16, 17, 18, 19, 44-47, 52-55, 57, 60, 61]. Recently, spatially discrete reaction–diffusion equations have received much attention in the research community of nonlinear dynamics [1, 2, 7, 9, 11, 12, 14, 20, 22, 26, 29, 31, 33, 36, 40-43, 50, 58, 59, 62, 65, 66]. They reproduce the dynamical phenomena of the corresponding partial differential equations. In addition, a number of new phenomena have been discovered that are not present in PDE's [1, 7, 9, 10, 12, 13, 15, 26, 33, 40, 43, 49].

Diffusively coupled bistable cells constitute some of the simplest reaction–diffusion systems. Nevertheless, their dynamics has many unexpected phenomena. In the previous paper, we have characterized stable stationary solutions of 1-d chains of

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diffusively coupled bistable cells with cubic and cubic-like nonlinearities. We have shown the presence of spatial chaos for any values of the diffusion constant, and we have shown the existence of a certain number of stationary stable solutions that have no counterparts in the corresponding PDE, in particular, pattern solutions. It is well-known that in the case of bistable cells with non-symmetric cubic nonlinearities there exist propagating kinks. Their propagation velocity depends on asymmetry of the cubic function. In the symmetric case considered in [9, 28, 42], however, kinks do not propagate, but constitute themselves stable stationary solutions.

In this paper, we extend the analysis of [49] to two-dimensional lattices with cubic and cubic-like nonlinearities. We prove existence of pattern solutions for this case, too. Furthermore, we show how initial conditions given by different stationary solutions evolve as propagating wave–fronts. Note that this happens in the case of a symmetric nonlinearity where, as usual, there are no propagating waves. We call this new type of waves traveling avalanche waves.

2 Reaction–diffusion systems

Nonlinear two dimensional spatially discrete reaction–diffusion systems with bistable nonlinearity are described by the following infinite system of ordinary differential equations on the lattice $\mathbb{Z}^2$:

$$\begin{gathered}\dot{u}_{n,m} = d(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) - f(u_{n,m}),\\ (m,n) \in \mathbb{Z}^2.\end{gathered} \tag{1}$$

Here d is a real parameter (the coupling or diffusion constant) that multiplies the discrete Laplacian, f is a bistable nonlinearity given by the cubic function

$$f(u) = u(u^2 - 1) \tag{2}$$

or the cubic-like piecewise-linear function

$$f(u) = u - \operatorname{sign} u = \begin{cases} u-1, & u > 0,\\ u+1, & u < 0,\\ \quad 0, & u = 0.\end{cases} \tag{3}$$

Eq. (1) has exact stationary stable solutions, $u_{n,m} \equiv 1$ and $u_{n,m} \equiv -1$. We will call them trivial stable solutions.

A nontrivial uniformly bounded stationary solution of Eq. (1), $u_{n,m}$, satisfies the condition $|u_{n,m}| < 1$.
This is an analogue of Lemma 3.1 [49].

A bounded stationary solution of Eq. (1), $u_{n,m}$, with cubic-like nonlinearity (3) is l_2-stable if and only if $u_{n,m} \neq 0$ for all $n, m \in \mathbb{Z}^2$. This is an analogue of Lemma 3.2 [49].

Let $u_{n,m}$ be a stable stationary solution of Eq. (1). The signature function $\hat{u}_{n,m} = \operatorname{sign} u_{n,m}$ is called the skeleton of the solution $u_{n,m}$. A stable stationary solution $u_{n,m}$ is uniquely defined by its skeleton $\hat{u}_{n,m} = \operatorname{sign} u_{n,m}$, and this is an analogue of Theorem 3.1 [49].

The proofs of the three results above are similar to the corresponding ones in [49] and, hence, are omitted.

3 Pattern solutions

A stationary solution of Eq. (1), $u_{n,m}$ is called a pattern solution if, for an arbitrary value of the coupling constant $d > 0$, the signature function $\hat{u}_{n,m} = \operatorname{sign} u_{n,m}$ is a skeleton of a stable solution of Eq. (1) [49].

Examples of periodic pattern solutions of Eq. (1) with nonlinearity (3) are the following.

a) A strip pattern. This solution $u_{n,m}$ does not depend on m and has period in n equal to $2p$,

$$\begin{aligned} u_{n,m} &= (-1)^n(1+4d)^{-1}, && \text{for } p=1,\\ u_{n,m} &= (-1)^{[n/2]}(1+2d)^{-1}, && \text{for } p=2, \end{aligned}$$

where $[n/2]$ denotes the integer part of the number $n/2$.

Fig. 1 shows a strip pattern for $p = 16$ and $d = 30$.

b) A checkerboard pattern. This solution has period in n and m equal to $2p$,

$$\begin{aligned} u_{n,m} &= (-1)^{n+m}(1+8d)^{-1}, && \text{for } p=1,\\ u_{n,m} &= (-1)^{[n/2]+[m/2]}(1+4d)^{-1}, && \text{for } p=2. \end{aligned}$$

Fig. 2 shows a checkerboard pattern for $p = 16$ and $d = 30$.

4 Definition of avalanche waves

Among nonstationary solutions of Eq. (1), an important role is played by stabilizing solutions, that is, solutions that tend to a stationary solution of Eq. (1) as $t \to \infty$. The stabilization process itself can generate propagating waves. We distinguish a new class of stabilizing solutions of Eq. (1), and call them avalanche waves. Such solutions, in two complementing regions of the plane $(m, n) \in \mathbb{Z}^2$, are close to two different stationary solutions. In one region, which contracts as t increases, the solution is close to a stationary pattern solution. In the other region, which expands as t increases, the solution approaches another stationary solution. The borderline that nominally separates these regions could form straight lines which make parallel movements with constant velocity. In this case, we will call it a plane avalanche wave.

If the borderline forms circles with radii increasing in time, we will call it a circular avalanche wave. One can also consider avalanche waves with other propagation fronts of avalanches, i. e., other shapes of regions where the pattern solution transforms into another stationary solution. Note that in the conclusion of [9] the authors briefly mention the possibility of existence of avalanche waves.

An exact definition of avalanche waves is given by the following.

Definition 1. Suppose we have two stationary solutions of Eq. (1), $u^{\text{st}}_{n,m}$ and $u^{\text{patt.}}_{n,m}$ which is a pattern solution. We will say that a solution $u^{\text{pl.av.w.}}_{n,m}(t)$ of Eq. (1) is a plane avalanche wave of a given pattern solution into a stationary solution $u^{\text{st}}_{n,m}$ and is traveling with velocity c in the direction $\nu = (\cos\varphi, \sin\varphi)$ in the plane $\mathbb{Z}^2$ if for any $\epsilon > 0$ there exists a number $R(\epsilon)$ such that

$$\begin{aligned} |u^{\text{pl.av.w.}}_{n.m}(t) - u^{\text{patt.}}_{n,m}| < \epsilon \quad &\text{if} \quad n\cos\varphi + m\sin\varphi \geq ct + R(\epsilon),\\ |u^{\text{pl.av.w.}}_{n.m}(t) - u^{\text{st}}_{n,m}| < \epsilon \quad &\text{if} \quad n\cos\varphi + m\sin\varphi \leq ct - R(\epsilon). \end{aligned} \tag{4}$$

We say that a solution $u^{\text{circ.av.w.}}_{n,m}(t)$ of Eq. (1) is a circular avalanche of a given pattern solution $u^{\text{patt.}}_{n,m}$ into a stationary solution $u^{\text{st}}_{n,m}$ and is traveling with velocity c if for any $\epsilon < 0$ there exists a number $R(\epsilon)$ such that

$$\begin{aligned} |u^{\text{circ.av.w.}}_{n,m}(t) - u^{\text{patt.}}_{n,m}| < \epsilon \quad \text{if} \quad \sqrt{n^2+m^2} \geq ct + R(\epsilon), \\ |u^{\text{circ.av.w.}}_{n,m}(t) - u^{\text{st}}_{n,m}| < \epsilon \quad \text{if} \quad \sqrt{n^2+m^2} \leq ct - R(\epsilon). \end{aligned} \tag{5}$$

In the case where $u^{\text{st}}_{n,m}$ is a trivial stable stationary solution, $u^{\text{st}}_{n,m} \equiv 1$ or $u^{\text{st}}_{n,m} \equiv -1$, we will simply call it an avalanche wave for a given pattern solution.

Fig. 3a - 3b show an avalanche wave in the case where a checkerboard pattern with $d = 10$ transforms into a broad soliton.

Fig. 4a - 4b show a circular avalanche wave when a checkerboard pattern transforms into the trivial solution $u_{n,m} \equiv 1$.

5 Plane avalanche waves for one-dimensional pattern solutions

Consider avalanche waves propagating in a certain direction $\nu = (\cos\varphi, \sin\varphi)$ at velocity $c > 0$ and such that, if $\xi = n\cos\varphi + m\sin\varphi \to \infty$, the solution $u_{n,m}$ is close to the pattern solution of type 1), and if $\xi \to -\infty$, the solutions approaches the stable solutions $u_{n,m} \to -1$. Thus, we will be looking for a solution of Eq. (1) that has the form

$$u_{n,m} = \begin{cases} a\Big(\dfrac{n\cos\varphi + m\sin\varphi}{\sqrt{d}} - ct\Big), & \text{if } n \text{ is even}, \\ b\Big(\dfrac{n\cos\varphi + m\sin\varphi}{\sqrt{d}} - ct\Big), & \text{if } n \text{ is odd}. \end{cases} \tag{6}$$

Boundary conditions for the functions $a(x)$ and $b(x)$ have the form

$$\begin{aligned} a(x) \to (1+4d)^{-1}, \quad b(x) \to -(1+4d)^{-1}, \qquad \text{if} \quad x \to \infty; \\ a(x) \to -1, \quad b(x) \to -1, \qquad \text{if} \quad x \to -\infty. \end{aligned} \tag{7}$$

By substituting (6) into (1), we get

$$\begin{aligned} -ca'(x) &= d\Big[b\Big(x + \frac{\cos\varphi}{\sqrt{d}}\Big) + b\Big(x - \frac{\cos\varphi}{\sqrt{d}}\Big) + a\Big(x + \frac{\sin\varphi}{\sqrt{d}}\Big) \\ &\quad + a\Big(x - \frac{\sin\varphi}{\sqrt{d}}\Big) - 4a(x)\Big] - a(x) + \operatorname{sign} a(x), \\ -cb'(x) &= d\Big[a\Big(x + \frac{\cos\varphi}{\sqrt{d}}\Big) + a\Big(x - \frac{\cos\varphi}{\sqrt{d}}\Big) + b\Big(x + \frac{\sin\varphi}{\sqrt{d}}\Big) \\ &\quad + b\Big(x - \frac{\sin\varphi}{\sqrt{d}}\Big) - 4b(x)\Big] - b(x) + \operatorname{sign} b(x). \end{aligned} \tag{8}$$

Boundary conditions (7) can be supplemented with the conditions

$$a(x) = \begin{cases} > 0, & \text{if } x > 0, \\ < 0, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \end{cases} \qquad b(x) < 0, \tag{9}$$

where, without loss of generality, we set $a(0) = 0$.

Instead of the functions $a(x)$ and $b(x)$, let us consider new functions $w(x)$ and $v(x)$ defined by

$$w(x) = a(x) + b(x), \qquad v(x) = a(x) - b(x). \tag{10}$$

Using (9) Eq. (8) can be written as

$$\begin{aligned} -cw'(x) &= d\Big[w\Big(x + \frac{\cos\varphi}{\sqrt{d}}\Big) + w\Big(x - \frac{\cos\varphi}{\sqrt{d}}\Big) + w\Big(x + \frac{\sin\varphi}{\sqrt{d}}\Big) \\ &\quad + w\Big(x - \frac{\sin\varphi}{\sqrt{d}}\Big) - 4w(x)\Big] - w(x) - 2\theta(-x), \\ cv'(x) &= d\Big[v\Big(x + \frac{\cos\varphi}{\sqrt{d}}\Big) + v\Big(x - \frac{\cos\varphi}{\sqrt{d}}\Big) - v\Big(x + \frac{\sin\varphi}{\sqrt{d}}\Big) \\ &\quad - v\Big(x - \frac{\sin\varphi}{\sqrt{d}}\Big) + 4v(x)\Big] + v(x) - 2\theta(x), \end{aligned} \tag{11}$$

where $\theta(x)$ is the Heaviside function.

The boundary conditions for w and v will become

$$\begin{gathered} w(x) \to 0 \quad \text{for} \quad x \to \infty, \qquad w(x) \to -2 \quad \text{for} \quad x \to -\infty; \\ v(x) \to 2(1+4d)^{-1} \quad \text{for} \quad x \to \infty, \qquad v(x) \to 0 \quad \text{for} \quad x \to -\infty; \\ w(0) = -v(0) = b(0). \end{gathered} \tag{12}$$

Solutions of (11) satisfying conditions (12) can be explicitly obtained by using the Fourier transform

$$w(x) = -2\theta(-x) + \frac{i}{\pi}\int e^{-i\lambda x}\Big(\frac{1}{L(\lambda)} - 1\Big)\frac{d\lambda}{\lambda}, \tag{13}$$

$$v(x) = \frac{2}{1+4d}\theta(x) + \frac{1}{i\pi}\int e^{-i\lambda x}\Big(\frac{1}{M(\lambda)} - \frac{1}{1+4d}\Big)\frac{d\lambda}{\lambda}, \tag{14}$$

where

$$L(\lambda) = 1 + i\lambda c + 4d\Big(\sin^2\frac{\lambda\cos\varphi}{2\sqrt{d}} + \sin^2\frac{\lambda\sin\varphi}{2\sqrt{d}}\Big), \tag{15}$$

$$M(\lambda) = 1 + 4d + i\lambda c + 2d\Big(\cos\frac{\lambda\cos\varphi}{\sqrt{d}} - \cos\frac{\lambda\sin\varphi}{\sqrt{d}}\Big). \tag{16}$$

For large d, the integrals in (13), (14) can be calculated by making the approximations

$$L(\lambda) = 1 + i\lambda c + \lambda^2, \qquad M(\lambda) = 1 + 4d + i\lambda c - \lambda^2\cos 2\varphi. \tag{17}$$

These solutions correspond to solutions of differential equations obtained from the differential difference equations (10) if the differences are replaced with derivatives up to the second order inclusive,

$$-cw' = w'' - w - 2\theta(-x) \tag{18}$$

$$cv' = (1+4d)v + \cos 2\varphi\, v'' - 2\theta(x). \tag{19}$$

Now a solution of problem (18), (12) can be found in an explicit form,

$$w(x) = \begin{cases} w(0)e^{-\frac{x}{p}}, & x > 0, \\ -2 + (2 + w(0))e^{px}, & x < 0, \end{cases} \tag{20}$$

where

$$p = \frac{2}{c+\sqrt{c^2+4}}, \qquad w(0) = -\Big(1+\frac{c}{2p}\Big)^{-1}. \tag{21}$$

To solve Eq. (19), consider the following three cases:

a) $\cos 2\varphi = 0$, i.e., $\varphi = \dfrac{\pi}{4}$;

b) $\cos 2\varphi < \dfrac{c^2}{4(1+4d)}$, i.e., the direction of propagation of the avalanche wave is close to the direction of the y-axis;

c) $\cos 2\varphi > \dfrac{c^2}{4(1+4d)}$, i.e., the direction of propagation of the avalanche wave is close to the direction of the x-axis.

In case a), when $\varphi = \dfrac{\pi}{4}$, Eq. (19) with conditions (12) has the solution

$$v(x) = \begin{cases} \dfrac{2}{1+4d}, & x > 0 \\ \dfrac{2}{1+4d} e^{\frac{1+4d}{c}x}, & x < 0. \end{cases} \tag{22}$$

Since $v(0) = 2(1+4d)^{-1}$, $w(0) = -(1+\frac{c}{2p})^{-1}$, and $w(0) = -v(0)$, the boundary condition (12) gives a value for the velocity of the avalanche wave,

$$c = 2\sqrt{d} - \frac{1}{2\sqrt{d}}. \tag{23}$$

Using (20) and (10) we have

$$a(x) = \begin{cases} \dfrac{1}{1+4d}(1-e^{-\frac{x}{p}}), & x > 0, \\ -(1-e^{px}) + \dfrac{1}{1+4d}(e^{\frac{1+4d}{c}x} - e^{p}x), & x < 0, \end{cases} \tag{24}$$

$$b(x) = \begin{cases} -\dfrac{1}{1+4d}(1+e^{-\frac{x}{p}}), & x > 0, \\ -(1-e^{px}) - \dfrac{1}{1+4d}(e^{\frac{1+4d}{c}x} + e^{px}), & x < 0, \end{cases} \tag{25}$$

In case b), we denote by q_+ and q_-, respectively, the positive and negative roots of the quadratic equation

$$\cos 2\varphi q^2 - cq + 1 + 4d = 0. \tag{26}$$

Then using (12) we can write the solution of Eq. (19) in the form

$$v(x) = \begin{cases} \dfrac{2}{1+4d} + \Big(v(0) - \dfrac{2}{1+4d}\Big)e^{q_- x}, & x > 0, \\ v(0)e^{q_+ x}, & x < 0, \end{cases} \tag{27}$$

where

$$b(0) = -v(0) = \frac{2q_-}{(1+4d)(q_+ - q_-)}, \tag{28}$$

and, if $\varphi > \frac{\pi}{4}$,

$$q_+ = \frac{1+4d}{q}, \qquad q_- = \frac{q}{\cos 2\varphi}, \qquad q = \frac{1}{2}(c + \sqrt{c^2 - 4(1+4d)\cos 2\varphi}). \tag{29}$$

Formulas (21), (28), (12) give the equation $w(0) = -v(0)$ which is used for finding the velocity of the avalanche waves in the case b),

$$\left[1 + \frac{c}{2p}\right]^{-1} = \frac{2|q_-|}{(1+4d)(q_+ - q_-)}. \tag{30}$$

This equation defines a value of the velocity c of the avalanche waves as a function of d and φ. Eq. (30) can be reduced to the following system of equations, which allows to analyze the dependence $c = c(d, \varphi)$,

$$\begin{aligned} c &= 4A[8A - 4(1+4d)\cos 2\varphi]^{-1/2}, \\ A &= B + [B^2 - 2\alpha d(1+4d)\cos 2\varphi]^{1/2}, \\ B &= d(\alpha + \cos 2\varphi) - \tfrac{1}{4}(\alpha - \cos 2\varphi), \\ \alpha &= [c + \sqrt{c^2 - 4(1+4d)\cos 2\varphi}] \cdot [c + \sqrt{c^2+4}]^{-1}, \end{aligned} \tag{31}$$

where $A = \frac{cq}{2}$ and $\alpha = pq$.

Let $d \ll 1$ and $\varphi > \frac{\pi}{4}$. Then we can put $\alpha = 1$ in (31) and obtain an approximate value for the velocity,

$$c = 4d(1 - \cot^4 \varphi)^{1/2}. \tag{32}$$

By comparing formulas (23) and (32) one can see that, if $\varphi = \frac{\pi}{4}$, avalanche waves exist only if $d > \frac{1}{4}$ and, if $\varphi = \frac{\pi}{2}$, they also exist for $d < \frac{1}{4}$.

If $d \gg 1$, velocity c can be approximately represented as $c = k(\varphi) \cdot d^{1/2}$, where the variable $k(\varphi)$ ranges over the interval $2 \leq k(\varphi) < 3$ if φ belongs to the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

In case c), the analysis is carried out similarly.

In this case, the real parts of the roots p_1 and p_2 of the characteristic equation (26) are positive and, hence, for $x > 0$,

$$v(x) = \frac{2}{1+4d}, \qquad x > 0. \tag{33}$$

This gives a value for velocity c of avalanche waves which does not depend on $\varphi \leq \frac{\pi}{4}$; it is given by (23).

Solution $v(x)$, for $x < 0$, has the form

$$v(x) = \frac{2}{1+4d}\left[\frac{p_2}{p_2 - p_1} e^{p_1 x} - \frac{p_1}{p_2 - p_1} e^{p_2 x}\right], \qquad x < 0, \tag{34}$$

where p_1 and p_2 are roots of the characteristic equation (26).

In particular, if $\varphi = 0$, we have

$$v(x) = \frac{2}{1+4d}\begin{cases} 1, & x > 0, \\ e^{(c/2)x}\left[\cos \beta x - \frac{c}{2}\frac{\sin \beta x}{\beta}\right], & x < 0, \end{cases}$$

where $\beta = \sqrt{1 + 4d - c^2/4}$ and the value of c is the same as in case a), i.e., given by formula (23).

Figures 5 and 6 show avalanche waves that move toward the x and y axes in the case of the simplest strip pattern with $p = 2$ and $d = 3$. Figure 7 shows avalanche waves that move in diagonal direction with the same pattern.

6 Plane avalanche waves for simple two-dimensional pattern solutions

Let us consider avalanche waves that propagate in the direction $\nu = (\cos\varphi, \sin\varphi)$ such that, if $\xi = n\cos\varphi + m\sin\varphi \to \infty$, the solution is close to a checkerboard pattern solution of type b). Thus we will be looking for a solution of Eq. (1) such that

$$u_{n,m} = \begin{cases} a\Big(\dfrac{n\cos\varphi + m\sin\varphi}{\sqrt{d}} - ct\Big), & n+m \text{ is even,} \\ b\Big(\dfrac{n\cos\varphi + m\sin\varphi}{\sqrt{d}} - ct\Big), & n+m \text{ is odd.} \end{cases} \tag{35}$$

Boundary conditions for the functions $a(x)$ and $b(x)$ are

$$\begin{gathered} a(x) \to (1+8d)^{-1} \quad \text{for} \quad x \to +\infty, \qquad a(x) \to -1 \quad \text{for} \quad x \to -\infty; \\ b(x) \to -(1+8d)^{-1} \quad \text{for} \quad x \to +\infty, \qquad b(x) \to -1 \quad \text{for} \quad x \to -\infty; \\ a(0) = 0, \qquad a(x) > 0 \quad \text{for} \quad x > 0, \qquad a(x) < 0 \quad \text{for} \quad x < 0; \\ b(x) < 0. \end{gathered} \tag{36}$$

Substituting (35) into (1) we get

$$\begin{aligned} -ca'(x) &= d\,\mathcal{L}\,b(x) + 4d(b(x) - a(x)) - a(x) + \operatorname{sign} x, \\ -cb'(n) &= d\,\mathcal{L}\,a(x) + 4d(a(x) - b(x)) - b(x) - 1, \end{aligned} \tag{37}$$

where $\mathcal{L}\,a(x) = a\Big(x+\dfrac{\cos\varphi}{\sqrt{d}}\Big)+a\Big(x-\dfrac{\cos\varphi}{\sqrt{d}}\Big)+a\Big(x+\dfrac{\sin\varphi}{\sqrt{d}}\Big)+a\Big(x-\dfrac{\sin\varphi}{\sqrt{d}}\Big)-4a(x).$

Replacing the functions a and b with w and v using (10), equations (37) can be written as

$$\begin{aligned} -cw'(x) &= d\,\mathcal{L}\,w(x) - w(x) - 2\theta(-x), \\ cv'(x) &= d\,\mathcal{L}\,v(x) + (1+8d)v(x) - 2\theta(x). \end{aligned} \tag{38}$$

Solution $w(x)$ of this problem has form (20). Solution $v(x)$, for $c^2 < 4(1+8d)$, has the form

$$v(x) = \begin{cases} \dfrac{2}{1+8d}, & x > 0, \\ \dfrac{2}{1+8d} e^{\frac{c}{2}x}\Big[\cos\beta x - \dfrac{c}{2}\dfrac{\sin\beta x}{\beta}\Big], & x < 0, \end{cases} \tag{39}$$

where $\beta = \sqrt{1+8d-\frac{c^2}{4}}$.

Going back to a and b from w and v we get

$$a(x) = \begin{cases} \dfrac{1}{1+8d}(1-e^{-\frac{x}{p}}), & x > 0, \\ -(1-e^{px}) + \dfrac{1}{1+8d}\Big\{e^{\frac{c}{2}x}\Big[\cos\beta x - \dfrac{c}{2}\dfrac{\sin\beta x}{\beta}\Big] - e^{px}\Big\}, & x < 0, \end{cases} \tag{40}$$

$$b(x) = \begin{cases} -\dfrac{1}{1+8d}(1+e^{-\frac{x}{p}}), & x > 0, \\ -(1-e^{px}) - \dfrac{1}{1+8d}\Big\{e^{\frac{c}{2}x}\Big[\cos\beta x - \dfrac{c}{2}\dfrac{\sin\beta x}{\beta}\Big] + e^{px}\Big\}, & x < 0. \end{cases} \tag{41}$$

By using the equation $w(0) = -v(0)$, we find the value for the velocity of the avalanche waves,

$$c = 2\sqrt{2d} - \frac{1}{2\sqrt{2d}}. \tag{42}$$

Fig. 8 shows the graph of an avalanche wave moving toward the y-axis in the case of a checkerboard pattern for $p = 2$, $d = 2$.

Note that, if $d \to \infty$, then approximately,

$$a(x) = b(x) = -\theta(-x)\Big[1 - e^{px}\Big],$$

and the avalanche waves have the following form, the same for the patterns of types a) and b),

$$u_{n,m} = -\theta(-x)(1 - e^{px}),$$

where $x = \dfrac{n\cos\varphi + m\sin\varphi}{\sqrt{d}} - ct$. However, the values of the velocity c differ in these two cases.

7 Avalanche waves with circular front

Consider now an avalanche wave with circular front in the case where the checkerboard pattern solution of type b) becomes the solution $u_{n,m} \equiv -1$.

We will suppose that

$$u_{n,m} = \begin{cases} a\Big(\dfrac{\sqrt{n^2+m^2}}{\sqrt{d}} - ct\Big), & n+m \text{ is even}, \\ b\Big(\dfrac{\sqrt{n^2+m^2}}{\sqrt{d}} - ct\Big), & n+m \text{ is odd}. \end{cases} \tag{43}$$

The functions $a(x)$ and $b(x)$ satisfy boundary condition (36).

Substituting (43) into (1) for $n^2 + m^2 \gg 1$ we get the following approximate differential equations:

$$\begin{aligned} -ca' &= b'' + 4d(b-a) - a + \operatorname{sign} x, \\ -cb' &= a'' + 4d(a-b) - b - 1. \end{aligned} \tag{44}$$

They coincide with the ones that have already been studied. Hence, an explicit expression for their solutions are given by formulas (40), (41), (32).

An avalanche wave with circular front for $d =$ is shown in Fig. 4 and, for larger values of $d = 40$, in Fig. 9 ($p = 2$).

8 Numerical experiments

Below we give results of numerical experiments carried out to study the propagation of avalanche waves in two dimensional lattices in the cases where the checkerboard pattern transforms into a cross (Fig. 10) or a sector (Fig. 11).

Numerical calculations show that, for reaction-diffusion system (1) with cubic nonlinearity (2), the qualitative behavior of propagation of avalanche waves, discussed above for piecewise linear nonlinearity (3), is preserved. However, for the cubic nonlinearity (2), one observes a number of special features that we will briefly discuss.

The simplest strip pattern has the form $u_{n,m} = (-1)^n\sqrt{1-4d}$ and exists only for $d < \frac{1}{4}$. The simplest checkerboard pattern has the form $u_{n,m} = (-1)^{n+m}\sqrt{1-8d}$ and exists only for $d < \frac{1}{8}$. For larger values of the coupling constant d, spatially periodic solutions have the following form:

$u_{n,m} = (-1)^n\phi(t, d)$ for the strip pattern and

$u_{n,m} = (-1)^{n+m}\phi(t, 2d)$ for the checkerboard solution,

where

$$\phi(t, d) = \phi(0)e^{-(4d-1)t}\Big[1 + \phi^2(0)\frac{1 - e^{-2(4d-1)t}}{4d-1}\Big]^{-1/2}.$$

This shows that, for large d, the patterns contract to the trivial solution $u_{n,m} \equiv 0$ as time increases. Thus, as avalanche waves propagate, the pattern layer decreases at the same time. Figures 12a, 12b and 12c show dynamics of periodic patterns, and figures 13a, 13b and 13c are graphs of the avalanche waves obtained from numerical experiments in the case of the cubic nonlinearity.

9 Conclusions

1. It was shown that, together with well-studied traveling waves in a spatially discrete bistable reaction-diffusion system that has a nonsymmetric nonlinearity, for $\int f(u)\,du \neq 0$, there can also exist waves of a new type even in the case where the nonlinearity $f(u)$ is given by an odd function of u. The authors called these waves avalanche waves, since their propagation mechanism is similar to that of falling dominoes or mountain rocks. For avalanche waves to propagate, the medium must have elements whose states are close to being unstable. For bistable reaction-diffusion systems, such a medium is formed by a pattern solution for large values of the coupling constant.
2. For the case of a piecewise linear nonlinearity, we make an analytical analysis of the form and velocities of the avalanche waves.
3. Numerical calculations for propagation of various traveling waves were carried out. It was shown that the qualitative features of avalanche waves are also preserved for a cubic nonlinearity.

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