CONVERGENCE OF DIRICHLET FORMS WITH CHANGING SPEED MEASURES ON $\mathbb{R}^D$

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Abstract. We consider a sequence of vaguely convergent measures $\mu_n = \sigma_n \, dx \to \sigma \, dx = \mu$ on $\mathbb{R}^d$ and a sequence of symmetric Dirichlet forms $\{\mathcal{E}_n\}$, $\mathcal{E}_n(f,g) = \sum_{i,j=1}^n \int_{\mathbb{R}^d} a_{i,j}^n \partial_i f \partial_j g \, dx$, where every $\mathcal{E}_n$ is defined on $L^2(\sigma_n \, dx)$. We apply a functional analytic theory recently developed by K. Kuwae and T. Shioya and obtain some new results about Mosco convergence of $\{\mathcal{E}_n\}$ and weak convergence of finite dimensional distributions of associated stochastic processes.

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1. Introduction

1.1. **Review of other results.** In this paper we consider some applications of a beautiful functional analytic theory developed in the recent work [10] of K. Kuwae and T. Shioya.

In their work they studied convergence of operators and quadratic forms which are not necessarily defined on the same Hilbert space. More precisely, K. Kuwae and T. Shioya considered a net of Hilbert spaces $\{H_\alpha\}$ over a field K, where $K = \mathbb{R}$ or $K = \mathbb{C}$, which converges in some sense to a Hilbert space H over K. They introduced some natural convergence $u_n \to u$ of vectors "along H_n", where $u_n \in H_n, u \in H$. They also defined strong convergence of operators $A_n \to A$ and Mosco convergence of quadratic forms $Q_n \to Q$, where $A_n \in L(H_n), A \in L(H)$ and Q_n is a quadratic form on H_n.

Recall that by the well-known Mosco theorem (see [17]) the Mosco convergence of quadratic forms $Q_n \to Q$ defined on the same Hilbert space H corresponds to the strong convergence of the associated semigroups. The main result of [10] (see Theorem 2.8) was a generalization of the Mosco theorem to the case when Q_n are defined on different spaces H_n. We stress that such a situation is typical for applications.

This new approach of Kuwae and Shioya is the starting point of our paper. We apply techniques developed by K. Kuwae and T. Shioya to get some new results about the Mosco convergence of Dirichlet forms on finite dimensional spaces. We also show how some classical results of this theory can be obtained in a more natural and easy way.

The study of the strong convergence of semigroups was motivated by the following problem from the theory of stochastic processes. Let $\{\mu_n\}$ be a sequence of measures of $\mathbb{R}^d$ which tends vaguely to a measure μ. For every n consider a Dirichlet form

$$\mathcal{E}_n(f,g) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^d} a_{i,j}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\, dx$$

on $L^2(\mu_n)$ and suppose that $\{\mathcal{E}_n\}$ converges in some (weak) sense to a Dirichlet form $\mathcal{E}$ (for example, $a_{i,j}^n \to a_{i,j}$ almost sure w.r.t. the Lebesgue measure). Let $(\Omega, \mathcal{F}, (X_t^n)_{t\geq 0}, (P_x^n))$ be the associated stochastic processes (see [18], Chapter IV). Under which condition do the finite dimensional distributions of $(X_t^n)_{t\geq 0}$ weakly converge to the finite dimensional distributions of $(X_t)_{t\geq 0}$?

The fist results on convergence of Dirichlet forms were obtained by S. Albeverio, R. Høegh-Krohn and L. Streit in [1] and by S. Albeverio, S. Kusuoka and L. Streit in [2]. In [1] and [2] the diagonal Dirichlet forms

$$\mathcal{E}_n(f,f) = \int_{\mathbb{R}} |\nabla f|^2 \varphi_n^2\, dx$$

were considered. The convergence result was proved in [1] under the condition $0 < \varphi_n \uparrow \varphi$ and $||\frac{\varphi_n}{\varphi}||_{L^\infty} \to 1$. The monotonicity condition was omitted in [2] and replaced by $\varphi_n \to \varphi$ and $\frac{1}{\varphi_n} \to \frac{1}{\varphi}$ in $L^2_{loc}(\,dx)$.

In the works [26] of M. Takeda and [15] of T. J. Lyons and T. S. Zhang some sufficient conditions for tightness of the measures

$$P_{\mu_n}^n = \int_{\mathbb{R}^d} P_x^n\, d\mu_n(x)$$

on $\Omega = C([0,\infty) \to \mathbb{R}^d)$ were obtained. A crucial point in [26] and [15] was the Lyons-Zheng decomposition method, developed by T. J. Lyons and W. Zheng in [16].

The following progress was made in the works [13], [14] [15], [22] of T. Lyons, M. Röckner and T. S. Zhang, where non-diagonal and non-symmetric case were considered. See also work [9] of M. Hino. In [22] the strong convergence of semigroups associated with the Dirichlet forms

$$\mathcal{E}_n(f,g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{i,j}^n \partial_i f \partial_j g \, dx + \sum_{i=1}^{d} \int_{\mathbb{R}^d} d_i^n f \partial_i g \, dx + \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i^n \partial_i f g \, dx + \int_{\mathbb{R}^d} c^n f g \, dx$$

was studied.

M. Röckner and T. S. Zhang proved in [25] that the laws of an infinite dimensional diffusion process can be obtained as a weak limit of laws of finite dimensional processes.

As mentioned before, in [17] U. Mosco obtained the fundamental result that the strong convergence of semigroups (defined on fixed Hilbert space) corresponds to some convergence of quadratic forms which is now called the Mosco convergence. This result allows to work direct with forms instead of semigroups. Applications of Mosco's theorem were made by T. Uemura in [27] and K. Kuwae and T. Uemura in [11], [12]. In [27] the result from [13] was generalized.

In [11] and [12] K. Kuwae and T. Uemura applied various analytic and probabilistic methods to the studying of the Mosco convergence of the forms

$$\mathcal{E}(f,g) = \frac{1}{2} \int_X \Gamma_n(f,g) \varphi_n^2 \, dm,$$

where Γ_n is a square field operator, X is a connected separable metric space and m is a σ-finite Borel measure. They obtained results about the weak convergence of measures on trajectories of the associated processes in terms of intrinsic pseudo metrics (see [4] for details).

1.2. **Organization of the paper. Main results.** In Section 2 we recall basic definitions and main results from [10]. Proofs of some crucial lemmas and theorems can be found in the Appendix.

Throughout the paper the following assumption holds:

Assumption I (Convergence of speed measures)

$\sigma_n > 0$ dx-a.e., $\sigma_n \in L^1_{loc}(dx)$, there exists a function σ such that $\sigma > 0$ dx-a.e., $\sigma \in L^1_{loc}(dx)$ and $\{\mu_n = \sigma_n \, dx\}$ tends to $\mu = \sigma \, dx$ vaguely, i.e.

$$\lim_n \int_{\mathbb{R}^d} \psi \sigma_n \, dx = \int_{\mathbb{R}^d} \psi \sigma \, dx$$

for every continuous ψ with compact support.

We denote this convergence by $\mu_n \to \mu$.

Let $H_n = L^2(\mu_n)$, $C = C_0^\infty(\mathbb{R}^d)$ and Φ_n be just the identical operators on $C_0^\infty(\mathbb{R}^d)$. Then $H_n \to H$ in the sense of Definition 2.1 and we define the space $\mathcal{H} = \bigcup_n H_n = \bigcup_n L^2(\mu_n)$. Note that Φ_n is well-defined because the measures μ_n have full support.

On every $L^2(\sigma_n\,dx)$ we are given a form $\mathcal{E}_n$,

$$\mathcal{E}_n(f,g) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^d} a_{i,j}^n \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\,dx = \int_{\mathbb{R}^d} (A_n(x)\nabla f, \nabla g)\,dx,$$

where $a_{i,j}^n = a_{j,i}^n$ are measurable functions and the matrix $(A_n(x))_{i,j} = a_{i,j}(x)$ is dx-a.e. non-negative. We denote by $(\mathcal{E}_n, \mathcal{D}(\mathcal{E}_n))$ and $(\mathcal{E}_n^+, \mathcal{D}(\mathcal{E}_n^+))$ the minimal and the maximal extension of $(\mathcal{E}_n, C_0^\infty(\mathbb{R}^d))$, respectively (see below for the precise definition). We denote the elements of the inverse matrix A_n^{-1} by $(a^{-1})_{i,j}^n$, $\left(A_n^{-1}(x)\right)_{ij} = (a^{-1})_{i,j}^n(x)$.

The main result of Section 3 is the following

Theorem 1.1. *Let $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{loc}(dx)$,*

$$\sup_n \int_\Omega \frac{dx}{\sigma_n} < \infty, \quad \sup_n \int_\Omega |(a^{-1})_{i,j}^n|\,dx < \infty, \quad i,j \in \{1,\cdots,d\} \tag{1}$$

for every bounded domain Ω and there exists a measurable mapping A from $\mathbb{R}^d$ to the space of non-negative symmetric $d\times d$-matrix such that $a_{i,j}, (a^{-1})_{i,j} \in L^1_{loc}(dx)$ and

$$a_{i,j}^n\,dx \to a_{i,j}\,dx, \quad (a^{-1})_{i,j}^n\,dx \to (a^{-1})_{i,j}\,dx$$

vaguely.

Let

$$\mathcal{E}(\varphi,\psi) = \int_{\mathbb{R}^d} (A\nabla\varphi, \nabla\psi)\,dx, \quad \varphi,\psi \in C_0^\infty(\mathbb{R}^d).$$

Suppose that the minimal and the maximal extensions of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\sigma\,dx)$ coincide. Then $\mathcal{E}_n \to \mathcal{E}$ and $\mathcal{E}_n^+ \to \mathcal{E}$ Mosco.

Let us compare this theorem with results of T. Lyons and T. S. Zhang, and T. Uemura. In [13] the following result was obtained. Suppose that

$$\frac{1}{c}|\xi|^2 \le (A_n(x)\xi,\xi) \le c|\xi|^2, \quad \frac{1}{c} \le \sigma_n(x) \le c \tag{2}$$

dx-a.e. for some positive constant c and every $\xi \in \mathbb{R}^d$, $\sigma_n(x) \to \sigma(x)$ dx-a.e. and $A_n(x) \to A(x)$ dx-a.e. in matrix norm. Let $h \in L^1_+(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$. T. Lyons and T. S. Zhang proved that the finite measures $M_n = \int_{\mathbb{R}^d} h\sigma_n^2 P_x^n\,dx$ do converge to $M = \int_{\mathbb{R}^d} h\sigma^2 P_x\,dx$ weakly in the space $C([0,\infty) \to \mathbb{R}^d)$ endowed with uniform topology. T. Uemura has shown in [27] that the bounds in (2) can be replaced with locally bounds.

In this paper we get the same result under the following weaker conditions: assume that $(A_n(x)\xi,\xi) \le c\sigma_n(x)|\xi|^2$ for every $\xi \in \mathbb{R}^d$ (this implies conservativeness of the corresponding process, see [26], Example 1), the measures $\{h(x)\sigma_n(x)\,dx\}$ are tight for some non-negative function $h(x)$ and the conditions of Theorem 1.1 are fulfilled. Then by Takeda's result we get that the measures $\{M_n = \int_{\mathbb{R}^d} h\sigma_n^2 P_x^n\,dx\}$ are tight.

Since the Mosco convergence of Dirichlet forms implies the vague convergence of finite dimensional distributions of the associated stochastic processes (see Section 6), Theorem 1.1 together with result of M. Takeda implies weak convergence $M_n \to M$. Note that in our theorem the bounds (2) are replaced with the local uniform integrability condition (1) and restrictions on convergence of matrix and speed measures are weaker.

In Section 3 we also give some sufficient conditions for existing of Mosco convergent subsequence of a sequence $\{\mathcal{E}_n\}$ (see Theorem 3.5). This result improves the classical results of S. Albeverio, S. Kusuoka and L. Streit in [2] (see Corollary 3.7 and Remark 3.8).

In Section 4 we consider the one-dimensional case. For every locally finite positive measure μ the following quadratic form on $L^2(\sigma\, dx)$ is defined:

$$\mathcal{E}_\mu(f) = \int_{\mathbb{R}} \left(\frac{\partial f_{ac}}{\partial \mu_{ac}}\right)^2 d\mu_{ac} + \left(\frac{\partial f_s}{\partial \mu_s}\right)^2 d\mu_s + \left(\frac{\partial f_d}{\partial \mu_d}\right)^2 d\mu_d < \infty.$$

Here $\frac{\partial f_{ac}}{\partial \mu_{ac}}$, $\frac{\partial f_s}{\partial \mu_s}$, $\frac{\partial f_d}{\partial \mu_d}$ are densities of the weak derivative of f w.r.t. absolutely continuous, singular and discrete part of μ, respectively (see below for the precise definitions).

We consider a sequence of forms $\{\mathcal{E}_n\}$, where

$$\mathcal{E}_n(f) = \int_{\mathbb{R}} (f')^2 \rho_n\, dx,$$

and every $\mathcal{E}_n$ is defined on $L^2(\sigma_n\, dx)$.

Theorem 1.2. *Assume that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ in $L^2_{loc}(dx)$, $\sup_n \int_\Omega \frac{dx}{\sigma_n} < \infty$ for every bounded Ω and there exists a locally finite measure μ on $\mathbb{R}$ with the absolutely continuous part $\frac{dx}{\tilde\rho}$ such that $\frac{dx}{\rho_n} \to \mu$ vaguely. Suppose in addition that there exists $\alpha > 0$ such that $\tilde\rho \le \alpha\sigma$ dx-a.e. Then $\mathcal{E}_n \to \mathcal{E}_\mu$ Mosco.*

This result is an essential generalization of Theorem 3.1 from [2], where only a partial case of Theorem 1.2 was considered. Theorem 1.2 shows that even in the one-dimensional case the Mosco limit of $\{\mathcal{E}_{\rho_n\, dx}\}$ may not coincide with $\mathcal{E}_{\rho\, dx}$, where ρ is the limit of ρ_n in $L^1_{loc}(\,dx)$. However, the Mosco limit of $\{\mathcal{E}_{\rho_n\, dx}\}$ has an explicit description.

In Section 5 we prove the following generalization of Theorem 1.1 for the gradient case. Denote by $\hat x^i$ the $d-1$-dimensional vector

$$\hat x^i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{d-1}.$$

Theorem 1.3. *Let $\rho_n = \sigma_n$, and suppose that for every $i \in \{1, \cdots, d\}$ there exist non-negative measurable functions $q^i_n(x)$, $p^i_n(\hat x^i)$ with the following properties:*

1) $\rho_n(x) = q^i_n(x) p^i_n(\hat x^i), \rho(x) = q^i(x) p^i(\hat x^i) \quad dx - a.e.$
2) *The measures $p^i_n(\hat x^i)\, d\hat x^i$, $p^i(\hat x^i)\, d\hat x^i$ are locally finite and $p^i_n(\hat x^i)\, d\hat x^i \to p^i(\hat x^i)\, d\hat x^i$ vaguely.*

3) *The measures* $\frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx$, $\frac{p^i(\hat{x}^i)}{q^i(x)}$, dx *are locally finite and*

$$\frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx \to \frac{p^i(\hat{x}^i)}{q^i(x)}\,dx$$

vaguely.

Suppose that the minimal and the maximal extensions of the form $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$,

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} (\nabla f, \nabla g)\rho\, dx$$

coincide. Then $\mathcal{E}_n^+ \to \mathcal{E}$ *Mosco, where* $\mathcal{E}_n^+$ *is the maximal extension of the form* $(\mathcal{E}_n, C_0^\infty(\mathbb{R}^d))$, $\mathcal{E}_n(f,g) = \int_{\mathbb{R}^d} (\nabla f, \nabla g)\rho_n\, dx$.

Theorem 1.3 is a generalization of the result of S. Albeverio, S. Kusuoka and L. Streit about the strong convergence of semigroups. Recall that in [2] the convergence result was obtained under conditions $\sqrt{\rho_n} \to \sqrt{\rho}$ and $\frac{1}{\sqrt{\rho_n}} \to \frac{1}{\sqrt{\rho}}$ in $L^2_{loc}(\,dx)$. We obtain the same result under the weaker conditions of Theorem 1.3 and the additional assumption that $\sqrt{\rho_n} \to \sqrt{\rho}$ in $L^2_{loc}(\,dx)$ (see Lemma 3.6 and Theorem 2.8). Example 6.2 shows that the condition $\frac{1}{\sqrt{\rho_n}} \to \frac{1}{\sqrt{\rho}}$ in $L^2_{loc}(dx)$ is not necessary for identifying the limit and it is even possible that $\frac{dx}{\rho_n}$ converges vaguely to a measure which differs from $\frac{dx}{\rho}$! Note, that Theorem 1.2 implies that such a situation is not possible in the one-dimensional case.

We have already seen in the one-dimensional case that the Mosco limit can essentially differ from the expected one. In Example 5.5 we give a multi-dimensional example. In this example the forms

$$\mathcal{E}_n(f,g) = \sum_{i=1}^{n} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} q_n^i(x_i) p_n^i(\hat{x}^i)\, dx$$

converges Mosco to the form

$$\tilde{\mathcal{E}}(f,g) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} \tilde{q}^i(x_i) p^i(\hat{x}^i)\, dx,$$

where $p_n^i \to p^i$ in $L^1_{loc}(\,d\hat{x}^i)$, $q_n^i \to q^i$ in $L^1_{loc}(\,dx_i)$ and $\frac{dx_i}{q_n^i(x_i)} \to \frac{dx_i}{\tilde{q}^i(x_i)}$ vaguely.

In Section 6 we consider examples and briefly discuss the vague convergence of the laws of the associated processes.

2. Functional-analytical preliminaries

In this section we give definitions and main results connected with the notion of convergent Hilbert spaces. All of them except of Definition 2.9 and Proposition 2.10 were introduced in [10].

All the Hilbert spaces in this paper assumed to be real ($K = \mathbb{R}$) and separable.

Definition 2.1. *We say that a sequence of Hilbert spaces* $\{H_n\}$ *converges to a Hilbert space* H *if there exists a dense subspace* $C \subset H$ *and a sequence of operators*

$$\Phi_n : C \to H_n$$

with the following property:

$$\lim_{n\to\infty} ||\Phi_n u||_{H_n} = ||u||_H \tag{3}$$

for every $u \in C$.

Definition 2.2. *(Strong convergence) We say that a sequence of vectors* $\{u_n\}$ *with* $u_n \in H_n$ *strongly converges to a vector* $u \in H$ *if there exists a sequence* $\{\tilde{u}_m\} \subset C$ *with the following properties:*

$$||\tilde{u}_m - u||_H \to 0$$

$$\lim_m \overline{\lim_n} ||\Phi_n \tilde{u}_m - u_n||_{H_n} = 0.$$

Definition 2.3. *(Weak convergence) We say that a sequence of vectors* $\{u_n\}$, $u_n \in H_n$ *weakly converges to* $u \in H$ *if*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}$, $v_n \in H_n$ *strongly tending to* $v \in H$.

Definition 2.4. *We say that a sequence of bounded operators* $\{B_n\}$, $B_n \in L(H_n)$ *strongly converges to an operator* $B \in L(H)$ *if for every sequence* $\{u_n\}$, $u_n \in H_n$ *strongly tending to* $u \in H$ *the sequence* $\{B_n u_n\}$ *strongly tends to* Bu.

We define the space $\mathcal{H} := \bigcup_n H_n$ as the disjoint union of H_n and define convergence in $\mathcal{H}$ as in Definition 2.2. Now we consider convergence of quadratic forms in $\mathcal{H}$. Recall that a quadratic form is a bilinear mapping $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$, where $\mathcal{D}(\mathcal{E}) \subset H$ is some subspace of H. We will only consider non-negative and symmetric quadratic forms. Recall that a form $\mathcal{E}$ is closed if $\mathcal{D}(\mathcal{E})$ equipped with the inner product $\mathcal{E}^1(u) = (u, u)_H + \mathcal{E}(u)$ is complete. We identify a quadratic form $\mathcal{E}$ with the function

$$\mathcal{E}(u) : u \to \begin{cases} \mathcal{E}(u, u), \ u \in \mathcal{D}(\mathcal{E}) \\ \quad \infty, \ u \notin \mathcal{D}(\mathcal{E}). \end{cases}$$

It is well-know fact that $\mathcal{E}$ is closed if and only if $\mathcal{E} : H \to \overline{\mathbb{R}}$ is lower-semicontinuous (see [17]).

Definition 2.5. *We say that a sequence* $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ *of quadratic forms Mosco converges to a quadratic form* $\mathcal{E}$ *on* H *if the following conditions hold:*

(M1) *If a sequence* $\{u_n\}$ *with* $u_n \in H_n$ *weakly converges to* $u \in H$ *then*

$$\mathcal{E}(u) \le \underline{\lim}_n \mathcal{E}_n(u_n)$$

(M2) *For every* $u \in H$ *there exists a strongly convergent sequence* $u_n \to u$ *with* $u_n \in H_n$ *such that*

$$\mathcal{E}(u) = \lim_n \mathcal{E}_n(u_n).$$

With every closed form $\mathcal{E}$ we associate a non-negative self-adjoint operator $-A$ with $\mathcal{D}(A) = \mathcal{D}(\mathcal{E})$, such that $\mathcal{E}(u,v) = (-Au, v)$, $u, v \in \mathcal{D}(\mathcal{E})$. We will denote the associated semigroup e^{tA}, $t \geq 0$ by $\{T_t\}$ and the resolvent $(\beta - A)^{-1}$, $\beta > 0$, by $\{G_\beta\}$.

Definition 2.6. *We say that a sequence $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ of quadratic forms Γ-converges to a quadratic form $\mathcal{E}$ if the following conditions hold:*

(G1) *If a sequence $\{u_n\}$ with $u_n \in H_n$ strongly converges to $u \in H$ then*

$$\mathcal{E}(u) \leq \underline{\lim}_n \mathcal{E}_n(u_n)$$

(G2) *For every $u \in H$ there exists a strongly convergent sequence $u_n \to u$ with $u_n \in H_n$ such that*

$$\mathcal{E}(u) = \lim_n \mathcal{E}_n(u_n).$$

Clearly, Mosco convergence implies Γ-convergence. Note that Γ-convergence can be defined for arbitrary functions on a topological space with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ (see, for instance [6]) and every Γ-limit is lower-semicontinuous. In particular, it means that if the form $\mathcal{E}$ is a Γ-limit, then $\mathcal{E}$ is closed.

It is a well-known fact that every sequence of functions on a second-countable space with values in $\overline{\mathbb{R}}$ has a Γ-convergent subsequence (see [6], Theorem 8.5). In particular, the following Theorem holds (see [10] Theorem 2.3):

Theorem 2.7. *Every sequence $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ of quadratic forms has a Γ-convergent subsequence whose Γ-limit is a closed quadratic form on H.*

The main result of [10] was the following generalization of the Mosco Theorem.

Theorem 2.8. *(Mosco, Kuwae, Shioya) Let $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ be a sequence of closed forms and $\mathcal{E}$ be a closed form on H. The following are all equivalent:*

1. *$\{\mathcal{E}_n\}$ Mosco converges to $\mathcal{E}$*
2. *$\{G_{n,\beta}\}$ strongly converges to G_β for every $\beta > 0$*
3. *$\{T_{n,t}\}$ strongly converges to T_t for every $t > 0$.*

Definition 2.9. *Suppose that we are given a convergent sequence of Hilbert spaces $H_n \to H$ and a sequence of closed quadratic forms $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$. We say that a sequence of pairs $\{(H_n, \mathcal{E}_n)\}$ converges to $\{(H, \mathcal{E})\}$ if*

$$\Phi_n(C) \subset \mathcal{D}(\mathcal{E}_n) \quad \textit{for every } n,$$

$C \subset \mathcal{D}(\mathcal{E})$ is dense in $(\mathcal{D}(\mathcal{E}), \mathcal{E}^1)$ and

$$\lim_n \mathcal{E}_n(\Phi_n(u)) = \mathcal{E}(u)$$

for every $u \in C$.

If we have a convergent sequence $\{(H_n, \mathcal{E}_n)\}$ then we can consider the sequence $\{\mathcal{D}(\mathcal{E}_n)\}$ as a sequence of Hilbert spaces with inner products $\mathcal{E}_n^1(u) = (u,u)_{H_n} + \mathcal{E}_n(u)$ which converges to $\mathcal{D}(\mathcal{E})$. Let us denote the space $\bigcup_n \mathcal{D}(\mathcal{E}_n)$ by $\mathcal{H}^1$. Evidently, if

$u_n \to u$ strongly in $\mathcal{H}^1$, then $u_n \to u$ strongly in $\mathcal{H}$. The same is not true for the weak convergence. More precisely, we have the following

Proposition 2.10. *Let $\{(H_n, \mathcal{D}(\mathcal{E}_n))\}$ converges to $(H, \mathcal{D}(\mathcal{E}))$ in the sense of Definition 2.9. Suppose that for every $\mathcal{H}$-weakly convergent sequence $\{u_n\} \to u$ such that $\sup_n \mathcal{E}^1_n(u_n) < \infty$ one has that $u \in \mathcal{D}(\mathcal{E})$ and $u_n \to u$ weakly in $\mathcal{H}^1$. Then $\mathcal{E}_n \to \mathcal{E}$ Mosco and $G_{n,1}(v_n) \to G_1(v)$ in $\mathcal{H}^1$ if $v_n \to v$ in $\mathcal{H}$.*

Proof. Let us verify condition (M1) of Definition 2.5. Let $\{u_n\}$, $u_n \in H_n$ be a sequence which converges weakly to $u \in H$ in $\mathcal{H}$. If $\underline{\lim}_n \mathcal{E}_n(u_n) \leq \infty$, the conditions of the proposition imply that one can extract a subsequence (denoted again by u_n) such that $u_n \to u$ in $\mathcal{H}^1$ weakly. Let us take a sequence $\{v_n\}$, $v_n \in \mathcal{D}(\mathcal{E}_n)$ such that $v_n \to u$ strongly in $\mathcal{H}^1$. Then $\mathcal{E}(u)^2 = \lim_n |\mathcal{E}(u_n, v_n)|^2 \leq \underline{\lim}_n \mathcal{E}(u_n)\mathcal{E}(v_n) \leq \mathcal{E}(u)\underline{\lim}_n \mathcal{E}(u_n)$, hence by Corollary 7.4.1

$$\mathcal{E}(u) \leq \underline{\lim}_n \mathcal{E}_n(u_n).$$

The case $\underline{\lim}_n \mathcal{E}_n(u_n) = \infty$ is obvious. Condition (M1) is proved.

Now we verify condition (M2). Let $u \in \mathcal{D}(\mathcal{E})$. Then by Corollary 7.4.2 there exists a sequence $u_n \in \mathcal{D}(\mathcal{E}_n)$ which strongly converges to u in $\mathcal{H}^1$ and, consequently, in $\mathcal{H}$. Hence, by Corollary 7.4.1 $||u_n||^2_{H_n} + \mathcal{E}_n(u_n) \to ||u||^2_H + \mathcal{E}(u)$ and $||u_n||_{H_n} \to ||u||_H$. Consequently, $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. Let $u \notin \mathcal{D}(\mathcal{E})$. Then condition (M1) implies that $\lim_n \mathcal{E}_n(u_n) = \infty$ for every sequence $\{u_n\}$ strongly (even weakly) tending to u.

Now we prove the second statement. Let $v_n \to v$ strongly in $\mathcal{H}$. Let us take a sequence $\{u_n\}$, $u_n \in H_n$ such that $u_n \to u \in \mathcal{D}(\mathcal{E})$ weakly in $\mathcal{H}^1$. By the condition of the proposition $u_n \to u$ weakly in $\mathcal{H}$. Then $(G_1(v_n), u_n)_{H^1_n} = (v_n, u_n)_{H_n} \to (v, u)_H = (G_1(v), u)_{H^1}$. It follows from Corollary 7.5 that $G_{1,n}(v_n) \to G_1(v)$ strongly in $\mathcal{H}^1$. □

3. Convergence of non-diagonal forms

In this section we study convergence of (non-diagonal) d-dimensional symmetric Dirichlet forms $\{\mathcal{E}_n\}$. Every $\mathcal{E}_n$ is defined on the Hilbert space $L^2(\sigma_n\, dx)$.

Let $A_n(x)$ be a symmetric, non-negative dx-a.e. invertible $d\times d$-matrix $\big(A_n(x)\big)_{ij} = a^n_{ij}(x)$ depending on x. We denote the elements of the inverse matrix A_n^{-1} by $(a^{-1})^n_{i,j}$, $\big(A_n^{-1}(x)\big)_{ij} = (a^{-1})^n_{i,j}(x)$.

For an arbitrary function f let us define the set

$$R(f) = \{x : \exists \varepsilon, \int_{|x-y|\leq\varepsilon} \frac{dy}{|f(y)|} < \infty\}.$$

Evidently, R is the largest open set V such that $\frac{1}{f} \in L^1_{loc}(V)$. If $A(x)$ is a symmetric non-negative $d \times d$ matrix depending on x, we define $R(A)$ as a largest open set V such that $(a^{-1})^n_{i,j} \in L^1_{loc}(V)$ for $i,j \in \{1, \cdots, d\}$. Note that the condition $(a^{-1})^n_{i,i} \in L^1_{loc}(V)$, $i \in \{1, \cdots, d\}$ implies that $(a^{-1})^n_{i,j} \in L^1_{loc}(V)$ for $i,j \in \{1, \cdots, d\}$, since $|(a^{-1})^n_{i,j}| \leq \sqrt{(a^{-1})^n_{i,i}(a^{-1})^n_{j,j}}$.

Recall that we are given a sequence of convergent Hilbert spaces $\{H_n\} = \{L^2(\mu_n)\} = \{L^2(\sigma_n\, dx)\}$ and Assumption I from the Introduction holds.

Lemma 3.1. *Consider the following statements:*

(1) $f_n \to f$ *strongly (weakly) in* $L^2(\,dx)$

(2) $\frac{f_n}{\sqrt{\sigma}_n} \to \frac{f}{\sqrt{\sigma}}$ *strongly (weakly) in* $\mathcal{H}$.

Assume that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ *weakly in* $L^2_{loc}(\,dx)$. *Then (1) implies (2) for the strong convergence and (2) implies (1) for the weak convergence. If, in addition,* $\sqrt{\sigma_n} \to \sqrt{\sigma}$ *strongly in* $L^2_{loc}(\,dx)$ *then (1) and (2) are equivalent.*

Proof. Suppose first that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ weakly in $L^2_{loc}(\,dx)$. Let $f_n \to f$ in $L^2(\,dx)$. Let us find a sequence $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n\sqrt{\sigma} \to f$ in $L^2(\,dx)$. Then $\varphi_n \to \frac{f}{\sqrt{\sigma}}$ in $L^2(\sigma\, dx)$. Hence

$$\lim_m \overline{\lim_n} \int_{\mathbb{R}^d} \left(\varphi_m - \frac{f_n}{\sqrt{\sigma}_n}\right)^2 \sigma_n(x)\, dx$$

$$= \lim_m \overline{\lim_n} \int_{\mathbb{R}^d} (\varphi_m\sqrt{\sigma}_n - f_n)^2\, dx = \lim_m (\varphi_m\sqrt{\sigma} - f)^2\, dx = 0.$$

Now suppose that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ strongly in $L^2_{loc}(\,dx)$ and $\frac{f_n}{\sqrt{\sigma}_n} \to \frac{f}{\sqrt{\sigma}}$ strongly in $\mathcal{H}$. Then there exist a sequence of C_0^∞-functions $\{\varphi_n\}$ such that $\varphi_n\sqrt{\sigma} \to f$ in $L^2(\,dx)$ and $\lim_m \overline{\lim}_n \int_{\mathbb{R}^d} (\varphi_m\sqrt{\sigma}_n - f_n)^2\, dx = 0$. Since $\lim_m \lim_n \varphi_m\sqrt{\sigma}_n = f$ (all the limits are $L^2(\,dx)$-limits), one can find a subsequence $n(k)$ such that $\varphi_k\sqrt{\sigma}_{n(k)}$ tends to f and $\lim_k ||\varphi_k\sqrt{\sigma}_{n(k)} - f_{n(k)}||_{L^2(\,dx)} = 0$. Hence $f_{n(k)} \to f$ in $L^2(\,dx)$. Since we can do the same with every subsequence of $\{f_n\}$, we get that $f_n \to f$ in $L^2(\,dx)$.

The case of the weak convergence follows easily from the statement of the lemma for the strong convergence. □

Lemma 3.2. *Let* $\sqrt{\sigma_n} \to \sqrt{\sigma}$ *weakly in* $L^2_{loc}(dx)$, $f_n \to f$ *weakly in* $\mathcal{H}$ *and* $f_n \to \tilde{f}$ dx*-almost sure. Then* $f = \tilde{f}$ dx*-a.e.*

Proof. Let us fix $N > 0$ and a bounded Borel set B. Denote

$$A_N = \{x : \sup_n |f_n(x)| < N\}.$$

Note that by Lemma 3.1 $f_n\sqrt{\sigma_n} \to f\sqrt{\sigma}$ weakly in $L^2(\,dx)$. Hence

$$\int_{A_N\cap B} f_n\sqrt{\sigma_n}\, dx \to \int_{A_N\cap B} f\sqrt{\sigma}\, dx.$$

But in the other hand $f_n\chi_{A_N\cap B} \to \tilde{f}\chi_{A_N\cap B}$ in $L^2(\,dx)$, hence $\int_{A_N\cap B} f_n\sqrt{\sigma_n}\, dx \to \int_{A_N\cap B} \tilde{f}\sqrt{\sigma}\, dx$. This implies that $\tilde{f} = f$ a.e. on A_N. The claim follows from the fact that $\bigcup_N A_N$ has full Lebesgue measure. □

Remark 3.3. *Let $\sqrt{\sigma_n} \to \sqrt{\sigma}$ strongly in $L^2_{loc}(dx)$. Then by Lemma 3.1 one can introduce a natural metric on $\mathcal{H}$ such that the convergence in $\mathcal{H}$ coincides with the convergence in this metric. Indeed, for $f \in H_n$ and $g \in H_m$, $n, m \in \mathbb{N} \cup \{\infty\}$ we define*

$$d(f, g) = \sqrt{|\delta_n - \delta_m|^2 + ||f\sqrt{\sigma}_n - g\sqrt{\sigma}_m||^2_{L^2(dx)}},$$

where $H_\infty = H$, $\sigma_\infty = \sigma$, $\delta_n = \frac{1}{n}$, $\delta_\infty = 0$.

Consider the following sequence of symmetric Dirichlet forms:

$$\mathcal{E}_n(f, g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a^n_{ij}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^d) \tag{4}$$

defined first on $C_0^\infty(\mathbb{R}^d)$. Here $a^n_{ij}(x)$ are Borel local integrable functions and $a^n_{i,j} = a^n_{j,i}$. We will denote by $A_n(x)$ the symmetric $d \times d$-matrix $\big(A_n(x)\big)_{ij} = a^n_{ij}(x)$ depending on x. Then (4) can be written as $\mathcal{E}_n(f, g) = \int_{\mathbb{R}^d} \big(A_n \nabla f, \nabla g\big) \, dx$. We suppose that A_n is dx-a.e. invertible and non-negative.

The following Assumption holds throughout the section and ensures the closability of $(\mathcal{E}_n, C_0^\infty(\mathbb{R}^d))$. Note that this Assumption is weaker than the standard closability conditions: $A_n > \operatorname{diag}\{\rho_1, \cdots, \rho_n\}$ and $dx(\mathbb{R}^d \setminus R(\rho_i)) = 0$, $i \in \{1, \cdots, d\}$.

Assumption II

$$R(A) \subset R(\sigma), \quad R(A_n) \subset R(\sigma_n), \quad dx(\mathbb{R}^d \setminus R(A_n)) = dx(\mathbb{R}^d \setminus R(A)) = 0.$$

Lemma 3.4. *The form $(\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$*

$$\mathcal{D}(\mathcal{E}^+) = \{f : f \in L^2(\sigma \, dx), \quad f \text{ admits weak derivatives } \partial_i f \text{ in } R(A)$$
$$\text{for every } i \in \{1, \cdots, d\}, \quad \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial f(x)}{\partial x_j} \, dx < \infty\},$$

$$\mathcal{E}^+(f, g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} \, dx, \quad f, g \in \mathcal{D}(\mathcal{E}^+).$$

is closed.

Proof. Let $f_n \to f$ in $L^2(\sigma \, dx)$, $f_n \in \mathcal{D}(\mathcal{E}^+)$ and $\mathcal{E}^+(f_n - f_m) \to 0$ as $n, m \to 0$. Let us fix a bounded domain $\Omega \subset R(\sigma)$. From the inequality

$$\Big(\int_\Omega |f - f_n| \, dx\Big)^2 \leq \int_\Omega (f - f_n)^2 \sigma \, dx \int_\Omega \frac{dx}{\sigma}$$

we get that $f_n \to f$ in $L^1_{loc}(R(\sigma))$.

Fix a bounded domain $\Omega \subset R(A)$. Now let us show that $\nabla(f_n - f_m) \to 0$ in $L^1_{loc}(\Omega)$ as $n, m \to \infty$. Denote by L_∞ the space of functions $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $||f||_{L_\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^d}\sqrt{(f(x),f(x))} < \infty$. Let $A = Q^TQ$, where Q is a positive symmetric matrix. We get

$$\begin{aligned}
\Big(\int_\Omega |\nabla(f_n-f_m)|\,dx\Big)^2 &= \Big(\sup_{||v||_{L_\infty}\le 1}\int_\Omega (\nabla(f_n-f_m),v)\,dx\Big)^2 \\
&= \Big(\sup_{||v||_{L_\infty}\le 1}\int_\Omega (Q\nabla(f_n-f_m),(Q^{-1})^Tv)\,dx\Big)^2 \\
&\le \int_\Omega (Q\nabla(f_n-f_m),Q\nabla(f_n-f_m))\,dx \sup_{||v||_{L_\infty}\le 1}\int_\Omega ((Q^{-1})^Tv,(Q^{-1})^Tv)\,dx \\
&= \int_\Omega (A\nabla(f_n-f_m),\nabla(f_n-f_m))\,dx \sup_{||v||_{L_\infty}\le 1}\int_\Omega (A^{-1}v,v)\,dx.
\end{aligned}$$

Obviously, $\sup_{||v||_{L_\infty}\le 1}\int_\Omega(A^{-1}v,v)\,dx < \infty$ in virtue of Assumption II. Hence $\int_\Omega|\nabla(f_n-f_m)|\,dx \to 0$ as $n,m\to\infty$ and $\partial_i f_n \to g_i$ in $L^1_{loc}(\Omega)$ for every $i\in\{1,\cdots d\}$. Let $\varphi\in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\varphi)\subset\Omega$. Since $\int_{\mathbb{R}^d} f\partial_i\varphi\,dx = \lim_n\int_{\mathbb{R}^d} f_n\partial_i\varphi\,dx = -\lim_n\int_{\mathbb{R}^d}\partial_i f_n\varphi\,dx = -\int_{\mathbb{R}^d} g_i\varphi\,dx$, we get that f admits weak derivatives in $R(A)$. Applying a standard diagonal procedure we may assume that $\nabla f_n\to\nabla f$ dx-a.e. Hence by Fatou's lemma $\mathcal{E}^+(f_n-f)\le \underline{\lim}_m\mathcal{E}^+(f_n-f_m)$ which is arbitrary small if n is large. The proof is complete. □

Let us denote by $\mathcal{D}(\mathcal{E}_n)$ the completion of $C_0^\infty(\mathbb{R}^d)$ w.r.t. $(\mathcal{E}^1_n(u))^{\frac12} = \big[||u||_{L^2(\mu_n)} + \mathcal{E}_n(u)\big]^{\frac12}$. The extension of $(\mathcal{E}_n, C_0^\infty(\mathbb{R}^d))$ to $\mathcal{D}(\mathcal{E}_n)$ is a Dirichlet form which is closed in $L^2(\sigma\,dx)$. We will denote it in the following again by $\mathcal{E}_n$ or $(\mathcal{E}_n,\mathcal{D}(\mathcal{E}_n))$. This is the so-called "minimal" extension. The form $\mathcal{E}_n^+$ is called the "maximal" extension.

Recall that a symmetric closed form Q is called a Dirichlet form if for every normal contraction T, i.e. for every function $T:\mathbb{R}\to\mathbb{R}$ with $T(0)=0$, $|T(s)-T(t)|\le|s-t|$ and $f\in\mathcal{D}(Q)$ one has $T(f)\in\mathcal{D}(Q)$ and $Q(T(f))\le Q(f)$ (see [18], I.4). It is well-known that the forms $\mathcal{E}_n$ and $\mathcal{E}_n^+$ are Dirichlet forms (see [8], [18]).

The following theorem gives us sufficient conditions for existence of a Mosco convergent subsequence of a sequence of Dirichlet forms. The idea of the proof is to show that the Γ-limit of a subsequence (which always exists by Theorem 2.7) is in fact the Mosco limit. Note that condition (5) can be regarded as some kind of "uniform closability" of $\{\mathcal{E}_n\}$.

Theorem 3.5. *Suppose that Assumptions I-II hold and there exist a sequence Ω_N of increasing bounded domains such that $dx(\mathbb{R}^d\setminus\bigcup_N\Omega_N)=0$ and for every N*

$$\sup_n\int_{\Omega_N}\frac{dx}{\sigma_n}<\infty,\quad \sup_n\int_{\Omega_N}|(a^{-1})^n_{i,j}|\,dx<\infty,\quad i,j\in\{1,\cdots,d\}. \tag{5}$$

Assume in addition that the sequence $\{\mu_n=\sigma_n\,dx\}$ satisfies one of the following conditions:

(i) *Every $\mu_n=\sigma_n\,dx$ is a finite measure and $\sqrt{\sigma_n}\to\sqrt{\sigma}$ in $L^2(dx)$*

(ii) $\sqrt{\sigma_n} \to \sqrt{\sigma}$ *in* $L^2_{loc}(dx)$ *and there exists a constant* $C > 0$ *such that*

(6)
$$\sum_{i,j=1}^{n} a^n_{i,j}(x)\xi_i\xi_j \le C\sigma_n(x)||\xi||^2$$

for every n *and for every vector* $\xi \in \mathbb{R}^d$.

Let $(\mathcal{E}_n, \mathcal{D}(\mathcal{E}_n))$ *and* $(\mathcal{E}^+_n, \mathcal{D}(\mathcal{E}^+_n))$ *be, accordingly, the minimal and the maximal extension of* $(\mathcal{E}_n, C_0^\infty(\mathbb{R}^d))$ *in* $L^2(\sigma_n\, dx)$. *Then there exists a subsequence* $\{n_k\}$ *and Dirichlet forms* $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}}^+$ *on* $L^2(\sigma\, dx)$ *such that* $\mathcal{E}_{n_k} \to \tilde{\mathcal{E}}$ *and* $\mathcal{E}^+_{n_k} \to \tilde{\mathcal{E}}^+$ *Mosco.*

Proof. We give the proof only for the case $\{\mathcal{E}_n\}$ because the case of $\{\mathcal{E}^+_n\}$ can be proved in the same way.

By Theorem 2.7 one can find a subsequence (denoted again by $\{\mathcal{E}_n\}$), which Γ-converges to a closed symmetric form $\tilde{\mathcal{E}}$. Let us show that in fact $\mathcal{E}_n \to \tilde{\mathcal{E}}$ Mosco. We have to prove that if $f_n \to f$ weakly, then $\tilde{\mathcal{E}}(f) \le \underline{\lim}_n \mathcal{E}_n(f_n)$. Let $\underline{\lim}_n \mathcal{E}_n(f_n) \le \infty$ (the other case is trivial).

Let us fix some N. In the same way as in Lemma 3.4 we show that

$$\Big(\int_{\Omega_N} |f_n|\, dx\Big)^2 \le \int_{\Omega_N} f_n^2\sigma_n\, dx \int_{\Omega_N} \frac{1}{\sigma_n}\, dx$$

and

$$\Big(\int_{\Omega_N} |\nabla f_n|\, dx\Big)^2 \le \Big[\int_{\Omega_N} (A_n\nabla f_n, \nabla f_n)\, dx\Big] \sup_{||v||_{L_\infty}\le 1} \int_{\Omega_N} (A_n^{-1}v, v)\, dx.$$

By virtue of (5) it is obvious that one can extract a subsequence from $\{f_n\}$, which is bounded in $H^{1,1}(\Omega_N)$. By the compactness of the embedding of $H^{1,1}(\Omega_N)$ to $L^1(\Omega_N)$ (see, e.g. [19]) we get a subsequence (denoted again by $\{f_n\}$) which converges to some function in $L^1_{loc}(\Omega_N)$. Taking into account that $dx(\mathbb{R}^d\setminus\bigcup_N \Omega_N) = 0$ it follows that (by a standard diagonal procedure) we can find a subsequence which converges to some function $\tilde{f}$ dx-a.e. on the whole space $\mathbb{R}^d$. By Lemma 3.2 we have $f = \tilde{f}$ dx-a.e.

Now suppose that (i) is fulfilled. Let $T^k(t) = (t \wedge k) \vee (-k)$. Note that T^k is a normal contraction. Since $f_n \to f$ dx-a.e. and $\sqrt{\sigma_n} \to \sqrt{\sigma}$ in $L^2(\, dx)$ one easily proves that $T^k(f_n)\sqrt{\sigma_n} \to T^k(f)\sqrt{\sigma}$ in $L^2(\, dx)$ as $n \to \infty$. By Lemma 3.1 $T^k(f_n) \to T^k(f)$ strongly in $\mathcal{H}$ and

$$\tilde{\mathcal{E}}(T^k(f)) \le \underline{\lim}_n \mathcal{E}_n(T^k(f_n)) \le \underline{\lim}_n \mathcal{E}_n(f_n).$$

Note that $T^k(f) \to f$ in $L^2(\sigma\, dx)$ as $k \to \infty$. Hence by closedness of $\tilde{\mathcal{E}}$ we get

$$\tilde{\mathcal{E}}(f) \le \underline{\lim}_k \tilde{\mathcal{E}}(T^k(f)) \le \underline{\lim}_n \mathcal{E}_n(f_n).$$

Now let us show that $\tilde{\mathcal{E}}$ is a Dirichlet form. To this end let us fix $f \in \mathcal{D}(\tilde{\mathcal{E}})$ and choose $\{f_n\}$ such that $f_n \to f$ strongly and $\mathcal{E}_n(f_n) \to \tilde{\mathcal{E}}(f)$. In the same way as above we show that $\{f_n^+ \wedge 1\}$ tends strongly to $f^+ \wedge 1$. By the contraction property

$\tilde{\mathcal{E}}(f^+ \wedge 1) \le \underline{\lim}_n \mathcal{E}_n(f_n^+ \wedge 1) \le \underline{\lim}_n \mathcal{E}_n(f_n) = \tilde{\mathcal{E}}(f)$. This implies that $\tilde{\mathcal{E}}$ is a Dirichlet form (see [18]).

Suppose that (ii) is fulfilled. Let us take a sequence of C_0^∞-functions $\{\varphi_k\}$ such that $\varphi_k \le 1$, $\varphi_k(x)$ tends to 1 for every x and $|\nabla \varphi_k| \le \frac{1}{k}$. By Corollary 4.5 of [18] $\varphi_k T^k(f_n) \in \mathcal{D}(\mathcal{E}_n)$. Observe that $\varphi_k T^k(f_n)\sqrt{\sigma_n} \to \varphi_k T^k(f)\sqrt{\sigma}$ in $L^2(\,dx)$ as $n \to \infty$.

Hence

$$\varphi_k T^k(f_n) \to \varphi_k T^k(f), \quad n \to \infty$$

strongly in $\mathcal{H}$ and consequently

$$\tilde{\mathcal{E}}(\varphi_k T^k(f)) \le \underline{\lim}_n \mathcal{E}_n(\varphi_k T^k(f_n)),$$

$$\mathcal{E}_n(\varphi_k T^k(f_n)) =$$

$$\sum_{i,j=1}^n \int_{\mathbb{R}^d} a_{i,j}^n \frac{\partial (T^k(f_n)\varphi_k)}{\partial x_i} \frac{\partial (T^k(f_n)\varphi_k)}{\partial x_j}\, dx = \sum_{i,j=1}^n \int_{\mathbb{R}^d} a_{i,j}^n \frac{\partial T^k(f_n)}{\partial x_i} \frac{\partial T^k(f_n)}{\partial x_j} \varphi_k^2\, dx$$
$$+ 2\sum_{i,j=1}^n \int_{\mathbb{R}^d} a_{i,j}^n T^k(f_n)\varphi_k \frac{\partial T^k(f_n)}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j}\, dx + \sum_{i,j=1}^n \int_{\mathbb{R}^d} a_{i,j}^n \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} (T^k(f_n))^2\, dx.$$

The first term does not exceed $\mathcal{E}_n(T^k(f_n))$ and, consequently, $\mathcal{E}_n(f_n)$. The third term can be estimated by $\frac{C}{k^2}||f_n||^2_{L^2(\sigma_n\, dx)}$ in virtue of (6). By Cauchy inequality the absolute value of the second term does not exceed $\frac{2\sqrt{C}}{k}||f_n||_{L^2(\sigma_n\, dx)}\sqrt{\mathcal{E}_n(f_n)}$. Hence $\overline{\lim}_k \tilde{\mathcal{E}}(\varphi_k T^k(f)) \le \underline{\lim}_n \mathcal{E}_n(f_n)$. Observe that by Lebesgue Theorem $\varphi_k T^k(f) \to f$ in $L^2(\sigma\, dx)$ as $k \to \infty$. Since $\tilde{\mathcal{E}}$ is closed we get that

$$\tilde{\mathcal{E}}(f) \le \underline{\lim}_k \tilde{\mathcal{E}}(\varphi_k T^k(f)) \le \underline{\lim}_n \mathcal{E}_n(f_n).$$

The contraction property of $\tilde{\mathcal{E}}$ can be shown the same way as above. The proof is complete. □

In the previous Theorem we established existence of Mosco-limit points of $\{\mathcal{E}_n\}$. But even if $\mathcal{E}_n(\varphi) \to \mathcal{E}(\varphi)$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ and some form $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$, the Mosco limit does not coincide in general with the closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$. In the following section we show that Mosco limits can be quite irregular. The following theorem gives us sufficient conditions for identifying $\mathcal{E}$ with the Mosco-limit. Note that in this theorem we omit conditions (i) and (ii), but we give the proof under other geometric restrictions, namely we require that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and $(\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$ coincide. Some sufficient condition for $(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$ can be found in [5], [7], [23], [24].

Proof of Theorem 1.1: Let $f_n \to f$ weakly in $\mathcal{H}$ and $\underline{\lim}_n \mathcal{E}_n^+(f_n) < \infty$.

Let us fix a bounded domain Ω and take $g \in L_\infty$ (see the proof of Theorem 3.5) with $\text{supp}(g) \subset \Omega$. Let $A_n = (Q_n)^T Q_n$, where Q_n is a symmetric non-negative matrix. We have

$$(\int_{\mathbb{R}^d}(\nabla f_n, g)\,dx)^2 = (\int_{\mathbb{R}^d}(Q_n\nabla f_n, (Q_n^{-1})^T g)\,dx)^2 \le$$

$$\Big(\int_{\mathbb{R}^d}(Q_n\nabla f_n, Q_n\nabla f_n)\,dx\Big)\int_{\mathbb{R}^d}((Q_n^{-1})^T g, (Q_n^{-1})^T g)\,dx = \mathcal{E}_n^+(f_n)\int_{\mathbb{R}^d}(A_n^{-1}g, g)\,dx.$$

Let us extract a subsequence (denoted again by $\{f_n\}$) such that $\sup_n \mathcal{E}_n^+(f_n) < \infty$. In the same way as in Lemma 3.4 we get the estimation

$$(\int_{\mathbb{R}^d}|\nabla f_n|\,dx)^2 \le \sup_n \mathcal{E}^+(f_n) \sup_{n,||g||_{L_\infty}}\int_{\mathbb{R}^d}(A_n^{-1}g, g)\,dx.$$

From (1) we get that $\{\nabla f_n\}$ is bounded in $L^1(\Omega)$. In the same way as in Theorem 3.5 we find a subsequence, which is bounded in $H^{1,1}(\Omega)$ for every bounded domain Ω. By the compact embedding theorem and a standard diagonal procedure one can find a subsequence (denoted again by $\{f_n\}$) which converges to some function $\tilde{f}$ in $L^1_{loc}(\,dx)$. By Lemma 3.2 $f = \tilde{f}$ dx-a.e.

Since the sequence $\{\partial_i f_n\}$ is uniformly bounded for every $1 \le i \le d$ in $L^1(\Omega)$, one can extract a subsequence (denoted again by $\{\partial_i f_n\}$) such that $\{\partial_i f_n\}$ tends vaguely to a locally finite measure m^i.

In the same way as above we get that for every continuous mapping $g : \mathbb{R}^d \to \mathbb{R}^d$ with compact support

$$\Big(\sum_i \int_{\mathbb{R}^d} g_i\,dm^i(x)\Big)^2 \le \underline{\lim}_n \mathcal{E}_n^+(f_n)\int_{\mathbb{R}^d}(A^{-1}g, g)\,dx. \tag{7}$$

This implies that every m^i is absolutely continuous w.r.t. Lebesgue measure. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Note that $\int_{\mathbb{R}^d} f\partial_i\varphi\,dx = \lim_n \int_{\mathbb{R}^d} f_n\partial_i\varphi\,dx = -\lim_n \int_{\mathbb{R}^d}\partial_i f_n\varphi\,dx = -\int_{\mathbb{R}^d}\varphi\,dm^i(x)$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$. It means that f admits partial weak derivatives and $m^i = \partial_i f\,dx$. So, we get $(\int_{\mathbb{R}^d}(\nabla f, g)\,dx)^2 \le \underline{\lim}_n \mathcal{E}_n^+(f_n)\int_{\mathbb{R}^d}(A^{-1}g, g)\,dx$.

Let us find a sequence of continuous mappings with compact supports g_n^N such that $g_n^N \to A^T\nabla f\chi_{B_N}$ w.r.t. the norm $g \to \int_{\mathbb{R}^d}|(\nabla f, g)|\,dx + \sqrt{\int_{\mathbb{R}^d}(A^{-1}g, g)\,dx}$, where

$$B_N = \{x : |x| \le N, (A\nabla f, \nabla f) \le N\}.$$

From 7 we get that

$$\int_{B_N}(A\nabla f, \nabla f)\,dx \le \underline{\lim}_n \mathcal{E}_n^+(f_n).$$

Tending N to infinity we get that $\int_{\mathbb{R}^d}(A\nabla f, \nabla f)\,dx \le \underline{\lim}_n \mathcal{E}_n^+(f_n) \le \underline{\lim}_n \mathcal{E}_n(f_n)$. Condition (M1) is fulfilled for $\{\mathcal{E}_n\}$ as well as for $\{\mathcal{E}_n^+\}$.

Condition (M2) is obviously fulfilled, since $C = C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}(\mathcal{E}^+) = \mathcal{D}(\mathcal{E})$, hence $\{(L^2(\sigma_n\,dx), \mathcal{E}_n)\}$ and $\{(L^2(\sigma_n\,dx), \mathcal{E}_n^+)\}$ converge to $(L^2(\sigma\,dx), \mathcal{E})$ in the sense of Definition 2.9 and (M2) can be proved in the same way as in the proof of Proposition 2.10. □

Now consider a special case $\sigma_n = \rho_n^i = \rho_n$. Consider the semigroup $P_{n,t}$ in $L^2(\rho_n\, dx)$ associated with the form $\mathcal{E}^+_{\rho_n}$, where

$$\mathcal{E}_{\rho_n}(f,g) = \int_{\mathbb{R}^d} (\nabla f, \nabla g)\rho_n\, dx, \quad f,g \in C_0^\infty(\mathbb{R}^d)$$

and set

$$T_{n,t} = \varphi_n P_{n,t} \varphi_n^{-1},$$

where $\varphi_n = \sqrt{\rho_n}$, so that $T_{n,t} = e^{-tH_n}$, where H_n is called a (generalized) Schrödinger Hamiltonian, formally given by $H_n = -\Delta + \frac{\Delta\varphi_n}{\varphi_n}$. This semigroup acts on $L^2(\,dx)$. The following Lemma is just a trivial corollary of Lemma 3.1.

Lemma 3.6. *Let $\sqrt{\rho_n} \to \sqrt{\rho}$ in $L^2_{loc}(\,dx)$. Then $\{T_{n,t}\}$ converges strongly to T_t in $L^2(\,dx)$ if and only if $\{P_{n,t}\}$ converges strongly to P_t in $\mathcal{H}$.*

We get the following corollary from Lemma 3.6, Theorem 3.5, Theorem 1.1 and Theorem 2.8.

Corollary 3.7. *Let $\sqrt{\rho_n} \to \sqrt{\rho}$ in $L^2_{loc}(\,dx)$ strongly and there exists an increasing sequence of bounded domains Ω_N such that $dx(\mathbb{R}^d \setminus \bigcup_N \Omega_N) = 0$ and*

$$\sup_n \int_{\Omega_N} \frac{dx}{\rho_n}\, dx < \infty$$

for every N. Then there exists a semigroup T_t on $L^2(\,dx)$ and a subsequence n_k such that $T_{n_k,t} \to T_t$ strongly. Suppose in addition that $R(\rho_n) = R(\rho) = \mathbb{R}^d$, $\frac{dx}{\rho_n}$ tends vaguely to $\frac{dx}{\rho}$ and the maximal and the minimal extensions of the form $(\mathcal{E}_\rho, C_0^\infty(\mathbb{R}^d))$ coincide. Then $T_t = \sqrt{\rho} P_t \sqrt{\rho^{-1}}$, where P_t is the semigroup associated with the form $\mathcal{E}_\rho$.

Remark 3.8. *This result improves well-known results from [2]. The existence of the subsequence which has a limit semigroup was obtained in [2] under the following assumption: there exist a sequence of bounded domains Ω_N, such that $\bigcup_N \Omega_N$ has full Lebesgue measure and $\frac{\chi_{\Omega_N}}{\sqrt{\rho_n}} \to \frac{\chi_{\Omega_N}}{\sqrt{\rho}}$ in $L^2_{loc}(\,dx)$ for every N. If, moreover, $\frac{1}{\sqrt{\rho_n}} \to \frac{1}{\sqrt{\rho}}$ in $L^2_{loc}(dx)$, the limit can be identified with $\sqrt{\ }\overline{\rho} P_t \sqrt{\rho^{-1}}$, where P_t is the semigroup associated with the maximal extension of $(\mathcal{E}_\rho, C_0^\infty(\mathbb{R}^d))$.*

In the fourth section we prove even stronger result about identifying the Mosco limit in the diagonal case (see Theorem 1.3). We will show that if the dimension of the space is more than one, then the condition "$\frac{dx}{\rho_n} \to \frac{dx}{\rho}$ vaguely" is not necessary for identifying the limit with $\mathcal{E}_\rho$ (see Example 6.2).

4. One-dimensional case.

Let μ be a positive, locally finite Borel measure on $\mathbb{R}$ with full support. Let $\mu = \mu_{ac} + \mu_s + \mu_d$ be the decomposition of μ in absolutely continuous, singular and discrete parts and S_{ac}, S_s and S_d be Borel disjoint sets such that $\mu_{ac}(\mathbb{R} \setminus S_{ac}) = \mu_s(\mathbb{R} \setminus S_s) = \mu_d(\mathbb{R} \setminus S_d) = 0$.

Consider a function f which is supposed to have finite variation on every bounded set. Let

$$f = f_{ac} + f_s + f_d$$

be a representation of f as a sum of absolutely continuous, singular and discrete (jump) parts (which are uniquely defined up to a constant). We assume that f_{ac}, f_s and f_d admit weak derivatives which are absolutely continuous w.r.t. μ_{ac}, μ_s and μ_d, respectively. The Radon-Nicodym densities of these derivatives we will denote by $\frac{\partial f_{ac}}{\partial \mu_{ac}}$, $\frac{\partial f_s}{\partial \mu_s}$ and $\frac{\partial f_d}{\partial \mu_d}$.

We define $\mathcal{D}(\mathcal{E}_\mu)$ as a set of functions with the above properties such that

$$\mathcal{E}_\mu(f) = \int_{\mathbb{R}} \Big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\Big)^2 d\mu_{ac} + \Big(\frac{\partial f_s}{\partial \mu_s}\Big)^2 d\mu_s + \Big(\frac{\partial f_d}{\partial \mu_d}\Big)^2 d\mu_d < \infty.$$

Note that for an absolutely continuous function f one has: $\mathcal{E}_\mu(f) = \int_{\mathbb{R}} (f'\tilde{\rho})^2 \frac{1}{\tilde{\rho}}\, dx = \int_{\mathbb{R}} (f')^2 \tilde{\rho}\, dx$.

We will show in this section that $\mathcal{E}_\mu$ is closed in $L^2(\sigma\, dx)$ (Lemma 4.1).

The following assumptions hold throughout the section.

Assumption III. *ρ_n and σ_n are non-negative locally integrable functions such that $R(\rho_n) = R(\sigma_n) = \mathbb{R}^d$.*

Assumption IV. *There exists a locally finite measure μ on $\mathbb{R}$ with the absolutely continuous part $\frac{dx}{\tilde{\rho}}$ such that $\tilde{\rho} > 0$ dx-a.e., $\frac{1}{\tilde{\rho}} \in L^1_{loc}(\,dx)$ and $\frac{dx}{\rho_n} \to \mu$ vaguely.*

We consider a sequence of forms $\{\mathcal{E}_n\}$, where

$$\mathcal{E}_n(f) = \int_{\mathbb{R}} (f')^2 \rho_n\, dx,$$

and every $\mathcal{E}_n$ is defined on $L^2(\sigma_n\, dx)$. Under Assumption III all $\mathcal{E}_n$ are closed.

Proof of Theorem 1.2: Condition (M2) follows from Proposition 4.4. Condition (M1) we get from Proposition 4.3 and Lemma 3.2. □

Lemma 4.1. *$\mathcal{E}_\mu$ is closed in $L^2(\sigma\, dx)$.*

Proof. Let $f_n \to f$ in $L^2(\sigma\, dx)$, $f_n \in \mathcal{D}(\mathcal{E}_\mu)$ and $\mathcal{E}_\mu(f_n - f_m) \to 0$, $n, m \to \infty$.

Since the sequences $\frac{\partial (f_n)_{ac}}{\partial \mu_{ac}}$, $\frac{\partial (f_n)_s}{\partial \mu_s}$ and $\frac{\partial (f_n)_d}{\partial \mu_d}$ are fundamental in $L^2(\mu_{ac})$, $L^2(\mu_s)$ and $L^2(\mu_d)$, respectively, there exist $g_{ac} \in L^2(\mu_{ac})$, $g_s \in L^2(\mu_s)$ and $g_d \in L^2(\mu_d)$, such that $\frac{\partial (f_n)_{ac}}{\partial \mu_{ac}} \to g_{ac}$ in $L^2(\mu_{ac})$, $\frac{\partial (f_n)_s}{\partial \mu_s} \to g_s$ in $L^2(\mu_s)$ and $\frac{\partial (f_n)_d}{\partial \mu_d} \to g_d$ in $L^2(\mu_d)$.

Let us take $\varphi \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset [-N, N]$. The condition $\frac{1}{\sigma} \in L^1_{loc}(\,dx)$ implies that $f_n \to f$ in $L^1_{loc}(\,dx)$, hence $\int_{\mathbb{R}} f_n \varphi'\, dx \to \int_{\mathbb{R}} f \varphi'\, dx$. But in the other hand

$$\begin{aligned}\int_{\mathbb{R}} f\varphi' \, dx &= \lim_n \int_{\mathbb{R}} f_n\varphi' \, dx \\ &= -\lim_n \Big[\int_{\mathbb{R}} \frac{\partial (f_n)_{ac}}{\partial \mu_{ac}}\varphi \, d\mu_{ac} + \int_{\mathbb{R}} \frac{\partial (f_n)_s}{\partial \mu_s}\varphi \, d\mu_s + \int_{\mathbb{R}} \frac{\partial (f_n)_d}{\partial \mu_d}\varphi \, d\mu_d\Big] \\ &= -\int_{\mathbb{R}} g_{ac}\varphi \, d\mu_{ac} - \int_{\mathbb{R}} g_s\varphi \, d\mu_s - \int_{\mathbb{R}} g_d\varphi \, d\mu_d.\end{aligned}$$

Now let us take $\tilde{f} = -\int_{-N}^{x} g_{ac}\, d\mu_{ac} - \int_{-N}^{x} g_s\, d\mu_s - \int_{-N}^{x} g_d\, d\mu_d$. One easily get by Fubini Theorem that $\int_{\mathbb{R}}(f-\tilde{f})\varphi' \, dx = 0$. Hence $f - \tilde{f} = c$ dx-a.e. on $[-N, N]$. Consequently, f has finite variation on $[-N, N]$ and f_{ac}, f_s and f_d admit weak derivatives $g_{ac}\, d\mu_{ac}$, $g_s\, d\mu_s$ and $g_d\, d\mu_d$, respectively. The prove is complete.

□

Now we consider a set $\tilde{C} \subset \mathcal{D}(\mathcal{E}_\mu)$ consisting of the functions of the type:

$$f(x) = \int_{-\infty}^{x} \sum_{i=1}^{m} c_i \chi_{A_i} \, d\mu, \tag{8}$$

where $A_i = [a_i, b_i]$ are bounded closed intervals with the property $\mu(a_i) = \mu(b_i) = 0$ and c_i are positive numbers such that $\sum_{i=1}^{m} c_i \mu(\chi_{A_i}) = 0$. Obviously, $\tilde{C}$ is dense in the subspace $\{f : f \in L^2(\mu), \int_{\mathbb{R}} f\, d\mu = 0\} \subset L^2(\mu)$.

Lemma 4.2. *Suppose that there exists $\alpha > 0$ such that $\tilde{\rho} \le \alpha\sigma$ dx-a.e. Then for every $f \in \mathcal{D}(\mathcal{E}_\mu)$ there exists a sequence of functions $\{f_n\}$, $f_n \in \tilde{C}$ such that $f_n \to f$ in $L^2(\sigma\, dx)$ and $\mathcal{E}_\mu(f_n - f) \to 0$.*

Proof. Suppose first that $\mathrm{supp}(f) \in [-N, N]$. Then

$$f(x) = \int_{-N}^{x} \frac{\partial f_{ac}}{\partial \mu_{ac}} \, d\mu_{ac} + \int_{-N}^{x} \frac{\partial f_s}{\partial \mu_s} \, d\mu_s + \int_{-N}^{x} \frac{\partial f_d}{\partial \mu_d} \, d\mu_d$$

dx-a.e. (here $\int_{-N}^{x} g\, dm := \int_{\mathbb{R}} g\chi_{[-N,x]}\, dm$). Let us find a sequence of functions $\{f_n\}$, $f_n \in \tilde{C}$, $f_n = \int_{-\infty}^{x} \psi_n\, d\mu$ with $\mathrm{supp}(\psi_n) \in [-N, N]$ such that

$$\psi_n = \sum_{i=1}^{m} c_i^n \chi_{A_i} \tag{9}$$

and

$$\psi_n \to \frac{\partial f_{ac}}{\partial \mu_{ac}} \chi_{S_{ac}} + \frac{\partial f_s}{\partial \mu_s} \chi_{S_s} + \frac{\partial f_d}{\partial \mu_d} \chi_{S_d}$$

in $L^2(\,d\mu)$. This is possible, since $\int_{-N}^{\infty} \left(\frac{\partial f_{ac}}{\partial \mu_{ac}} \chi_{S_{ac}} + \frac{\partial f_s}{\partial \mu_s} \chi_{S_s} + \frac{\partial f_d}{\partial \mu_d} \chi_{S_d} \right) d\mu = 0$. Evidently, $f_n = \int_{-N}^{x} \psi_n\, d\mu \to f(x)$ dx-a.e. and the sequence $\{f_n\}$ is uniformly bounded by $\sup_n \left(\sqrt{\mu[-N, N] \int_{\mathbb{R}} \psi_n^2\, d\mu} \right)$. Hence $f_n \to f$ in $L^2(\sigma\, dx)$. The property $\mathcal{E}_\mu(f - f_n) \to 0$ is obvious.

Let f have no compact support. Consider a sequence of C_0^∞-functions g_k such that $|g_k|$ and $|g_k'|$ are uniformly bounded, $g_k(x) \to 1$ and $g_k'(x) \to 0$ for every x. Let $f_k = f g_k$. Evidently $f_k \to f$ in $L^2(\sigma\, dx)$. Further we get

$$\mathcal{E}_\mu(f-f_k) = \int_{\mathbb{R}} (1-g_k)^2 \Big[\Big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\Big)^2 d\mu_{ac} + \Big(\frac{\partial f_s}{\partial \mu_s}\Big)^2 d\mu_s + \Big(\frac{\partial f_d}{\partial \mu_d}\Big)^2 d\mu_d\Big] + \int_{\mathbb{R}} (g_k')^2 f^2 \tilde{\rho}\, dx.$$

Taking into account that $f \in L^2(\sigma\, dx)$ and $\tilde{\rho} \le \alpha\sigma$ we get that $\mathcal{E}_\mu(f - f_k) \to 0$. Hence the general case follows readily from the case then f has a compact support. $\square$

Proposition 4.3. *In the situation of Theorem 1.2 suppose that $\{f_n\}$, $f_n \in \mathcal{D}(\mathcal{E}_n)$ is a sequence of functions such that $\sup_n \int_{\mathbb{R}} f_n^2 \sigma_n\, dx + \underline{\lim}_n \int_{\mathbb{R}} (f_n')^2 \rho_n\, dx < \infty$. Then one can extract a subsequence from $\{f_n\}$ which tends in $L^1_{loc}(dx)$ to a function f such that $f \in \mathcal{D}(\mathcal{E}_\mu)$ and*

$$\int_{\mathbb{R}} \Big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\Big)^2 d\mu_{ac} + \Big(\frac{\partial f_s}{\partial \mu_s}\Big)^2 d\mu_s + \Big(\frac{\partial f_d}{\partial \mu_d}\Big)^2 d\mu_d \le \underline{\lim}_n \int_{\mathbb{R}} (f_n')^2 \rho_n\, dx. \tag{10}$$

Proof. In the same way as in Theorem 1.1 we show that there is a subsequence (denoted again by $\{f_n\}$) such that $f_n \to f$ in $L^1_{loc}(dx)$ for some function f and f possesses a weak derivatives f' such that f' is a locally finite measure and

$$\Big(\int_{\mathbb{R}} g\, df'(x)\Big)^2 \le \underline{\lim}_n \int_{\mathbb{R}} (f_n')^2 \rho_n\, dx \int_{\mathbb{R}} g^2\, d\mu(x) \tag{11}$$

for every continuous function g with compact support. By Lebesgue Theorem this inequality remains valid for bounded Borel functions with compact support. Hence f' is absolutely continuous w.r.t. μ.

Let us take

$$g_N(x) = \Big[\frac{\partial f_{ac}}{\partial \mu_{ac}} \chi_{S_{ac}} + \frac{\partial f_s}{\partial \mu_s} \chi_{S_s} + \frac{\partial f_d}{\partial \mu_d} \chi_{S_d}\Big] \chi_{A_N},$$

where $A_N = [-N, N] \cap \{x : \big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\big)^2 + \big(\frac{\partial f_s}{\partial \mu_s}\big)^2 + \big(\frac{\partial f_d}{\partial \mu_d}\big)^2 \le N\}$. Note that

$$\int_{\mathbb{R}} g_N\, df'(x) = \int_{\mathbb{R}} g_N^2\, d\mu(x) = \int_{A_N} \Big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\Big)^2 d\mu_{ac} + \Big(\frac{\partial f_s}{\partial \mu_s}\Big)^2 d\mu_s + \Big(\frac{\partial f_d}{\partial \mu_d}\Big)^2 d\mu_d.$$

This and inequality (11) imply that

$$\int_{A_N} \Big(\frac{\partial f_{ac}}{\partial \mu_{ac}}\Big)^2 d\mu_{ac} + \Big(\frac{\partial f_s}{\partial \mu_s}\Big)^2 d\mu_s + \Big(\frac{\partial f_d}{\partial \mu_d}\Big)^2 d\mu_d \le \underline{\lim}_n \int_{\mathbb{R}} (f_n')^2 \rho_n\, dx.$$

Tending N to infinity we get the desired inequality. $\square$

Proposition 4.4. *Let $\sqrt{\sigma_n} \to \sqrt{\sigma}$ in $L^2_{loc}(dx)$ and there exists $\alpha > 0$ such that $\tilde{\rho} \le \alpha\sigma$ dx-a.e. Then for every $f \in \mathcal{D}(\mathcal{E}_\mu)$ there exists a sequence of functions $\{f_n\}$, $f_n \in \mathcal{D}(\mathcal{E}_n)$ such that $f_n \to f$ in $\mathcal{H}$ and $\mathcal{E}_n(f_n) \to \mathcal{E}_\mu(f)$.*

Proof. Let us verify the claim for $f \in \tilde{C}$. Let f have the form (8). Then $\int_{A_i} \frac{dx}{\rho_n} \to \mu(A_i)$. This implies that there exist sequences c_i^n such that $c_i^n \to c_i$ and

$$\int_{\mathbb{R}} \sum_{i=1}^{m} c_i^n \chi_{A_i} \frac{dx}{\rho_n} = 0.$$

The functions

$$f_n(x) = \int_{-\infty}^{x} \sum_{i=1}^{m} c_i^n \chi_{A_i} \frac{dx}{\rho_n}$$

are uniformly bounded, have common compact support and tend to f dx-a.e. Hence $\lim_n \int_{\mathbb{R}} (f_n \sqrt{\sigma_n} - f\sqrt{\sigma})^2 \, dx \to 0$. By Lemma 3.1 $f_n \to f$ in $\mathcal{H}$. In addition

$$\mathcal{E}_\mu(f_n) = \sum_{i,j=1}^{m} \int_{\mathbb{R}} c_i^n c_j^n \chi_{A_i} \chi_{A_j} \frac{dx}{\rho_n} \to \sum_{i,j=1}^{m} \int_{\mathbb{R}} c_i c_j \chi_{A_i} \chi_{A_j} \, d\mu = \mathcal{E}_\mu(f).$$

Now we consider an arbitrary function $f \in L^2(\sigma \, dx)$. By Lemma 3.1 the space $\mathcal{H} = \bigcup_n L^2(\sigma_n \, dx)$ is metrizable with the metric d. By Lemma 4.2 we construct a sequence $\{\tilde{f}_n\}$, $\tilde{f}_n \in \tilde{C}$ such that $\tilde{f}_n \to f$ in $L^2(\sigma \, dx)$ and $\mathcal{E}_\mu(\tilde{f}_n) \to \mathcal{E}_\mu(f)$. Then we find for every n a sequence $\{\tilde{f}_{m,n}\}$, $\tilde{f}_{m,n} \in H_m$ such that $\tilde{f}_{m,n} \to \tilde{f}_n$ in $\mathcal{H} = \bigcup_n L^2(\sigma_n \, dx)$ and $\mathcal{E}_m(\tilde{f}_{m,n}) \to \mathcal{E}_\mu(\tilde{f}_n)$ if $m \to \infty$. For every $\tilde{f}_n$ let $\{M(n)\}$ be a sequence of natural numbers such that $M(n+1) > M(n)$, $d(\tilde{f}_n, \tilde{f}_{m,n}) \le \frac{1}{n}$ and $|\mathcal{E}_m(\tilde{f}_{m,n}) - \mathcal{E}_\mu(\tilde{f}_n)| \le \frac{1}{n}$ for every $m > M(n)$. Now we construct the following sequence: $f_m = \tilde{f}_{m,k(m)}$, where $k(m)$ is chosen in such a way that $M(k(m)) < m \le M(k(m)+1)$ if $m > M(2)$ and $k(m) = 1$ if $m \le M(2)$. The sequence $\{f_m\}$ possesses the desired properties. The proof is complete. $\square$

5. Diagonal case

In this section we study more detailed the diagonal case, give new sufficient conditions for identifying the Mosco limit and construct an interesting example, when the Mosco limit does not coincide with the expected one.

We consider the following Dirichlet form on $L^2(\mu_n) = L^2(\sigma_n \, dx)$

$$\mathcal{E}_n(f,g) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \rho_n^i \, dx$$

(i.e. $a_{i,i}^n = \rho_n^i$, $a_{i,j}^n = 0$, if $i \ne j$).

We denote by $\hat{x}^i$ the $d-1$-dimensional vector $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$.

Throughout the section we assume that the following representation of $\rho_n^i(x)$, $\rho^i(x)$ is given:

$$\rho_n^i(x) = q_n^i(x) p_n^i(\hat{x}^i), \quad \rho^i(x) = q^i(x) p^i(\hat{x}^i). \tag{12}$$

Here q_n^i and p_n^i are measurable non-negative functions such that the measure $p_n^i(\hat{x}^i)\, d\hat{x}^i$ is locally finite. A natural example of decomposition (12) is the following: $\rho_n^i \, dx$ is a probability measure, p_n^i is the density of $\rho_n^i\,dx \circ \pi_i^{-1}$, where π_i is the orthogonal projection on the hyperplane $\{x : x_i = 0\}$, and $q_n^i(x) = q_n^i(x_i, \hat{x}^i)$ is the density of the one-dimensional conditional measure in the point $\hat{x}^i$.

Throughout the section we assume that the following assumptions hold:

Assumption V

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{d} \rho_n^i.$$

Assumption VI

$$\rho_n^i\,dx \to \rho^i\,dx, \quad p_n^i\,d\hat{x}^i \to p^i\,d\hat{x}^i \quad \textit{vaguely.}$$

Assumption VII *For $p_n^i\,d\hat{x}^i$-a.e. $\hat{x}^i$*

$$\frac{1}{q_n^i(x_i, \hat{x}^i)} \in L^1_{loc}(\,dx_i).$$

In the following proposition we consider a sequence of partial Dirichlet forms

$$\mathcal{E}_n^{i,+}(f,g) = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\rho_n^i\,dx \tag{13}$$

with the domain of definition

$\mathcal{D}(\mathcal{E}_n^{i,+}) := \{f \in L^2(\sigma_n\,dx)$, for $p_n^i(\hat{x}^i)\,d\hat{x}^i$-almost every $\hat{x}^i$ f has an absolutely continuous $(\,dx_i)$-version $\tilde{f}$ and $\frac{\partial f}{\partial x_i} := \left(\frac{d\tilde{f}}{dx_i}\right) \in L^2(\rho_n^i\,dx)$ $\}$.

Lemma 5.1. *The form (13) is closed.*

Proof. Note that if $f_k \to f$ in $L^2(\sigma_n\,dx)$, Assumption V implies that $f_k \to f$ in $L^2(\rho_n\,dx)$. The claim follows from Theorem 3.2 of [3]. □

Proposition 5.2. *Suppose that for some i the measure $\frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx$ is locally finite and there exists a function $\tilde{q}^i$, such that the measure $\frac{p^i(\hat{x}^i)}{\tilde{q}^i(x)}\,dx$ is locally finite and*

$$\frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx \to \frac{p^i(\hat{x}^i)}{\tilde{q}^i(x)}\,dx$$

vaguely.

Let $\{f_n\}$, $f_n \in \mathcal{D}(\mathcal{E}_n^{i,+})$ be a sequence of functions such that

$$\sup_n \int_{\mathbb{R}^d} f_n^2\sigma_n\,dx + \underline{\lim}_n \int_{\mathbb{R}^d}\left(\frac{\partial f_n}{\partial x_i}\right)^2 \rho_n^i\,dx < \infty.$$

Then there exist a function f^i such that for some subsequence of $\{f_n p_n^i\,dx\}$ (denoted again by $\{f_n p_n^i\,dx\}$) one has:

$$f_n p_n^i\,dx \to f^i p^i\,dx$$

vaguely and for $p^i(\hat{x}_i)\,d\hat{x}_i$-almost every $\hat{x}_i$ the function f^i admits an absolutely continuous dx_i-version $\tilde{f}^i(.,\hat{x}_i)$ such that

$$\int_{\mathbb{R}^d}\left(\frac{\partial \tilde{f}^i}{\partial x_i}\right)^2 \tilde{q}^i p^i\,dx \le \underline{\lim}_n \mathcal{E}_n^{i,+}(f_n).$$

Proof. Let us choose a subsequence (denoted again by $\{f_n\}$) such that $c_i := \sup_n \int_{\mathbb{R}^d}\left(\frac{\partial f_n}{\partial x_i}\right)^2 \rho_n^i\,dx < \infty$ for every i.

Let Ω be a bounded domain. By Cauchy inequality:

$$\Big[\int_\Omega \left|\left(\frac{\partial f_n}{\partial x_i}\right)\right| p_n^i(\hat{x}^i)\,dx_i\,d\hat{x}^i\Big]^2 \le \int_{\mathbb{R}^d}\left(\frac{\partial f_n}{\partial x_i}\right)^2 \rho_n^i\,dx \int_\Omega \frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx \le c_i \int_\Omega \frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx.$$

Since the measures $\frac{p_n^i(\hat{x}^i)}{q_n^i(x)}\,dx$ converge vaguely, by a usual diagonal procedure one can extract from $\{\frac{\partial f_n}{\partial x_i} p_n^i\,dx\}$ a subsequence of measures (denoted again by $\{\frac{\partial f_n}{\partial x_i} p_n^i\,dx\}$) which converges vaguely to a measure m.

Now let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then

(14)
$$\Big[\int_{\mathbb{R}^d}\varphi\,dm\Big]^2 = \lim_n \Big[\int_{\mathbb{R}^d}\varphi \frac{\partial f_n}{\partial x_i} p_n^i\,dx_i\,d\hat{x}^i\Big]^2 \le c_i \lim_n \int_{\mathbb{R}^d}\varphi^2 \frac{p_n^i}{q_n^i}\,dx = c_i \int_{\mathbb{R}^d}\varphi^2 \frac{p^i}{\tilde{q}^i}\,dx.$$

This implies that m is absolutely continuous w.r.t. $p^i(\hat{x^i})\,dx$, hence $\rho^i\,dx$.

Note that the condition $\sup_n \|f_n\|^2_{L^2(\sigma_n\,dx)} < \infty$ implies that $\sup_n \|f_n\|^2_{L^2(\rho_n^i\,dx)} < \infty$ for every $i \in \{1,\cdots,d\}$. Hence, we can do the same procedure with the sequence of measures $\{f_n p_n^i\,dx\}$. Finally we get that there exist measurable functions f^i and g^i such that

$$f_n p_n^i\,dx \to f^i p^i\,dx, \qquad \left(\frac{\partial f_n}{\partial x_i}\right) p_n^i\,dx \to g^i p^i\,dx$$

vaguely.

Let us take $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^{d-1}}\Big[\int_{\mathbb{R}}\varphi g^i\,dx_i\Big]p^i\,d\hat{x}^i = \lim_n \int_{\mathbb{R}^{d-1}}\Big[\int_{\mathbb{R}}\varphi\left(\frac{\partial f_n}{\partial x_i}\right)\,dx_i\Big]p_n^i\,d\hat{x}^i =$$
$$-\lim_n \int_{\mathbb{R}^{d-1}}\Big[\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x_i} f_n\,dx_i\Big]p_n^i\,d\hat{x}^i = -\int_{\mathbb{R}^{d-1}}\Big[\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x_i} f^i\,dx_i\Big]p^i\,d\hat{x}^i.$$

We get by a slightly modification of Theorem 1.1.3.1 in [19] that for $p^i(\hat{x}^i)\,d\hat{x}^i$-a.e. $\hat{x}^i$ the function $f^i(.,\hat{x}^i)$ has an absolutely continuous dx_i-version $\tilde{f}^i(.,\hat{x}^i)$ such that $\frac{\partial \tilde{f}^i(x_i,\hat{x}^i)}{\partial x_i} = g^i(x_i,\hat{x}^i)$.

From (14) we get:

$$\Big[\int_{\mathbb{R}^d}\varphi\left(\frac{\partial \tilde{f}^i}{\partial x_i}\right) p^i(\hat{x}^i)\,dx\Big]^2 \le c_i \int_{\mathbb{R}^d}\varphi^2 \frac{p^i(\hat{x}^i)}{\tilde{q}^i(x)}\,dx.$$

Then choosing an appropriative sequence of C_0^∞-functions we prove the desired inequality in the same way as in Theorem 1.1. $\square$

Now we define $(\mathcal{E}_n^+, \mathcal{D}(\mathcal{E}_n^+))$ as the sum of partial forms $\mathcal{E}_n^+ = \sum_{i=1}^d \mathcal{E}_n^{i,+}$ with the domain of definition $\mathcal{D}(\mathcal{E}_n^+) = \bigcap_{i=1}^d \mathcal{D}(\mathcal{E}_n^{i,+})$. This form is closed (see [3]) and $\mathcal{D}(\mathcal{E}_n^+)$ is called the "maximal" domain. The "minimal" domain $\mathcal{D}(\mathcal{E}_n)$ is obtained as usual by completion of $C_0^\infty(\mathbb{R}^d)$ w.r.t. $\left[\mathcal{E}_n(u) + (u,u)_{L^2(\sigma_n\,dx)}\right]^{\frac{1}{2}}$.

Proof of Theorem 1.3 Let $f_n \to f$ weakly and $\underline{\lim}_n \mathcal{E}_n^+(f_n) < \infty$. For a fixed i let f^i be a function from Proposition 5.2. Let us prove that $f = f^i \ \rho\, dx$ -a.e. Indeed, let $\psi_n = \frac{\varphi}{q_n^i(x_i,\hat{x}^i)}$, where $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let us show that $\psi_n \to \frac{\varphi}{q^i} = \psi$ strongly. Indeed, let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \psi_n \tilde{\varphi} \rho_n\, dx = \int_{\mathbb{R}^d} \frac{\varphi}{q_n^i(x_i,\hat{x}^i)} \tilde{\varphi} \rho_n\, dx = \int_{\mathbb{R}^d} \varphi \tilde{\varphi} p_n^i(\hat{x}^i)\, dx_i\, d\hat{x}^i$$
$$\to \int_{\mathbb{R}^d} \varphi \tilde{\varphi} p^i(\hat{x}^i)\, dx_i\, d\hat{x}^i = \int_{\mathbb{R}^d} \psi \tilde{\varphi} \rho\, dx$$

and

$$\int_{\mathbb{R}^d} \psi_n^2 \rho_n\, dx = \int_{\mathbb{R}^d} \frac{\varphi^2}{(q_n^i)^2(x_i,\hat{x}^i)} \rho_n\, dx = \int_{\mathbb{R}^d} \varphi^2 \frac{p_n^i}{q_n^i}\, dx \to \int_{\mathbb{R}^d} \varphi^2 \frac{p^i}{q^i}\, dx = \int_{\mathbb{R}^d} \psi^2 \rho\, dx.$$

By Lemma 7.6 we have that $\psi_n \to \psi$ strongly. Hence $\int_{\mathbb{R}^d} f_n \psi_n \rho_n\, dx \to \int_{\mathbb{R}^d} f \psi \rho\, dx$. But $\int_{\mathbb{R}^d} f_n \psi_n \rho_n\, dx = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f_n \varphi\, dx_i p_n^i\, d\hat{x}^i \to \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f^i \varphi\, dx_i p^i\, d\hat{x}^i = \int_{\mathbb{R}^d} f^i \psi \rho\, dx$. Hence $f = f^i$ a.e. with respect to $\rho\, dx = \sigma\, dx$. By Proposition 5.2 we get that

$$\mathcal{E}^+(f) \le \sum_{i=1}^d \underline{\lim}_n \mathcal{E}_n^{i,+}(f_n) \le \underline{\lim}_n \sum_{i=1}^d \mathcal{E}_n^{i,+}(f_n) \le \underline{\lim}_n \mathcal{E}_n^+(f_n).$$

The condition (M1) is proved.

The condition (M2) can be proved as in Theorem 1.1. $\square$

Remark 5.3. *Note that Proposition 5.2 can be generalized to the case then μ_n don't have densities w.r.t. Lebesgue measure and even to the infinite dimensional case. As a consequence, some Mosco convergence results for partial Dirichlet forms in the infinite dimensional case on varying L^2-spaces can be proved. These topics will be studied in a subsequent paper of the author. Note, that convergence of all partial forms and convergence of the gradient form do not in general imply each other. Consider the following interesting example. O. Pugachev has recently constructed an example of a function ρ, such that the partial forms $(\mathcal{E}_i, C_0^\infty(\mathbb{R}^d))$, $\mathcal{E}_i(f,g) = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} \rho\, dx$ are not closable and the form $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$, $\mathcal{E}(f,g) = \int_{\mathbb{R}^d} (\nabla f, \nabla g) \rho\, dx$ is closable in $L^2(\rho\, dx)$ (see [20]). Then the sequence $\{\tilde{\mathcal{E}}_{i,n}\} = \{\mathcal{E}_i\}$ does not converge Mosco to $\mathcal{E}_i$, since every Mosco limit is closed, but the sequence $\{\tilde{\mathcal{E}}_n\} = \{\mathcal{E}\}$ evidently converges Mosco to the closure of $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\rho\, dx)$.*

In the rest of the section we analyze a partial case then every ρ_n^i has the form $\rho_n^i(x) = p_n^i(x_i)q_n^i(\hat{x}^i)$. In this case we can give a simple description of the Mosco limit. However, this example shows that the general situation turns out to be quite complicated.

Let us define $\tilde{C}_0^\infty(\mathbb{R}^d) \subset C_0^\infty(\mathbb{R}^d)$ as the space of function of the type $\tilde{C}_0^\infty(\mathbb{R}^d) = \{f : f = \sum_{k=1}^m \prod_{i=1}^d f_k^i(x_i), \quad m \in \mathbb{N}\}$, where $f_k^i \in C_0^\infty(\mathbb{R})$. The following Lemma is well-known and can be proved with usual technique of mollifying functions.

Lemma 5.4. *For every $f \in C_0^\infty(\mathbb{R}^d)$ there exists a sequence of functions $\{f_n\}$, $f_n \in \tilde{C}_0^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $\partial_i f_n \to \partial_i f$ uniformly on $\mathbb{R}^d$ as $n \to \infty$ for every $i \in \{1, \cdots, d\}$.*

Example 5.5. *Suppose that the function $q_n^i(x) = q_n^i(x_i)$ depends only on x_i and the following assumptions hold:*

1) *$q_n^i \to q^i$ in $L_{loc}^1(dx_i)$.*
2) *There exist a function $\tilde{q}^i$ such that $\frac{1}{\tilde{q}^i} \in L_{loc}^1(dx_i)$ and $\frac{dx_i}{q_n^i} \to \frac{dx_i}{\tilde{q}^i}$ vaguely.*
3) *The sequence $\{p_n^i\}$ is uniformly bounded on every bounded set and $p_n^i \to p^i$ $d\hat{x}^i$-almost everywhere.*
4) *For every $i \in \{1, \cdots, d\}$ and every bounded domain $\Omega \in \mathbb{R}^{d-1}$*

$$\sup_n \int_\Omega \frac{d\hat{x}^i}{p_n^i(\hat{x}^i)} < \infty.$$

Let

$$\tilde{\mathcal{E}}(\varphi, \psi) = \int_{\mathbb{R}^d} \sum_{i=1}^d \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_i} \tilde{q}^i(x_i) p^i(\hat{x}^i)\, dx, \quad \varphi, \psi \in C_0^\infty(\mathbb{R}^d). \tag{15}$$

Suppose that the minimal and the maximal extensions of $(\tilde{\mathcal{E}}, C_0^\infty(\mathbb{R}^d))$ in $L^2(\sigma\, dx)$ coincide. Then $\mathcal{E}_n^+ \to \tilde{\mathcal{E}}$ Mosco.

Proof.

First we show that condition (M1) is fulfilled. Let $f_n \to f$ weakly in $\mathcal{H}$. Note that the conditions of the example imply that $\sqrt{\sigma_n} \to \sqrt{\sigma}$ in $(L_{loc}^2 dx)$ and $\sup_n \int_\Omega \frac{dx}{\rho_n^i} < \infty$, $\sup_n \int_\Omega \frac{dx}{\sigma_n} < \infty$, for every i and every bounded domain Ω. Then in the same way as in Theorem 3.5 we show that there exists a subsequence which converges to a function $\tilde{f}$ in $L_{loc}^1(dx)$. By Lemma 3.2 $f = \tilde{f}$ dx-a.e. By Proposition 5.2 there exists a subsequence of $\{f_n\}$ and a function f^i such that $f_n p_n^i\, dx$ converges vaguely to $f^i p^i$. Hence, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have $\int_{\mathbb{R}^d} f_n \varphi p_n^i\, dx \to \int_{\mathbb{R}} f^i \varphi p^i\, dx$. In the other hand, from the convergence $f_n \to f$ in $L_{loc}^1(dx)$ we get that $\int_{\mathbb{R}} f_n \varphi\, dx_i \to \int_{\mathbb{R}} f \varphi\, dx_i$ in $L_{loc}^1(d\hat{x}^i)$. Then the condition 3) implies that $\int_{\mathbb{R}^d} f_n \varphi p_n^i\, dx \to \int_{\mathbb{R}} f \varphi p^i\, dx$. From conditions 3), 4) we get that $\frac{1}{p^i} \in L_{loc}^1(d\hat{x}^i)$, hence $f = f^i$ dx-a.e. By Proposition 5.2 $\int_{\mathbb{R}^d} \left(\frac{\partial f}{\partial x_i}\right)^2 \tilde{q}^i p^i\, dx \leq \underline{\lim}_n \mathcal{E}_n^{i,+}(f_n)$. Since we can do the same for every i, we get that

$f \in \mathcal{D}(\tilde{\mathcal{E}^+})$. The desired inequality we get by summarizing in index i and applying properties of $\underline{\lim}_n$. (M1) is proved.

Let us prove (M2). We define by $\mathcal{F}$ the following set of functions:

$$\mathcal{F} = \{f : f = \sum_{k=1}^{m} f_k, \quad f_k = \prod_{i=1}^{d} \psi_k^i(x_i)\},$$

where every ψ_k^i has the form

$$\psi_k^i = \int_{-\infty}^{x_i} \sum_{j=1}^{N(i,k)} c_{j,k}^i \chi_{A_{j,k}^i} \frac{dx_i}{\tilde{q}^i},$$

where $A_{j,k}^i = [a_{j,k}^i, b_{j,k}^i]$ and $\sum_{j=1}^{N(i,k)} c_{j,k}^i \int_{A_{j,k}^i} \frac{dx_i}{\tilde{q}^i} = 0$. Note, that if we show that $\mathcal{F}$ is dense in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$ and for every $f \in \mathcal{F}$ there exists a sequence of functions $\{f_n\}$ such that $f_n \in \mathcal{D}(\mathcal{E}_n^+)$, $f_n \to f$ in $\mathcal{H}$, $\mathcal{E}_n^+(f_n) \to \tilde{\mathcal{E}}(f)$, the condition (M2) can be proved in the same way as in Proposition 4.4 .

Let us fix $f \in \mathcal{F}$, $f = \sum_{k=1}^m f_k$, $f_k = \prod_{i=1}^d \psi_k^i(x_i)$ and take $f_n = \sum_{k=1}^m f_{n,k}$, where $f_{n,k} = \prod_{i=1}^d \psi_{n,k}^i(x_i)$ and

$$\psi_{n,k}^i = \int_{-\infty}^{x_i} \sum_{j=1}^{N(i,k)} c_{n,j,k}^i \chi_{A_{j,k}^i} \frac{dx_i}{q_n^i},$$

such that $\sum_{j=1}^{N(i,k)} c_{n,j,k}^i \int_{A_{j,k}^i} \frac{dx_i}{q_n^i} = 0$. In addition we require that $c_{n,j,k}^i \to c_{j,k}^i$. It is possible, since $\int_{A_{j,k}^i} \frac{dx_i}{q_n^i} \to \int_{A_{j,k}^i} \frac{dx_i}{\tilde{q}^i}$. It is obvious that the functions f_n are uniformly bounded and converges pointwise. Hence $f_n \to f$ in $\mathcal{H}$ by Lemma 3.1. Let us prove that $\mathcal{E}_n^+(f_n) \to \tilde{\mathcal{E}}(f)$. Since $\mathcal{E}_n^+(f_n) = \sum_{k,l=1}^m \mathcal{E}_n^+(f_{n,k}, f_{n,l})$, it is enough to prove that $\mathcal{E}_n^+(f_{n,k}, f_{n,l}) \to \tilde{\mathcal{E}}(f_k, f_l)$. Indeed,

$$\mathcal{E}_n^+(f_{n,k}, f_{n,l}) = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial f_{n,k}}{\partial x_i} \frac{\partial f_{n,l}}{\partial x_i} q_n^i(x_i) p_n^i(\hat{x}_i)\, dx =$$

$$\sum_{i=1}^{d} \int_{\mathbb{R}} \frac{\partial \psi_{n,k}^i(x_i)}{\partial x_i} \frac{\partial \psi_{n,l}^i(x_i)}{\partial x_i} q_n^i(x_i)\, dx_i \int_{\mathbb{R}^{d-1}} \prod_{i_1 \neq i} \psi_{n,k}^{i_1}(x_{i_1}) \prod_{i_2 \neq i} \psi_{n,l}^{i_2}(x_{i_2}) p_n^i(\hat{x}_i)\, d\hat{x}^i.$$

We get that

$$\int_{\mathbb{R}} \frac{\partial \psi_{n,k}^i(x_i)}{\partial x_i} \frac{\partial \psi_{n,l}^i(x_i)}{\partial x_i} q_n^i(x_i)\, dx_i = \sum_{j_1=1}^{N(i,k)} \sum_{j_2=1}^{N(i,l)} \int c_{n,j_1,k}^i c_{n,j_2,l}^i \chi_{A_{j_1,k}^i} \chi_{A_{j_2,l}^i} \frac{dx_i}{q_n^i}$$

$$\to \sum_{j_1=1}^{N(i,k)} \sum_{j_2=1}^{N(i,l)} \int c_{j_1,k}^i c_{j_2,l}^i \chi_{A_{j_1,k}^i} \chi_{A_{j_2,l}^i} \frac{dx_i}{\tilde{q}^i} = \int_{\mathbb{R}} \frac{\partial \psi_k^i(x_i)}{\partial x_i} \frac{\partial \psi_l^i(x_i)}{\partial x_i} \tilde{q}^i(x_i)\, dx_i$$

and

$$\int_{\mathbb{R}^{d-1}} \prod_{i_1\neq i} \psi^{i_1}_{n,k}(x_{i_1}) \prod_{i_2\neq i} \psi^{i_2}_{n,l}(x_{i_2}) p^i_n(\hat{x}^i)\, d\hat{x}^i$$

tends to

$$\int_{\mathbb{R}^{d-1}} \prod_{i_1\neq i} \psi^{i_1}_{k}(x_{i_1}) \prod_{i_2\neq i} \psi^{i_2}_{l}(x_{i_2}) p^i(\hat{x}^i)\, d\hat{x}^i,$$

since $\psi^{i_1}_{n,k} \to \psi^{i_1}_k$, $\psi^{i_2}_{n,l} \to \psi^{i_2}_l$ for every x, $\psi^{i_1}_{n,k}, \psi^{i_2}_{n,l}$ are uniformly bounded, have common compact support and $p^i_n \to p^i$ in $L^1_{loc}(\mathbb{R}^{d-1})$.

Now let us show that $\mathcal{F}$ is dense in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$. Let $f \in \tilde{C}^\infty_0(\mathbb{R}^d)$, $f = \sum_{k=1}^m f_k$, where $f_k = \prod_{i=1}^d f^i_k(x_i)$, $f^i_k \in C^\infty_0(\mathbb{R})$. Denote by $I(f)$, $f \in C^\infty_0(\mathbb{R})$ the minimal segment $I = [a, b]$ such that $\mathrm{supp}(f) \in I$. Let us choose for every f^i_k a sequence $g^i_{n,k} = \sum_{j=1}^{m(n,i,k)} c^i_{n,j,k}\chi_{A^i_{n,j,k}}$ with the following properties: $A^i_{n,j,k} \subset I(f^i_k)$, $\int_{I(f^i_k)} \frac{g^i_{n,k}}{\tilde{q}^i} = 0$, and $g^i_{k,n} \to (f^i_k)'\tilde{q}^i$ in $L^2(\frac{dx_i}{\tilde{q}^i})$. One easily proves that $\tilde{f}_n = \sum_{k=1}^m \prod_{i=1}^d \int_{-\infty}^{x_i} \frac{g^i_{n,k}(x_i)}{\tilde{q}^i}\, dx_i$ tends to f in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$. Since $C^\infty_0(\mathbb{R}^d)$ is dense in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$ by condition of the example, Lemma 5.4 implies that $\tilde{C}^\infty_0(\mathbb{R}^d)$ is also dense in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$. Hence, $\mathcal{F}$ is dense in $(\mathcal{D}(\tilde{\mathcal{E}}), \tilde{\mathcal{E}}^1)$. The claim is proved. □

6. Examples. Convergence of the laws of the processes.

Example 6.1. *Let $d = 1$ and $\mathcal{E}_n(f) = \int_{\mathbb{R}} (f'(x))^2 \rho_{1,n}(x)\, dx$, where*

$$\rho_{1,n}(x) = \begin{cases} \frac{1}{n}, & x \in A_n \\ 1, & x \notin A_n \end{cases},$$

$A_n = \bigcup_{k,l}[k + \frac{l}{n} - \frac{1}{n^2}, k + \frac{l}{n}]$, $k \in \mathbb{Z}$, $1 \le l \le n$. The form $(\mathcal{E}_n, C^\infty_0(\mathbb{R}))$ is closable on $L^2(\rho_{1,n}\, dx)$. Evidently, $\sqrt{\rho_{1,n}} \to 1$ in $L^2_{loc}(\, dx)$ and $\frac{dx}{\rho_{1,n}} \to 2\, dx$ weakly. By Theorem 1.2 the sequence $\{\mathcal{E}_n\}$ converges Mosco to the form

$$\frac{1}{2}\int_{\mathbb{R}} (f'(x))^2\, dx$$

on $L^2(\, dx)$.

A similar situation on the plane gives us another result.

Example 6.2. *Let $d = 2$ and $\mathcal{E}_n(f) = \int_{\mathbb{R}^2} (\nabla f(x_1, x_2))^2 \rho_{2,n}(x_1, x_2)\, dx_1\, dx_2$, where*

$$\rho_{2,n}(x_1, x_2) = \begin{cases} \frac{1}{n^2}, & (x_1, x_2) \in A_n \\ 1, & (x_1, x_2) \notin A_n \end{cases},$$

$A_n = \bigcup_{k_1,l_1}\bigcup_{k_2,l_2}[k_1 + \frac{l_1}{n} - \frac{1}{n^2}, k_1 + \frac{l_1}{n}][k_2 + \frac{l_2}{n} - \frac{1}{n^2}, k_2 + \frac{l_2}{n}]$, $k_1, k_2 \in \mathbb{Z}$, $1 \le l_1, l_2 \le n$. The form $(\mathcal{E}_n, C^\infty_0(\mathbb{R}^2))$ is closable on $L^2(\rho_n\, dx_1\, dx_2)$. Evidently, $\sqrt{\rho_{2,n}} \to 1$ in

$L^2_{loc}(\,dx_1\,dx_2)$ *and* $\frac{dx_1\,dx_2}{\rho_{2,n}} \to 2\,dx_1\,dx_2$ *weakly. Theorem 1.1 is not applicable in this situation. Represent* $\rho_{2,n}(x_1,x_2)$ *as*

$$\rho_{2,n}(x_1,x_2) = q^1_n(x_1,x_2)p^1_n(x_2),$$

where $p^1_n(x_2)$ *equals to the one-dimensional function* $\rho_{1,n}(x_2)$ *from Example 6.1 and* $q^1_n(x_1,x_2) = \frac{\rho_{2,n}(x_1,x_2)}{p^1_n(x_2)}$. *Let us do the same procedure with the second coordinate. One easily verifies that the functions* q^1_n *and* p^1_n *satisfy the conditions of Theorem 1.3 with* $p^1 = 1$, $q^1 = \tilde{q}^1 = 1$ *(the same is true for* p^2_n *and* q^2_n*). Hence* $\{\mathcal{E}_n\}$ *converges Mosco to the form*

$$\int_{\mathbb{R}^2} (\nabla f(x_1,x_2))^2\,dx_1\,dx_2$$

on $L^2(\,dx_1\,dx_2)$.

Example 6.3. *Let* $d = 2$ *and* $\rho_n(x_1x_2) = p_n(x_1)p_n(x_2)$, *where* p_n *is equal to the function* $\rho_{1,n}$ *from Example 6.1. It follows from Example 5.5 that the sequence* $\{\mathcal{E}_n\}$, *where* $\mathcal{E}_n(f) = \int_{\mathbb{R}^2}(\nabla f(x_1,x_2))^2\rho_n(x_1x_2)\,dx_1\,dx_2$, *tends Mosco to the form*

$$\frac{1}{2}\int_{\mathbb{R}^2} (\nabla f(x_1,x_2))^2\,dx_1\,dx_2$$

on $L^2(\,dx_1\,dx_2)$.

Now let us briefly discuss the convergence of the laws of the associated processes. Let $\{\mathcal{E}_n\}$ be a Mosco convergent sequence of Dirichlet forms on $\mathcal{H} = \bigcup_n L^2(\mathbb{R}^d;\mu_n)$ and $(\Omega, \mathcal{F}, (X^n_t)_{t\geq 0}, (P^n_x))$ be the associated stochastic processes. We suppose that $\Omega = C([0,\infty) \to \mathbb{R}^d)$, i.e. the trajectories of every $(X^n_t)_{t\geq 0}$ are continuous and that the process $(X^n_t)_{t\geq 0}$ is conservative. Then the finite dimensional distributions of the measure

$$P^n_{\mu_n} = \int_{\mathbb{R}^d} P^n_x \mu_n(\,dx)$$

vaguely converge to $P_\mu = \int_{\mathbb{R}^d} P_x\mu(\,dx)$.

Indeed, it follows just from the formula

$$\int f_0(X^n_0)f_1(X^n_{t_1})f_2(X^n_{t_1+t_2})\cdots f_m(X^n_{t_1+\cdots+t_m})\,dP_{\mu_n} =$$

$$\int f_0 T^n_{t_1}(f_1 T^n_{t_2}(f_2\cdots T^n_{t_m}(f_m))\cdots)\,d\mu_n,$$

applied to $f_0, f_1, \cdots, f_m \in C^\infty_0(\mathbb{R}^d)$, and the strong convergence of T^n_t in $\mathcal{H}$.

7. Appendix

In order to make the paper more self-contained we summarize in this section the main results from [10]. We prove also some new useful facts (see Proposition 7.2, Lemma 7.6).

Lemma 7.1. *Let $\{u_n\}$, $\{v_n\}$ be two sequences of vectors in $\mathcal{H}$ with $u_n, v_n \in H_n$, and let $u \in H$. Suppose that $u_n \to u$ in $\mathcal{H}$. Then $v_n \to u$ in $\mathcal{H}$ if and only if $||u_n - v_n||_{H_n} \to 0$.*

Proof. The "if"-part is trivial. Let us prove the "only if"-part. We find sequences of vectors $\{\tilde{u}_m\} \subset H$ and $\{\tilde{v}_m\} \subset H$ such that $\tilde{u}_m \to u$, $\tilde{v}_m \to u$ in H and

$$\lim_m \overline{\lim_n} ||\Phi_n \tilde{u}_m - u_n||_{H_n} = \lim_m \overline{\lim_n} ||\Phi_n \tilde{v}_m - v_n||_{H_n} = 0.$$

Since $||u_n - v_n||_{H_n} \leq ||\Phi_n \tilde{u}_m - u_n||_{H_n} + ||\Phi_n \tilde{v}_m - v_n||_{H_n} + ||\Phi_n(\tilde{v}_m - \tilde{u}_m)||_{H_n}$ we have that

$$\overline{\lim_n} ||u_n - v_n||_{H_n} \leq \overline{\lim_n} ||\Phi_n \tilde{u}_m - u_n||_{H_n} + \overline{\lim_n} ||\Phi_n \tilde{v}_m - v_n||_{H_n} + ||\tilde{v}_m - \tilde{u}_m||_H$$

for every m. Tending m to infinity we obtain our claim. □

Proposition 7.2. *Assume that all H_n are infinite dimensional. Then there exists a metric d on $\mathcal{H}$ such that the convergence in d coincides with the strong convergence and there exists an isometry $\Psi : (\mathcal{H}, d) \to I \times l^2$, where $I = \{0\} \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\} \subset \mathbb{R}$.*

Proof. Let us construct an orthonormal basis $\{e_n\}$ in H consisting of vectors from C using a standard orthogonalization procedure. Let us denote by L_k the linear span $< e_1, \cdots, e_k >$. Note that by (3) the sequences of matrices $\{M_n = (M_n)_{ij} = (\Phi_n(e_i), \Phi_n(e_j))_{H_n}\} \subset L(\mathbb{R}^k)$ tends to $E_k = \text{diag}\{1, \cdots, 1\} \in L(\mathbb{R}^k)$. This implies that one can find a sequence of increasing numbers $n(k)$ such that for every n from $[n(k), n(k+1))$ there exists a linear isometry $\Psi_n : H \to H_n$ such that the norm of operator

$$(\Psi_n - \Phi_n)|_{L_k} : L_k \to H_n$$

is less than $\frac{1}{k}$. For the finite sequence of numbers $\{n : n < n(1)\}$ we just fix some arbitrary isometries $\Psi_n : H \to H_n$ and take $\Psi_0 = Id : H_0 \to H_0$, where $H_0 = H$. Evidently for every $v \in L_k$ we have $\lim_n ||(\Phi_n - \Psi_n)v||_{H_n} = 0$. Let us fix a vector $u = \sum_{i=1}^{\infty} \alpha_i e_i \in H$. We show that $\Psi_n u$ converges to u in $\mathcal{H}$. Indeed, take $\tilde{u}_m = \sum_{i=1}^{m} \alpha_i e_i$. Then $\tilde{u}_m \to u$ in H and $\lim_m \overline{\lim_n} ||\Phi_n \tilde{u}_m - u_n||_{H_n} = \lim_m \overline{\lim_n} ||(\Phi_n - \Psi_n) \sum_{i=1}^{m} \alpha_i e_i - \Psi_n \sum_{i=m+1}^{\infty} \alpha_i e_i||_{H_n} = \lim_m || \sum_{i=m+1}^{\infty} \alpha_i e_i||_H = 0$. Lemma 7.1 implies that a sequence of vectors $v_n = \sum_{i=1}^{\infty} \alpha_{i,n} \Psi_n(e_i)$ converges to v if and only if $\lim_n \sum_{i=1}^{\infty} (\alpha_{i,n} - \alpha_i)^2 = 0$. It implies that

$$d(v, w) = \sqrt{|\delta_n - \delta_m|^2 + \sum_{i=1}^{\infty} (v_i - w_i)^2},$$

where $v = \sum_{i=1}^{\infty} v_i \Psi_n(e_i) \in H_n$, $w = \sum_{i=1}^{\infty} w_i \Psi_m(e_i) \in H_m$ and $\delta_n = \frac{1}{n}$, $\delta_0 = 0$ is a desired metric and the following mapping is a desired isometry:

$$\Psi(v) = \{\delta_n\} \times (v_i)_{i \in \mathbb{N}}.$$

□

Remark 7.3. *It follows from Proposition 7.2 that the space* $(\mathcal{H}, d)$ *can be isometrically embedded into the Hilbert space* $\tilde{l^2} = \mathbb{R} \times l^2$ *such that*

$$\Psi(H_n) = \{v \in \tilde{l^2}, (v, e_0)_{\tilde{l^2}} = \delta_n\},$$

where e_0 *is a unitary vector which is orthogonal to* l^2. *The strong and weak convergence in* $\mathcal{H}$ *correspond to the strong and weak convergence in* $\tilde{l^2}$. *This observation implies immediately the usual properties of the strong and weak convergence summarized in the following Corollaries. Note that our proof is given only for the case, when all* H_n *are infinite dimensional, but one can readily verify that the same remains valid for the finite dimensional case.*

Corollary 7.4. 1. *Let* $\{u_n\}$ *be a sequence with* $u_n \in H_n$ *and* $u_n \to u$ *strongly. Then* $||u_n||_{H_n} \to ||u||_H$.

2. *For every* $u \in H$ *there exists a sequence* $\{u_n\}$ *with* $u_n \in H_n$ *such that* $u_n \to u$ *strongly.*

Corollary 7.5. 1. *Let* $\{u_n\}$ *be a sequence with* $u_n \in H_n$. *If the sequence of norms* $||u_n||_{H_n}$ *is bounded, there exists a weakly convergent subsequence of* $\{u_n\}$.

2. *Let* $\{u_n\}$, $u_n \in H_n$ *be a sequence which weakly converges to* $u \in H$. *Then*

$$\sup_n ||u_n||_{H_n} < \infty, \quad ||u||_H \leq \underline{\lim}_n ||u_n||_{H_n}.$$

Moreover, $u_n \to u$ *strongly if and only if*

$$||u||_H = \lim_n ||u_n||_{H_n}.$$

3. *The sequence* $\{u_n\}$ *tends to* u *strongly if and only if*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every $\{v_n\}$ *weakly tending to* v *in* $\mathcal{H}$.

The following lemma gives a simple criterion for the strong convergence in $\mathcal{H}$.

Lemma 7.6. *A sequence of vectors* $\{u_n\}$, $u_n \in H_n$ *converges to a vector* $u \in H$ *if and only if* $\|u_n\|_{H_n} \to \|u\|_H$ *and* $(u_n, \Phi_n(\varphi))_{H_n} \to (u, \varphi)_H$ *for every* $\varphi \in C$.

Proof. The "only if"-part follows from the previous corollaries and the fact that $\Phi_n(\varphi) \to \varphi$ strongly. Let us prove the "if"-part. Indeed, let $\varphi_m \to u$ in H, $\varphi_m \in C$. Then

$$\lim_m \overline{\lim}_n \|u_n - \Phi_n(\varphi_m)\|_{H_n} = \lim_m (\|u\|_H^2 - 2(u, \varphi_m)_H + \|\varphi_m\|_H^2)^{\frac{1}{2}} = \lim_m \|u - \varphi_m\|_H = 0.$$

The proof is complete. □

Proof of the relations "(1) $\Longrightarrow$ (2)" and "(2) $\Longrightarrow$ (3)" of Theorem 2.8: Let us show that (1) $\to$ (2). Let $\{u_n\}$ be a strongly convergent sequence $u_n \in H_n$ and $u_n \to u$. Denote $z_n = G_{n,\beta} u_n$, $z = G_\beta u$. The vector z_n is characterized as the unique minimizer of $\mathcal{E}_n(v) + \beta(v, v)_{H_n} - 2(u_n, v)_{H_n}$ over H_n.

Since the norm of $G_{n,\beta}$ is bounded by $\frac{1}{\beta}$, by Corollary 7.5.1 there exists a subsequence of $\{z_n\}$, still denoted $\{z_n\}$ in the following, that converges weakly to a vector $\tilde{z} \in H$. For an arbitrary given $v \in H$ by condition (M2) of Definition 2.5 we can find a sequence $\{v_n\} \to v$, $v_n \in H_n$ such that $\lim_n \mathcal{E}_n(v_n) = \mathcal{E}(v)$. Since for every n,

$$\mathcal{E}_n(z_n) + \beta(z_n, z_n)_{H_n} - 2(u_n, z_n)_{H_n} \leq \mathcal{E}_n(v_n) + \beta(v_n, v_n)_{H_n} - 2(u_n, v_n)_{H_n}$$

by taking condition (M1), Corollary 7.4.1 and Corollary 7.5.2 into account, we find in the limit

$$\mathcal{E}(\tilde{z}) + \beta(\tilde{z}, \tilde{z})_H - 2(u, \tilde{z})_H \leq \mathcal{E}(v) + \beta(v, v)_H - 2(u, v)_H$$

therefore $\tilde{z} = G_\beta(u)$. By the uniqueness of such $\tilde{z}$ it proves that $\{z_n\}$ converges to z weakly in $\mathcal{H}$. We now prove that $(z_n, z_n)_{H_n} \to (z, z)_H$. We choose $\{v_n\} \to z$, $v_n \in H_n$ such that $\lim_n \mathcal{E}_n(v_n) = \mathcal{E}(z)$, therefore, by rewriting the first inequality above as

$$\mathcal{E}_n(z_n) + \beta||z_n - \frac{u_n}{\beta}||^2_{H_n} \leq \mathcal{E}_n(v_n) + \beta||v_n - \frac{u_n}{\beta}||^2_{H_n}$$

we get in the limit again by condition (M1) and Corollary 7.4.1 $\overline{\lim}_n ||z_n - \frac{u_n}{\beta}||^2_{H_n} \leq ||z - \frac{u}{\beta}||^2_H$, hence $||z_n - \frac{u_n}{\beta}||^2_{H_n} \to ||z - \frac{u}{\beta}||^2_H$, and this together with Corollary 7.4.1 and Corollary 7.5.2 concludes the proof.

Now we prove that (2) $\Longrightarrow$ (3). Let E_n be the spectral measure of $-A_n$. Then

$$G_{n,\beta} = \int_0^\infty (\beta + \lambda)^{-1}\, dE_n(\lambda),$$

$$T_{n,t} = \int_0^\infty e^{-t\lambda}\, dE_n(\lambda).$$

Let $\mathbb{A}$ be a set of continuous functions on $[0, \infty)$, vanishing on infinity such that $\varphi(-A_n)$ converges strongly to $\varphi(-A)$ if $\varphi \in \mathbb{A}$. Clearly, $\mathbb{A}$ is algebra. Since

$$||\varphi(-A_n)||_{L(H_n)} \leq \sup_{\lambda \geq 0} |\varphi(\lambda)|,$$

one get that $\mathbb{A}$ is closed with respect to the uniform norm. Note that the functions $(\beta + \lambda)^{-1}$, $\beta > 0$ separate the points of $\mathbb{R}^+$. Hence by Stone-Weierstrass Theorem (see Theorem IV.9 in [21]) $\mathbb{A}$ consists of all continuous functions vanishing in infinity, in particular $e^{-t\lambda} \in \mathbb{A}$, $t > 0$. □

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