SAMPLE PATH LARGE DEVIATIONS FOR DIFFUSION PROCESSES ON CONFIGURATION SPACES OVER A RIEMANNIAN MANIFOLD

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ABSTRACT. In this paper, we establish a sample path large deviation principle for a class of diffusion processes on configuration spaces over a Riemannian manifold. The rate funtional turns out to be the energy of the paths associated to the L^2-Wasserstein distance.

CONTENTS

1. INTRODUCTION

Since Varadhan's work [Var67] on large deviations for the small time asymptotics for diffusion processes a large number of papers has been devoted to this subject. For diffusions on finite dimensional state spaces we would like to mention here particularly Norris's work [Nor97] and also the references therein. In recent years, small time large deviations of diffusions have also been studied in infinite dimensions, see [Fan94, FZ99, AZ02, Hin02, Ram01, Sch96, Zha00]).

In this paper we prove a small time large deviation principle for a class of diffusions on configuration space (i.e., infinite particle systems in continuum) on the sample path level. The paper in the literature, which is closest to our situation, is the paper by Schied for the case of the super-Brownian motion. Our diffusions, however, take values in Γ_M, i.e., the space of all $\mathbf{Z}_+ \cup \{+\infty\}$-valued Radon measures on a finite dimensional, connected complete Riemannian manifold M. So, (at least

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if M is not compact) the diffusions on Γ_M can be heuristically written as

$$X_t = \sum_{i=1}^{\infty} \delta_{X_t^i},$$

representing interacting random particles. One way to construct such a diffusion is to use the theory of Dirichlet forms (see, for example, [MR00, AKR96a]). The geometry and analysis on configuration space Γ_M was carried out in [AKR96a, AKR96b, AKR98a, AKR98b]. The intrinsic metric of the associated Dirichlet form was identified as an L^2-Wasserstein type distance in [RS99]. As said before in this paper, we establish a sample path large deviation principle for the diffusion process $X_t, t \geq 0$, on the path space $\Omega := C([0,1] \to \Gamma_M)$. Our strategy is to first establish the principle for the so called Brownian motion on configuration space i.e., the independent particle process on M, which is the harder part, and then to obtain the large deviation principle for more general diffusions via Girsanov transformation. The rate functional turns out to be the energy of the paths associated to the L^2-Wasserstein type distance from [RS99]. When the manifold M is the real line, the small time asymptotics (not the sample path large deviation) was analyzed in [Zha01].

We want to emphasize that in the case of the independent particle process we do not use its construction by Dirichlet forms, but rather the pathwise construction from [KLR00], which (as shown in [KLR00]) is possible for an explicitly described set $\Gamma_\infty \subset \Gamma_M$ of initial configurations. Γ_∞ is also an invariant set for the process, i.e., $(X_t)_{t\geq 0}$ stays in Γ_∞ for all times. We, therefore, can prove the sample path large deviation principle for all initial conditions $\gamma \in \Gamma_\infty$. We also use the metric d_∞ introduced on Γ_∞ in [KLR00] in a decisive way. d_∞ induces a stronger topology on Γ_∞ than the vague topology and is crucially used in Section 6 below (cf. Theorem 6.8).

The paper is organized as follows. In Section 2 we present our framework giving all conditions on M used below. We also recall relevant definitions and results from [KLR00]. In Section 3, we prove exponential estimates which are necessary for the sequel. Section 4 is devoted to the upper bound estimates for finite dimensional projections of the diffusion. In Section 5 the rate functional is identifed. The lower bound estimates for finite dimensional projections are discussed in Section 6. The sample path large deviation principle is finally proved in Section 7. In Section 8, we establish the large deviation principle for a more general class of diffusions.

2. Framework

Let M be a complete, connected Riemannian manifold as in the introduction. For simplicity we assume that M has dimension bigger than two. Let $p_t(x,y)$ denote the heat kernel on M. Throughout the paper, we assume that the manifold M satisfies the following conditions:

A.1. For any $\delta > 0$ there exists a constant $c_1(\delta)$ such that

$$\frac{\exp\left[-\frac{d(x,y)^2}{(2-\delta)t}\right]}{c_1(\delta)m(B(y,\sqrt{t}))} \leq p_t(x,y) \leq c_1(\delta)\frac{\exp\left[-\frac{d(x,y)^2}{(2+\delta)t}\right]}{m(B(y,\sqrt{t}))} \tag{2.1}$$

for all $x, y \in M$, $t \leq 2T$, where $p_t(x,y)$ denotes the heat kernel, $B(y,\sqrt{t})$ denotes the geodesic ball centered at y with radius $\sqrt{t}$.

A.2. For some fixed point $x_0 \in M$, there exist $c_{x_0} > 0$, $N \in \mathbf{N}$ such that

$$m(B(x_0, r)) \leq c_{x_0} r^N, \ r > 0, \tag{2.2}$$

where $m(dx)$ denotes for the Riemannian volume on M.

A.3. For any $r > 0$,

$$\inf_{x \in M} m(B(x, r)) > 0. \tag{2.3}$$

Remark 2.1. (i) Condition (A.1) is satisfied if, for example, the Ricci curvature is bounded from below, see [Stu92]. (A.3) holds if M has bounded geometry. We refer the reader to [Dav89] for more details.

(ii) We note that (e.g. by the proof of [KLR00, Lemma 8.2]) (A.1)-(A.3) imply the conditions (C.1), (C.2), (C.3) in [KLR00] imposed there for one of the main results, namely Corollary 8.1 and Remark 8.3 in that paper, which we shall use below in a crucial way.

Let Γ_M be the space of all $\mathbf{Z}_+ \cup \{+\infty\}$-valued Radon measures on M. Equipped with the vague topology Γ_M is a Polish space. The set of all $\gamma \in \Gamma_M$ such that $\gamma(\{x\}) \in \{0, 1\}$ is called the configuration space over M. For simplicity, we also call Γ_M configuration space over M. The geometry and analysis on configuration space has been developed in [AKR96a, AKR96b, AKR98a, AKR98b]. Let us recall some results and definitions from these papers. For $f \in C_0(M)$ (the space of all continuous functions on M having compact support), set

$$\langle f, \gamma \rangle := \int_M f(x)\, \gamma(dx) = \sum_{x \in \gamma} f(x).$$

Define the space of smooth cylindrical functions on Γ_M, $\mathcal{F}C_b^\infty$, as the set of functions on Γ_M of the form

$$u(\gamma) = F\big(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle\big), \ \gamma \in \Gamma_M, \tag{2.4}$$

for some $n \in \mathbf{N}$, $F \in C_b^\infty(R^n)$, and $f_1, \ldots, f_n \in C_0^\infty(M)$. For u as in (2.1) define its gradient ∇u as a mapping from $\Gamma_M \times M$ to TM (the tangent bundle of M):

$$\nabla u(\gamma, x) := \sum_{i=1}^n \partial_i F\big(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle\big) \nabla f_i(x), \ \gamma \in \Gamma_M, \ x \in M.$$

Here ∂_i denotes partial derivative with respect to the i-th coordinate, and ∇ is the usual gradient on M.

Denote by m the Riemannian volume on M. Let π be the Poisson measure on Γ_M with intensity m, i.e., the unique measure on Γ_M whose Laplace transform is given by

$$\int_{\Gamma_M} e^{\langle f, \gamma \rangle}\, \pi(d\gamma) = \exp\left(\int_M (e^{f(x)} - 1)\, m(dx)\right)$$

for all $f \in C_0(M)$. Introduce the pre-Dirichlet form :

$$\mathcal{E}_0(u, v) := \int_{\Gamma_M} \langle \nabla u, \nabla v \rangle_\gamma\, \pi(d\gamma) \tag{2.5}$$

$$u, v \in \mathcal{F}C_b^\infty,$$

where $\langle \nabla u, \nabla v \rangle_\gamma = \int_M \nabla u(\gamma, x) \cdot \nabla v(\gamma, x) \gamma(dx)$. It has been shown in [AKR98a] and [MR00] that $\mathcal{E}_0(u, v) = \int_{\Gamma_M} \langle \nabla u, \nabla v \rangle_\gamma \pi(d\gamma)$ is closable on $L^2(\Gamma_M, \pi)$ and its closure, denoted by $(\mathcal{E}, D(\mathcal{E}))$, is a quasi-regular Dirichlet form. Thus by the theory of Dirichlet forms (see [MR92]), there exists a diffusion process $\mathcal{M} := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_\gamma, \gamma \in \Gamma_M\}$ associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, where $\Omega := C([0, \infty) \to \Gamma_M)$ is the canonical path space. The diffusion $\mathcal{M}$ itself is also called Brownian motion on the configuration space. It was also proved in [AKR98a] that starting with π as initial distribution the process has the same law as the well known independent particle process (already studied by Doob in [Doo53]). Correspondingly, for $\gamma \in \Gamma_M$ P_γ should be the distribution of the process

$$X_t^\gamma := \sum_{x\in\gamma} \delta_{B_t^x},\ t \geq 0, \tag{2.6}$$

where $(B_t^x)_{t\geq 0}$ are independent Brownian motions starting at $x \in \gamma$ and δ_x denotes Dirac measure at x. However, it is easy to see that this is not true for any $\gamma \in \Gamma_M$. For example, even when $M = \mathbf{R}$, $X_t = \sum_i \delta_{B_t^i}$ with initial configuration $\gamma = \sum_{i=1}^\infty \delta_{\log(i)}$ will not be a Radon measure on $\mathbf{R}$ for some $t > 0$.

In [KLR00, Corollary 8.1 and Remark 8.3] it was proved that $(X_t)_{t\geq 0}$ defined in (2.6), however, a.s. does take values in Γ_M for all $t \geq 0$ if one starts from points in a paricular set Γ_∞ and in fact $(X_t)_{t\geq 0}$ stays in Γ_∞ for all $t \geq 0$. Let us recall the definition of Γ_∞ from [KLR00]. First we fix a base point x_0 in M once and for all. Let $d(x,y)$ denote the Riemannian distance on M. For each positive integer m, we define the functional

$$B_m(\gamma) := \Big\langle \exp\Big[-\frac{1}{m}d(x_0,\cdot)\Big],\gamma\Big\rangle = \sum_{x\in\gamma}\exp\Big[-\frac{1}{m}d(x_0,x)\Big],\ \gamma\in\Gamma_M, \tag{2.7}$$

and define Γ_m by

$$\Gamma_m := \big\{\gamma \in \Gamma_M \bigm| \gamma(\{x\}) \in \{0,1\} \text{ for all } x \in M \text{ and } B_m(\gamma) < \infty\big\}. \tag{2.8}$$

Set

$$\Gamma_\infty := \bigcap_{m=1}^\infty \Gamma_m. \tag{2.9}$$

From now on we shall always use the version $(X_t^\gamma)_{t\geq 0}$ defined as in (2.6) above, for $\gamma \in \Gamma_\infty$, of the process $\mathcal{M}$ constructed by Dirichlet form methods. For topological reasons we shall, however, consider $(X_t^\gamma)_{t\geq 0}$ to take values in the bigger space $\Gamma_M \supset \Gamma_\infty$, since Γ_∞ with the vague topology is not Polish.

In Section 8, we shall present how our results on the sample path large deviations can be extended to other diffusions on Γ_M. Let us describe the latter here.

Let $\psi \in D(\mathcal{E})$ such that $0 < \psi$ and $\int \psi^2 d\pi = 1$. As in [Ebe96] we define a new Dirichlet form on $L^2(\Gamma_M, \psi^2 d\pi)$ by

$$\mathcal{E}_\psi(u,v) := \int_{\Gamma_M} \langle \nabla u, \nabla v\rangle_\gamma \psi^2(\gamma)\,\pi(d\gamma), \tag{2.10}$$

$$D(\mathcal{E}_\psi) = \overline{\mathcal{D}}^{\mathcal{E}_{\psi,1}},$$

where $\mathcal{E}_{\psi,1}(u,u) := \mathcal{E}_\psi(u,u) + \int u^2(\gamma)\psi^2(\gamma)\pi(d\gamma)$ and $\mathcal{D}$ is given by

$$\mathcal{D} := \Big\{u \in D(\mathcal{E}) \Bigm| \int \big(\Gamma(u,u)(\gamma) + u^2(\gamma)\big)\psi^2(\gamma)\,\pi(d\gamma) < \infty\Big\}.$$

The diffusion process associated with the new Dirichlet form $(\mathcal{E}_\psi, D(\mathcal{E}_\psi))$ will be denoted by $\mathcal{M}_\psi := \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, Q_\gamma, \gamma \in \Gamma_M\}$, which are no longer indepedent Brownian particles.

For $u \in D(\mathcal{E}_\psi)$, set $\Gamma(u,u)(\gamma) := \langle \nabla u, \nabla u\rangle_\gamma$. Recall that the intrinsic metric of the Dirichlet form $(\mathcal{E}_\psi, D(\mathcal{E}_\psi))$ is defined by

$$\varrho(\gamma,\eta) := \sup\big\{u(\gamma) - u(\eta) \bigm| u \in D(\mathcal{E}_\psi) \cap C(\Gamma_M) \text{ and } \Gamma(u,u) \leq 1\big\}$$

for $\gamma, \eta \in \Gamma_M$. It was proved in [RS99] that ϱ is a Wasserstein type distance on Γ_M given by

$$\varrho(\gamma,\eta) := \inf\left\{\sqrt{\int_{M\times M} \frac{1}{2} d(x,y)^2\, \omega(dx,dy)}\ \middle|\ \omega \in \Gamma_{\gamma\times\eta}\right\}, \tag{2.11}$$

where $\Gamma_{\gamma\times\eta}$ denotes the set of $\omega \in \Gamma_{M\times M}$ having marginals γ and η, and $d(x,y)$ stands for the Riemannian distance on M. We emphasize that ϱ is independent of ψ.

3. Some preliminary exponential estimates

From this section until the end of Section 7, ψ is assumed to be one.

Lemma 3.1. *Let $T > 0$. Then there exist $c_1(T)$ and $c_2(T)$ such that*

$$P\big(d(x, B_t^x) > r\big) \le c_1(T) \exp\Big(-\frac{c_2(T) r^2}{t}\Big), \text{ for all } r > 0,\ t \le T,\ x \in M, \tag{3.1}$$

where B_t^x stands for the Brownian motion on M starting from x.

Proof. By condition A.1, for any $\delta > 0$ there exists a constant $c_1(\delta)$ such that

$$\frac{\exp\Big[-\frac{d(x,y)^2}{(2-\delta)t}\Big]}{c_1(\delta) m(B(y,\sqrt{t}))} \le p_t(x,y) \le c_1(\delta) \frac{\exp\Big[-\frac{d(x,y)^2}{(2+\delta)t}\Big]}{m(B(y,\sqrt{t}))} \tag{3.2}$$

for all $x, y \in M$, $t \le 2T$.
Choose $\delta, \delta_1, \delta_2 \in (0,\infty)$ such that $1 \le a := \frac{2+\delta}{2-\delta_2}\frac{1}{1-\delta_1} \le 2$. It follows from (3.2) that

$$\begin{aligned}
P\big(d(x,B_t^x) > r\big) &= \int_{d(x,y)>r} p_t(x,y)\, m(dy) \\
&\le \int_{d(x,y)>r} c_1(\delta) \frac{\exp\Big[-\frac{d(x,y)^2}{(2+\delta)t}\Big]}{m(B(y,\sqrt{t}))}\, m(dy) \\
&\le \exp\Big[-\frac{\delta_1 r^2}{(2+\delta)t}\Big] \int_{d(x,y)>r} c_1(\delta) \frac{\exp\Big[-\frac{(1-\delta_1)d(x,y)^2}{(2+\delta)t}\Big]}{m(B(y,\sqrt{t}))}\, m(dy) \\
&= \exp\Big[-\frac{\delta_1 r^2}{(2+\delta)t}\Big] \int_{d(x,y)>r} c_1(\delta) \frac{\exp\Big[-\frac{d(x,y)^2}{(2-\delta_2)\tilde{t}}\Big]}{m(B(y,\sqrt{t}))}\, m(dy) \\
&\quad (\text{where } \tilde{t} := (\tfrac{2+\delta}{2-\delta_2}\tfrac{1}{1-\delta_1})t = at \le 2T.) \\
&\le \exp\Big[-\frac{\delta_1 r^2}{(2+\delta)t}\Big] \int_{d(x,y)>r} c_1(\delta) c_1(\delta_2) \frac{m\big(B(y,\sqrt{\tilde{t}})\big)}{m(B(y,\sqrt{t}))} \frac{1}{c_1(\delta_2)} \frac{\exp\Big[-\frac{d(x,y)^2}{(2-\delta_2)\tilde{t}}\Big]}{m\big(B(y,\sqrt{\tilde{t}})\big)}\, m(dy) \\
&\le c c_1(\delta) c_1(\delta_2) \exp\Big[-\frac{\delta_1 r^2}{(2+\delta)t}\Big] \exp\big[(d-1)(\sqrt{a}-1)\sqrt{t}\big] \int_{d(x,y)>r} p_{\tilde{t}}(x,y)\, m(dy) \\
&\le c_1 \exp\Big[-\frac{\delta_1 r^2}{(2+\delta)t}\Big]
\end{aligned}$$

where we have used the inequality:

$$\frac{m\big(B(y,\sqrt{\tilde{t}})\big)}{m(B(y,\sqrt{t}))} \le c \exp\big[(d-1)(\sqrt{a}-1)\sqrt{t}\big]$$

which can be found, for example, in [GW00]. □

Corollary 3.2. *Let $T>0$. Then there exists constants $c_1(T), c_2(T)>0$ such that for all $s\in(0,T]$*

$$P\Big(\sup_{0\le t\le s} d(x,B_t^x)>r\Big)\le c_1(T)\exp\Big(-\frac{c_2(T)r^2}{s}\Big) \text{ for all } x\in M,\ r>0.$$

Proof. The corollary follows from Lemma 3.1 above and Lemma 8.1 in [KLR00]. □

4. Large deviation estimates for finite dimensional projections: upper bounds

Throughout this section, we fix a finite partition $D=\{0=t_0<t_1<t_2<\cdots<t_n=1\}$ of $[0,1]$. Let Y^ε be the random vector $Y^\varepsilon=(X_{\varepsilon t_1}, X_{\varepsilon t_2},\dots,X_{\varepsilon t_n})$ Let $\mathcal{X}$ denote the set of all signed Radon measures on M. Equip $\mathcal{X}$ with the vague topology generated by

$$\big\{U_{f,\alpha}=\{\nu\in\mathcal{X}\;\big|\;|\langle f,\nu\rangle-\alpha|<\delta\},\ f\in C_0(M),\ \alpha\in\mathbf{R},\ \delta>0\big\}.$$

$\mathcal{X}$ is a locally convex topological vector space with its topological dual $\mathcal{X}^*$ being identified as $C_0(M)$. It is well known that Γ_M is a closed subspace of $\mathcal{X}$. Denote by $B(x_0,r)$ the geodesic ball $\{x\,|\,d(x_0,x)<r\}$ of M.

Lemma 4.1. *For any $\delta>0$, there exists $c_3(\delta)>0$ such that for all $r>0$, $\varepsilon\le 1$, $c>0$,*

$$E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}c\chi_{B(x_0,r)}(B_\varepsilon)}\big]\le 1+c_3(\delta)\exp\Big[-\frac{((d(\gamma_0^i,x_0)-r)^2-2(2+\delta)c)}{2(2+\delta)\varepsilon}\Big], \tag{4.1}$$

where $P_{\gamma_0^i}$ is the law of the Brownian motion on M starting from γ_0^i.

Proof. By (3.2),

$$\begin{aligned}E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}c\chi_{B(x_0,r)}(B_\varepsilon)}\big]&=e^{\frac{1}{\varepsilon}c}\int_{B(x_0,r)}p_\varepsilon(\gamma_0^i,y)\,m(dy)+\int_{B(x_0,r)^c}p_\varepsilon(\gamma_0^i,y)\,m(dy)\\ &\le 1+e^{\frac{1}{\varepsilon}c}\int_{B(x_0,r)}c_1(\delta)\frac{\exp\Big[-\frac{d(\gamma_0^i,y)^2}{(2+\delta)\varepsilon}\Big]}{m(B(y,\sqrt{\varepsilon}))}\,m(dy).\end{aligned} \tag{4.2}$$

Note that for $y\in B(x_0,r)$, we have $d(\gamma_0^i,y)>d(\gamma_0^i,x_0)-r$. Using the lower bound in (3.2) as in Lemma 3.1, we see that (4.2) is dominated by

$$\begin{aligned}1+e^{\frac{1}{\varepsilon}c}&\exp\Big[-\frac{(d(\gamma_0^i,x_0)-r)^2}{2(2+\delta)\varepsilon}\Big]\int_{B(x_0,r)}c_1(\delta)\frac{\exp\Big[-\frac{d(\gamma_0^i,y)^2}{2(2+\delta)\varepsilon}\Big]}{m(B(y,\sqrt{\varepsilon}))}\,m(dy)\\ &\le 1+e^{\frac{1}{\varepsilon}c}\exp\Big[-\frac{(d(\gamma_0^i,x_0)-r)^2}{2(2+\delta)\varepsilon}\Big]\int_{B(x_0,r)}c_3(\delta)p_{\varepsilon}(\gamma_0^i,y)\,m(dy)\\ &\le 1+c_3(\delta)\exp\Big[-\frac{\big((d(\gamma_0^i,x_0)-r)^2-2(2+\delta)c\big)}{2(2+\delta)\varepsilon}\Big],\end{aligned}$$

which proves the assertion. □

Define $\mathcal{X}^n=\mathcal{X}\times\mathcal{X}\times\cdots\times\mathcal{X}$. Then it follows that

$$\begin{aligned}(\mathcal{X}^n)^*&=\mathcal{X}^*\oplus\mathcal{X}^*\oplus\cdots\oplus\mathcal{X}^*\\ &=C_0(M)\oplus C_0(M)\oplus\cdots\oplus C_0(M).\end{aligned} \tag{4.3}$$

For $x,x_1,x_2,\dots,x_n\in M$, set

$$h_x(x_1,x_2,\dots,x_n):=\frac{1}{2}\sum_{k=1}^n\frac{1}{(t_k-t_{k-1})}d(x_k,x_{k-1})^2,\ \text{where } x_0=x.$$

Lemma 4.2. *Let $F = (f_1, f_2, \dots, f_n) \in (\mathcal{X}^n)^*$. Then*

$$\begin{aligned}\Lambda(F) &:= \lim_{\varepsilon\to 0} \varepsilon \log P_{\gamma_0}\left[e^{\frac{1}{\varepsilon}F(Y^\varepsilon)}\right]\\ &= \int_M \sup_{(x_1,x_2,\dots,x_n)}\left[\sum_{k=1}^n f_k(x_k) - h_x(x_1,x_2,\dots,x_n)\right]\gamma_0(dx). \end{aligned} \tag{4.4}$$

Proof. By the independence,

$$\begin{aligned}\Lambda(F) &= \lim_{\varepsilon\to 0}\varepsilon\log E_{\gamma_0}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n\langle f_k, X_{\varepsilon t_k}\rangle}\right]\\ &= \lim_{\varepsilon\to 0}\varepsilon\log E_{\gamma_0}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n\sum_{i=1}^\infty f_k(B^i_{\varepsilon t_k})}\right]\\ &= \lim_{\varepsilon\to 0}\varepsilon\log E_{\gamma_0}\left[e^{\sum_{i=1}^\infty\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B^i_{\varepsilon t_k})}\right]\\ &= \lim_{\varepsilon\to 0}\varepsilon\log \Pi_{\gamma_0^i\in\gamma_0} E_{\gamma_0^i}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B^i_{\varepsilon t_k})}\right]\\ &= \lim_{\varepsilon\to 0}\sum_i\varepsilon\log E_{\gamma_0^i}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\right]. \end{aligned} \tag{4.5}$$

By the large deviation principle of Brownian motion (see, for example, [Aze80]), it follows that

$$\lim_{\varepsilon\to 0}\varepsilon\log E_{\gamma_0^i}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\right] = \sup_{(x_1,x_2,\dots,x_n)}\left[\sum_{k=1}^n f_k(x_k) - h_{\gamma_0^i}(x_1,x_2,\dots,x_n)\right].$$

Taking the limit inside the series in (4.5), we get

$$\begin{aligned}\Lambda(F) &= \sum_i \sup_{(x_1,x_2,\dots,x_n)}\left[\sum_{k=1}^n f_k(x_k) - h_{\gamma_0^i}(x_1,x_2,\dots,x_n)\right]\\ &= \int_M \sup_{(x_1,x_2,\dots,x_n)}\left[\sum_{k=1}^n f_k(x_k) - h_x(x_1,x_2,\dots,x_n)\right]\gamma_0(dx). \end{aligned} \tag{4.6}$$

It now remains to justify that we can take the limit inside the series. We suppose that the supports of f_k, $0 \le k \le n$, are contained in $B(x_0, r_0)$ for some r_0. By Lemma 4.1, we have

$$\begin{aligned}E_{\gamma_0^i}\left[e^{\pm\frac{n}{\varepsilon}f_k(B_{\varepsilon t_k})}\right] &\le E_{\gamma_0^i}\left[e^{\frac{n}{\varepsilon}\|f_k\|_\infty\chi_{B(x_0,r_0)}(B_{\varepsilon t_k})}\right]\\ &\le 1 + c_3(\delta)\exp\left[-\frac{\left((d(\gamma_0^i,x_0)-r_0)^2 - 2(2+\delta)nt_k\|f\|_\infty\right)}{2(2+\delta)\varepsilon t_k}\right]. \end{aligned}$$

Hence by Schwartz's inequality,

$$\begin{aligned}\left(E_{\gamma_0^i}\left[e^{\frac{n}{\varepsilon}f_k(B_{\varepsilon t_k})}\right]\right)^{-1} &\le E_{\gamma_0^i}\left[e^{-\frac{n}{\varepsilon}f_k(B_{\varepsilon t_k})}\right]\\ &\le 1 + c_3(\delta)\exp\left[-\frac{\left((d(\gamma_0^i,x_0)-r_0)^2 - 2(2+\delta)nt_k\|f_k\|_\infty\right)}{2(2+\delta)\varepsilon t_k}\right]. \end{aligned}$$

Using Hölder's inequality we have that

$$\begin{aligned}&E_{\gamma_0^i}\left[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\right]\\ &\quad\le \prod_{k=1}^n\left(1 + c_3(\delta)\exp\left[-\frac{\left((d(\gamma_0^i,x_0)-r_0)^2 - 2(2+\delta)nt_k\|f_k\|_\infty\right)}{2(2+\delta)\varepsilon t_k}\right]\right)^{\frac{1}{n}}. \end{aligned}$$

Similarly,

$$\begin{aligned}&\big(E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\big]\big)^{-1}\\&\qquad\le\prod_{k=1}^n\bigg(1+c_3(\delta)\exp\bigg[-\frac{\big((d(\gamma_0^i,x_0)-r_0)^2-2(2+\delta)nt_k\|f_k\|_\infty\big)}{2(2+\delta)\varepsilon t_k}\bigg]\bigg)^{\frac{1}{n}}.\end{aligned}$$

Hence it follows that for $\varepsilon\le 1$,

$$\begin{aligned}&\Big|\log E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\big]\Big|\\&\qquad\le\sum_{k=1}^n\frac{1}{n}\log\bigg(1+c_3(\delta)\exp\bigg[-\frac{\big((d(\gamma_0^i,x_0)-r_0)^2-2(2+\delta)nt_k\|f_k\|_\infty\big)}{2(2+\delta)\varepsilon t_k}\bigg]\bigg)\\&\qquad\le\sum_{k=1}^n c_3(\delta)\frac{1}{n}\exp\bigg[-\frac{\big((d(\gamma_0^i,x_0)-r_0)^2-2(2+\delta)nt_k\|f_k\|_\infty\big)}{2(2+\delta)\varepsilon t_k}\bigg].\end{aligned}$$

Since $\sum_{i=1}^\infty\exp[-\frac{1}{m}d(x_0,\gamma_0^i)]<\infty$ for all m, the above estimates show that

$$\sum_i\varepsilon\log E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}\sum_{k=1}^n f_k(B_{\varepsilon t_k})}\big]$$

converges absolutely and uniformly with respect to ε, which justifies to take the limit inside the series. □

Proposition 4.3. *Let μ_ε be the law of Y^ε on $\mathcal{X}^n$ under P_{γ_0}. Then $\{\mu_\varepsilon,\varepsilon\in(0,1]\}$ is exponentially tight, namely, for any $L>0$, there exists a compact subset $K_L\subset\mathcal{X}^n$ such that*

$$\limsup_{\varepsilon\to0}\varepsilon\log P_{\gamma_0}(Y^\varepsilon\in K_L^c)\le -L \tag{4.7}$$

Proof. We first prove that the law of X_ε is exponentially tight. So, let $\varepsilon\in(0,1]$. Note that a set of the form

$$K_{\{L_n\}}=\bigcap_n\{\mu\in\mathcal{X}\mid|\mu|(B(x_0,n))\le L_n\} \tag{4.8}$$

with $L_n\in(0,\infty)$, is relatively compact. Given $\hat L>0$, we will choose L_n so that

$$\limsup_{\varepsilon\to0}\varepsilon\log P_{\gamma_0}\big(X_\varepsilon\in K_{\{L_n\}}^c\big)\le-\hat L. \tag{4.9}$$

Let $m_n:=\sharp\{i\mid d(\gamma_0^i,x_0)\le n+\sqrt{2(2+\delta)}\}$. By Lemma 4.1,

$$E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}\chi_{B(x_0,n)}(B_\varepsilon)}\big]\le 1+c_3\exp\bigg[-\frac{\big((d(\gamma_0^i,x_0)-n)^2-1\big)}{2(2+\delta)\varepsilon}\bigg].$$

Hence for all $\varepsilon \le 1$,

$$
\begin{aligned}
&P_{\gamma_0}\big(X_\varepsilon(B(x_0,n)) > L_n\big)\\
&\quad \le e^{-\frac{L_n}{\varepsilon}} E_{\gamma_0}\big[e^{\frac{1}{\varepsilon}\sum_{i=1}^\infty \chi_{B(x_0,n)}(B^i_\varepsilon)}\big]\\
&\quad = e^{-\frac{L_n}{\varepsilon}} \prod_{i=1}^\infty E_{\gamma_0^i}\big[e^{\frac{1}{\varepsilon}\chi_{B(x_0,n)}(B^i_\varepsilon)}\big]\\
&\quad \le e^{-\frac{L_n}{\varepsilon}} \prod_{i=1}^\infty \left(1 + c_3(\delta)\exp\left[-\frac{\big((d(\gamma_0^i,x_0)-n)^2 - 2(2+\delta)\big)}{2(2+\delta)\varepsilon}\right]\right)\\
&\quad \le e^{-\frac{L_n}{\varepsilon}} \prod_{(d(\gamma_0^i,x_0)-n)^2 - 2(2+\delta) > 0} \left(1 + c_3(\delta)\exp\left[-\frac{\big((d(\gamma_0^i,x_0)-n)^2 - 2(2+\delta)\big)}{2(2+\delta)\varepsilon}\right]\right)\\
&\quad \times \prod_{|d(\gamma_0^i,x_0)-n| \le \sqrt{2(2+\delta)}} \left(1 + c_3(\delta)\exp\left[\frac{1}{\varepsilon}\right]\right)\\
&\quad \le e^{-\frac{L_n}{\varepsilon}} \exp\Big\{c_3(\delta)\sum_i e^{-\frac{((d(\gamma_0^i,x_0)-n)^2-2(2+\delta))}{2(2+\delta)\varepsilon}}\Big\}(1 + c_3(\delta)e^{\frac{1}{\varepsilon}})^{m_n}\\
&\quad \le e^{-\frac{L_n}{\varepsilon}} \exp\Big\{c_3(\delta)\sum_i e^{-\frac{((d(\gamma_0^i,x_0)-n)^2-2(2+\delta))}{2(2+\delta)}}\Big\}(1 + c_3(\delta))^{m_n} e^{\frac{m_n}{\varepsilon}}\\
&\quad = \exp\Big\{-\frac{1}{\varepsilon}(L_n - m_n) + c_3(\delta)\sum_i e^{-\frac{((d(\gamma_0^i,x_0)-n)^2-2(2+\delta))}{2(2+\delta)}} + m_n\log(1+c_3(\delta))\Big\}.
\end{aligned}
$$

Define

$$L_n := \hat{L} + m_n + c_3(\delta)\sum_i e^{-\frac{(d(\gamma_0^i,x_0)-n)^2-2(2+\delta))}{2(2+\delta)}} + m_n\log(1+c_3(\delta)) + n$$

and define $K_{\hat{L}} := K_{\{L_n\}}$ where $K_{\{L_n\}}$ is defined as in (4.8). Then we have

$$
\begin{aligned}
P_{\gamma_0}\big(X_\varepsilon \in (K^c_{\{L_n\}})\big) &\le \sum_{n=1}^\infty P_{\gamma_0}\big(X_\varepsilon(B(x_0,n)) > L_n\big)\\
&\le \sum_{n=1}^\infty \exp\Big\{-\frac{\hat{L}+n}{\varepsilon}\Big\} \le \frac{1}{e-1}e^{-\frac{\hat{L}}{\varepsilon}}.
\end{aligned}
$$

This implies (4.9).

Let $L > 0$ and choose $\hat{L}$ such that $\inf_{1\le k\le n}(\frac{\hat{L}}{t_k}) > L$. Let $K_{\hat{L}}$ be defined as above. Put

$$K_L = (K_{\hat{L}})^n$$

Then K_L is compact and

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon\log P_{\gamma_0}(Y^\varepsilon \in K_L^c) &= \limsup_{\varepsilon\to 0}\varepsilon\log P_{\gamma_0}\left(\bigcup_{k=1}^n (X_{\varepsilon t_k} \in K^c_{\hat{L}})\right)\\
&\le \limsup_{\varepsilon\to 0}\varepsilon\log\left(n\max_{1\le k\le n} P_{\gamma_0}(X_{\varepsilon t_k} \in K^c_{\hat{L}})\right)\\
&= \max_{1\le k\le n}\frac{1}{t_k}\left(\limsup_{\varepsilon\to 0}\varepsilon t_k\log P_{\gamma_0}(X_{\varepsilon t_k} \in K^c_{\hat{L}})\right)\\
&\le \max_{1\le k\le n}\frac{1}{t_k}(-\hat{L}) \le -L.
\end{aligned}
$$

□

Define, for $(\eta_1, \dots, \eta_n) \in \mathcal{X}^n$,

$$I_{\gamma_0}(\eta_1, \dots \eta_n) := \sup_{(f_1,\dots,f_n)\in C_0(M)^n} \Bigg(\sum_{k=1}^n \langle f_k, \eta_k \rangle$$

$$- \int_M \sup_{(x_1,x_2,\dots,x_n)} \left[\sum_{k=1}^n f_k(x_k) - h_x(x_1, x_2, \dots, x_n) \right] \gamma_0(dx) \Bigg). \tag{4.10}$$

Theorem 4.4. *Let μ_ε be the law of Y^ε on $\mathcal{X}^n$ under P_{γ_0}. Then, for any closed subset $F \subset \mathcal{X}^n$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) \le - \inf_{(\eta_1,\dots,\eta_n)\in F} I_{\gamma_0}(\eta_1, \dots, \eta_n).$$

Proof. The assertion follows by a combination of Lemma 4.2, Proposition 4.3 above and Theorem 4.5.20 in [DZ92]. □

5. Identification of the rate functional

The main task of this section is to identify the rate function $I_{\gamma_0}(\eta_1, \dots, \eta_n)$ as $\sum_{k=1}^n \frac{1}{(t_k - t_{k-1})} \varrho(\eta_k, \eta_{k-1})^2$ with $\eta_0 = \gamma_0$. This turns out to be highly non-trivial. We will also show that the rate functional is good. Let $\mathcal{X}^+$ denote the set of positive Radon measures on M. The definition of ϱ in (2.11) extends to $\mathcal{X}^+$ naturally with Γ_M replaced by $\mathcal{X}^+$. Below, unless otherwise stated we fix $\gamma_0 \in \mathcal{X}^+$. We also fix positive numbers $a_1, a_2, \dots, a_n$. Introduce a functional $H(\gamma_1, \dots, \gamma_n)$ on $(\mathcal{X}^+)^n$ by

$$H(\gamma_1, \dots, \gamma_n) = \sum_{k=1}^n a_k \varrho(\gamma_k, \gamma_{k-1})^2. \tag{5.1}$$

Proposition 5.1. *H is convex and lower semi-continuous on $(\mathcal{X}^+)^n$, equipped with the product topology of vague convergence.*

Proof. Let $\gamma_1, \gamma_2, \bar\gamma_1, \bar\gamma_2 \in \chi^+$. For any $0 < \alpha < 1$, we first show that

$$\varrho\big(\alpha\gamma_1 + (1-\alpha)\gamma_2, \alpha\bar\gamma_1 + (1-\alpha)\bar\gamma_2\big)^2 \le \alpha\varrho(\gamma_1, \bar\gamma_1)^2 + (1-\alpha)\varrho(\gamma_2, \bar\gamma_2)^2. \tag{5.2}$$

To this end, we may and will assume $\varrho(\gamma_1, \bar\gamma_1) < \infty$ and $\varrho(\gamma_2, \bar\gamma_2) < \infty$. By Lemma 4.1 in [RS99] (although the Lemma was stated for Γ_M, its proof works also for $\mathcal{X}^+$), there exist $\eta, \bar\eta \in \mathcal{X}^+(M \times M)$ (the set of all positive Radon measures on $M \times M$) with

$$\eta(dx, M) = \gamma_1(dx), \eta(M, dy) = \bar\gamma_1(dy), \bar\eta(dx, M) = \gamma_2(dx), \bar\eta(M, dy) = \bar\gamma_2(dy)$$

and such that

$$\varrho(\gamma_1, \bar\gamma_1)^2 = \frac{1}{2} \int_{M\times M} d(x,y)^2 \, \eta(dx, dy) \tag{5.3}$$

$$\varrho(\gamma_2, \bar\gamma_2)^2 = \frac{1}{2} \int_{M\times M} d(x,y)^2 \, \bar\eta(dx, dy). \tag{5.4}$$

Define $\eta^* = \alpha\eta + (1-\alpha)\bar\eta$. Then,

$$\eta^*(dx, M) = \alpha\gamma_1(dx) + (1-\alpha)\gamma_2(dx)$$
$$\eta^*(M, dy) = \alpha\bar\gamma_1(dy) + (1-\alpha)\bar\gamma_2(dy).$$

Thus,

$$
\begin{aligned}
\varrho\big(\alpha\gamma_1+&(1-\alpha)\gamma_2, \alpha\bar{\gamma}_1+(1-\alpha)\bar{\gamma}_2\big)^2\\
&\leq \int_{M\times M} d(x,y)^2\,\eta^*(dx,dy)\\
&= \alpha\int_{M\times M} d(x,y)^2\,\eta(dx,dy) + (1-\alpha)\int_{M\times M} d(x,y)^2\,\bar{\eta}(dx,dy)\\
&= \alpha\varrho(\gamma_1,\bar{\gamma}_1)^2 + (1-\alpha)\varrho(\gamma_2,\bar{\gamma}_2)^2, \qquad (5.5)
\end{aligned}
$$

which proves (5.2). Let $\vec{\gamma}_1 = (\gamma_1^1,\dots,\gamma_n^1)$, $\vec{\gamma}_2 = (\gamma_1^2,\dots,\gamma_n^2) \in (\mathcal{X}^+)^n$. It follows from (5.5) that

$$
\begin{aligned}
H(\alpha\vec{\gamma}_1 + (1-\alpha)\vec{\gamma}_2) &= \sum_{k=1}^n a_k\varrho\big(\alpha\gamma_k^1+(1-\alpha)\gamma_k^2, \alpha\gamma_{k-1}^1+(1-\alpha)\gamma_{k-1}^2\big)^2\\
&\leq \alpha\sum_{k=1}^n a_k\varrho(\gamma_k^1,\gamma_{k-1}^1)^2 + (1-\alpha)\sum_{k=1}^n a_k\varrho(\gamma_k^2,\gamma_{k-1}^2)^2\\
&= \alpha H(\vec{\gamma}_1) + (1-\alpha)H(\vec{\gamma}_2), \qquad (5.6)
\end{aligned}
$$

showing that $H(\cdot)$ is convex. Next we will prove that H is lower semi-continuous. Let $\{\vec{\gamma}_m, m\geq 1\}$ be a sequence of elements in $(\mathcal{X}^+)^n$ converging vaguely to $\vec{\gamma}$. We need to show that

$$
H(\vec{\gamma}) \leq \liminf_{m\to\infty} H(\vec{\gamma}_m). \qquad (5.7)
$$

For this purpose, we may and will assume that the limit $\lim_{m\to\infty} H(\vec{\gamma}_m)$ exists and is finite as well as each $H(\vec{\gamma}_m)$. Write $\vec{\gamma}_m = (\gamma_1^m,\gamma_2^m,\dots,\gamma_n^m)$ and $\vec{\gamma} = (\gamma_1,\gamma_2,\dots,\gamma_n)$. By Lemma 4.1 in [RS99] there exists $\eta_{k,k-1}^m \in \mathcal{X}^+(M\times M)$ with $\eta_{k,k-1}^m(dx,M) = \gamma_k^m(dx)$, $\eta_{k,k-1}^m(M,dy) = \gamma_{k-1}^m(dy)$, such that

$$
\varrho(\gamma_k^m,\gamma_{k-1}^m)^2 = \int_{M\times M} d(x,y)^2\,\eta_{k,k-1}^m(dx,dy). \qquad (5.8)
$$

Thus,

$$
\lim_{m\to\infty} H(\vec{\gamma}_m) = \lim_{m\to\infty}\sum_{k=1}^n a_k\int_{M\times M} d(x,y)^2\,\eta_{k,k-1}^m(dx,dy). \qquad (5.9)
$$

Let Π_1 be the projection operator from $M\times M$ to M defined by $\Pi_1(x,y)=x$. For any compact set $K\subset M\times M$ and $k\geq 1$, we have

$$
\sup_m \eta_{k,k-1}^m(K) \leq \sup_m \prod_1^* \eta_{k,k-1}^m\Big(\prod_1(K)\Big) = \sup_m \gamma_k^m\Big(\prod_1(K)\Big) < \infty
$$

since γ_k^m converges vaguely to γ_k as $m\to\infty$. This implies that for each $1\leq k\leq n$, the family $\{\eta_{k,k-1}^m, m\geq 1\}$ is relatively compact with respect to the topology of vague convergence. Now, choose a common subsequence $\{m_l\}$ such that

$$
\lim_{l\to\infty}\eta_{k,k-1}^{m_l} = \eta_{k,k-1}^0 \qquad (5.10)
$$

vaguely for each $1\leq k\leq n$. Next we prove that

$$
\eta_{k,k-1}^0(dx,M) = \gamma_k(dx),\ \eta_{k,k-1}^0(M,dy) = \gamma_{k-1}(dy)
$$

Notice that this is not automatically a consequence of the vague convergence. We only prove one of them, say, $\eta_{k,k-1}^0(dx,M) = \gamma_k(dx)$, the other is proved analogously. Choose a sequence $\{\phi_j(y), j\geq 1\}$ of continuous functions on M satisfying

$0 \le \phi_j(y) \le 1$, $\phi_j(y) = 1$ on $B(x_0, j)$, $\phi_j(y) = 0$ on $B(x_0, j+1)^c$. Take $f \in C_0(M)$. Suppose $\mathrm{supp}[f] \subset B(x_0, m_0)$ for some m_0. Then

$$\begin{aligned}\int_{M\times M} f(x)\,\eta^0_{k,k-1}(dx,dy) &= \lim_{j\to\infty}\int_{M\times M} f(x)\phi_j(y)\,\eta^0_{k,k-1}(dx,dy)\\ &= \lim_{j\to\infty}\lim_{l\to\infty}\int_{M\times M} f(x)\phi_j(y)\,\eta^{m_l}_{k,k-1}(dx,dy). \qquad (5.11)\end{aligned}$$

But,

$$\begin{aligned}&\limsup_{l\to\infty}\left|\int_{M\times M} f(x)\phi_j(y)\,\eta^{m_l}_{k,k-1}(dx,dy) - \int_{M\times M} f(x)\,\eta^{m_l}_{k,k-1}(dx,dy)\right|\\ &\le \limsup_{l\to\infty}\int_{M\times M}|f(x)|(1-\phi_j(y))\,\eta^{m_l}_{k,k-1}(dx,dy)\\ &\le 2\|f\|_\infty \limsup_{l\to\infty}\int_{M\times M}\chi_{B(x_0,m_0)}(x)\chi_{B(x_0,j)^c}(y)\,\eta^{m_l}_{k,k-1}(dx,dy)\\ &\le 2\|f\|_\infty \sup_l \int_{\{d^2(x,y)\ge (j-m_0)^2\}}\eta^{m_l}_{k,k-1}(dx,dy)\\ &\le 2\|f\|_\infty \frac{1}{(j-m_0)^2}\sup_l\int_{M\times M} d^2(x,y)\,\eta^{m_l}_{k,k-1}(dx,dy) \le \frac{M}{(j-m_0)^2} \qquad (5.12)\end{aligned}$$

for some $M > 0$. Combining (5.11) and (5.12) we arrive at

$$\begin{aligned}&\int_{M\times M} f(x)\,\eta^0_{k,k-1}(dx,dy) = \lim_{j\to\infty}\lim_{l\to\infty}\int_{M\times M} f(x)\phi_j(y)\,\eta^{m_l}_{k,k-1}(dx,dy)\\ &= \limsup_{l\to\infty}\int_{M\times M} f(x)\,\eta^{m_l}_{k,k-1}(dx,dy) = \lim_{l\to\infty}\int_M f(x)\,\gamma^{m_l}_k(dx) = \int_M f(x)\,\gamma_k(dx).\end{aligned}$$

Since f was arbitrary, we conclude that $\eta^0_{k,k-1}(dx, M) = \gamma_k(dy)$.
Since $\int_{M\times M} d^2(x,y)\,\eta(dx,dy)$ is obvisouly lower semi-continuous with respect to η, it follows that

$$\begin{aligned}H(\vec\gamma) &\le \sum_{k=1}^n a_k \int_{M\times M} d^2(x,y)\,\eta^0_{k,k-1}(dx,dy)\\ &\le \sum_{k=1}^n a_k \liminf_{l\to\infty}\int_{M\times M} d^2(x,y)\,\eta^{m_l}_{k,k-1}(dx,dy)\\ &\le \liminf_{l\to\infty}\sum_{k=1}^n a_k \int_{M\times M} d^2(x,y)\,\eta^{m_l}_{k,k-1}(dx,dy)\\ &= \lim_{l\to\infty} H(\vec\gamma_{m_l}). \qquad (5.13)\end{aligned}$$

□

Lemma 5.2. *Let $f_1, f_2, \ldots, f_n \in C_0(M)$. Then*

$$g(x) := \sup_{x_1,x_2,\ldots,x_n}\left(\sum_{k=1}^n f_k(x_k) - h_x(x_1,x_2,\ldots,x_n)\right),\ x \in M, \qquad (5.14)$$

is lower semi-continuous and has compact support.

Proof. Clearly g is lower semi-continuous. We need to show $g(x) = 0$ outside some sufficiently big compact subset. Let K be a compact subset of M that contains the support of f_k for all $1 \le k \le n$. We may only consider $x \in K^c$. For such x, we have that $g(x) \ge 0$ since $\sum_{k=1}^n f_k(x_k) - h_x(x_1, x_2, \ldots, x_n) = 0$ for $x_1 = x_2 = \cdots =$

$x_n = x$. Furthermore, denoting by C the suppremum of the function $\sum_{k=1}^n f_k(\cdot)$, we have

$$\begin{aligned} g(x) &\le \sup_{(x_1,\dots,x_n)\in(K^c\times\cdots\times K^c)^c} \Big(\sum_{k=1}^n f_k(x_k) - h_x(x_1,x_2,\dots,x_n)\Big) \\ &\quad \vee \sup_{(x_1,\dots,x_n)\in K^c\times\cdots\times K^c} \big(-h_x(x_1,x_2,\dots,x_n)\big) \\ &\le \sup_{(x_1,\dots,x_n)\in(K^c\times\cdots\times K^c)^c} \big(C - h_x(x_1,x_2,\dots,x_n)\big) \vee 0 \\ &\le \bigvee_{k=1}^n \sup_{(x_1,\dots,x_n)\,:\,x_k\in K} \big(C - h_x(x_1,x_2,\dots,x_n)\big) \vee 0. \end{aligned} \tag{5.15}$$

Thus, it suffices to prove that for each $1 \le k \le n$,

$$g_k(x) := \sup_{(x_1,\dots,x_n)\,:\,x_k\in K} \big(C - h_x(x_1,x_2,\dots,x_n)\big) \vee 0 = 0 \tag{5.16}$$

outside some compact subset. Since $x_k \in K$, we can find a compact subset F_{k-1} such that for $x_{k-1} \in F_{k-1}^c$,

$$C - h_x(x_1,x_2,\dots,x_n) \le C - \frac{1}{2}\frac{1}{t_k - t_{k-1}} d(x_k,x_{k-1})^2 < 0.$$

Therefore, $g_k(x)$ can be written as

$$g_k(x) = \sup_{\substack{(x_1,\dots,x_n)\,:\,x_k\in K,\\ x_{k-1}\in F_{k-1}}} \big(C - h_x(x_1,x_2,\dots,x_n)\big) \vee 0.$$

Repeating the same arguments, we can find compact subsets $F_{k-2}, F_{k-3}, \dots, F_1$ such that

$$\begin{aligned} g_k(x) &= \sup_{\substack{(x_1,\dots,x_n)\,:\,x_k\in K,\\ x_{k-1}\in F_{k-1},\dots,x_1\in F_1}} \big(C - h_x(x_1,x_2,\dots,x_n)\big) \vee 0 \\ &\le \sup_{x_1\in F_1} \Big(C - \frac{1}{2}\frac{1}{t_1 - t_0} d(x,x_1)^2\Big) \vee 0. \end{aligned} \tag{5.17}$$

The latter is clearly zero for x outside some sufficiently big compact subset. This completes the proof of the lemma. □

Lemma 5.3. *Let $f_1, f_2, \dots, f_n \in C_0(M)$. Then for $\gamma_0 \in \Gamma_M$*

$$\begin{aligned} &\sup_{\gamma_1,\gamma_2,\dots,\gamma_n\in\Gamma_M} \Big(\sum_{k=1}^n \langle f_k,\gamma_k\rangle - \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\gamma_k,\gamma_{k-1})^2\Big) \\ &\qquad = \int_M \sup_{x_1,x_2,\dots,x_n} \Big(\sum_{k=1}^n f_k(x_k) - h_x(x_1,x_2,\dots,x_n)\Big)\gamma_0(dx). \end{aligned} \tag{5.18}$$

Proof. Given $\gamma_1,\gamma_2,\dots,\gamma_n \in \Gamma_M$ with $\varrho(\gamma_k,\gamma_{k-1}) < \infty$, $1 \le k \le n$. Write $\gamma_0 = \sum_{i=1}^\infty \delta_{\gamma_0^i}$, $\gamma_k = \sum_{i=1}^\infty \delta_{\gamma_k^i}$. Following [RS99, Lemma 4.1(iv)], renumbering if necessary, we may assume

$$\varrho(\gamma_k,\gamma_{k-1})^2 = \frac{1}{2}\sum_{i=1}^\infty d(\gamma_k^i,\gamma_{k-1}^i)^2. \tag{5.19}$$

Thus,

$$\sum_{k=1}^{n}\langle f_k,\gamma_k\rangle-\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}\varrho(\gamma_k,\gamma_{k-1})^2$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{n}f_k(\gamma_k^i)-\sum_{i=1}^{\infty}\frac{1}{2}\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}d(\gamma_k^i,\gamma_{k-1}^i)^2$$

$$=\sum_{i=1}^{\infty}\left(\sum_{k=1}^{n}f_k(\gamma_k^i)-h_{\gamma_0^i}(\gamma_1^i,\gamma_2^i,\dots,\gamma_n^i)\right)$$

$$\le\sum_{i=1}^{\infty}\sup_{x_1,x_2,\dots,x_n}\left(\sum_{k=1}^{n}f_k(x_k)-h_{\gamma_0^i}(x_1,x_2,\dots,x_n)\right)$$

$$=\int_M\sup_{x_1,x_2,\dots,x_n}\left(\sum_{k=1}^{n}f_k(x_k)-h_x(x_1,x_2,\dots,x_n)\right)\gamma_0(dx). \tag{5.20}$$

(Note that the last integral exists by Lemma 5.2)
Hence,

$$\sup_{\gamma_1,\gamma_2,\dots,\gamma_n\in\Gamma_M}\left(\sum_{k=1}^{n}\langle f_k,\gamma_k\rangle-\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}\varrho(\gamma_k,\gamma_{k-1})^2\right)$$

$$\le\int_M\sup_{x_1,x_2,\dots,x_n}\left(\sum_{k=1}^{n}f_k(x_k)-h_x(x_1,x_2,\dots,x_n)\right)\gamma_0(dx) \tag{5.21}$$

To prove the dual inequality, we again set

$$g(x):=\sup_{x_1,x_2,\dots,x_n}\left(\sum_{k=1}^{n}f_k(x_k)-h_x(x_1,x_2,\dots,x_n)\right).$$

Choose a compact subset K containing the supports of g and f_k for all $1\le k\le n$. This is possible due to Lemma 5.2. Moreover, there exists an integer N_0 such that $\gamma_0^i\in K^c$ for $i>N_0$. Now, for any $\varepsilon>0$, there is $(x_1^i,\dots,x_n^i)\in M\times M\times\cdots\times M$ such that

$$\sum_{k=1}^{n}f_k(x_k^i)-h_{\gamma_0^i}(x_1^i,x_2^i,\dots,x_n^i)>\sup_{x_1,x_2,\dots,x_n}\left(\sum_{k=1}^{n}f_k(x_k)-h_{\gamma_0^i}(x_1,x_2,\dots,x_n)\right)-\frac{\varepsilon}{N_0}. \tag{5.22}$$

Define $\gamma_k^i:=x_k^i$, $i\le N_0$, $\gamma_k^i:=\gamma_0^i$, $i>N_0$, and set $\gamma_k:=\sum_{i=1}^{\infty}\delta_{\gamma_k^i}$. We have setting $x_0^i:=\gamma_0^i$

$$\sum_{k=1}^{n}\langle f_k,\gamma_k\rangle-\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}\varrho(\gamma_k,\gamma_{k-1})^2$$

$$=\sum_{k=1}^{n}\sum_{i=1}^{N_0}f_k(\gamma_k^i)-\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}\varrho(\gamma_k,\gamma_{k-1})^2 \tag{5.23}$$

$$\ge\sum_{k=1}^{n}\sum_{i=1}^{N_0}f_k(x_k^i)-\sum_{k=1}^{n}\frac{1}{t_k-t_{k-1}}\frac{1}{2}\sum_{i=1}^{N_0}d(x_k^i,x_{k-1}^i)^2$$

$$=\sum_{i=1}^{N_0}\left(\sum_{k=1}^{n}f_k(x_k^i)-h_{\gamma_0^i}(x_1^i,x_2^i,\dots,x_n^i)\right)$$

$$>\sum_{i=1}^{N_0}g(\gamma_0^i)-\varepsilon=\int_M g(x)\,\gamma_0(dx)-\varepsilon. \tag{5.24}$$

Since ε is arbitrary, the dual inequality follows and the lemma is proved. □

Proposition 5.4. *$I_{\gamma_0}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\gamma_k, \gamma_{k-1})^2$ for all $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma_M$.*

Proof. By Lemma 5.3,

$$\begin{aligned} I_{\gamma_0}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \sup_{(f_1,\ldots,f_n)\in C_0(M)^n} &\Bigg(\sum_{k=1}^n \langle f_k, \gamma_k\rangle \\ &- \sup_{\tilde\gamma_1,\tilde\gamma_2,\ldots,\tilde\gamma_n\in\Gamma_M} \Bigg(\sum_{k=1}^n \langle f_k, \tilde\gamma_k\rangle - \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\tilde\gamma_k, \tilde\gamma_{k-1})^2 \Bigg)\Bigg). \end{aligned} \tag{5.25}$$

Since for $f_1, \ldots f_n \in C_0(M)$

$$\begin{aligned} \sum_{k=1}^n \langle f_k, \gamma_k\rangle - \sup_{\tilde\gamma_1,\tilde\gamma_2,\ldots,\tilde\gamma_n\in\Gamma_M} &\Bigg(\sum_{k=1}^n \langle f_k, \tilde\gamma_k\rangle - \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\tilde\gamma_k, \tilde\gamma_{k-1})^2 \Bigg) \\ &\le \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\gamma_k, \gamma_{k-1})^2, \end{aligned} \tag{5.26}$$

we have

$$I_{\gamma_0}(\gamma_1, \gamma_2, \ldots, \gamma_n) \le \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\gamma_k, \gamma_{k-1})^2. \tag{5.27}$$

On the other hand, by the general theorem on the inverse Legendre transform of convex functions (see [Dav89, Lemma 4.5.8]) and Proposition 5.1, 5.3, we see that

$$\begin{aligned} \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\gamma_k, \gamma_{k-1})^2 = \sup_{(f_1,\ldots,f_n)\in C_0(M)^n} &\Bigg\{ \sum_{k=1}^n \langle f_k, \gamma_k\rangle \\ &- \sup_{\tilde\gamma_1,\tilde\gamma_2,\ldots,\tilde\gamma_n\in\chi^+(M)} \Bigg(\sum_{k=1}^n \langle f_k, \tilde\gamma_k\rangle - \sum_{k=1}^n \frac{1}{t_k - t_{k-1}} \varrho(\tilde\gamma_k, \tilde\gamma_{k-1})^2 \Bigg)\Bigg\} \end{aligned}$$

which is smaller than $I_{\gamma_0}(\gamma_1, \gamma_2, \ldots, \gamma_n)$. □

Proposition 5.5. *For any $\gamma_0 \in \Gamma_M$ the rate functional I_{γ_0} is good on Γ_M^n, i.e., for any $L > 0$ the level set*

$$R_L := \left\{ (\gamma_1, \ldots, \gamma_n) \in \Gamma_M^n \;\middle|\; I_{\gamma_0}(\gamma_1, \gamma_2, \ldots, \gamma_n) \le L \right\}$$

is compact in Γ_M^n.

Proof. By Proposition 5.1 the rate functional I_{γ_0} is lower semi-continuous on Γ_M^n. So,we only need to show that evey level set is relatively compact in Γ_M^n. So, let $L > 0$. It is sufficient to prove that for each $1 \le k \le n$ the set

$$R_L^k := \Pi_k(R_L),$$

is relatively compact, where $\Pi_k : \Gamma_M^n \to \Gamma_M$ is the natural projection on the $k-$th component. Let $r_0 > 0$, $\gamma \in \Gamma_M$. The for any $\omega \in \Gamma_{\gamma_0 \times \gamma}$,

$$\begin{aligned}
\gamma(B(x_0, r_0)) &= \int_{M\times M} \chi_{B(x_0,r_0)}(y)\, \omega(dx, dy) \\
&= \int_{M\times M} \chi_{B(x_0,r_0+\sqrt{2})}(x)\chi_{B(x_0,r_0)}(y)\, \omega(dx, dy) \\
&\quad + \int_{M\times M} \chi_{B(x_0,r_0+\sqrt{2})^c}(x)\chi_{B(x_0,r_0)}(y)\, \omega(dx, dy) \\
&\le \int_{M\times M} \chi_{B(x_0,r_0+\sqrt{2})}(x)\, \omega(dx, dy) + \frac{1}{2}\int_{M\times M} d(x,y)^2\, \omega(dx, dy) \\
&= \gamma_0(B(x_0, r_0 + \sqrt{2})) + \int_{M\times M} d(x,y)^2\, \omega(dx, dy). \qquad (5.28)
\end{aligned}$$

Taking the infimum over ω we get

$$\gamma(B(x_0, r_0)) \le \gamma_0(B(x_0, r_0 + \sqrt{2})) + \varrho(\gamma_0, \gamma)^2.$$

If $\gamma = \gamma_k \in R_L^k$, then by definition there exist $\gamma_1, \dots, \gamma_{k-1}, \gamma_{k+1}, \dots, \gamma_n \in \Gamma_M$ so that $(\gamma_1, \dots, \gamma_n) \in R_L$. Hence (since $\sum_{l=1}^n (t_l - t_{l-1}) = 1$)

$$\varrho(\gamma_0, \gamma_k)^2 \le \left(\sum_{l=1}^n \varrho(\gamma_l, \gamma_{l-1})\right)^2 \le \sum_{l=1}^n \frac{1}{t_l - t_{l-1}}\, \varrho(\gamma_l, \gamma_{l-1})^2 \le L.$$

Thus,

$$\sup_{\gamma_k \in R_L^k} \gamma_k(B(x_0, r_0)) \le \gamma_0(B(x_0, r_0 + \sqrt{2})) + L$$

Since r_0 was arbitrary, this implies that R_L^k is relatively compact in $\mathcal{X}^+$. Since Γ_M is Polish, it is closed in $\mathcal{X}^+$, so the assertion follows. □

6. Large deviation estimates for finite dimensional projections: the lower bounds

The lower bound holds under even a little stronger topology on Γ_M which was introduced in [KLR00], namely, the topology on Γ_∞ induced by the following metric:

$$d_\infty(\gamma_1, \gamma_2) := d_v(\gamma_1, \gamma_2) + \sum_{m=1}^\infty 2^{-m} \frac{|B_m(\gamma_1) - B_m(\gamma_2)|}{(1 + |B_m(\gamma_1) - B_m(\gamma_2)|)}, \qquad (6.1)$$

where d_v is any metric compatible with the vague topology.
Let U be an d_v-open neighborhood in Γ_M described by

$$U = \left\{ \gamma \in \Gamma_M \,\middle|\, \gamma(\partial W_r) = 0,\ \gamma|_{W_r} = \sum_{i=1}^n \delta_{x_i} \text{ with } \sum_{i=1}^n d(x_i, y_i)^2 < \delta_0 \right\}, \qquad (6.2)$$

where $W_r := B(x_0, r)$, $r > 0$, $y_1, \dots, y_n$ are fixed points in W_r, and $\delta_0 > 0$.

Proposition 6.1. *Let $\gamma_0 \in \Gamma_\infty$. Then, for any $\delta_1 > 0$ and distinct integers $i_1, \dots, i_n$,*

$$\begin{aligned}
&\liminf_{\varepsilon\to 0, \gamma\to\gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in U) \\
&\ge -\frac{1}{2}\sum_{j=1}^n d(y_j, \gamma_0^{i_j})^2 (1 + \delta_1) - \frac{1}{2}\left(\frac{1}{\delta_1} + 1\right)\delta_0 - \sum_{k\notin\{i_1, i_2, \dots i_n\}} \frac{1}{2} d(\gamma_0^k, \bar{W}_r^c)^2, \qquad (6.3)
\end{aligned}$$

where $d(\gamma_0^k, \bar{W}_r^c)$ denotes the distance from γ_0^k to $\bar{W}_r^c$ and $\gamma \to \gamma_0$ with respect to d_∞, in Γ_∞.

Proof. Let $\{\gamma_m, m \geq 1\}$ be any sequence in Γ_∞ such that $d_\infty(\gamma_m, \gamma_0) \to 0$. It is sufficient to show (6.3) for such a sequence. Write $\gamma_m = \sum_{i=1}^\infty \delta_{\gamma_m^i}$ and $\gamma_0 = \sum_{i=1}^\infty \delta_{\gamma_0^i}$. Rearranging elements in γ_m if necessary, we may assume that $\lim_{m\to\infty} \gamma_m^i = \gamma_0^i$ for $i \geq 1$ (see the proof of Theorem 6.1 in [KLR00]). For distinct integers $i_1, i_2, i_3, \ldots, i_n$, define

$$A_{i_1,i_2,\ldots,i_n} := \Bigg\{ B_\varepsilon^{i_1} \in W_r, \ldots, B_\varepsilon^{i_n} \in W_r,\ \sum_{j=1}^n d(B_\varepsilon^{i_j}, y_j)^2 < \delta_0, \\ B_\varepsilon^k \in \bar{W}_r^c,\ k \notin \{i_1, \ldots, i_n\} \Bigg\}.$$

Then

$$\{X_\varepsilon \in U\} = \bigcup_{i_1,i_2\cdots,i_n} A_{i_1,i_2\cdots,i_n},$$

hence,

$$P_{\gamma_m}(X_\varepsilon \in U) = \sum_{i_1,i_2,\ldots,i_n}^\infty P_{\gamma_m}(A_{i_1,i_2,\ldots,i_n}).$$

So, for any distinct integers $i_1, i_2, i_3, \ldots, i_n$ we have

$$\begin{aligned} P_{\gamma_m}(X_\varepsilon \in U) &\geq P_{\gamma_m}(A_{i_1,i_2,\ldots,i_n}) \\ &= \int\cdots\int_{\substack{\{\sum_{j=1}^n d(x_j,y_j)^2<\delta_0, \\ x_j \in W_r\}}} \prod_{j=1}^n p_\varepsilon(\gamma_m^{i_j}, x_j)\, m(dx_1)\, m(dx_2) \ldots m(dx_n) \\ &\quad \times \prod_{k\notin\{i_1,i_2,\ldots,i_n\}} P_{\gamma_m^k}(B_\varepsilon \in \bar{W}_r^c) \\ &= a_\varepsilon^{i_1,i_2,\ldots,i_n} \times b_\varepsilon^{i_1,i_2,\ldots,i_n}. \end{aligned}$$

Note that for any $\delta_1 > 0$,

$$\sum_{j=1}^n d(x_j, \gamma_m^{i_j})^2 \leq \left(1 + \frac{1}{\delta_1}\right) \sum_{j=1}^n d(x_j, y_j)^2 + (1 + \delta_1) \sum_{j=1}^n d(y_j, \gamma_m^{i_j})^2.$$

This and (3.2) imply

$$\begin{aligned} a_\varepsilon^{i_1,i_2,\ldots,i_n} &\geq \int\cdots\int_{\substack{\{\sum_{j=1}^n d^2(x_j,y_j)<\delta_0, \\ x_j \in W_r\}}} \prod_{j=1}^n \frac{\exp\left[-\frac{d(\gamma_m^{i_j},x_j)^2}{(2-\delta)\varepsilon}\right]}{c_1(\delta) m(B(\gamma_m^{i_j}, \sqrt{\varepsilon}))}\, m(dx_1)\, m(dx_2) \ldots m(dx_n) \\ &\geq \int\cdots\int_{\substack{\{\sum_{j=1}^n d^2(x_j,y_j)<\delta_0, \\ x_j \in W_r\}}} \exp\left(-\frac{(1+\frac{1}{\delta_1})\delta_0}{(2-\delta)\varepsilon}\right) \prod_{j=1}^n \frac{1}{c_1(\delta) m(B(\gamma_m^{i_j}, \sqrt{\varepsilon}))} \\ &\qquad \exp\left(-(1+\delta_1)\frac{\sum_{j=1}^n d(y_j, \gamma_m^{i_j})^2}{(2-\delta)\varepsilon}\right) m(dx_1)\, m(dx_2) \ldots m(dx_n). \end{aligned}$$

Since $\lim_{\varepsilon\to 0, m\to\infty} \varepsilon \log\left(\prod_{j=1}^n \frac{1}{c_1(\delta) m(B(\gamma_m^{i_j}, \sqrt{\varepsilon}))}\right) = 0$, we have

$$\liminf_{\varepsilon\to 0, m\to\infty} \varepsilon \log a_\varepsilon^{i_1,i_2,\ldots,i_n} \geq -\frac{1}{2-\delta}\left(\frac{1}{\delta_1} + 1\right)\delta_0 - (1+\delta_1)\frac{1}{2-\delta}\sum_{j=1}^n d(y_j, \gamma_0^{i_j})^2.$$

To treat $b_\varepsilon^{i_1,i_2,\ldots,i_n}$, we need the following

Lemma 6.2. *Let $c_1(1)$, $c_2(1)$ be as in Lemma 3.1 (for $T := 1$). Then*

$$\left|\log P\big(d(x_0, B^x_\varepsilon) > r\big)\right| \le \frac{c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}{1 - c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)} \tag{6.4}$$

for all $r > 0$ and $x \in M$ with $d(x_0, x) > 2r$ and $\varepsilon \le 1$.

Proof. Note that $d(x_0, B^x_\varepsilon) > d(x_0, x) - d(x, B^x_\varepsilon)$. Using $d(x_0, x) > 2r$ and Lemma 3.1 it follows that

$$\begin{aligned} P\big(d(x_0, B^x_\varepsilon) > r\big) &\ge P\left(d(x, B^x_\varepsilon) \le \frac{1}{2} d(x_0, x)\right) \\ &= 1 - P\left(d(x, B^x_\varepsilon) > \frac{1}{2} d(x_0, x)\right) \\ &\ge 1 - c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big). \end{aligned}$$

Hence,

$$\begin{aligned} \left|\log P\big(d(x_0, B^x_\varepsilon) > r\big)\right| &= -\log P\big(d(x_0, B^x_\varepsilon) > r\big) \\ &\le \log\left(\frac{1}{1 - c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}\right) \\ &= \log\left(1 + \frac{c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}{1 - c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}\right) \\ &\le \frac{c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}{1 - c_1(1)\exp\big(-c_2(1)d(x_0,x)^2\big)}, \end{aligned}$$

which proves the assertion. □

Corollary 6.3. *We have*

$$\lim_{\varepsilon \to 0, m \to \infty} \varepsilon \log b_\varepsilon^{i_1, i_2, \ldots, i_n} = -\frac{1}{2} \sum_{k \notin \{i_1, \ldots, i_n\}} d(\gamma_0^k, \bar{W}^c_r)^2. \tag{6.5}$$

Proof. Note that

$$\varepsilon \log b_\varepsilon^{i_1, i_2, \ldots, i_n} = \sum_{k \notin \{i_1, \ldots, i_n\}} \varepsilon \log P_{\gamma^k_m}\big(d(x_0, B_\varepsilon) > r\big).$$

By the large deviation principle of Brownian motion,

$$\lim_{\varepsilon \to 0, m \to \infty} \varepsilon \log P_{\gamma^k_m}\big(d(x_0, B_\varepsilon) > r\big) = -\frac{1}{2} d(\gamma_0^k, \bar{W}^c_r)^2.$$

Since $\gamma_0 \in \Gamma_\infty$ and $\gamma_m \to \gamma_0$ with respect to d_∞, (6.5) now follows from Lemma 6.2 and the dominated convergence theorem. □

Using Corollary 6.3 and letting $\delta \to 0$ we get Proposition 6.1. □

Proposition 6.4. *Let $\gamma_0 \in \Gamma_\infty$. Suppose that $O \subset \Gamma_\infty$ is an open subset w.r.t. d_∞. Then,*

$$\liminf_{\varepsilon \to 0, \gamma \to \gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in O) \ge -\inf_{\gamma \in O} \varrho(\gamma_0, \gamma)^2,$$

where $\gamma \to \gamma_0$ with respect to d_∞ in Γ_∞.

We first prove the analogue of Proposition 6.4 for the vague topology.

Proposition 6.5. *Let $\gamma_0 \in \Gamma_\infty$. Suppose that $O \subset \Gamma_M$ is an open subset w.r.t. the vague topology. Then,*

$$\liminf_{\varepsilon \to 0, \gamma \to \gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in O) \ge -\inf_{\gamma \in O} \varrho(\gamma_0, \gamma)^2,$$

where $\gamma \to \gamma_0$ with respect to d_∞ in Γ_∞.

Proof. It is sufficient to prove

$$\liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in O) \geq -\varrho(\gamma_0,\hat\gamma)^2$$

for any $\hat\gamma \in O$. So, fix $\hat\gamma \in O$. Without loss of generality, we can assume that

$$\varrho(\gamma_0,\hat\gamma)^2 = \frac{1}{2}\sum_{i=1}^{\infty} d(\hat\gamma^i,\gamma_0^i)^2 < \infty.$$

Choose an increasing sequence W_{r_n} of geodesic balls such that $\bigcup W_{r_n} = M$ and

$$\hat\gamma|_{W_{r_n}} = \sum_{i=1}^{m_n} \delta_{\hat\gamma^i},\ \hat\gamma(\partial W_{r_n}) = 0.$$

Let $\delta_l, l \geq 1$, be a sequence of positive numbers converging to zero. Set

$$U_{n,l} := \left\{\gamma \in \Gamma_M \;\middle|\; \gamma(\partial W_{r_n}) = 0,\ \gamma|_{W_{r_n}} = \sum_{i=1}^{m_n} \delta_{x_i} \text{ with } \sum_{i=1}^{m_n} d(x_i,\hat\gamma^i)^2 < \delta_l\right\}.$$

Obviously, $\{U_{n,m}\,|\,n,m \in \mathbf{N}\}$ form a basis of neighbourhoods for $\hat\gamma$ in the vague topology. Since O is open, there exist n_0, l_0 such that $U_{n,l} \subset O$ for $n \geq n_0$, $l \geq l_0$. By Proposition 6.1 it holds that for $n \geq n_0$, $l \geq l_0$ and any $\delta_1 > 0$,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in O)\\
&\quad\geq \liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in U_{n,l})\\
&\quad\geq -\frac{1}{2}\sum_{i=1}^{m_n} d(\hat\gamma^i,\gamma_0^i)^2(1+\delta_1) - \frac{1}{2}\left(\frac{1}{\delta_1}+1\right)\delta_l - \sum_{k=m_n+1}^{\infty} \frac{1}{2} d(\gamma_0^k, \bar W_{r_n}^c)^2.
\end{aligned}$$

First letting $l \to \infty$ and then $\delta_1 \to 0$, we obtain that

$$\liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon \log P_\gamma(X_\varepsilon \in O) \geq -\frac{1}{2}\sum_{i=1}^{m_n} d(\hat\gamma^i,\gamma_0^i)^2 - \sum_{k=m_n+1}^{\infty} \frac{1}{2} d(\gamma_0^k, \bar W_{r_n}^c)^2. \tag{6.6}$$

By the choice of W_{r_n}, $d(\gamma_0^k, \bar W_{r_n}^c) \leq d(\gamma_0^k, \hat\gamma^k)$ for $k \geq m_n + 1$. Hence, it follows from (6.6) that

$$\liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon ln P_{\gamma_m}(X_\varepsilon \in O) \geq -\frac{1}{2}\sum_{i=1}^{\infty} d(\hat\gamma^i,\gamma_0^i)^2 = -\varrho(\gamma_0,\hat\gamma)^2,$$

which completes the proof. □

Let $\hat\gamma \in \Gamma_\infty$. To prove Proposition 6.4 we have to consider d_∞-neighbourhoods of $\hat\gamma$ of the form:

$$U(\hat\gamma, n, \bar\delta) := \left\{\gamma = \sum \delta_{\gamma^i} \;\middle|\; |B_n(\gamma) - B_n(\hat\gamma)| < \bar\delta\right\}. \tag{6.7}$$

and intersections of finitely many of them with vaguely open sets in Γ_M.

Lemma 6.6. *Let $\gamma_0 \in \Gamma_\infty$. Then*

$$\liminf_{\varepsilon\to 0,\gamma\to\gamma_0} \varepsilon \log P_\gamma\big(X_\varepsilon \in U(\hat\gamma, n, \bar\delta)\big) \geq -\varrho(\gamma_0,\hat\gamma)^2, \tag{6.8}$$

where $\gamma \to \gamma_0$ with respect to d_∞ in Γ_∞.

Proof. Let $\{\gamma_m\}$ be a sequence such that $d_\infty(\gamma_m,\gamma_0) \to 0$. It suffices to show (6.8) for such a sequence. We may assume $\varrho(\gamma_0,\hat\gamma) < \infty$ and

$$\varrho(\gamma_0,\hat\gamma)^2 = \sum_{i=1}^{\infty} \frac{1}{2} d(\gamma_0^i,\hat\gamma^i)^2,$$

where $\gamma_0 = \sum \delta_{\gamma_0^i}$, $\hat{\gamma} = \sum \delta_{\hat{\gamma}^i}$. For the numeration of $\gamma_0 = \sum \delta_{\gamma_0^i}$, by [KLR00] there exists a numeration of γ_m, say $\gamma_m = \sum \delta_{\gamma_m^i}$, such that for $i \geq 1$, $\gamma_m^i \to \gamma_0^i$ as $m \to \infty$. From now on, we stick to such a numeration. For any $N \geq 1$,we have

$$\begin{aligned}\sum_{i=N+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0,\gamma_m^i)\Big] &= B_{2n}(\gamma_m) - \sum_{i=1}^{N} \exp\Big[-\frac{1}{2n}d(x_0,\gamma_m^i)\Big] \\ &\to B_{2n}(\gamma_0) - \sum_{i=1}^{N} \exp\Big[-\frac{1}{2n}d(x_0,\gamma_0^i)\Big] \\ &= \sum_{i=N+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0,\gamma_0^i)\Big]\end{aligned}$$

as $m \to \infty$.
So we can choose N_0 so that

$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0,\hat{\gamma}^i)\Big] < \frac{\bar{\delta}}{4},$$
$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0,{\gamma_0}^i)\Big] < \frac{\bar{\delta}}{4},$$
$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0,{\gamma_m}^i)\Big] < \frac{\bar{\delta}}{4}$$

for all $m \geq 1$.
Define

$$\begin{aligned}U(\hat{\gamma},n,\bar{\delta},1) = \Big\{\gamma = \delta_{\gamma^i} \;\Big|\; &\exists\, i_1,\dots,i_{N_0}\ s.t. \\ &\sum_{j\notin\{i_1,\dots,i_{N_0}\}} \exp\Big[-\frac{1}{n}d(x_0,\gamma^j)\Big] < \frac{\bar{\delta}}{4}, \\ &\Big|\sum_{k=1}^{N_0}\exp\Big[-\frac{1}{n}d(x_0,\gamma^{i_k})\Big] - \sum_{k=1}^{N_0}\exp\Big[-\frac{1}{n}d(x_0,\hat{\gamma}^k)\Big]\Big| < \frac{\bar{\delta}}{4}\Big\}.\end{aligned}$$

It is easy to see that

$$U(\hat{\gamma},n,\bar{\delta},1) \subset U(\hat{\gamma},n,\bar{\delta}).$$

Furthermore, there exists $\tilde{\delta} > 0$ such that for $\delta_1 \leq \tilde{\delta}$,

$$U(\hat{\gamma},n,\bar{\delta},\delta_1) \subset U(\hat{\gamma},n,\bar{\delta},1),$$

where

$$\begin{aligned}U(\hat{\gamma},n,\bar{\delta},\delta_1) = \Big\{\gamma = \delta_{\gamma^i} \;\Big|\; &\exists\, i_1,\dots,i_{N_0}\ s.t. \\ &\sum_{j\notin\{i_1,\dots,i_{N_0}\}} \exp\Big[-\frac{1}{n}d(x_0,\gamma^j)\Big] < \frac{\bar{\delta}}{4},\ \sum_{k=1}^{N_0} d(\gamma^{i_k},\hat{\gamma}^k)^2 < \delta_1\Big\}.\end{aligned}$$

Now it is enough to prove that

$$\liminf_{\varepsilon\to 0, m\to\infty} \varepsilon \log P_{\gamma_m}\big(X_\varepsilon \in U(\hat{\gamma},n,\bar{\delta},1)\big) \geq -\varrho(\gamma_0,\hat{\gamma})^2. \tag{6.9}$$

For $\delta_1 \le \tilde{\delta}$, we have

$$\begin{aligned}
&P_{\gamma_m}\big(X_\varepsilon \in U(\hat{\gamma}, n, \bar{\delta}, 1)\big) \\
&\qquad \ge P_{\gamma_m}\big(X_\varepsilon \in U(\hat{\gamma}, n, \bar{\delta}, \delta_1)\big) \\
&\qquad \ge P_{\gamma_m}\bigg(\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{n} d(x_0, {B_\varepsilon}^i)\Big] < \frac{\bar{\delta}}{4},\ \sum_{i=1}^{N_0} d({B_\varepsilon}^i, \hat{\gamma}^i)^2 < \delta_1\bigg)
\end{aligned}$$

(where ${B_\varepsilon}^i$ is the Brownian motion starting at γ_m^i)

$$\begin{aligned}
&\qquad = P_{\gamma_m}\bigg(\sum_{i=1}^{N_0} d({B_\varepsilon}^i, \hat{\gamma}^i)^2 < \delta_1\bigg) \times P_{\gamma_m}\bigg(\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{n} d(x_0, {B_\varepsilon}^i)\Big] < \frac{\bar{\delta}}{4}\bigg) \\
&\qquad := a_\varepsilon^m \times b_\varepsilon^m.
\end{aligned}$$

Note that for any $\delta_2 > 0$,

$$\sum_{i=1}^{N_0} d(x^i, \gamma_m^i)^2 \le \Big(1 + \frac{1}{\delta_2}\Big) \sum_{i=1}^{N_0} d(x^i, \hat{\gamma}^i)^2 + (1+\delta_2) \sum_{i=1}^{N_0} d(\hat{\gamma}^i, \gamma_m^i)^2.$$

Using

$$a_\varepsilon^m = \int \cdots \int_{\sum_{i=1}^{N_0} d(x^i, \hat{\gamma}^i)^2 < \delta_1} \prod_{i=1}^{N_0} p_\varepsilon(\gamma_m^i, x^i)\, m(dx^1) \dots m(dx^{N_0}),$$

as in the proof of Proposition 6.1, we have

$$\liminf_{\varepsilon \to 0, m \to \infty} \varepsilon \log a_\varepsilon^m \ge -\frac{1}{2-\delta}\Big(1 + \frac{1}{\delta_2}\Big)\delta_1 - \frac{1}{2-\delta}(1+\delta_2) \sum_{i=1}^{N_0} d(\gamma_0^i, \hat{\gamma}^i)^2. \tag{6.10}$$

Next we are going to show that

$$\liminf_{\varepsilon \to 0, m \to \infty} \varepsilon \log b_\varepsilon^m = 0.$$

For this end, it is sufficient to establish

$$\liminf_{\varepsilon \to 0, m \to \infty} b_\varepsilon^m > 0. \tag{6.11}$$

By the choice of N_0, it is easy to see that

$$\begin{aligned}
\Big\{\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{n} d(x_0, {B_\varepsilon}^i)\Big] < \frac{\bar{\delta}}{4}\Big\} &\supset \bigcap_{i=N_0+1}^{\infty} \Big\{d(x_0, {B_\varepsilon}^i) > \frac{1}{2} d(x_0, \gamma_m^i)\Big\} \\
&\supset \bigcap_{i=N_0+1}^{\infty} \Big\{d(\gamma_m^i, {B_\varepsilon}^i) < \frac{1}{2} d(x_0, \gamma_m^i)\Big\}
\end{aligned}$$

since $d(x_0,\gamma_m^i)\le d(x_0,{B_\varepsilon}^i)+d(\gamma_m^i,{B_\varepsilon}^i)$.
Hence, by Lemma 3.1, for $\varepsilon\le 1$,

$$\begin{aligned}
b_\varepsilon^m &= P_{\gamma_m}\Big(\sum_{i=N_0+1}^\infty \exp\Big[-\frac{1}{n}d(x_0,{B_\varepsilon}^i)\Big]<\frac{\bar\delta}{4}\Big)\\
&\ge \prod_{i=N_0+1}^\infty P_{\gamma_m^i}\Big(d(\gamma_m^i,{B_\varepsilon}^i)<\frac12 d(x_0,\gamma_m^i)\Big)\\
&\ge \prod_{i=N_0+1}^\infty \Big(1-c_1\exp\Big(-\frac{c_2 d(x_0,\gamma_m^i)^2}{4\varepsilon}\Big)\Big)\\
&\ge \prod_{i=N_0+1}^\infty \Big(1-c_1\exp\Big(-\frac{c_2 d(x_0,\gamma_m^i)^2}{4}\Big)\Big)\\
&= \exp\Big[\sum_{i=N_0+1}^\infty \log\Big(1-c_1\exp\Big(-\frac{c_2 d(x_0,\gamma_m^i)^2}{4}\Big)\Big)\Big]\\
&\ge \exp\Big[-\frac12\sum_{i=1}^\infty c_1\exp\Big(-\frac{c_2 d(x_0,\gamma_m^i)^2}{4}\Big)\Big].
\end{aligned}$$

The last expression is bigger than some positive constant independent of m since $\sup_m B_n(\gamma_m)<\infty$ for all $n\ge 1$. Therefore, (6.11) follows.
Combining (6.10) and (6.11) we arrive at

$$\liminf_{\varepsilon\to0,\gamma\to\gamma_0}\varepsilon\log P_\gamma\big(X_\varepsilon\in U(\hat\gamma,n,\bar\delta)\big)\ge -\frac{1}{2-\delta}\Big(1+\frac{1}{\delta_2}\Big)\delta_1-\frac{1}{2-\delta}(1+\delta_2)\sum_{i=1}^{N_0}d(\gamma_0^i,\hat\gamma^i)^2.$$

Letting first $\delta_1\to0$, then $\delta_2\to0$ and finally $\delta\to0$ we get

$$\liminf_{\varepsilon\to0,\gamma\to\gamma_0}\varepsilon\log P_\gamma\big(X_\varepsilon\in U(\hat\gamma,n,\bar\delta)\big)\ge -\frac12\sum_{i=1}^{N_0}d(\gamma_0^i,\hat\gamma^i)^2\ge-\varrho(\gamma_0,\hat\gamma)^2.$$

This completes the proof. □

We will sketch the case for the intersection of two d_∞-neighbourhoods in the next lemma.
Let $z_0\in\mathbf{R}$, $\bar\delta>0$. Define

$$U(z_0,n,\bar\delta):=\Big\{\gamma=\sum\delta_{\gamma^i}\ \Big|\ |B_n(\gamma)-z_0|<\bar\delta\Big\}. \tag{6.12}$$

Let U be an d_v- open neighborhood in Γ_M described by

$$U=\Big\{\gamma\in\Gamma_M\ \Big|\ \gamma(\partial W_{r_0})=0,\ \gamma|_{W_{r_0}}=\sum_{i=1}^{n_0}\delta_{x_i}\ \text{with}\ \sum_{i=1}^{n_0}d(x_i,y_i)^2<\delta_0\Big\}, \tag{6.13}$$

where $W_{r_0}:=B(x_0,r_0)$, $r_0>0$, $y_1,\dots,y_{n_0}$ are fixed points in W_{r_0}, and $\delta_0>0$. Set $V=U(z_0,n,\bar\delta)\cap U$.

Lemma 6.7. *Let $\gamma_0\in\Gamma_\infty$. Then for any $\hat\gamma\in V$, we have*

$$\liminf_{\varepsilon\to0,\gamma\to\gamma_0}\varepsilon\log P_\gamma(X_\varepsilon\in V)\ge-\varrho(\gamma_0,\hat\gamma)^2, \tag{6.14}$$

where $\gamma\to\gamma_0$ with respect to d_∞ in Γ_∞.

Proof. Let $\{\gamma_m\}$ be a sequence such that $d_\infty(\gamma_m,\gamma_0)\to0$. It suffices to show (6.14) for such a sequence. Again, we may assume $\varrho(\gamma_0,\hat\gamma)<\infty$ and

$$\varrho(\gamma_0,\hat\gamma)^2=\sum_{i=1}^\infty\frac12 d(\gamma_0^i,\hat\gamma^i)^2,$$

where $\gamma_0 = \sum \delta_{\gamma_0^i}$, $\hat{\gamma} = \sum \delta_{\hat{\gamma}^i}$. For the numeration of $\gamma_0 = \sum \delta_{\gamma_0^i}$, by [KLR00] there exists a numeration of γ_m, say $\gamma_m = \sum \delta_{\gamma_m^i}$, such that for $i \geq 1$, $\gamma_m^i \to \gamma_0^i$ as $m \to \infty$. We will stick to such a numeration. Let $\delta_1 := \bar{\delta} - |B_n(\hat{\gamma}) - z_0|$. Since $\hat{\gamma} \in U(z_0, n, \bar{\delta})$, $\delta_1 > 0$. Choose $\delta_2 < \frac{\delta_1}{4}$ such that $\exp(-\frac{1}{2n}d(x_0, x)) < \delta_2$ implies $d(x_0, x) > r_0$, where r_0 is as in the definition of U. On the other hand, since $\hat{\gamma} \in U$, there exist $\hat{\gamma}^{i_k} \in \hat{\gamma}$, $k = 1, \ldots, n_0$, such that

$$\hat{\gamma}|_{W_{r_0}} = \sum_{k=1}^{n_0} \delta_{\hat{\gamma}^{i_k}}, \ \sum_{k=1}^{n_0} d(\hat{\gamma}^{i_k}, y_k)^2 < \delta_0.$$

Now, arguing as in the proof of Proposition 6.6, we can choose $N_0 \geq i_{n_0}$ so that

$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0, \hat{\gamma}^i)\Big] < \frac{\bar{\delta}}{4},$$

$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0, {\gamma_0}^i)\Big] < \frac{\bar{\delta}}{4},$$

$$\sum_{i=N_0+1}^{\infty} \exp\Big[-\frac{1}{2n}d(x_0, {\gamma_m}^i)\Big] < \frac{\bar{\delta}}{4},$$

for all $m \geq 1$. For such N_0, there exists $\tilde{\delta} > 0$ such that for all $\delta_3 \leq \tilde{\delta}$ and $(z_1, z_2, \ldots, z_{N_0}) \in M^{N_0}$, $|\sum_{i=1}^{N_0} d(z_i, \hat{\gamma}^i)^2 < \delta_3$ implies

(i) $|\sum_{i=1}^{N_0} \exp(-\frac{1}{2n}d(x_0, z_i) - \sum_{i=1}^{N_0} \exp(-\frac{1}{2n}d(x_0, \hat{\gamma}^i)| < \delta_1$,
(ii) $\sum_{k=1}^{n_0} d(z_{i_k}, y_k)^2 < \delta_0$,
(iii) $\sharp\{i; z_i \in W_{r_0}\} = n_0$.

So, for $\delta_3 \leq \tilde{\delta}$,

$$U(\hat{\gamma}, n, \delta_1, \delta_3) \subset V,$$

where

$$U(\hat{\gamma}, n, \delta_1, \delta_3) = \Big\{\gamma = \sum_i \delta_{\gamma^i} \ \Big| \ \exists\, i_1, \ldots, i_{N_0} \ s.t.$$
$$\sum_{j \notin \{i_1, \ldots, i_{N_0}\}} \exp\Big[-\frac{1}{2n}d(x_0, \gamma^j)\Big] < \frac{\bar{\delta}}{4}, \ \sum_{k=1}^{N_0} d(\gamma^{i_k}, \hat{\gamma}^k)^2 < \delta_1\Big\}.$$

Now it is enough to prove that

$$\liminf_{\varepsilon \to 0, m \to \infty} \varepsilon \log P_{\gamma_m}\big(X_\varepsilon \in U(\hat{\gamma}, n, \delta_1, \delta_3)\big) \geq -\varrho(\gamma_0, \hat{\gamma})^2. \tag{6.15}$$

The rest of the proof is exactly the same as the corresponding proof of Proposition 6.6. So, we omit it here. □

Finite intersections can be done similarly. Since arbitrary finite intersections of d_∞-neighbourhoods as above form a base for the d_∞-neighbourhoods of any $\hat{\gamma} \in \Gamma_\infty$, we can then prove Proposition 6.4 in the same way as Proposition 6.5.

Theorem 6.8. *Let $\gamma_0 \in \Gamma_\infty$ and let μ_ε be the law of Y^ε on Γ_M^n under P_{γ_0}. Then for any subset $G \subset \Gamma_M^n$, which is open with respect to the product topology induced by d_v,*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_{(\gamma_1, \ldots, \gamma_n) \in G} I_{\gamma_0}(\gamma_1, \ldots, \gamma_n)$$

Proof. It suffices to show that for any $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in G$ with $I_{\gamma_0}(\gamma_1, \ldots, \gamma_n) < \infty$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -I_{\gamma_0}(\gamma_1, \ldots, \gamma_n).$$

Obviously, if $\gamma, \tilde{\gamma} \in \Gamma_M$ such that $\varrho(\gamma, \tilde{\gamma}) < \infty$ and $\gamma \in \Gamma_\infty$, then $\tilde{\gamma} \in \Gamma_\infty$. So, we may assume that $\gamma_1, \ldots, \gamma_n \in \Gamma_\infty$, since so is γ_0. For $\gamma \in \Gamma_\infty$, let $P_t(\gamma, U) := P_\gamma(X_t \in U)$ be the transition function. For a d_∞-open neighbourhood O_n of γ_n in Γ_∞, it follows from Proposition 6.4 that

$$\liminf_{\varepsilon \to 0, \gamma \to \gamma_{n-1}} \varepsilon \log P_{\varepsilon(t_n - t_{n-1})}(\gamma, O_n) \geq -\frac{\varrho(\gamma_{n-1}, \gamma_n)^2}{t_n - t_{n-1}},$$

where $\gamma \to \gamma_{n-1}$ with respect to d_∞ in Γ_∞. Thus, given $\delta_n > 0$, there exist $\varepsilon_n > 0$ and a d_∞- open neighbourhood O_{n-1} of γ_{n-1} in Γ_∞ such that

$$P_{\varepsilon(t_n - t_{n-1})}(\gamma, O_n) \geq \exp\left[-\frac{1}{\varepsilon}\left(\frac{\varrho(\gamma_{n-1}, \gamma_n)^2}{t_n - t_{n-1}} - \delta_n\right)\right] \text{ for } \varepsilon \leq \varepsilon_n,\ \gamma \in O_{n-1}. \tag{6.16}$$

The same arguments imply that for $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_{n-1} > 0$ there exist d_∞-open neighbourhoods O_i of γ_i in Γ_∞ and $\varepsilon_i > 0$, $i = 1, 2, \ldots, n-1$, such that

$$P_{\varepsilon(t_i - t_{i-1})}(\gamma, O_i) \geq \exp\left[-\frac{1}{\varepsilon}\left(\frac{\varrho(\gamma_{i-1}, \gamma_i)^2}{t_i - t_{i-1}} - \delta_i\right)\right] \text{ for } \varepsilon \leq \varepsilon_i,\ \gamma \in O_{i-1}. \tag{6.17}$$

Moreover, making them smaller if necessary, we may assume $O_1 \times O_2 \times \cdots \times O_n \subset G$. Then for $\varepsilon \leq \varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_n$, we have

$$\begin{aligned}
\mu_\varepsilon(G) &\geq \mu_\varepsilon(O_1 \times O_2 \times \cdots \times O_n) \\
&= \int_{O_1} P_{\varepsilon t_1}(\gamma_0, d\eta_1)\Big(\int_{O_2} P_{\varepsilon(t_2 - t_1)}(\eta_1, d\eta_2) \times \\
&\quad \cdots \times \int_{O_{n-1}} P_{\varepsilon(t_{n-1} - t_{n-2})}(\eta_{n-2}, d\eta_{n-1}) \times P_{\varepsilon(t_n - t_{n-1})}(\eta_{n-1}, O_n) \\
&\geq \exp\left[-\frac{1}{\varepsilon}\left(\frac{\varrho(\gamma_{n-1}, \gamma_n)^2}{t_n - t_{n-1}} - \delta_n\right)\right] \int_{O_1} P_{\varepsilon t_1}(\gamma_0, d\eta_1)\Big(\int_{O_2} P_{\varepsilon(t_2 - t_1)}(\eta_1, d\eta_2) \times \\
&\quad \cdots \times \int_{O_{n-2}} P_{\varepsilon(t_{n-2} - t_{n-3})}(\eta_{n-3}, d\eta_{n-2}) P_{\varepsilon(t_{n-1} - t_{n-2})}(\eta_{n-2}, O_{n-1}) \\
&\geq \cdots \\
&\geq \exp\left[-\frac{1}{\varepsilon}\left(\sum_{k=1}^n \frac{\varrho(\gamma_{k-1}, \gamma_k)^2}{t_k - t_{k-1}} + \sum_{k=1}^n \delta_k\right)\right].
\end{aligned} \tag{6.18}$$

Therefore,

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -I_{\gamma_0}(\gamma_1, \ldots, \gamma_n) - \sum_{k=1}^n \delta_k.$$

Letting $\sum_{k=1}^n \delta_k \to 0$, the theorem follows. □

7. The sample path large deviations

In this section we fix $\gamma_0 \in \Gamma_\infty$. Let P^ε denote the law of $(X_{\varepsilon t})_{t \in [0,1]}$ on $C_{\gamma_0}([0,1] \to \Gamma_M)$, i.e., the set of all continuous paths $t \to \omega_t$ from $[0,1]$ to Γ_M such that $\omega_0 = \gamma_0$. We equip $C_{\gamma_0}([0,1] \to \Gamma_M)$ with the topology of uniform convergence, where Γ_M is equipped with the vague topology. For $\omega \in C_{\gamma_0}([0,1] \to \Gamma_M)$, define

$$I(\omega) := \sup_{0 = t_0 < t_1 < \cdots < t_n = 1} \left\{\sum_{k=1}^n \frac{\varrho(\omega_{t_{k-1}}, \omega_{t_k})^2}{t_k - t_{k-1}}\right\}, \tag{7.1}$$

where the supremum is taken over all finite partitions of the interval $[0,1]$. $I(\omega)$ is the energy of ω associated to the Wasserstein type distance ϱ. The function I is obviously lower semicontinuous. Furthermore, by [RS99, Lemma 4.1(vii)] closed ϱ-balls are compact. Therefore, since for $\omega \in \{I \leq \text{const.}\}$ we have $\omega([0,1]) \subset$

$\{\varrho(\gamma_0,\cdot) < \infty\}$ and since on $\{\varrho(\gamma_0,\cdot) < \infty\}$ the ϱ-topology is stronger than the d_v-topology, Arzela's Theorem implies that $\{I \le \text{const.}\}$ is compact in $C_{\gamma_0}([0,1] \to \Gamma_M)$.

Theorem 7.1. *$\{P^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle on $C_{\gamma_0}([0,1] \to \Gamma_M)$ with good rate function $I(\cdot)$ given in (7.1), i.e.,*

(i) for any closed subset $C \subset C_{\gamma_0}([0,1] \to \Gamma_M)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log P^\varepsilon(C) \le -\inf_{\omega\in C} I(\omega).$$

(ii) for any open subset $O \subset C_{\gamma_0}([0,1] \to \Gamma_M)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log P^\varepsilon(O) \ge -\inf_{\omega\in O} I(\omega).$$

Proof. The results in Section 4 and 6 imply (see Theorem 4.6.1 in [DZ92]) that $\{P^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle under a weaker topology on $C_{\gamma_0}([0,1] \to \Gamma_M)$ with the same good rate function $I(\cdot)$. So, the assertion follows from the exponential tightness proven in Proposition 7.3 below. □

We need the following lemma.

Lemma 7.2. *Let $\hat{B}$ denote a real valued standard Brownain motion. Let c_0 be a constant satisfying*

$$A := E\Big[\exp\Big(c_0 \sup_{0\le u\le 1} \hat{B}_u^2\Big)\Big] < \infty.$$

For $f \in C_0^2(M)$, set $\lambda_f = \frac{c_0}{2\|\nabla f\|_\infty^2 + 1}$. Then for $0 \le s < t \le 1$, $\varepsilon > 0$,

$$E\bigg[\exp\bigg(\lambda_f \frac{\big(f(B_{\varepsilon t}) - f(B_{\varepsilon s})\big)^2}{\varepsilon(t-s)}\bigg)\bigg] \le A \exp(\|\Delta f\|_\infty^2). \tag{7.2}$$

Proof. By Ito's formula,

$$f(B_{\varepsilon t}) - f(B_{\varepsilon s}) = M_{\varepsilon t}^f - M_{\varepsilon s}^f + \int_{\varepsilon s}^{\varepsilon t} \frac{1}{2}\Delta f(B_u)\, du, \tag{7.3}$$

where M^f is a martingale with $\langle M^f\rangle_t = \int_0^t |\nabla f|^2(B_u)\, du$. This implies

$$\frac{\big(f(B_{\varepsilon t}) - f(B_{\varepsilon s})\big)^2}{\varepsilon(t-s)} \le 2\frac{(M_{\varepsilon t}^f - M_{\varepsilon s}^f)^2}{\varepsilon(t-s)} + \|\Delta f\|_\infty^2. \tag{7.4}$$

By the martingale representation theorem, there is a real-valued Brownian motion $\hat{B}$ such that $M_{\varepsilon t}^f - M_{\varepsilon s}^f = \hat{B}_{\langle M^f\rangle_{\varepsilon t} - \langle M^f\rangle_{\varepsilon s}}$. Thus, it follows from (7.4) that

$$\begin{aligned}
\frac{\big(f(B_{\varepsilon t}) - f(B_{\varepsilon s})\big)^2}{\varepsilon(t-s)} &\le 2\frac{\big(\hat{B}_{\int_{\varepsilon s}^{\varepsilon t}|\nabla f|^2(B_u)du}\big)^2}{\varepsilon(t-s)} + \|\Delta f\|_\infty^2 \\
&\le 2 \sup_{0\le u\le \|\nabla f\|_\infty^2 \varepsilon(t-s)} \frac{\hat{B}_u^2}{\varepsilon(t-s)} + \|\Delta f\|_\infty^2 \qquad (7.5)\\
&= 2\|\nabla f\|_\infty^2 \sup_{0\le u\le 1} \tilde{B}_u^2 + \|\Delta f\|_\infty^2, \qquad (7.6)
\end{aligned}$$

where $\tilde{B}_u := \frac{1}{\sqrt{\varepsilon(t-s)}}\hat{B}_{\varepsilon(t-s)u}$ is again a real-valued Brownian motion and we used the scaling invariance property of standard Brownian motion in the last step. Hence,

$$\begin{aligned}
E\bigg[\exp\bigg(\lambda_f \frac{\big(f(\hat{B}_{\varepsilon t}) - f(\hat{B}_{\varepsilon s})\big)^2}{\varepsilon(t-s)}\bigg)\bigg] &\le E\Big[\exp\Big(\lambda_f 2\|\nabla f\|_\infty^2 \sup_{0\le u\le 1} \tilde{B}_u^2 + \|\Delta f\|_\infty^2\Big)\Big] \\
&\le A\, \exp(\|\Delta f\|_\infty^2). \qquad (7.7)
\end{aligned}$$

□

Let P^ε be the law of $(X_{\varepsilon t})_{t\in[0,1]}$ on $C_{\gamma_0}([0,1]\to\Gamma_M)$.

Proposition 7.3. *$\{P^\varepsilon, \varepsilon > 0\}$ is exponentially tight on $C([0,1]\to\Gamma_M)$, i.e., for any $L>0$, there exists a compact subset $K_L \subset C_{\gamma_0}([0,1]\to\Gamma_M)$ satisfying*

$$\limsup_{\varepsilon\to\infty} \log P^\varepsilon(K_L^c) \le -L. \tag{7.8}$$

Proof. Fix a countable, dense subset $\{f_n, n\ge 1\}$ of $C_0(M)$ such that $f_n\in C_0^2(M)$ and the family $\{f_n, n\ge 1\}$ is closed under addition. For $L_n>0$, $n\in\mathbf{N}$, define a subset of $C_{\gamma_0}([0,1]\to\chi)$ by

$$\begin{aligned} K_{\{L_n\}} := \bigcap_{n=1}^{\infty}\{\omega\in C_{\gamma_0}([0,1]\to\Gamma_M) \mid \\ |\langle\omega_t, f_n\rangle - \langle w_s, f_n\rangle| \le L_n|t-s|^{\frac12-\delta},\ 0\le s,\ t\le 1\}\end{aligned} \tag{7.9}$$

where $0<\delta<\frac12$ is fixed. It is known from [Kal97] that K is compact. For $L>0$, we will choose L_n properly so that (7.8) is satisfied with $K_L := K_{\{L_n\}}$. Suppose $\mathrm{supp}[f_n]\subset B(x_0,N_n)$. Select an integer k_n such that

$$d(x_0,\gamma_0^j)\ge (N_n + 1\sqrt{2(n+L)})\vee(2N_n+1)$$

for all $j\ge k_n$. This is possible as $\lim_{j\to\infty} d(x_0,\gamma_0^j)=+\infty$. Set

$$c_\delta := \sup_{0\le u\le 1}|\log(u^2)u^\delta|,\ \lambda_n := \frac{c_0}{(2\|\nabla f_n\|_\infty^2+1)k_n}.$$

where c_0 is as in Lemma 7.2. Now, choose L_n such that

$$\left(\frac{\sqrt{\lambda_n}L_n}{8c_\delta}-1\right)^2 \ge L+n+k_n\|\Delta f_n\|_\infty^2 + k_n\log(A), \tag{7.10}$$

where A is as in Lemma 7.1. Define $K_L := K_{\{L_n\}}$ with $K_{\{L_n\}}$ as in (7.9). We claim that K_L satisfies (7.8). To prove this we set

$$A_n := \{\omega\in C_{\gamma_0}([0,1]\to\Gamma_M) \mid |\langle\omega_t,f_n\rangle - \langle w_s,f_n\rangle| \le L_n|t-s|^{\frac12-\delta},\ 0\le s,\ t\le 1\}.$$

Then

$$P^\varepsilon(K_L^c) \le \sum_{n=1}^{\infty} P^\varepsilon(A_n^c). \tag{7.11}$$

Since the support of f_n is contained in $B(x_0,N_n)$, we have

$$\begin{aligned} P^\varepsilon(A_n^c) &= P_{\gamma_0}\left(A_n^c \bigcap\left\{\inf_{j\ge k_n}\inf_{u\le 1} d(x_0, B^j_{\varepsilon u})\le N_n\right\}\right) \\ &\quad + P_{\gamma_0}\left(A_n^c \bigcap\left\{\inf_{j\ge k_n}\inf_{u\le 1} d(x_0, B^j_{\varepsilon u}) > N_n\right\}\right) \\ &\le P_{\gamma_0}\left(\inf_{j\ge k_n}\inf_{u\le 1} d(x_0,B^j_{\varepsilon u})\le N_n\right) \\ &\quad + P_{\gamma_0}\left(\left|\sum_{j=1}^{k_n} f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\right| > L_n|t-s|^{\frac12-\delta} \text{ for some } 0\le s\le t\le 1\right) \\ &=: I_n^\varepsilon + II_n^\varepsilon. \end{aligned} \tag{7.12}$$

Clearly,

$$
\begin{aligned}
I_n^\varepsilon &\le \sum_{j\ge k_n}^{\infty} P_{\gamma_0}\Big(\inf_{u\le 1} d(x_0, B^j_{\varepsilon u}) < N_n + 1\Big)\\
&= \sum_{j\ge k_n}^{\infty} P_{\gamma_0}\Big(\inf_{u\le 1} d(x_0, B^j_{\varepsilon u}) \le N_n + 1,\ \sup_{u\le 1} d(\gamma_0^j, B^j_{\varepsilon u}) \ge d(x_0,\gamma_0^j) - \inf_{u\le 1} d(x_0, B^j_{\varepsilon u})\Big)
\end{aligned}
$$

(7.13)

$$
\le \sum_{j\ge k_n}^{\infty} P_{\gamma_0}\Big(\sup_{u\le 1} d(\gamma_0^j, B^j_{\varepsilon u}) \ge d(x_0,\gamma_0^j) - N_n - 1\Big).
$$

By Corollary 3.2 and setting $c_1 := c_1(1)$, $c_2 := c_2(1)$,

(7.14)

$$
P_{\gamma_0}\Big(\sup_{u\le 1} d(\gamma_0^j, B^j_{\varepsilon u}) \ge d(x_0,\gamma_0^j) - N_n - 1\Big) \le c_1 \exp\bigg(-c_2 \frac{\big(d(x_0,\gamma_0^j) - N_n - 1\big)^2}{\varepsilon}\bigg).
$$

Combining (7.13), (7.14) and the choice of k_n we arrive at

$$
\begin{aligned}
I_n^\varepsilon &\le \sum_{j\ge k_n}^{\infty} c_1 \exp\bigg(-c_2 \frac{\big(d(x_0,\gamma_0^j) - N_n - 1\big)^2}{\varepsilon}\bigg)\\
&= \sum_{j\ge k_n}^{\infty} c_1 \exp\Big(-\frac{(n+L)}{\varepsilon}\Big) \exp\bigg(-c_2 \frac{1}{2}\frac{\big(d(x_0,\gamma_0^j) - N_n - 1\big)^2}{\varepsilon}\bigg)\\
&\le c_1 \exp\Big(-\frac{(n+L)}{\varepsilon}\Big) \sum_{j\ge k_n}^{\infty} \exp\Big(-c_2 \frac{1}{8}\frac{d(x_0,\gamma_0^j)^2}{\varepsilon}\Big)
\end{aligned}
$$

$$
\le \hat{c}_1 \exp\Big(-\frac{(n+L)}{\varepsilon}\Big), \tag{7.15}
$$

since $\gamma_0 \in \Gamma_\infty$. For the term II_n^ε, we note that by Lemma 7.2,

$$
\begin{aligned}
&E\bigg[\exp\bigg(\lambda_n \frac{\big|\sum_{j=1}^{k_n} f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\big|^2}{\varepsilon(t-s)}\bigg)\bigg]\\
&\qquad \le E\bigg[\exp\bigg(\lambda_n k_n \sum_{j=1}^{k_n} \frac{\big(f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\big)^2}{\varepsilon(t-s)}\bigg)\bigg]\\
&\qquad = \prod_{j=1}^{k_n} E\bigg[\exp\bigg(\lambda_n k_n \frac{\big(f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\big)^2}{\varepsilon(t-s)}\bigg)\bigg]
\end{aligned}
$$

$$
\le A^{k_n} \exp(k_n \|\Delta f_n\|_\infty^2). \tag{7.16}
$$

Set

$$
D_{\varepsilon,n} := \int_0^1\int_0^1 \exp\bigg(\lambda_n \frac{\big|\sum_{j=1}^{k_n} f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\big|^2}{|t-s|}\bigg)\, dt\, ds.
$$

Then (7.16) implies that

$$
E\big[D_{\varepsilon,n}^{\frac{1}{\varepsilon}}\big] \le A^{k_n} \exp\big(k_n \|\Delta f_n\|_\infty^2\big). \tag{7.17}
$$

Set $p(u) := u^{\frac{1}{2}}$. Then by Garsia's Lemma,

$$\begin{aligned}\left|\sum_{j=1}^{k_n} f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\right| &\le \frac{8}{\sqrt{\lambda_n}} \int_0^{|t-s|} \left(\log\left(\frac{D_{\varepsilon,n}}{u^2}\right)\right)^{\frac{1}{2}} dp(u) \\ &\le \frac{8c_\delta}{\sqrt{\lambda_n}}\left(\sqrt{\log(D_{\varepsilon,n})}+1\right)|t-s|^{\frac{1}{2}-\delta}. \qquad (7.18)\end{aligned}$$

It follows that

$$\begin{aligned} II^\varepsilon_n &= P_{\gamma_0}\left(\left|\sum_{j=1}^{k_n} f_n(B^j_{\varepsilon t}) - f_n(B^j_{\varepsilon s})\right| > L_n|t-s|^{\frac{1}{2}-\delta} \text{ for some } 0\le s\le t\le 1\right) \\ &\le P_{\gamma_0}\left(\frac{8c_\delta}{\sqrt{\lambda_n}}\left(\sqrt{\log(D_{\varepsilon,n})}+1\right) > L_n\right) \\ &= P_{\gamma_0}\left\{D^{\frac{1}{\varepsilon}}_{\varepsilon,n} > \exp\left(\frac{\left(\frac{L_n\sqrt{\lambda_n}}{8c_\delta}-1\right)^2}{\varepsilon}\right)\right\} \\ &\le \exp\left(-\frac{\left(\frac{L_n\sqrt{\lambda_n}}{8c_\delta}-1\right)^2}{\varepsilon}\right) E\left[D^{\frac{1}{\varepsilon}}_{\varepsilon,n}\right] \\ &\le \exp\left(-\frac{\left(\frac{L_n\sqrt{\lambda_n}}{8c_\delta}-1\right)^2}{\varepsilon}\right) A^{k_n}\exp\left(k_n\|\Delta f_n\|^2_\infty\right) \\ &= \exp\left(-\frac{\left(\frac{L_n\sqrt{\lambda_n}}{8c_\delta}-1\right)^2}{\varepsilon} + k_n\log(A) + k_n\|\Delta f_n\|^2_\infty\right) \qquad (7.19) \\ &\le \exp\left(-\frac{n+L}{\varepsilon}\right), \qquad (7.20)\end{aligned}$$

where we used the definition of L_n in the last step.
Combining (7.11), (7.15) and (7.20) gives

$$\begin{aligned} P^\varepsilon_{\gamma_0}(K^c_L) &\le (\hat{c}_1+1)\sum_{n=1}^\infty \exp\left(-\frac{n+L}{\varepsilon}\right) \\ &\le (\hat{c}_1+1)\exp\left(-\frac{L}{\varepsilon}\right)\sum_{n=1}^\infty \exp(-n). \qquad (7.21)\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0} \varepsilon \log P^\varepsilon_{\gamma_0}(K^c_L) \le -L. \qquad (7.22)$$

and the proof is complete. □

8. Interacting case

In this section, we will prove a sample path large deviation principle for the interacting random particles introduced in Section 2. Recall that $\mathcal{M}_\psi := \{\Omega, X_t, \mathcal{F}, \mathcal{F}_t, Q_\gamma, \gamma\in\Gamma_M\}$ denotes the diffusion associated with the following Dirichlet form:

$$\mathcal{E}_\psi(u,v) = \frac{1}{2}\int_{\Gamma_M} \langle\nabla u, \nabla v\rangle_\gamma \psi^2(\gamma)\,\pi(d\gamma),$$

$$D(\mathcal{E}_\psi) = \overline{\mathcal{D}}^{\mathcal{E}_{\psi,1}},$$

where $\mathcal{D}$ is given by

$$\mathcal{D} := \left\{u\in D(\mathcal{E}) \,\middle|\, \int \left(\Gamma(u,u)(\gamma) + u^2(\gamma)\right)\psi^2(\gamma)\,\pi(d\gamma) < \infty\right\}.$$

The following theorem is a special case of the general result obtained in [Ebe96].

Theorem 8.1. *The diffusion* $(\Omega, (X_t)_{t\geq 0}, \mathcal{F}, \mathcal{F}_t, Q_\gamma, \gamma \in \Gamma_M)$ *is given by*

$$dQ_\gamma|_{\mathcal{F}_t} = \exp\left\{ M_t^{\log\psi} - \frac{1}{2}\langle M^{\log\psi}\rangle_t \right\} dP_\gamma|_{\mathcal{F}_t},$$

where $M^{\log\psi}$ *stands for the martingale part of the Fukushima 's decomposition of the additive functional* $\log\psi(X_t) - \log\psi(X_0)$ *(see* [Fuk80]*) and*

$$\langle M^{\log\psi}\rangle_t = \int_0^t \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)\, ds$$

is the bracket, where $\Gamma(\psi,\psi)$ *is defined as in Section 2.*

Theorem 8.2. *Assume* $\int_{\Gamma_M} \exp(\delta \frac{\Gamma(\psi,\psi)}{\psi^2}(\gamma))\pi(d\gamma) < \infty$ *for some* $\delta > 0$. *Then there exists a subset* $F_M \subset \Gamma_\infty$ *with* $\pi(F_M) = 1$ *such that for* $\gamma_0 \in F_M$ *the following holds*

(i) for any closed subset $C \subset C_{\gamma_0}([0,1] \to \Gamma_M)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log Q_{\gamma_0}(X_{\varepsilon\cdot} \in C) \leq - \inf_{\omega\in C} I(\omega),$$

(ii) for any open subset $O \subset C_{\gamma_0}([0,1] \to \Gamma_M)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log Q_{\gamma_0}(X_{\varepsilon\cdot} \in O) \geq - \inf_{\omega\in O} I(\omega).$$

In particular, if $\frac{\Gamma(\psi,\psi)}{\psi^2}$ *is bounded,* F_M *can be chosen to be equal to* Γ_∞.

Proof. Let $P_\pi(\cdot) := \int_{\Gamma_M} P_\gamma(\cdot)\pi(d\gamma)$. Denote by E_π the expectation w.r.t. P_π. By Jensen's inequality, for any $\varepsilon > 0$

$$\begin{aligned} E_\pi\left[\exp\left(\delta \int_0^1 \frac{\Gamma(\psi,\psi)}{\psi^2}(X_{\varepsilon s})\, ds\right)\right] &\leq \int_0^1 E_\pi\left[\exp\left(\delta \frac{\Gamma(\psi,\psi)}{\psi^2}(X_{\varepsilon s})\right)\right] ds \\ &= \int_{\Gamma_M} \exp\left(\delta \frac{\Gamma(\psi,\psi)}{\psi^2}(\gamma)\right) \pi(d\gamma) < \infty. \end{aligned}$$

This implies that

$$F_1 := \left\{ \gamma \;\middle|\; E_\gamma\left[\exp\left(\delta \int_0^1 \frac{\Gamma(\psi,\psi)}{\psi^2}(X_{\frac{s}{k}})\, ds\right)\right] < \infty \text{ for all } k \geq 1 \right\}$$

has full π measure. Set $F_M := \Gamma_\infty \cap F_1$ and fix $\gamma_0 \in F_M$. Let us prove (ii) first. Assume O is open and set

$$Z_t := \exp\left\{ M_t^{\log\psi} - \frac{1}{2}\langle M^{\log\psi}\rangle_t \right\}.$$

We may assume $\lambda := \inf_{\omega\in O} I(\omega) < \infty$. For any $\delta_1 > 0$, it follows that

$$\begin{aligned} Q_{\gamma_0}(X_{\varepsilon\cdot} \in O) &= P_{\gamma_0}(Z_\varepsilon; X_{\varepsilon\cdot} \in O) \\ &\geq \exp\left(-\frac{\delta_1}{\varepsilon}\right) P_{\gamma_0}\left(Z_\varepsilon > \exp\left(-\frac{\delta_1}{\varepsilon}\right), X_{\varepsilon\cdot} \in O\right) \\ &\geq \exp\left(-\frac{\delta_1}{\varepsilon}\right)\left[P_{\gamma_0}(X_{\varepsilon\cdot} \in O) - P_{\gamma_0}\left(Z_\varepsilon \leq \exp\left(-\frac{\delta_1}{\varepsilon}\right)\right)\right] \qquad (8.1) \end{aligned}$$

where $P_{\gamma_0}(Z_\varepsilon \le \exp(-\frac{\delta_1}{\varepsilon}))$ can be estimated as follows: for $a > 0$,

$$\begin{aligned}
&P_{\gamma_0}\Big(Z_\varepsilon \le \exp\Big(-\frac{\delta_1}{\varepsilon}\Big)\Big)\\
&\quad= P_{\gamma_0}\Big(Z_\varepsilon^{-1} \ge \exp\Big(\frac{\delta_1}{\varepsilon}\Big)\Big)\\
&\quad= P_{\gamma_0}\Big[\exp\Big(-aM_\varepsilon^{\log\psi} + \frac{a}{2}\int_0^\varepsilon \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)\,ds\Big) \ge \exp\Big(\frac{a\delta_1}{\varepsilon}\Big)\Big]\\
&\quad\le \exp\Big(-\frac{a\delta_1}{\varepsilon}\Big)E_{\gamma_0}\Big[\exp\Big(-2aM_\varepsilon^{\log\psi} - 2a^2\int_0^\varepsilon \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)\,ds\Big)\Big]^{\frac{1}{2}}\\
&\qquad\times E_{\gamma_0}\Big[\exp\Big((2a^2+a)\int_0^\varepsilon \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)\,ds\Big)\Big]^{\frac{1}{2}}\\
&\quad\le \exp\Big(-\frac{a\delta_1}{\varepsilon}\Big)D_a(\varepsilon),
\end{aligned}\tag{8.2}$$

where

$$D_a(\varepsilon) := E_{\gamma_0}\Big[\exp\Big((2a^2+a)\int_0^\varepsilon \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)\,ds\Big)\Big]^{\frac{1}{2}},$$

and we have used the fact that $\exp\big(-2aM_\varepsilon^{\log\psi} - 2a^2\int_0^\varepsilon \frac{\Gamma(\psi,\psi)}{\psi^2}(X_s)ds\big)$ is a supermartingale.

On the other hand, for any $\delta_2 > 0$, by Theorem 7.1 there is $\varepsilon_1 > 0$ such that if $\varepsilon \le \varepsilon_1$,

$$P_{\gamma_0}(X_{\varepsilon\cdot} \in O) \ge \exp\Big(-\frac{\lambda+\delta_2}{\varepsilon}\Big).\tag{8.3}$$

Choose $a = \frac{\lambda+2\delta_2}{\delta_1}$. It follows from (8.1), (8.2) and (8.3) that if $\varepsilon \le \varepsilon_1$,

$$\begin{aligned}
Q_{\gamma_0}(X_{\varepsilon\cdot} \in O) &\ge \exp\Big(-\frac{\delta_1}{\varepsilon}\Big) \times \Big[\exp\Big(-\frac{\lambda+\delta_2}{\varepsilon}\Big) - D_a(\varepsilon)\exp\Big(-\frac{\lambda+2\delta_2}{\varepsilon}\Big)\Big]\\
&= \exp\Big(-\frac{\delta_1}{\varepsilon}\Big)\exp\Big(-\frac{\lambda+\delta_2}{\varepsilon}\Big)\Big[1-\exp\Big(-\frac{\delta_2}{\varepsilon}\Big)D_a(\varepsilon)\Big].
\end{aligned}\tag{8.4}$$

By the choice of γ_0, it is easy to see that

$$\lim_{\varepsilon\to0}\Big[1-\exp\Big(-\frac{\delta_2}{\varepsilon}\Big)D_a(\varepsilon)\Big] = 1.\tag{8.5}$$

Hence, we obtain from (8.4), (8.5) that

$$\liminf_{\varepsilon\to0} \varepsilon\log Q_{\gamma_0}(X_{\varepsilon\cdot} \in O) \ge -\delta_1 - \lambda - \delta_2.$$

Since δ_1, δ_2 were arbitrary, (ii) follows.

Let us now prove (i). Because of the choice of γ_0, by similar arguments as used above one can see that $\lim_{\varepsilon\to0} E_{\gamma_0}[Z_\varepsilon^p] = 1$ for any $p > 0$. For any $\delta_1 > 0$,$a > 0$, we have

$$\begin{aligned}
Q_{\gamma_0}(X_{\varepsilon\cdot} \in C) &= P_{\gamma_0}(Z_\varepsilon; X_{\varepsilon\cdot} \in C)\\
&= P_{\gamma_0}(Z_\varepsilon; X_{\varepsilon\cdot} \in C, Z_\varepsilon \le e^{\frac{\delta_1}{\varepsilon}}) + P_{\gamma_0}(Z_\varepsilon; X_{\varepsilon\cdot} \in C, Z_\varepsilon > e^{\frac{\delta_1}{\varepsilon}})\\
&\le P_{\gamma_0}(X_{\varepsilon\cdot} \in C)e^{\frac{\delta_1}{\varepsilon}} + E_{\gamma_0}[Z_\varepsilon^{a+1}]e^{-\frac{a\delta_1}{\varepsilon}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log Q_{\gamma_0}(X_{\varepsilon\cdot} \in C) \\
&\qquad \le \Big(\limsup_{\varepsilon\to 0} \varepsilon \log P_{\gamma_0}(X_{\varepsilon\cdot} \in C) + \delta_1\Big) \vee \Big(\limsup_{\varepsilon\to 0} \varepsilon \log E_{\gamma_0}[Z_\varepsilon^{a+1}] - a\delta_1\Big) \\
&\qquad \le \Big(-\inf_{\omega\in C} I(\omega) + \delta_1\Big) \vee (-a\delta_1).
\end{aligned}$$

First letting $a \to \infty$ and then $\delta_1 \to 0$, we prove (i). □

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