# Towards an $L^p$-potential theory for sub-Markovian semigroups: Variational inequalities and balayage theory

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**Abstract**: We give a new variational approach to $L^p$-potential theory for sub-Markovian semigroups. It is based on the observation that the Gâteaux-derivative of the corresponding $L^p$-energy functional is a monotone operator. This allows to apply the well established theory of Browder and Minty on monotone operators to the nonlinear problems in $L^p$-potential theory. In particular, using this approach it is possible to avoid any symmetry assumptions of the underlying semigroup. We prove existence of corresponding $(r,p)$-equilibrium potentials and obtain a complete characterization in terms of a variational inequality. Moreover we investigate associated potentials and encounter a natural interpretation of the so-called nonlinear potential operator in the context of monotone operators.

## 1. Introduction

Since the paper [3] of M. Fukushima it is known that up to a set of capacity zero we may associate a Hunt process with every (symmetric) $L^2$-sub-Markovian semigroup $(T_t^{(2)})_{t\geq 0}$, compare also the monographs [4], [7] and that of Z.-M. Ma and M. Röckner, M. [19]. The construction of the process relies much on the potential theory of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ assoiated with $(T_t^{(2)})_{t\geq 0}$, the exceptional sets are determined in terms of a capacity defined by using $(\mathcal{E}, D(\mathcal{E}))$. In fact, this potential theory is a Hilbert space theory and not only involved in the construction of the process but also of much use in the stochastic analysis for the process associated with $(T_t^{(2)})_{t\geq 0}$ or equivalently with $(\mathcal{E}, D(\mathcal{E}))$. From the very start of the theory one of its draw-back was clear (compared to the "Feller theory"), namely the existence of exceptional sets. One way to overcome the problem is to work with refinements or redefinitions as suggested by P. Malliavin [20] and taken up by M. Fukushima and coauthors, see [6] and [5], but also H. Kaneko [16]. This approach uses $(r,p)$-capacities and the $\Gamma$-transform $V_r^{(p)}$ of the semigroup $(T_t^{(2)})_{t\geq 0}$ which is assumed to be symmetric, hence analytic (by a result due to E.M. Stein [22]) in all $L^p$-spaces, $1 < p < \infty$. Although the technique of $(r,p)$-capacities uses implicitely an $L^p$-theory, only recently a systematic approach to an $L^p$-theory of sub-Markovian semigroups was initiated with the aim to have both, the analogue to the Maz'ya–Havin

[21], Adams–Hedberg [1] theory and the theory of Dirichlet forms, i.e. the variational theory.

This paper, a companion of the paper [15] with R. Schilling, wants to establish a variational approach to the $L^p$-theory for sub-Markovian semigroups with the aim to get for the capacities as well as the equilibrium potentials in the $L^p$-theory results completely analogous to those in the theory of Dirichlet forms. This leads to a non-linear potential theory.

As a starting point we take a general sub-Markovian $L^p$-semigroup on a completely metrizable metric space equipped with a Borel measure. Applications are especially sought for pseudo-differential operators generating Markov processes, see [10] – [11], the survey [12] as well as [14]. We consider the generalized Bessel potential spaces $\mathcal{F}_{r,p}$ defined by the $\Gamma$-transform (For a rather systematic study of the spaces $\mathcal{F}_{r,p}$ we refer to [13].). In section 2 we introduce the $L^p$-energy functional $E_{r,p}$ (see (2.6)) as the norm in the $\mathcal{F}_{r,p}$-space. Then objects in $L^p$-potential theory defined in a variational way like equilibrium potentials are given as minimizers of this functional over appropriate close convex sets.

Main results are the following:

- We show that $E_{r,p}$ is Gâteaux differentiable and moreover, as the basic observation, we prove that the Gâteaux derivative $\mathcal{A}_r^{(p)}$ (see (2.11)) is a nonlinear monotone operator in the sense of Browder and Minty (see Proposition 3.2).
- We characterize the minimizers of the energy functional $E_{r,p}$ over closed convex sets as the unique solution of a variational inequality involving the operator $\mathcal{A}_r^{(p)}$ (Theorem 4.3). This result is new in itself, independent of an application to Dirichlet forms.
- As a conclusion of the above results we obtain existence of $(r, p)$-equilibrium potentials (Theorem 5.1). This also improves previous results (see [6], [17]) in the sense that by this approach it is not necessary to assume that the semigroup is symmetric nor that the adjoint semigroup is sub-Markovian.
- We identify the so-called nonlinear potential operator (see (5.9)) as the continuous inverse of $\mathcal{A}_r^{(p)}$ by the Browder-Minty theorem. This operator was introduced in the classical context by Maz'ya and Havin and permits the representation of nonlinear potential by nonnegative measures (see Prop. 5.3 and Theorem 5.4).

Moreover we discuss in section 5 how the results on $(r, p)$-capacities and equilibrium potentials can be used to derive results in $L^p$-potential theory similar to the results in the theory of Dirichlet forms.

From the above it becomes clear that the monotone operator $\mathcal{A}_r^{(p)}$ playes an important role in nonlinear potential theory. We want to emphasize that besides the above results the main concern of this paper is to point out that this opens a remarkable connection to Browder Minty's theory of monotone operators. Therefore we included in section 3, which is devoted to the analysis of $\mathcal{A}_r^{(p)}$, an overview of basic concepts from the Browder-Minty theory and we investigate related continuity properties of $\mathcal{A}_r^{(p)}$.

## 2. Preliminaries. The functionals $E_{r,p}$ and $E_{r,p}^f$

We assume that the underlying space $X$ is a completely metrizable space equipped with a Borel measure $\mu$. Given an $L^p$-sub-Markovian semigroup $(T_t^{(p)})_{t\geq 0}$ on this space with generator $(A^{(p)}, D(A^{(p)}))$ we define its $\Gamma$-transform by

$$V_r^{(p)} u = \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t^{(p)} u \, dt \tag{2.1}$$

and the associated Bessel potential space by

$$\mathcal{F}_{r,p} = V_r^{(p)}(L^p), \qquad \|u\|_{\mathcal{F}_{r,p}} = \|f\|_{L^p} \text{ where } u = V_r^{(p)} f. \tag{2.2}$$

According to [2] we have

$$\mathcal{F}_{r,p} = D((\mathrm{id} - A^{(p)})^{r/2}) \tag{2.3}$$

and

$$V_r^{(p)} = (\mathrm{id} - A^{(p)})^{-r/2}, \tag{2.4}$$

note that for $(T_t^{(2)})_{t\geq 0}$ symmetric the result is best known for $A^{(2)}$ and it follows from spectral theory, compare [6], whereas in [2] a functional calculus related to Bernstein functions is used to prove (2.3) and (2.4) in the general situation.

We denote by $(T_t^{(p)^*})_{t\geq 0}$ the dual semigroup on $L^{p'}, p' = \frac{p}{p-1}$. Then $(T_t^{(p)^*})_{t\geq 0}$ is also positivity preserving, but we do not assume that it is sub-Markovian. Its generator is given by the adjoint operator $A^{(p)^*}$, i.e. we have

$$\int_X (A^{(p)^*} u) v \, \mu(dx) = \int_X u A^{(p)} v \, \mu(dx) \tag{2.5}$$

for all $u \in D(A^{(p)^*})$, $v \in D(A^{(p)})$, and $A^{(p)^*}$ generates the semigroup $(T_t^{(p)^*})_{t\geq 0}$.

We assume also that $C_0(X) \cap \mathcal{F}_{r,p}$ is dense in both, $(\mathcal{F}_{r,p}, \|\cdot\|_{\mathcal{F}_{r,p}})$ and $(C_\infty(X), \|\cdot\|_\infty)$. Furthermore, in this section we assume always that $p > 2$. On $\mathcal{F}_{r,p}$ we consider the functional

$$E_{r,p}(u) := \frac{1}{p}\|u\|_{\mathcal{F}_{r,p}}^p = \frac{1}{p}\int_X |(\mathrm{id} - A^{(p)})^{r/2} u|^p \mu(dx). \tag{2.6}$$

Note that for $p \geq 2$ the function $g(s) = |s|^p$, $s \in \mathbb{R}$ is a convex function with strictly monotone increasing derivative, i.e. it is a strictly convex function. Moreover, $g$ has a continuous second derivative and we have

$$g'(s) = ps|s|^{p-2}, \tag{2.7}$$

$$g''(s) = p(p-1)|s|^{p-2}, \tag{2.8}$$

and further we note the estimate

$$(s|s|^{p-2} - r|r|^{p-2})(s-r) \geq 2^{2-p}|s-r|^p \tag{2.9}$$

which holds for all $r, s \in \mathbb{R}$, see E. Zeidler [23], p.503. For shorthand we write sometimes

(2.10)

$$T_{r,p} := (\mathrm{id} - A^{(p)})^{r/2} \quad \text{and} \quad T^*_{r,p} := ((\mathrm{id} - A^{(p)})^{r/2})^* = ((\mathrm{id} - A^{(p)})^{r/2})^*.$$

**Lemma 2.1.** *The functional $E_{r,p} : \mathcal{F}_{r,p} \to \mathbb{R}$ is strictly convex, coercive, i.e. $\frac{E_{r,p}(u)}{\|u\|_{\mathcal{F}_{r,p}}} \to \infty$ as $\|u\|_{\mathcal{F}_{r,p}} \to \infty$, and Gâteaux differentiable. Its Gâteaux derivative at $u \in \mathcal{F}_{r,p}$ is given by*

$$\mathcal{A}_r^{(p)} u := (id - A^{(p)^*})^{r/2}(|(id - A^{(p)})^{r/2}u|^{p-2} \cdot (id - A^{(p)})^{r/2}u). \tag{2.11}$$

Clearly, for $A^{(p)} = \Delta$ the operator $\mathcal{A}_r^{(p)}$ turns out to be the $p$-Laplacian related to $\mathrm{id} - \Delta$.

*Proof.* The strict convexity is obvious and the coercivity follows from

$$\frac{E_{r,p}(u)}{\|u\|_{\mathcal{F}_{r,p}}} = \frac{1}{p}\|u\|^{p-1}_{\mathcal{F}_{r,p}}.$$

Now, fix $u, h \in \mathcal{F}_{r,p}$ and consider on $[0,1]$ the quotient

$$\begin{aligned} t \quad \mapsto \quad & \frac{E_{r,p}(u+th) - E_{r,p}(u)}{t} \\ = \quad & \frac{1}{p}\int_X \frac{|(\mathrm{id} - A^{(p)})^{r/2}(u+th)|^p - |(\mathrm{id} - A^{(p)})^{r/2}u|^p}{t}\mu(dx) \\ = \quad & \int_X ((1-\alpha)T_{r,p}(u+th) + \alpha T_{r,p}u) \cdot \\ & \cdot|(1-\alpha)T_{r,p}(u+th) + \alpha T_{r,p}u|^{p-2}T_{r,p}h\,\mu(dx) \end{aligned}$$

where we used for the last line the meanvalue theorem, $\alpha = \alpha(x) \in [0,1]$. Thus for $t \to 0$ we arrive at

$$\lim_{t\to 0}\frac{E_{r,p}(u+th) - E_{r,p}(u)}{t} = \int_X (T_{r,p}u)|T_{r,p}u|^{p-2}T_{r,p}h\,\mu(dx)$$

which yields

$$\lim_{t\to 0}\frac{E_{r,p}(u+th) - E_{r,p}(u)}{t} = \langle T^*_{r,p}(|T_{r,p}u|^{p-2}T_{r,p}u), h\rangle = \langle \mathcal{A}_r^{(p)}u, h\rangle,$$

and the lemma is proved. □

Note that a function $g$ is in $L^p$ if and only if $g \cdot |g|^{p-2}$ is in $L^{p'}$. From this it follows that $\mathcal{A}_r^{(p)}$ maps $\mathcal{F}_{r,p}$ into $\mathcal{F}_{r,p}^*$.

Now let $f \in L^{p'}$, $p' = \frac{p}{p-1}$, and define on $\mathcal{F}_{r,p}$

$$E_{r,p}^f(v) = E_{r,p}(v) - \int_X f v \, \mu(dx). \tag{2.12}$$

**Corollary 2.2.** *The functional $E_{r,p}^f$ on $\mathcal{F}_{r,p}$ is strictly convex, coercive and Gâteaux differentiable with Gâteaux derivative at $u \in \mathcal{F}_{r,p}$ given by*

$$\mathcal{A}_{r,f}^{(p)} u := \mathcal{A}_r^{(p)} u - f. \tag{2.13}$$

*Proof.* The strict convexity of $E_{r,p}^f$ follows since $E_{r,p}$ is strictly convex and $v \mapsto -\int_X f v \, \mu(dx)$ is linear. Furthermore we find

$$\begin{aligned} \frac{E_{r,p}^f(u)}{\|u\|_{\mathcal{F}_{r,p}}} &= \frac{1}{p}\|u\|_{\mathcal{F}_{r,p}}^{p-1} - \frac{1}{\|u\|_{\mathcal{F}_{r,p}}} \int f u \, \mu(dx) \\ &\geq \frac{1}{p}\|u\|_{\mathcal{F}_{r,p}}^{p-1} - \frac{\|f\|_{L^{p'}} \|u\|_{L^p}}{\|u\|_{\mathcal{F}_{r,p}}} \\ &\geq \frac{1}{p}\|u\|_{\mathcal{F}_{r,p}}^{p-1} - \|f\|_{L^{p'}}, \end{aligned}$$

implying the coercivity of $E_{r,p}^f$. Since

$$\frac{-\int_X f \cdot (u + th) \, \mu(dx) + \int_X f u \, \mu(dx)}{t} = -\int_X f h \, \mu(dx)$$

it follows from the preceding lemma that $E_{r,p}^f$ is Gâteaux differentiable with Gâteaux derivative (2.13). □

## 3. The operators $\mathcal{A}_r^{(p)}$ and $\mathcal{A}_{r,f}^{(p)}$

We start with recalling some definitions.

**Definition 3.1.** Let $Y$ be a reflexive, separable real Banach space and $K \subset Y$ a closed convex set. Further let $T : K \to Y^*$ be an operator.

**A.** We call $T$ **monotone** if

$$\langle Tu - Tv, u - v \rangle \geq 0 \tag{3.1}$$

for all $u, v \in K$.

**B.** The operator is called **strictly monotone** if

$$\langle Tu - Tv, u - v \rangle > 0 \tag{3.2}$$

for all $u, v \in K$ and $u \neq v$.

**C.** If there is a strictly increasing continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, $\gamma(0) = 0$ and $\lim\limits_{t\to\infty} \gamma(t) = \infty$, such that for all $u, v \in K$

$$\langle Tu - Tv, u - v\rangle \geq \gamma(\|u - v\|_y) \cdot \|u - v\|_y \tag{3.3}$$

holds, then $T$ is called **uniformly monotone**.

**D.** If $K$ is unbounded we say that $T$ is **coercive** with respect to $K$ if there is an element $\varphi \in K$ such that

$$\lim_{\substack{\|u\|_Y\to\infty \\ u\in K}} \frac{\langle Tu - T\varphi, u - \varphi\rangle}{\|u - \varphi\|_Y} = \infty. \tag{3.4}$$

Clearly, for a linear space $K$ we may pose the condition

$$\lim_{\substack{\|u\|_Y\to\infty \\ u\in K}} \frac{\langle Tu, u\rangle}{\|u\|_Y} = \infty, \tag{3.5}$$

compare Lemma 2.1. If $K$ is an unbounded set and $T$ uniformly monotone, then $T$ is coercive as well as strictly monotone, hence monotone.

**Proposition 3.2.** *Let $\mathcal{A}_r^{(p)}$ be as in* (2.11). *Then for any closed convex set $K \subset \mathcal{F}_{r,p}$ we have the estimate*

$$\langle \mathcal{A}_r^{(p)} u - \mathcal{A}_r^{(p)} v, u - v\rangle \geq 2^{-p+2} \|u - v\|^p_{\mathcal{F}_{r,p}}, \tag{3.6}$$

*implying that $\mathcal{A}_r^{(p)}$ is uniformly monotone and for every unbounded set $K$ coercive.*

*Proof.* From the definition of $\mathcal{A}_r^{(p)}$ we find

$$\langle \mathcal{A}_r^{(p)} u - \mathcal{A}_r^{(p)} v, u - v\rangle =$$
$$= \int_X \left[|(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2} \cdot (\mathrm{id} - A^{(p)})^{r/2} u - |(\mathrm{id} - A^{(p)})^{r/2} v|^{p-2} \cdot (\mathrm{id} - A^{(p)}) v\right] \cdot$$
$$\cdot((\mathrm{id} - A^{(p)})^{r/2} u - (\mathrm{id} - A^{(p)})^{r/2} v) \mu(dx)$$

and (2.9) yields

$$\begin{aligned} \langle \mathcal{A}_r^{(p)} u - \mathcal{A}_r^{(p)} v, u - v\rangle &\geq 2^{-p+2} \int_X |(\mathrm{id} - A^{(p)})^{r/2} (u - v)\|^p \mu(dx) \\ &= 2^{-p+2} \|u - v\|^p_{\mathcal{F}_{r,p}}. \end{aligned}$$

□

**Corollary 3.3.** *For every $f \in L^{p'}$, $p' = \frac{p}{p-1}$ it follows that*

$$\langle \mathcal{A}_{r,f}^{(p)} u - \mathcal{A}_{r,f}^{(p)} v, u - v\rangle \geq 2^{-p+2} \|u - v\|^p_{\mathcal{F}_{r,p}} \tag{3.7}$$

*holds.*

**Definition 3.4.** Let $Y$ be a reflexive, separable real Banach space. Further let $T : Y \to Y^*$ be an operator.

**A.** The operator $T$ is called **hemicontinuous** if for all $u, v \in Y$ and $h \in Y$ the function

$$s \mapsto \langle T(u+sv), h\rangle$$

is continuous on $[0,1]$.

**B.** If for every sequence $(u_\nu)_{\nu\in\mathbb{N}}$, $u_\nu \in Y$, which converges strongly to $u \in K$ it follows that $(Tu_\nu)_{\nu\in\mathbb{N}}$ converges weakly (in $Y^*$) to $Tu$, then $T$ is said to be **demicontinuous**.

**C.** The operator $T$ is **bounded** if it maps bounded sets onto bounded sets.

**Proposition 3.5.** *The operator $\mathcal{A}_r^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}_{r,p}^*$ is hemicontinuous,.*

*Proof.* The operator $(\mathrm{id} - A^{(p)})^{r/2} : \mathcal{F}_{r,p} \to L^p$ is continuous and for $u, v \in \mathcal{F}_{r,p}$, $0 \le s \le 1$, it follows for $s \to s_0$ that

$$\begin{aligned}&|(\mathrm{id} - A^{(p)})^{r/2}(u+sv)|^{p-2}(\mathrm{id} - A^{(p)})^{r/2}(u+sv)\\ &\qquad\to |(\mathrm{id} - A^{(p)})^{r/2}(u+s_0v)|^{p-2}(\mathrm{id} - A^{(p)})^{r/2}(u+s_0v)\end{aligned}$$

$\mu$–almost everywhere. In addition we find for all $s \in [0,1]$

$$\begin{aligned}&\big||(\mathrm{id} - A^{(p)})^{r/2}(u+sv)|^{p-2}(\mathrm{id} - A^{(p)})^{r/2}(u+sv)\big|\\ =\ &|(\mathrm{id} - A^{(p)})^{r/2}(u+sv)|^{p-1}\\ \le\ &c_p(|(\mathrm{id} - A^{(p)})^{r/2}u|^{p-1} + |(\mathrm{id} - A^{(p)})^{r/2}v|^{p-1}),\end{aligned}$$

thus the dominated convergence theorem implies

$$\begin{aligned}&|(\mathrm{id} - A^{(p)})^{r/2}(u+sv)|^{p-2}(\mathrm{id} - A^{(p)})^{r/2}(u+sv)\\ &\qquad\to |(\mathrm{id} - A^{(p)})^{r/2}(u+s_0v)|^{p-2}(\mathrm{id} - A^{(p)})^{r/2}(u+s_0v)\end{aligned}$$

in $L^{p'}$, which yields the hemi-continuity of $\mathcal{A}_r^{(p)}$. □

**Corollary 3.6. A.** *The operator $\mathcal{A}_r^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}_{r,p}^*$ is hemicontinuous.*

**B.** *The operator $\mathcal{A}_{r,f}^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}_{r,p}^*$ is hemicontinuous.*

Now we recall two fundamental results from F. Browder's and G. Minty's theory of monotone operators:

**Theorem 3.7.** *(Browder-Minty): Let $T : Y \to Y^*$ be a monotone, coercive, and hemicontinuous operator.*

**A.** *For every $f \in Y^*$ the set of solutions of*

$$Tv = f$$

*is non-empty, closed and convex.*

**B.** *If in addition $T$ is strictly monotone, then the inverse operator*

$$T^{-1} : Y^* \to Y$$

*exists, $T^{-1}$ is strictly monotone, demicontinuous and bounded.*

**C.** *If $T$ is even uniformly monotone, then $T^{-1}$ is continuous.*

Proofs of Theorem 3.7 are given in E. Zeidler [23], Theorem 26.A or in H. Gajewski et al. [8], chapter III.2.

**Theorem 3.8.** *For all $f \in \mathcal{F}^*_{r,p}$ there exists a unique solution $u \in \mathcal{F}_{r,p}$ of the equation*

$$\mathcal{A}^{(p)}_r u = f. \tag{3.8}$$

*Moreover, $\mathcal{A}^{(p)}_r : \mathcal{F}_{r,p} \to \mathcal{F}^*_{r,p}$ has a continuous inverse which is given by*

$$f \mapsto (id - A^{(p)})^{-r/2}(|(id - A^{(p)^*})^{-r/2} f|^{p'-2}(id - A^{(p)^*})^{-r/2} f) \tag{3.9}$$

*Proof.* It remains to prove (3.9). Note again that for a function $g$ it follows that $h = g|g|^{p-2} \in L^{p'}$ if and only if $g = h|h|^{p'-2} \in L^p$. Now, given $f \in \mathcal{F}^*_{r,p}$ we find a unique $u \in \mathcal{F}_{r,p}$ such that

$$\mathcal{A}^{(p)}_r u = (\mathrm{id} - A^{(p)^*})^{r/2}(|(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2}(\mathrm{id} - A^{(p)})^{r/2} u) = f,$$

hence

$$|(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2}(\mathrm{id} - A^{(p)})^{r/2} u = (\mathrm{id} - A^{(p)^*})^{-r/2} f \in L^{p'},$$

i.e.

$$(\mathrm{id} - A^{(p)})^{r/2} u = |(\mathrm{id} - A^{(p)^*})^{-r/2} f|^{p'-2}(\mathrm{id} - A^{(p)^*})^{-r/2} f \in L^p,$$

which leads to (3.9). □

## 4. On variational inequalities involving $\mathcal{A}^{(p)}_{r,f}$

Let $K \subset \mathcal{F}_{r,p}$, $p > 2$, be a closed convex set. In this section we want to exploit the properties of the functionals $E^f_{r,p}$ obtained so far in order to study the minimization problem

$$\inf\{E^f_{r,p}(u);\ u \in K\}, \tag{4.1}$$

where $f \in L^{p'}$, $p' = \frac{p}{p-1}$ is a given function.

For this purpose recall

**Proposition 4.1.** *Suppose that $E : M \mapsto \mathbb{R}$ is a convex coercive continuous functional on a nonempty closed convex subset $M$ of a reflexive Banach space. Then $E$ has a minimum on $M$.*
*If $E$ is moreover strictly monotone the minimum is attained in a unique point.*

For a proof of Proposition 4.1 see E. Zeidler [23], Theorem 25.D and Proposition 25.20 (i) as well as Corollary 25.15.

From Corollary 2.2 we know that $E^f_{r,p}$ is a strictly convex, coercive and moreover the continuity of $E^f_{r,p}$ follows immediately from the definition of $E_{r,p}$ and continuity of the embedding of $\mathcal{F}_{r,p}$ into $L^p$. Thus we have

**Theorem 4.2.** *For every closed convex set $K \subset \mathcal{F}_{r,p}$ the minimization problem*

$$inf\{E^f_{r,p}(u);\ u \in K\}$$

*has a unique solution.*

Next we we want to get a more precise description of the minimizer. Recall that $E^f_{r,p}$ is Gâteaux differentiable with monotone Gâteaux derivative $\mathcal{A}^{(p)}_{r,f}$ by Proposition 3.2. Suppose that $u \in K$ minimizes $E^f_{r,p}(u)$ in $K$ and for $0 \le t \le 1$ consider the function $(1-t)u + t\varphi \in K$, $\varphi \in K$. It follows that

$$\begin{aligned}
0 &\le \frac{d}{dt}\Big|_{t=0} E^f_{r,p}((1-t)u + t\varphi) \\
&= \frac{d}{dt}\Big|_{t=0} \left\{ \frac{1}{p} \int_X |(\mathrm{id} - A^{(p)})^{r/2}((1-t)u + t\varphi)|^p \mu(dx) \right\} \\
&\quad - \int_X f \cdot ((1-t)u + t\varphi)\mu(dx) \\
&= \langle \mathcal{A}^{(p)}_r u, \varphi - u\rangle - \int_X f(\varphi - u)\mu(dx) \\
&= \langle \mathcal{A}^{(p)}_{r,f} u, \varphi - u\rangle \qquad \text{for all } \varphi \in K.
\end{aligned}$$

Note moreover that the minimizer $u$ is the only element in $K$ satisfying this inequality, because for any $u_1, u_2 \in K$ having this property

$$\begin{aligned}
\langle \mathcal{A}^{(p)}_{r,f} u_1, u_1 - u_2\rangle &\le 0, \\
\langle -\mathcal{A}^{(p)}_{r,f} u_2, u_1 - u_2\rangle &\le 0,
\end{aligned}$$

hence

$$\langle \mathcal{A}^{(p)}_{r,f} u_1 - \mathcal{A}^{(p)}_{r,f} u_2, u_1 - u_2\rangle \le 0,$$

which implies $u_1 = u_2$ by the strict monotony of $\mathcal{A}^{(p)}_{r,f}$.

Thus, as we expect from the general theory of variational inequalities, see for example the monograph [18] of D. Kinderlehrer and G. Stampacchia, we obtain

**Theorem 4.3.** *An element $u \in K$ is the solution of the minimization problem* (4.1) *if and only if $u$ satisfies the variational inequality*

$$\langle \mathcal{A}^{(p)}_{r,f} u, \varphi - u\rangle \ge 0 \qquad \textit{for all } \varphi \in K. \tag{4.2}$$

In particular in case that $K$ is a linear space this means:

**Corollary 4.4.** *Suppose that $K$ is a closed linear subspace of $\mathcal{F}_{r,p}$. Then (4.1) has a unique solution $u \in K$ satisfying*

$$\langle \mathcal{A}^{(p)}_{r,f} u, \psi \rangle = 0 \tag{4.3}$$

*for all $\psi \in K$, i.e.*

$$\langle \mathcal{A}^{(p)}_{r} u, \psi \rangle = \int_X f\psi \, \mu(dx) \tag{4.4}$$

*for all $\psi \in K$.*

**Remark 4.5.** Note that inequality (4.2) and equality (4.4) may be written as

(4.5)
$$\int_X |(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2} (\mathrm{id} - A^{(p)})^{r/2} u \cdot (\mathrm{id} - A^{(p)})^{r/2} (\varphi - u) \mu(dx) \geq \int_X f \cdot (\varphi - u) \mu(dx)$$

and

(4.6)
$$\int_X |(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2} (\mathrm{id} - A^{(p)})^{r/2} u \cdot (\mathrm{id} - A^{(p)})^{r/2} \psi \, \mu(dx) = \int_X f\psi \, \mu(dx),$$

respectively.

In order to have a closer analogy to the theory of Dirichlet forms let us introduce on $\mathcal{F}_{r,p} \times \mathcal{F}_{r,p}$ the form

(4.7)
$$\mathcal{E}^{(p)}_r (u, v) := \int_X |(\mathrm{id} - A^{(p)})^{r/2} u|^{p-2} \cdot (\mathrm{id} - A^{(p)})^{r/2} u \cdot (\mathrm{id} - A^{(p)})^{r/2} v \, \mu(dx)$$

For $u, v \in \mathcal{F}_{r,p}$ we find

$$\mathcal{E}^{(p)}_r (u, u) = E_{r,p}(u) \geq 0 \tag{4.8}$$

and

$$\begin{aligned} |\mathcal{E}^{(p)}_r (u, v)| &\leq \| |(\mathrm{id} - A^{(p)})^{r/2} u|^{p-1} \|_{L^{p'}} \|(\mathrm{id} - A^{(p)})^{r/2} v\|_{L^p} \\ &= \|u\|^{p/p'}_{\mathcal{F}_{r,p}} \|v\|_{\mathcal{F}_{r,p}}, \end{aligned} \tag{4.9}$$

thus $\mathcal{E}^{(p)}_r : \mathcal{F}_{r,p} \times \mathcal{F}_{r,p} \to \mathbb{R}$ and it is continuous in the second argument for fixed first argument. Furthermore, from the proof of Proposition 3.5 it follows that the mapping

$$s \mapsto \mathcal{E}^{(p)}_r (u + sw, v), \quad u, w, v \in \mathcal{F}_{r,p} \tag{4.10}$$

is continuous from $\mathbb{R}$ to $\mathbb{R}$. Using $\mathcal{E}^{(p)}_r$ we find for $u, v \in \mathcal{F}_{r,p}$

$$\langle \mathcal{A}^{(p)}_r u, v \rangle = \mathcal{E}^{(p)}_r (u, v).$$

## 5. Applications to existence of $(r,p)$-equilibrium potentials

Let $G \subset X$ be an open set and define

$$C_{r,p}(G) := \{u \in \mathcal{F}_{r,p}\,;\ u \geq 1\ \mu - \text{a.e. on } G\}, \tag{5.1}$$

and the capacity

$$\begin{aligned} \operatorname{cap}_{r,p}(G) &:= \inf\{\|u\|^p_{\mathcal{F}_{r,p}}\,;\ u \in C_{r,p}(G)\} \\ &= \inf\{p \cdot E_{r,p}(u);\ u \in C_{r,p}(G)\}. \end{aligned} \tag{5.2}$$

We extend $\operatorname{cap}_{r,p}$ to arbitrary sets $A \subset X$ by

$$\operatorname{cap}_{r,p}(A) := \inf\{\operatorname{cap}_{r,p}(G);\ G \supset A \text{ and } G \text{ open}\}. \tag{5.3}$$

The results of the preceding section are immediately applicable to this minimization problem. Note that a different proof of the following theorem is given in [13], see also M. Fukushima and H. Kaneko [6] or M. Fukushima [5].

**Theorem 5.1.** *Let $2 < p < \infty$. For any open set $G \subset X$ with $cap_{r,p}(G) < \infty$ there exists a unique element $u_G \in \mathcal{F}_{r,p}$ such that $u_G \geq 1$ a.e. on $G$ and*

$$cap_{r,p}(G) = \|u_G\|^p_{\mathcal{F}_{r,p}} \tag{5.4}$$

*holds. Moreover, there exists $f \in L^p$, $f \geq 0$ a.e., with the property that $u_G = V_r^{(p)} f$.*

*Proof.* The first part of the theorem follows from Theorem 4.2 applied to the closed convex set $C_{r,p}(G)$.

For the second assertion note that the Γ-transform

$$V_r^{(p)} = T^{-1}_{r,p} : L^p \to \mathcal{F}_{r,p}$$

is a bijection. Let $f \in L^p$ the unique element such that $e_G = V_r^{(p)} f \in \mathcal{F}_{r,p}$ and decompose $f = f_+ - f_-$ into positive and negative part. Since $V_r^{(p)}$ is positivity preserving we have

$$V_r^{(p)} f_+ \geq V_r^{(p)} f = u_G \in C_{r,p}(G).$$

Hence $V_r^{(p)} f_+ \in C_{r,p}(G)$ and $\|V_r^{(p)} f_+\|^p_{\mathcal{F}_{r,p}} \geq \operatorname{cap}_{r,p}(G)$.

Suppose that $f_-$ is not identically 0 $\mu$-a.e. Then

$$\|V_r^{(p)} f_+\|^p_{\mathcal{F}_{r,p}} = \|f_+\|^p_{L^p} < \|f\|^p_{L^p} = \|V_r^{(p)} f\|^p_{\mathcal{F}_{r,p}} = \|u_G\|^p_{\mathcal{F}_{r,p}} = \operatorname{cap}_{r,p}(G),$$

which gives a contradiction. □

The function $u_G$ is called the $(r,p)$–**equilibrium potential**. Now, for $G \subset X$ open it is clear that $C_{r,p}(G)$ is a closed convex subset of $\mathcal{F}_{r,p}$. Combining Theorem 5.1 with the results of the previous section we find that $u_G$ satisfies the variational inequality

$$\langle p\mathcal{A}_r^{(p)} u_G, \varphi - u_G\rangle \geq 0$$

for all $\varphi \in C_{r,p}(G)$, hence

$$\langle \mathcal{A}_r^{(p)} u_G, \varphi - u_G \rangle \geq 0, \tag{5.5}$$

or

$$\mathcal{E}_r^{(p)}(u_G, \varphi - u_G) \geq 0 \tag{5.6}$$

for all $\varphi \in C_{r,p}(G)$.

Let $\psi \in \mathcal{F}_{r,p}$ such that $\psi|_G \geq 0$ a.e. It follows that $\varphi := u_G + \psi \in C_{r,p}(G)$ implying that

$$\langle \mathcal{A}_r^{(p)} u_G, \psi \rangle = \mathcal{E}_r^{(p)}(u_G, \psi) \geq 0 \tag{5.7}$$

for all $\psi \in \mathcal{F}_{r,p}, \psi|_G \geq 0$ a.e.

From this it is reasonable to adapt the following notion from the theory of Dirichlet forms.

**Definition 5.2.** A function $u \in \mathcal{F}_{r,p}$ is called a **potential** if

$$\mathcal{E}_r^{(p)}(u, \psi) \geq 0 \quad \text{for all } \psi \in \mathcal{F}_{r,p},\ \psi \geq 0. \tag{5.8}$$

We want to reformulate our results in a way that is close to the Riesz representation theorem for potentials in classical potential theory. For this purpose we call an element $\varphi \in \mathcal{F}_{r,p}^*$ positive if

$$\langle \varphi, \psi \rangle \geq 0 \quad \text{for all } \psi \in \mathcal{F}_{r,p},\ \psi \geq 0$$

and we write $\varphi \in (\mathcal{F}_{r,p})_+$.

Recall that the operator

$$\mathcal{A}_r^{(p)} : \mathcal{F}_{r,p} \to \mathcal{F}_{r,p}^*$$

is invertible and we calculated the inverse

$$U_r^{(p)} := (\mathcal{A}_r^{(p)})^{-1} : \mathcal{F}_{r,p}^* \to \mathcal{F}_{r,p}$$

in Theorem 3.8 as

$$U_r^{(p)} f = V_r^{(p)}(|V_r^{(p)*} f|^{p'-2} \cdot V_r^{(p)*} f). \tag{5.9}$$

The operator $U_r^{(p)}$ is called **nonlinear potential operator** and has been investigated before for instance by V.G. Maz'ya, V.P. Havin [21] and D.R Adams, L.I. Hedberg [1].

Since $\mathcal{E}_r^{(p)}(u, v) = \langle \mathcal{A}_r^{(p)} u, v \rangle$ we see imediately from the definition of $(\mathcal{F}_{r,p}^*)_+$:

**Proposition 5.3.** *A function $u \in \mathcal{F}_{r,p}$ is a potential if and only if*

$$u = U_r^{(p)} f$$

*for some $f \in (\mathcal{F}_{r,p}^*)_+$.*

Furthermore, as in the classical case, in many situations the positive elements of $\mathcal{F}_{r,p}^*$ can be identified as measures. The following theorem is proven using similar arguments as in T. Kazumi, I. Shigekawa [17], Theo. 3.1.

**Theorem 5.4.** *Assume that the capacity* $cap_{r,p}$ *is tight, i.e. for every* $\epsilon > 0$ *there is a compact set* $K \subset X$ *such that*

$$cap_{r,p}(X \setminus K) < \epsilon.$$

*Moreover assume that there is a function* $h \in \mathcal{F}_{r,p}$ *such that* $h \geq c > 0$. *Then for every* $f \in (\mathcal{F}^*_{r,p})_+$ *there is a measure* $\nu$ *on* $X$ *such that*

$$\langle f, v \rangle = \int_X v \, d\nu \quad \textit{for all } v \in \mathcal{F}_{r,p} \cap C_b(X).$$

To improve properties of the equilibrium potential $u_G$, $G \subset X$ open and $\text{cap}_{r,p}(G) < \infty$, we have to work with $(r,p)$–Dirichlet capacities, i.e. capacities in space where the truncation property holds.

**Definition 5.5.** *Let* $(\mathcal{H}(X), \|\dot{\|}_{\mathcal{H}})$ *be a subspace of* $L^p(X)$.
**A**. *We say that the* **Lipschitz functions** $T$ *with constant* 1 *operate on* $\mathcal{H}(X)$ *if*

$$u \in \mathcal{H}(X) \textit{ implies } T \circ u \in \mathcal{H}(X) \tag{5.10}$$

*and*

$$\|T \circ u\|_{\mathcal{H}} \leq c \|u\|_{\mathcal{H}} \tag{5.11}$$

*hold for all* $u \in \mathcal{H}(X)$ *and all* $T : \mathbb{R} \to \mathbb{R}$ *such that* $|T(x) - T(y)| \leq |x - y|$ *and* $T(0) = 0$.
**B**. *If the Lipschitz functions operate on* $\mathcal{H}(X)$ *and (5.9) holds with* $c = 1$, *then we say the* **truncation property** *holds for* $\mathcal{H}(X)$.

We refer to the paper [9] of F. Hirsch where a detailed discussion of the truncation property in $\mathcal{F}_{r,p}$ is given.

Now suppose that $\mathcal{F}_{r,p}$ has the truncation property. Note that this is a serious restriction. Further let $G \subset X$ be an open set with $\text{cap}_{r,p}(G) < \infty$ and $u_G$ its equilibrium potential. It follows that $(0 \vee u_G) \wedge 1 \in C_{r,p}(G)$ and

$$\|((0 \vee u_G) \wedge 1)\|^p_{\mathcal{F}_{r,p}} \leq \|u_G\|^p_{\mathcal{F}_{r,p}},$$

implying that $u_G = (0 \vee u_G) \wedge 1$, i.e. $0 \leq u_G \leq 1$ $\mu$-a.e. and $u_G|_G = 1$ $\mu$-a.e.

Further we claim that $u_G$ is the unique element in $\mathcal{F}_{r,p}$ satisfying $u_G = 1 \ \mu - \text{a.e. on} \, G$ such that

$$\mathcal{E}^{(p)}_r(u_G, v) \geq 0 \quad \text{for all } v \in \mathcal{F}_{r,p}, \ v \geq 0 \ \mu - \text{a.e. on} \, G. \tag{5.12}$$

From (5.6) we know already that for $u_G$ inequality (5.12) holds. Now let $u_G \in \mathcal{F}_{r,p}$, $u_G \leq 1$ $\mu$-a.e. on $G$ and $\mathcal{E}^{(p)}_r(u_G, v) \geq 0$ for all $v \in \mathcal{F}_{r,p}$, $v \geq 0$ on $G$. It follows that $w - u_G \geq 0$ $\mu$-a.e. on $G$ for all $w \in C_{r,p}(G)$, thus

$$\mathcal{E}^{(p)}_r(u_G, w - u_G) \geq 0$$

and the variational inequality characterisation of the equilibrium potential yields the result. In particular, it follows that for $v \in \mathcal{F}_{r,p}$ such that $v = 1\ \mu -$ a.e. on $G$ it follows that

$$\mathcal{E}_r^{(p)}(e_G, v) = \mathrm{cap}_{r,p}(G).$$

Indeed, for $v = 1\ \mu-$a.e. on $G$ we find that $\pm(e_G - v) \geq 0\ \mu-$a.e. on $G$ implying

$$O \leq \pm\mathcal{E}_r^{(p)}(e_G, e_G - v) = \pm(\mathcal{E}_r^{(p)}(e_G, e_G) - \mathcal{E}(e_G, v)),$$

i.e.

$$\mathrm{cap}_{r,p}(G) \leq \mathcal{E}(e_G, v) \leq \mathrm{cap}_{r,p}(G).$$

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