On the heat equation with positive generalized stochastic process potential

Habib **Ouerdiane**
Département de Mathématiques, Faculté des Sciences de Tunis.
Université de Tunis El Manar. Campus Universitaire,
1060 Tunis, Tunisie

José Luis **Silva**
University of Madeira, CCM 9000-390 Funchal, Portugal

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Abstract

In this paper we study the stochastic heat equation with potential and initial condition as generalized stochastic processes. For positive generalized stochastic process potential $(V(t))_{t\geq 0}$ and initial condition f the solution is given as a convergent series of integrals. Our approach is based on the convolution calculus on a suitable distribution space.

Keywords: Generalized functions, convolution calculus, stochastic heat equation, generalized stochastic process.

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1 Introduction

The aim of this work is to study the solution of the following Cauchy problem corresponding to the heat equation

$$\begin{cases} \frac{\partial}{\partial t}X(t,x,\omega) = a\Delta X(t,x,\omega) + X(t,x,\omega) * V(t,x,\omega) \\ X(0,x,\omega) = f(x,\omega) \end{cases}, \qquad (1)$$

where $a \in \mathbb{R}_+$, $t \in [0, \infty)$ is the time parameter, $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ is the spatial variable, $r \in \mathbb{N}$, $\Delta = \sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^r$, $\omega = (\omega_1, \ldots, \omega_d)$ is the stochastic vector variable in the tempered Schwartz distribution space $S'_d := S'(\mathbb{R}, \mathbb{R}^d)$ with the standard Gaussian measure, $d \in \mathbb{N}$, $*$ is the convolution product between generalized functions (see Subsection 2.2 below) and the initial condition f as well as the potential V are stochastic generalized functions.

The Cauchy problem (1) was analyzed by many authors from different point of view, see e.g., [1], [6], and references therein. Often in the literature is used the Wick product $\diamondsuit$ (see [7] for this notion) instead of convolution product $*$ proposed here.

Recently Ouerdiane et al. [13] obtained the solution of (1) in terms of the convolution exponential as a well defined generalized function in a suitable distribution space, see Theorem 3.1 below. The main result of this paper is to prove that for positive generalized stochastic processes $(V(t))_{t\geq 0}$ and initial condition f the solution is given in terms of a convergent series of integrals. In the case of time independent and deterministic potential V then it agrees with the results of, e.g., [1].

The starting point is the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \gamma) \supset \mathcal{F}_\theta(\mathcal{N}'),$$

where $\mathcal{N}'$ is a the dual of a complex nuclear Fréchet space $\mathcal{N}$, θ is a Young function, γ is the usual Gaussian measure on $\mathcal{M}'$ which corresponds to the real part of $\mathcal{N}'$. The test function space $\mathcal{F}_\theta(\mathcal{N}')$ is defined as the space of all holomorphic functions on $\mathcal{N}'$ with an exponential growth condition of order θ. The generalized function space $\mathcal{F}'_\theta(\mathcal{N}')$ represents the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$. In the following we will choose the nuclear space $\mathcal{N} = (S_d \times \mathbb{R}^r)_{\mathbb{C}}$, the complexification of the real nuclear space $S_d \times \mathbb{R}^r$, which is adapted to our situation.

Using the Laplace transform $\mathcal{L}$ we may define the convolution of two generalized functions $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi \cdot \mathcal{L}\Psi)$$

which allows us to introduce the convolution exponential of Φ denoted by $\exp^* \Phi$ as an element in $\mathcal{F}'_\varphi(\mathcal{N}')$, where the Young function $\varphi = (e^{\theta^*})^*$ and

$$\theta^*(x) := \sup_{y \geq 0}(yx - \theta(y)) \tag{2}$$

denotes the polar function associated to θ, see e.g., [8].

For positive generalized stochastic process $V = (V(t))_{t\geq 0}$ there exists a family of Radon measures $\mu = (\mu_t)_{t\geq 0}$ (see e.g., [12]) on $\mathcal{M}'$ which represents V and, therefore, the Fourier transform of μ_t, $t \geq 0$ is given by

$$\langle\langle V(t), \exp(i\xi)\rangle\rangle = \hat{\mu}_t(\xi) = \int_{\mathcal{M}'} \exp(i\langle y, \xi\rangle) d\mu_t(y),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ and corresponds to the extension of the inner product of $L^2(\mathcal{M}', \gamma)$. Under these hypothesis we prove that for every test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and all $u \in \mathcal{N}'$ we have

$$\left(\exp^*\left(\int_0^t V_s ds\right) * \varphi\right)(u)$$
$$= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \varphi(u + y_1 + \ldots + y_n) d\mu_{s_1}(y_1) \ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n$$

which connects the convolution calculus and convergent series of integrals. We will use this equality to write the solution $X(t,x)$ of (1) for deterministic potential V and suitable choice of φ as

$$X(t,x) = (4a\pi t)^{-r/2} \int_{\mathbb{R}^r} f(y) e^{\frac{|x-y|^2}{4at}} dy$$
$$+ (4a\pi t)^{-r/2} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathbb{R}^r)^n} \int_{\mathbb{R}^r} f(y) e^{\frac{|x+y_1+\ldots+y_n-y|^2}{4at}} dy d\mu_{s_1}(y_1) \ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n.$$

We would like to stress that the positivity of the potential V implies the following property for μ: for each $t \geq 0$ there exists $n \in \mathbb{N}$, $m > 0$ with $\mu_t(\mathcal{M}_{-n}) = 1$ and μ_t satisfies the integrability condition,

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n})) d\mu_t(y) < \infty. \tag{3}$$

If V is deterministic and time independent, then the corresponding measure μ which verify (3) implies that V belongs to the so-called Albeverio-Høegh-Krohn class, see [1]. This class of potentials was studied by Asai et al. [2] and Kuna et al. [9] for the Schrödinger equation in connection with Feynman integrals. Our method may also be applied to solve the Cauchy problem corresponding to the Schrödinger equation if we replace a by $i\frac{\hbar}{2m}$, where $\hbar$ is the Plank's constant divided by 2π and m is the mass of the non relativistic particle, see Remark 3.8 for more details.

2 Preliminaries

2.1 Test and generalized functions spaces

In this section we introduce the framework need later on. The starting point is the real Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. More precisely, if $(f, x) = ((f_1, \ldots, f_d), (x_1, \ldots, x_r)) \in \mathcal{H}$, then

$$|(f,x)|^2 := \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(u)du + \sum_{i=1}^{r} x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \mathcal{M}. \tag{4}$$

The pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle (\omega, x), (\xi, y) \rangle := (\omega, \xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x, y)_{\mathbb{R}^r}$, $(\omega, x) \in \mathcal{M}'$ and $(\xi, y) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_n(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \bigcap_{n=0}^{\infty} \mathcal{M}_n,$$

where $S_n(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2_{\mathbb{R}^r}$, see e.g., [5] or [3] and references therein. We will consider the complexification of the triple (4) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{5}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos's theorem via its characteristic functional: for every $(\xi, p) \in \mathcal{M}$

$$C_\gamma(\xi, p) = \int_{\mathcal{M}'} \exp(i\langle (\omega, x), (\xi, p) \rangle) d\gamma((\omega, x)) = \exp(-\frac{1}{2}(|\xi|^2 + |p|^2)).$$

In order to solve the Cauchy problem (1) we need to introduce an appropriate space of generalized functions. We borrow this construction from [10]. Let $\theta = (\theta_1, \theta_2) : \mathbb{R}_+^2 \to \mathbb{R}$, $(t_1, t_2) \mapsto \theta_1(t_1) + \theta_2(t_2)$ where θ_1, θ_2 are two Young functions, i.e., $\theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ continuous convex strictly increasing function and

$$\lim_{t\to\infty} \frac{\theta_i(t)}{t} = \infty, \quad \theta_i(0) = 0, \quad i = 1, 2.$$

For every pair $m = (m_1, m_2)$ with $m_1, m_2 \in]0, \infty[$, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{N}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f : \mathcal{N}_{-n} \to \mathbb{C}, \text{ entire}, \ \|f\|_{\theta,m,n} = \sup_{z\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta(m|z|_{-n})) < \infty\},$$

where for each $z = (\omega, x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Now we consider as test function space the space of entire functions on $\mathcal{N}'$ of (θ_1, θ_2)-exponential growth and minimal type

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to construct the triple of the complex Hilbert space $L^2(\mathcal{M}', \gamma)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need another condition on the pair of Young functions (θ_1, θ_2). Namely,

$$\lim_{t\to\infty} \frac{\theta_i(t)}{t^2} < \infty, \quad i = 1, 2. \tag{6}$$

This is enough to obtain the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{7}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \gamma)$ endowed with the inductive limit topology.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. First we define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ the exponential function $\exp((\xi, p))$ is a well defined element in $\mathcal{F}_\theta(\mathcal{N}')$, see [4]. The Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ is defined by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\!\langle \Phi, \exp((\xi, p)) \rangle\!\rangle. \tag{8}$$

We are ready to state to characterization theorem, see e.g., [4] and [13].

Theorem 2.1 *The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g,\ |g(\xi,p)| \le k\exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)),\ (\xi,p)\in\mathcal{N}_n.$$

2.2 The Convolution Product $*$

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator. Let us define the convolution between a generalized and a test function. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\!\langle \Phi, \tau_{-(\omega,x)}\varphi\rangle\!\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega+\eta, x+y).$$

It is not hard the see that $\Phi * \varphi \in \mathcal{F}_\theta(\mathcal{N}')$. The convolution product is given in terms of the dual pairing as $(\Phi * \varphi)(0,0) = \langle\!\langle \Phi, \varphi\rangle\!\rangle$ for any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given. Then $\Phi * \Psi$ is defined as

$$\langle\!\langle \Phi * \Psi, \varphi\rangle\!\rangle := \langle\!\langle \Phi, \Psi * \varphi\rangle\!\rangle,\ \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{9}$$

This definition of convolution product for generalized functions will be used on Section 3 in order to write the solution of the stochastic heat equation given in (1). We have the following equality, see [13], Proposition 3.3:

$$\Phi * \exp((\xi,p)) = (\mathcal{L}\Phi)(\xi,p)\exp((\xi,p)),\ (\xi,p)\in\mathcal{N}.$$

As a consequence of the above equality and the definition (9) we obtain that

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi,\ \Phi,\Psi \in \mathcal{F}'_\theta(\mathcal{N}') \tag{10}$$

which says that the Laplace transform maps the convolution product in $\mathcal{F}'_\theta(\mathcal{N}')$ into the usual pointwise product in the algebra of functions $\mathcal{G}_{\theta^*}(\mathcal{N})$. Therefore we may use Theorem 2.1 to define convolution product between two generalized functions as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi\mathcal{L}\Psi).$$

Relation (10) allows us to define the convolution exponential of a generalized function. In fact, for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we may easily check that $\exp(\mathcal{L}\Phi) \in \mathcal{G}_{e^{\theta^*}}(\mathcal{N})$. Using the inverse Laplace transform and the fact that any Young function θ verify the property $(\theta^*)^* = \theta$ we obtain that $\mathcal{L}^{-1}(\mathcal{G}_{e^{\theta^*}}(\mathcal{N})) = \mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$. Now we give the definition of the convolution exponential of $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$, denoted by $\exp^* \Phi$

$$\exp^* \Phi := \mathcal{L}^{-1}(\exp(\mathcal{L}\Phi)).$$

Notice that $\exp^* \Phi$ is well defined element in $\mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$ and therefore the distribution $\exp^* \Phi$ is given in terms of a convergent series

$$\exp^* \Phi = \delta_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\Phi^{*n}, \tag{11}$$

where Φ^{*n} is the convolution of Φ with itself n times, $\Phi^{*0} := \delta_0$ by convention with δ_0 denoting the Dirac distribution at 0.

3 Applications to the heat equation

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi(t),\ t \geq 0\} \subset \mathcal{F}'_\theta(\mathcal{N}')$. The process $\Phi(t)$ is said to be continuous if the map $t \mapsto \Phi(t)$ is continuous. In order to introduce generalized stochastic integrals, we use the characterization theorem for sequences of generalized functions, see [11], Theorem 3. For a given continuous generalized stochastic process $(X(t))_{t\geq 0}$ we define the stochastic generalized process

$$Y(t,x,\omega) = \int_0^t X(s,x,\omega)ds \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X(s,x,\omega)ds\right)(\xi,p) := \int_0^t \mathcal{L}X(s,p,\xi)ds. \tag{12}$$

The process $Y(t,x,\omega)$ is differentiable and we have $\frac{\partial}{\partial t}Y(t,x,\omega) = X(t,x,\omega)$. The details of the proof can be seen in [13], Proposition 4.11. The main result in [13] is stated in the following theorem.

Theorem 3.1 *1. The Cauchy problem (1) has an unique solution $X(t)$ which is a generalized $\mathcal{F}'_\beta(\mathcal{N}')$-valued stochastic process, where the Young function β is given by $\beta = (e^{\theta^*})^*$. Moreover, the solution $X(t)$ is given explicitly by*

$$X(t,\omega,x) = f(\omega,x) * \exp^*\left(\int_0^t V(s)(\omega,x)ds\right) * \gamma_{2at}, \qquad (13)$$

where γ_{2at} is Gaussian measure on $\mathbb{R}^r$ with variance $2at$.

2. If the potential V and the initial condition f do not depend on the random parameter ω then the solution of (1) is given by

$$X(t,x) = (g(t,\cdot) * \gamma_{2at})(x), \qquad (14)$$

where g is equal to

$$g(t,x) = f(x)\exp\left(\int_0^t V(s,x)ds\right).$$

3.1 The solution of heat equation as limit of integrals

In this section we will write the solution of the Cauchy problem (1) as a limit of convergent series of integrals. To this end, we choose the potential $V = (V(t))_{t\geq 0}$ as a positive generalized stochastic process represented by the family of Radon measures $(\mu_t)_{t\geq 0}$, i.e., for any $t \geq 0$

$$\langle\langle V(t),\varphi\rangle\rangle = \int_{\mathcal{M}'} \varphi(y)d\mu_t(y), \quad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

Moreover the measure μ_t verify the following integrability condition: there exists $n \in \mathbb{N}$ and $m > 0$ with $\mu_t(\mathcal{M}_{-n}) = 1$ such that

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n}))d\mu_t(y) < \infty. \qquad (15)$$

Lemma 3.2 *For each Radon measure μ on $\mathcal{M}'$ verifying (15) and all $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and $u = (x, \omega) \in \mathcal{N}' = \mathcal{M}' + i\mathcal{M}'$ we have*

$$((\exp^* \mu) * \varphi)(u) = \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \varphi(u + y_1 + \ldots + y_n) d\mu(y_1) \ldots d\mu(y_n). \tag{16}$$

Proof. First we compute $\mu^{*n} * \varphi$, for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and $n = 2$. For any $u \in \mathcal{N}'$ we have

$$\begin{aligned}
((\mu * \mu) * \varphi)(u) &= (\mu * (\mu * \varphi))(u) \\
&= \langle\langle \mu, \tau_{-u}(\mu * \varphi) \rangle\rangle \\
&= \int_{\mathcal{M}'} \tau_{-u}(\mu * \varphi)(y_1) d\mu(y_1) \\
&= \int_{\mathcal{M}'} (\mu * \varphi)(u + y_1) d\mu(y_1). \\
&= \int_{\mathcal{M}'} \left(\int_{\mathcal{M}'} \varphi(u + y_1 + y_2) d\mu(y_2) \right) d\mu(y_1).
\end{aligned}$$

Now, using iteratively this procedure on the equality (11) we obtain for every $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and any $u \in \mathcal{N}'$

$$\begin{aligned}
&((\exp^* \mu) * \varphi)(u) \qquad (17) \\
&= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} (\mu^{*n} * \varphi)(u) \\
&= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^*} \varphi(u + y_1 + \ldots + y_n) d\mu(y_1) \ldots d\mu(y_n).
\end{aligned}$$

This proves the desired result. ■

Lemma 3.3 *Let $(V(s))_{s \geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ be a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s \geq 0}$. The for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have*

$$\begin{aligned}
\left\langle\left\langle \int_0^t V(s) ds, \varphi \right\rangle\right\rangle &= \int_0^t \langle\langle V(s), \varphi \rangle\rangle ds \qquad (18) \\
&= \int_0^t \left(\int_{\mathcal{M}'} \varphi(y) d\mu_s(y) \right) ds,
\end{aligned}$$

Moreover, we have

$$\left\langle\!\!\left\langle \exp^*\left(\int_0^t V(s)ds\right),\varphi\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \exp^*\left(\int_0^t \mu_s ds\right),\varphi\right\rangle\!\!\right\rangle. \tag{19}$$

Proof. In fact equality (18) is nothing but the definition (12) with $\varphi = \exp((\xi,p))$. Therefore by a limit procedure we get the required result (18) for general test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$.

To prove equality (19) we proceed in two steps: first we notice that for every $s \geq 0$ $V(s) * V(s)$ is represented by $\mu_s * \mu_s$ because for all $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$

$$\begin{aligned}\langle\!\langle V(s) * V(s),\varphi\rangle\!\rangle &= \langle\!\langle V(s), V(s)*\varphi\rangle\!\rangle \\ &= \int_{\mathcal{M}'} (V(s)*\varphi)(x)d\mu_s(x). \\ &= \int_{\mathcal{M}'}\left(\int_{\mathcal{M}'}\varphi(x+y)d\mu_s(y)\right)d\mu_s(x) \\ &= \langle\!\langle \mu_s * \mu_s,\varphi\rangle\!\rangle.\end{aligned}$$

Iterating this process we obtain

$$\langle\!\langle \exp^* V(s),\varphi\rangle\!\rangle = \langle\!\langle \exp^* \mu_s,\varphi\rangle\!\rangle. \tag{20}$$

Then equality (19) is a consequence of (18) and (20). ■

We now use the last two lemmas to derive the following corollary.

Corollary 3.4 *Let $(V(s))_{s\geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ be a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s\geq 0}$. Then for any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$, $u \in \mathcal{N}'$ holds*

$$\left(\exp^*\left(\int_0^t V(s)ds\right) * \varphi\right)(u) \tag{21}$$
$$= \varphi(u) + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{[0,t]^n}\int_{(\mathcal{M}')^n}\varphi(u+y_1+\ldots+y_n)d\mu_{s_1}(y_1)\ldots d\mu_{s_n}(y_n)ds_1\ldots ds_n.$$

We are now ready to write the solution (13) of the Cauchy problem (1) as a convergent series of integrals. We will apply the preceding corollary with φ of the following form

$$\varphi_t(x) = (\gamma_{2at} * f)(x) = (4a\pi t)^{-r/2}\int_{\mathbb{R}^r} f(y)e^{-\frac{|x-y|^2}{4at}}dy, \quad x \in \mathbb{R}^r,$$

where the initial condition f is a given function.

Proposition 3.5 *Let V, f be deterministic functions. The solution of the Cauchy problem (1) admits the following representation*

$$\begin{aligned} X_t(x) &= \\ &= \varphi_t(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathbb{R}^r)^n} \varphi_t(x + y_1 + \ldots + y_n) d\mu_{s_1}(y_1) \ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n \\ &= (4a\pi t)^{-r/2} \int_{\mathbb{R}^r} f(y) e^{-\frac{|x-y|^2}{4at}} dy \\ &+ (4a\pi t)^{-r/2} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathbb{R}^r)^n} \int_{\mathbb{R}^r} f(y) e^{-\frac{|x+y_1+\ldots+y_n-y|^2}{4at}} dy d\mu_{s_1}(y_1) \ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n. \end{aligned}$$

If the potential V is time independent and $r = 1$ then the solution is given by

$$\begin{aligned} X_t(x) &= (4a\pi t)^{-1/2} \int_{\mathbb{R}} f(y) e^{-\frac{|x-y|^2}{4at}} dy \\ &+ (4\pi a t)^{-1/2} \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^n} \int_{\mathbb{R}} f(y) e^{-\frac{(x+y_1+\ldots+y_n-y)^2}{4at}} dy d\mu(y_1) \ldots d\mu(y_n). \end{aligned}$$

We are no going to obtain an analogous result of Proposition 3.5 for initial condition from our generalized function space $\mathcal{F}'_{\theta^*}(\mathcal{N}')$. Before let us define the adjoint of the translation operator applied to a generalized function form $\mathcal{F}'_{\theta^*}(\mathcal{N}')$. For any $x \in \mathcal{N}'$ and $\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ we define the generalized function $\tau^*_{-x}\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ as

$$\langle\langle \tau^*_{-x}\Phi, \varphi \rangle\rangle = \langle\langle \Phi, \tau_{-x}\varphi \rangle\rangle, \quad \forall \varphi \in \mathcal{F}_{\theta^*}(\mathcal{N}').$$

Now we generalize Lemma 3.2.

Lemma 3.6 *Let μ be a Radon measure on $\mathcal{M}'$ fulfilling condition (15), then for every distribution $\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ we have*

$$(\exp^* \mu) * \Phi = \Phi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1) \ldots d\mu(y_n), \tag{22}$$

where for every $n = 1, 2, \ldots$ the distribution $\int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1) \ldots d\mu(y_n)$

is defined for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ *as*

$$\begin{aligned}
&\left\langle\!\!\left\langle \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n), \varphi \right\rangle\!\!\right\rangle \\
&= \int_{(\mathcal{M}')^n} \langle\!\langle \tau^*_{-y_1-\ldots-y_n}\Phi, \varphi\rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n) \\
&= \int_{(\mathcal{M}')^n} \langle\!\langle \Phi, \tau_{-y_1-\ldots-y_n}\varphi\rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n) \\
&= \int_{(\mathcal{M}')^n} (\Phi * \varphi)(y_1+\ldots+y_n) d\mu(y_1)\ldots d\mu(y_n).
\end{aligned}$$

Proof. Equality (22) may be derived as follows: for any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ definition (9) gives

$$\langle\!\langle (\exp^* \mu) * \Phi, \varphi\rangle\!\rangle = \langle\!\langle \exp^* \mu, \Phi * \varphi\rangle\!\rangle.$$

Now we use the relation between the convolution product and dual pairing to obtain

$$\langle\!\langle \exp^* \mu, \Phi * \varphi\rangle\!\rangle) = ((\exp^* \mu) * (\Phi * \varphi))(0).$$

Applying Lemma 3.2 with $\Phi * \varphi$ replacing φ yields

$$\begin{aligned}
&\langle\!\langle (\exp^* \mu) * \Phi, \varphi\rangle\!\rangle \\
&= (\Phi * \varphi)(0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathcal{M}')^n} (\Phi * \varphi)(0 + y_1 + \ldots + y_n) d\mu(y_1)\ldots d\mu(y_n) \\
&= \langle\!\langle \Phi, \varphi\rangle\!\rangle + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathcal{M}')^n} \langle\!\langle \Phi, \tau_{-y_1-\ldots-y_n}\varphi\rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n) \\
&= \langle\!\langle \Phi, \varphi\rangle\!\rangle + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathcal{M}')^n} \langle\!\langle \tau^*_{-y_1-\ldots-y_n}\Phi, \varphi\rangle\!\rangle d\mu(y_1)\ldots d\mu(y_n).
\end{aligned}$$

■

Theorem 3.7 *Let* $(V(t))_{t\geq 0}$ *be a positive generalized stochastic process represented by the family of Radon measures* $(\mu_t)_{t\geq 0}$ *on* $\mathcal{M}'$ *which verify the integrability condition (15). If the initial condition* f *is a generalized function in* $\mathcal{F}'_{\theta^*}(\mathcal{N}')$*, then the solution of the Cauchy problem (1) is given by*

$$\left(\exp^* \int_0^t V(s)ds\right) * \Psi = \Psi + \sum_{n=1}^\infty \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Psi d\mu_{s_1}(y_1)\ldots d\mu_{s_n}(y_n) ds_1 \ldots ds_n,$$

where Ψ is the distribution given by $f * (\gamma_{2at} \otimes \delta_0)$, here γ_{2at} is the Gaussian measure on $\mathbb{R}^r$ with variance $2at$ and δ_0 is the Dirac measure on S'_d.

Proof. The prove can be easily derived from (21) and (22). ■

Remark 3.8 *The Cauchy problem corresponding to the Schrödinger equation is*

$$\begin{cases} i\hbar\frac{\partial}{\partial t}X(t,x) = -\frac{\hbar^2}{2m}\Delta X(t,x) + X(t,x) * V(t,x) \\ X(0,x) = f(x) \end{cases}.$$

In our framework this corresponds to choose $a = i\frac{\hbar}{2m}$ and interpret the measure γ_{2at} as a generalized function defined for any test function $\varphi \in \mathcal{F}_\theta(\mathbb{C}^r)$ by

$$(\gamma_{2at} * \varphi)(x) = (2\pi it\hbar/m)^{-r/2} \int_{\mathbb{R}^r} \varphi(y) e^{im\frac{|x-y|^2}{2t\hbar}} dy.$$

Remark 3.9

1. *We would like to mention that the spaces $\mathcal{F}_\theta(\mathcal{N}')$ and its dual $\mathcal{F}'_\theta(\mathcal{N}')$ are independent of the Gaussian measure γ. For another probability measure P on $\mathcal{M}'$ we can construct the analogous Gelfand triple as in (7) changing in an appropriate way the condition on the Young function θ in (6).*

2. *If one wants to handle potential not as generalized functions as we do in this paper but as an ordinary function, e.g., $v \in \mathcal{F}_\theta(\mathcal{N}')$, then we may identify v with the generalized function $vP \in \mathcal{F}'_\theta(\mathcal{N}')$. In fact, for every test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have*

   $$\langle\langle vP, \varphi \rangle\rangle = \langle\langle P, v\varphi \rangle\rangle = \int_{\mathcal{M}'} v(x)\varphi(x) dP(x)$$

   and this obviously defines a linear continuous functional on $\mathcal{F}_\theta(\mathcal{N}')$. Notice that in this case $vP * \varphi$ coincides with the usual convolutions $(v * \varphi)_P$ between functions with respect to the measure P.

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References

[1] S. Albeverio and R. Høegh-Krohn, *Mathematical theory of Feynman path integrals*, Lecture Notes in Math., vol. 523, Springer-Verlag, Berlin, Heidelberg and New York, 1976.

[2] N. Asai, I. Kubo, and H.-H. Kuo, *Feynman integrals associated with Albeverio-Høegh-Krohn potentials*, Stochastics in Finite and Infinite Dimensions : In Honor of Gopinath Kallianpur (Boston ; Basel ; Berlin) (T. Hida, R. L. Karandikar, H. Kunita, B. S. Rajput, S. Watanabe, and J. Xiong, eds.), Trends in Mathematics, Birkhäuser, 2001, pp. 29–48.

[3] Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite-dimensional analysis*, vol. 1, Kluwer Academic Publishers, Dordrecht, 1995, Em Russo.

[4] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui, *Un théorème de dualité entre espace de fonctions holomorphes à croissance exponentielle*, J. Funct. Anal. **171** (2000), no. 1, 1–14.

[5] T. Hida, H. H. Kuo, J. Potthoff, and L. Streit, *White noise. an infinite dimensional calculus*, Kluwer Academic Publishers, Dordrecht, 1993.

[6] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang, *Stochastic partial differential equations: a modeling, white noise functional approach*, Birkhäuser, Boston, Basel, Berlin, 1996.

[7] Yu. G. Kondratiev, P. Leukert, and L. Streit, *Wick calculus in Gaussian analysis*, Acta Appl. Math. **44** (1996), 269–294.

[8] M-A. Krasnosel'ski and Ya-B. Ritickili, *Convex functions and orliez spaces*, P. Nordhoff. Itd, Groningen, The Nethelands, 1961.

[9] T. Kuna, L. Streit, and W. Westerkamp, *Feynman integrals for a class of exponentially growing potentials*, J. Math. Phys. **39** (1999), no. 9, 4476–4491.

[10] H. Ouerdiane, *Nuclear * agebra of entire functions and applications*, Tech. Report 00-05-15, BiBoS University of Bielefeld, 2000.

[11] H. Ouerdiane and A. Rezgui, *Representation integrale de fonctionnlles analytiques*, Physics and Geometry: New Interplays (Canada), Canadian Mathematical Society, 1999.

[12] ______, *Un théorème de Bochner-Minlos avec une condition d'intégrabilité*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), no. 2, 297–302.

[13] Habib Ouerdiane and José Luis Silva, *Generaliezed Feymann-Kac formula with stochastic potential*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), no. 2, 1–13.