Harnack and Functional Inequalities for Generalized Mehler Semigroups

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Abstract

Harnack inequalities are established for a class of generalized Mehler semigroups, which in particular imply upper bound estimates for the transition density. Moreover, Poincaré and log-Sobolev inequalities are proved in terms of estimates for the square field operators. Furthermore, under a condition, well-known in the Gaussian case, we prove that generalized Mehler semigroups are strong Feller. The results are illustrated by concrete examples. In particular, we show that a generalized Mehler semigroup with an α-stable part is not hyperbounded but exponentially ergodic, and that the log-Sobolev constant obtained by our method in the special Gaussian case can be sharper than the one following from the usual curvature condition. Moreover, a Harnack inequality is established for the generalized Mehler semigroup associated with the Dirichlet heat semigroup on $(0,1)$. We also prove that this semigroup is not hyperbounded.

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1 Introduction and main results

Harnack and functional inequalities for the transition semigroup P_t of Ornstein-Uhlenbeck processes on infinite dimensional (e.g. separable Hilbert) spaces E, i.e. solution to stochastic differential equations of type

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}W_t$$

(where $A : E \to E$ is a linear operator generating a C_0-semigroup $T_t := \mathrm{e}^{tA}, t \geq 0$, on E and $(W_t)_{t\geq 0}$ is an E-valued Brownian motion), have been studied quite intensively in the literature (cf. e.g. [1, 2, 4, 5, 10, 13, 16, 17] and references therein). The dimension-free Harnack inequality introduced in [24] and the Poincaré or log-Sobolev inequalities have been established on $L^2(E, \mu)$, where μ is a symmetrizing or merely invariant probability measure for P_t. The fact that P_t may have an invariant measure makes Ornstein-Uhlenbeck processes in many respects better reference processes in infinite dimensions than Brownian motion (cf. [11]). This holds at least, if one wants to analyze diffusions. If one is interested in merely cadlag processes in infinite dimensions, i.e. if one wants to include models with possible jumps for the trajectories, suitable corresponding reference processes are processes solving stochastic differential equations of type

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}Y_t,$$

where now $(Y_t)_{t\geq 0}$ is a Levy process on E (i.e. has stationary independent increments and starts at zero), completely determined by a respective negative definite function $\lambda : E \to \mathbb{C}$. Their corresponding transition semigroups P_t have the explicit form (1.1) below (determined by T_t and λ) and are called *generalized Mehler semigroups*. They as well admit invariant probability measures in many cases and are thus of high interest. They have been studied in several papers [8, 9, 12, 13, 19, 20].

The present paper is devoted to prove Harnack and functional inequalities (as Poincaré and log-Sobolev inequalities) for such generalized Mehler semigroups. The main results rely in part on their explicit (algebraically beautiful) form (1.1) below and on an explicit formula for the square field operators of P_t which was recently proved in [20] (cf. (1.9) below). Below we summarize our results and applications (including stochastic heat equations with Levy noise) precisely. To keep preliminaries to a minimum, from now on we do not refer to infinite dimensional processes solving the above stochastic equations, but define P_t independently just in terms of T_t and a negative definite function $\lambda : E \to \mathbb{C}$.

Let E be a (real) separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and Borel σ-field $\mathcal{B}(E)$. Let $(T_t)_{t\geq 0}$ be a C_0-(i.e., strongly continuous) semigroup of bounded linear operators on E generated by A. Throughout the paper, we identify E with its dual space. Let $(\mu_t)_{t\geq 0}$ be a family of probability measures on E satisfying

$$\mu_{t+s} = (\mu_t \circ T_s^{-1}) * \mu_s \quad \text{for all } s, t \geq 0.$$

Let $\mathcal{B}_b(E)$ denote the set of all bounded measurable real-valued functions on E. Then the generalized Mehler semigroup associated with T_t and μ_t is given by (see [8] for details)

$$P_t f(x) = \int_E f(T_t x + y)\mu_t(\mathrm{d}y), \quad f \in \mathcal{B}_b(E),\ x \in E. \tag{1.1}$$

It is shown in [8] that, up to some regularity conditions, the characteristic function of μ_t must have the form

$$\hat{\mu}_t(\xi) := \exp\left[-\int_0^t \lambda(T_s^*\xi)\mathrm{d}s\right], \quad \xi \in E, t > 0, \tag{1.2}$$

where T_s^* denotes the adjoint operator of T_s and λ is a negative definite function on E with $\lambda(0) = 0$. We refer to [8, 12, 14, 19, 20] for the background literature and further known results on generalized Mehler semigroups.

In this paper apart from Section 4 we consider the case where λ is Sazonov-continuous (cf. e.g. [23]). But the main results can also be applied to the non-Sazonov-continuous case via an approximation procedure as made in Section 4 below. By the Sazonov-continuity, λ has a unique Levy-Khinchin representation (see e.g. [21, Theorem VI.4.10])

$$\lambda(\xi) = -i\langle\xi, a\rangle + \frac{1}{2}\langle\xi, R\xi\rangle - \int_E \left(\mathrm{e}^{i\langle\xi,x\rangle} - 1 - \frac{i\langle\xi, x\rangle}{1 + \|x\|^2}\right)M(\mathrm{d}x), \quad \xi \in E, \tag{1.3}$$

where $a \in E, R$ is a symmetric trace class operator on E, and M is a Levy measure on $\mathcal{B}(E)$, i.e. $M(\{0\}) = 0$ and $\int_E (1 \wedge \|x\|^2)M(\mathrm{d}x) < \infty$.

As mentioned above we aim to establish Harnack and functional inequalities for P_t. Since E can be infinite-dimensional, it turns out that the Li-Yau type Harnack inequality involving dimensions does not work. Therefore, we follow the line of [24] to establish a dimension-free Harnack inequality. To this end, we exploit an explicit formula for the underlying square field operators obtained recently in [20, Proposition 4.1]. Thus, we make use of the following assumptions made in [19, 20].

(H_1) P_t has an invariant probability measure μ.

(H_2) There exists $(x_n)_{n\geq 1} \subset E$ consisting of eigenvectors of A^* and separating the points of E.

Assumption (H_2) holds if, in particular, A is self-adjoint with compact resolvent. Assumption (H_1) holds provided the following (stronger) condition holds (see [14, Theorem 3.1])

(H_1'): (i) $T_t x \to 0$ as $t \to \infty$ for all $x \in E$;

(ii) $\sup_{t>0} \mathrm{Tr}\big(\int_0^t T_s R T_s^* \mathrm{d}s\big) < \infty$;

(iii) $\int_0^\infty \mathrm{d}s \int_E (1 \wedge \|T_s x\|^2)M(\mathrm{d}x) < \infty$,

(iv) $a_\infty := \lim_{t\to\infty} \big\{ \int_0^t T_s a \mathrm{d}s + \int_0^t \mathrm{d}s \int_E T_s x \big(\frac{1}{1+\|T_s x\|^2} - \frac{1}{1+\|x\|^2}\big) M(\mathrm{d}x) \big\}$ exists.

In this case P_t has a unique invariant probability measure μ with characteristic function

$\hat{\mu}(\xi) := \mathrm{e}^{-\lambda_\infty(\xi)}$, where λ_∞ is given by (1.3) with $a_\infty, R_\infty := \int_0^\infty T_s R T_s^* \mathrm{d}s$ and $M_\infty := \int_0^\infty M \circ T_s^{-1} \mathrm{d}s$ replacing a, R and M respectively. Indeed, for any $f \in C_b(E)$ and any $x \in E$ one has $P_t f(x) \to \mu(f)$ as $t \to \infty$ (cf. the proof of [13, Theorem 3.1]).

Let

$$\rho(x,y) := \begin{cases} \inf\{\|z\| : z \in (R^{1/2})^{-1}(x-y)\}, & \text{if } x - y \in R^{1/2}E, \\ \infty, & \text{otherwise.} \end{cases}$$

In fact, there exists an alternative definition of $\rho(x,y)$. Let G be the orthogonal complement of $\mathrm{Ker} R^{1/2}$, and let $S := R^{1/2}|_G : G \to R^{1/2}G =: H_0$. Then S is a continuous isomorphism (see [20]) and $(H_0, \langle\cdot,\cdot\rangle_{H_0})$ is a Hilbert space with $\langle v, w\rangle_{H_0} := \langle S^{-1}v, S^{-1}w\rangle$ for $v, w \in H_0$. It is clear that $\rho(x,y) = \|x-y\|_{H_0} := \langle x-y, x-y\rangle_{H_0}^{1/2}$ if $x - y \in H_0$.

We shall use the following two conditions.

(H_3) $T_t R E \subset R^{1/2}E$ and there is a strictly positive $h_1 \in C[0,\infty)$ such that

$$\|T_t R x\|_{H_0} \le \sqrt{h_1(t)}\, \|Rx\|_{H_0} \quad x \in E, t \ge 0.$$

(H_4) There is a strictly positive $h_2 \in C[0,\infty)$ such that $M \circ T_t^{-1} \le h_2(t) M, \quad t \ge 0.$

Our first main result is the following.

Theorem 1.1. *Assume $(H_1), (H_2)$ and (H_3). Then*

$$|P_t f(x)|^2 \le \left[P_t f^2(y)\right] \exp\left[\frac{\rho(x,y)^2}{\int_0^t h_1(s)^{-1}\mathrm{d}s}\right], \quad f \in C_b(E), x, y \in E, t > 0. \tag{1.4}$$

If in particular $M = 0$, then for any $p > 1$ and any $f \in C_b(E)$,

$$|P_t f(x)|^p \le \left[P_t |f|^p(y)\right] \exp\left[\frac{2\rho(x,y)^2}{p(p-1)\int_0^t h_1(s)^{-1}\mathrm{d}s}\right], \quad x, y \in E, t > 0. \tag{1.5}$$

Corollary 1.2. *Assume $(H_1), (H_2)$ and (H_3). Assume further that*

$$T_t E \subset R_t^{1/2} E, \quad t > 0, \tag{1.6}$$

where $R_t := \int_0^t T_s R T_s^* \mathrm{d}s$. Let $\|\cdot\|_p$ denote the norm in $L^p(\mu)$. Then:

(1) P_t is strong Feller, in particular, $P_t(x, \mathrm{d}y)$ has a density $p_t(x,y)$ w.r.t. μ for $x \in \mathrm{supp}\mu$ (where as usual $\mathrm{supp}\mu$ is the smallest closed set in E whose complement has μ-measure zero). Furthermore, (1.4) (resp. (1.5) if $M = 0$) holds for all $f \in \mathcal{B}_b(E)$ and one has

$$\|p_t(x,\cdot)\|_2 \le \left\{\int_E \exp\left[-\frac{\rho(x,y)^2}{\int_0^t h_1(s)^{-1}\mathrm{d}s}\right]\mu(\mathrm{d}y)\right\}^{-1/2}, \quad x\in \mathrm{supp}\mu, t>0, \tag{1.7}$$

where as usual we set $\frac{1}{0} := \infty$, *so the right-hand side is equal to* ∞ *if* $\mu(x+H_0)=0$. *If in particular* $M=0$, *then for any* $p,p'>1$ *with* $p^{-1}+p'^{-1}=1$,

$$\|p_t(x,\cdot)\|_{p'} \le \left\{\int_E \exp\left[-\frac{2\rho(x,y)^2}{p(p-1)\int_0^t h_1(s)^{-1}\mathrm{d}s}\right]\mu(\mathrm{d}y)\right\}^{-1/p}, \quad x\in E, t>0. \tag{1.8}$$

(2) *If there exists* $t>0$ *such that* $T_tR^{1/2}E$ *is dense in* E, *then* $\mathrm{supp}\mu = E$ *and* μ *is the unique invariant probability measure of* P_t.

It has been proved in [27] (see also [11]) that when $M=0$ (1.6) is equivalent to the strong Feller property of P_t.

To prove the Harnack inequality (1.4), we need an estimate for the square field operator of P_t, obtained in Lemma 2.1 below. This extends known gradient estimates of diffusion semigroups on manifolds.

Let us recall the formula for the square field operator obtained in [20]. Let W_0 be the space of functions f of the form

$$f(x) = F(\langle\xi_1,x\rangle,\cdots,\langle\xi_m,x\rangle), \quad x\in E,$$

for some $m\ge 1$ and $F\in\mathcal{S}(\mathbb{R}^m,\mathbb{C})$ (i.e. the Schwartz space of complex-valued functions, "rapidly decreasing" at infinity as well as their derivatives). Let W be the real-valued elements of W_0. Then W is dense in $L^p(\mu)$ for any $p\ge 1$ and is a core of $\mathcal{D}(L)$, the $L^2(\mu)$-domain of the generator L of P_t, see [19]. Moreover, according to [20, Proposition 4.1], for any $f\in W$,

$$\Gamma(f,f) := Lf^2 - 2fLf = \langle RDf, Df\rangle + \int_E [f(\cdot)-f(\cdot+y)]^2 M(\mathrm{d}y), \tag{1.9}$$

where Df denotes the Fréchet derivative of f. Note that by [19, Theorem 1.1 (i)], $\Gamma(f,f)\in C_b(E)$ for $f\in W$.

As another application of our estimate for the square field operator, we have the following Poincaré and log-Sobolev inequalities.

Theorem 1.3. *Assume* $(H_1),(H_2),(H_3)$ *and* (H_4). *Then for* $C(t) := \left(\int_0^t h_1(s)\mathrm{d}s\right)\vee\left(\int_0^t h_2(s)\mathrm{d}s\right)$, *where* $h_1:=0$ *if* $R=0$,

$$P_tf^2 - (P_tf)^2 \le C(t)P_t\Gamma(f,f), \quad t>0, f\in W. \tag{1.10}$$

If in particular $M = 0$, then

$$P_t(f^2 \log f^2) - (P_t f^2) \log P_t f^2 \le \left(2\int_0^t h_1(s)\mathrm{d}s\right) P_t\Gamma(f,f), \quad t > 0, f \in W. \tag{1.11}$$

Corollary 1.4. *Assume that $(H_1'), (H_2), (H_3)$ and (H_4) hold.*
(1) *If $C(\infty) < \infty$, then*

$$\mu(f^2) - \mu(f)^2 \le C(\infty)\mu(\Gamma(f,f)), \quad f \in W.$$

Moreover,

$$\mu((P_t f - \mu(f))^2) \le \mathrm{e}^{-2t/C(\infty)}\mu((f^2 - \mu(f))^2), \quad t \ge 0,\ f \in L^2(\mu). \tag{1.12}$$

If in particular $M = 0$, then

$$\mu(f^2 \log f^2) \le \left(2\int_0^\infty h_1(t)\mathrm{d}t\right)\mu(\Gamma(f,f)), \quad f \in W,\ \mu(f^2) = 1.$$

(2) *If $M = 0$ and there exists $t > 0$ such that $\|R_\infty^{-1/2} T_t R_\infty^{1/2}\|^2 := \varepsilon_t < 1$, then*

$$\mu(f^2 \log f^2) \le \left(\frac{2}{1-\varepsilon_t}\int_0^t h_1(s)\mathrm{d}s\right)\mu(\Gamma(f,f)), \quad f \in W,\ \mu(f^2) = 1.$$

Consequently, if in addition (1.6) holds, then P_t is compact in $L^p(\mu)$ for all $t > 0$ and hence the L^p-essential spectrum of L is empty.

For applications of Corollary 1.4 we refer to Example 3.1 below. Finally, we formulate a result on the Harnack inequality relying on conditions on μ_t and T_t only without referring to R and M. This result mainly applies in finite-dimensional cases, cf. Examples 3.2 and 3.3 below.

Theorem 1.5. *Assume that $\mu_t \circ \theta_{T_t x}^{-1}$ is absolutely continuous with respect to μ_t for all $x \in E$ and all $t > 0$, where $\theta_x(y) := x + y, y \in E$. Let*

$$\eta_t(x,z) := \frac{\mathrm{d}\mu_t \circ \theta_{T_t x}^{-1}}{\mathrm{d}\mu_t}(z).$$

If there exist $p' \in (1, \infty]$ and $t > 0$ such that

$$\Phi_{t,p'}(x) := \|\eta_t(x,\cdot)\|_{L^{p'}(\mu_t)} < \infty, \tag{1.13}$$

then

$$|P_t f(x)|^p \le P_t|f|^p(y)\Phi_{t,p'}(x-y)^p, \quad f \in \mathcal{B}_b(E), x, y \in E, \tag{1.14}$$

where $p \in [1, \infty)$ *such that* $p^{-1} + p'^{-1} = 1$*. If, in addition, there exists* $\varepsilon > 0$ *such that*

$$C(t, p', \varepsilon) := \int_E \frac{\mu(\mathrm{d}x)}{[\int \Phi_{t,p'}(x-y)^{-p}\mu(\mathrm{d}y)]^{1+\varepsilon}} < \infty, \tag{1.15}$$

then

$$\|P_t\|_{p\to(1+\varepsilon)p} \le C(t, p', \varepsilon)^{1/(1+\varepsilon)p}. \tag{1.16}$$

Proposition 1.6. *Consider the situation of Theorem 1.5. If* $C(t, p', 0) < \infty$ *for some* $t > 0$ *and* $p' \in (1, \infty]$*, then* P_s *is compact on* $L^p(\mu)$ *for any* $s > t$*.*

The above results are proved in the next section and are illustrated by some examples presented in Section 3: Example 3.1 contains some infinite-dimensional models with Poincaré and log-Sobolev inequalities; Example 3.2 shows that the generalized Mehler semigroups determined by α-stable measures are not hyperbounded, but satisfy a Poincaré inequality and a Harnack type inequality; a class of hyperbounded non-Gaussian semigroups are presented in Example 3.3; furthermore it is shown in Example 3.4 that the log-Sobolev constant given in Corollary 1.4 can be sharper than the one obtained using the usual curvature condition. Finally, the generalized Mehler semigroup on $E := L^2((0,1); \mathrm{d}x)$ associated with the Dirichlet heat semigroup $T_t := \mathrm{e}^{\Delta t}$ on $(0,1)$ is studied in Section 4. We prove that this semigroup satisfies a Harnack inequality, but is not hyperbounded.

2 Proofs

To prove Theorem 1.1, we need the following two lemmas.

Lemma 2.1. (1) *If* (H_3) *holds and* $P_t W \subset W$ *for all* $t > 0$*, then*

$$\sqrt{\langle RDP_t f, DP_t f\rangle} \le \sqrt{h_1(t)} P_t \sqrt{\langle RDf, Df\rangle}, \quad t \ge 0, f \in W. \tag{2.1}$$

(2) *If* (H_4) *holds, then for any* $t \ge 0, f \in W$ *and* $x \in E$*,*

$$\int_E [P_t f(x+y) - P_t f(x)]^2 M(\mathrm{d}y) \le h_2(t) P_t \Big\{ \int_E [f(\cdot + y) - f(\cdot)]^2 M(\mathrm{d}y) \Big\}(x). \tag{2.2}$$

Proof. Let $f \in W, t > 0$. It is easy to see that $\|Df\|$ is bounded, hence because $P_t f \in W$ we conclude by Lebesgue's dominated convergence theorem that for all $x, z \in E$

$$\langle DP_t f(x), z\rangle = \int \langle Df(T_t x + y), T_t z\rangle \mu_t(\mathrm{d}y) = P_t(\langle Df(\cdot), T_t z\rangle)(x).$$

Hence, since by assumption $T_t RE \subset R^{1/2}E$, we obtain that

$$\langle Rz, DP_t f\rangle \leq P_t(\|R^{1/2}Df\| \cdot \|S^{-1}T_t Rz\|) \leq \sqrt{h_1(t)\langle z, Rz\rangle} P_t \sqrt{\langle RDf, Df\rangle}.$$

Choosing $z := DP_t f(x), x \in E$, the first assertion follows.

Next, we have

$$\begin{aligned}
&\int_E [P_t f(x+y) - P_t f(x)]^2 M(\mathrm{d}y) \\
&= \int_E \left\{ \int_E [f(T_t x + T_t y + z) - f(T_t x + z)]\mu_t(\mathrm{d}z) \right\}^2 M(\mathrm{d}y) \\
&\leq \int_{E\times E} [f(T_t x + y + z) - f(T_t x + z)]^2 \mu_t(\mathrm{d}z)(M \circ T_t^{-1})(\mathrm{d}y) \\
&\leq h_2(t) P_t \left\{ \int_E [f(\cdot + y) - f(\cdot)]^2 M(\mathrm{d}y) \right\}(x).
\end{aligned}$$

Hence (2.2) holds. □

Lemma 2.2. *If there exists $p \in [1, \infty)$ and a measurable function $\Phi : E \times E \to (0, \infty)$ such that*

$$|P_t f(x)|^p \leq [P_t |f|^p(y)]\Phi(x, y), \quad x, y \in E, f \in \mathcal{B}_b(E). \tag{2.3}$$

Then P_t has transition density $p_t(x, y)$ w.r.t. μ satisfying

$$\|p_t(x, \cdot)\|_{p'} \leq \left\{ \int_E \frac{\mu(\mathrm{d}y)}{\Phi(x, y)} \right\}^{-1/p}, \quad x \in E, \tag{2.4}$$

where $p' := \frac{p}{p-1}$.

Proof. Let $x \in E$ be fixed. For any $A \in \mathcal{B}(E)$ with $\mu(A) = 0$, (2.3) yields that

$$\left(\mu_t \circ \theta_{T_t x}^{-1}(A)\right)^p \leq [P_t 1_A(y)]\Phi(x, y).$$

Then

$$\left(\mu_t\circ\theta_{T_tx}^{-1}(A)\right)^p\int_E\frac{\mu(\mathrm{d}y)}{\Phi(x,y)}\le\int_E[P_t1_A(y)]\mu(\mathrm{d}y)=\mu(A)=0.$$

Therefore, $P_t1_A(x)=\mu_t\circ\theta_{T_tx}^{-1}(A)=0$ and hence $\mu_t\circ\theta_{T_tx}^{-1}$ is absolutely continuous w.r.t. μ. Let $p_t(x,\cdot)$ denote the Radon-Nikodym derivative. Dividing (2.3) by $\Phi(x,y)$ and integrating with respect to $\mu(\mathrm{d}x)$ we thus obtain by (H_1) for all $f\in\mathcal{B}_b(E)$

$$\langle p_t(x,\cdot),f\rangle_{L^2(\mu)}\le\|f\|_p\left[\int_E\frac{\mu(\mathrm{d}y)}{\Phi(x,y)}\right]^{-1/p}.$$

Thus, $\|p_t(x,\cdot)\|_{p'}\le\left\{\int_E\Phi(x,y)^{-1}\mu(\mathrm{d}y)\right\}^{-1/p}$. □

Proof of Theorem 1.1. We first prove (1.4) for nonnegative $f\in W$. For $\varepsilon\in(0,1)$, let $P_t^{(\varepsilon)}$ be the generalized Mehler semigroup determined by T_t and λ_ε, where λ_ε is given by (1.3) with M replaced by

$$M_\varepsilon(\mathrm{d}x):=1_{\{a:\varepsilon\le\|a\|\le 1/\varepsilon\}}(x)M(\mathrm{d}x).$$

By [19, Theorem 1.3 (i) and Proposition 3.3] one has $P_{t,\varepsilon}W\subset W$ for any $t\ge0$. For $x\ne y$ with $\rho(x,y)<\infty$, by (2.1)

$$\begin{aligned}\langle DP_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f)^2,y-x\rangle&=\inf_{z\in(R^{1/2})^{-1}(x-y)}\langle DP_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f)^2,R^{1/2}z\rangle\\&\le\rho(x,y)\sqrt{\langle RDP_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f)^2,DP_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f)^2\rangle}\\&\le 2\rho(x,y)h_1(t-s)P_{t-s}^{(\varepsilon)}\big\{(P_s^{(\varepsilon)}f)\sqrt{\langle RDP_s^{(\varepsilon)}f,DP_s^{(\varepsilon)}f\rangle}\big\}.\end{aligned}\tag{2.5}$$

Let

$$x_s:=x+\frac{(y-x)\int_0^sh_1(t-u)^{-1}\mathrm{d}u}{\int_0^th_1(u)^{-1}\mathrm{d}u},$$
$$\phi(s):=\log P_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f+\varepsilon)^2(x_s),\quad s\in[0,t].$$

It follows from [19, Theorem 1.1 (ii)], (1.9) and (2.5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(s)&=\frac{-P_{t-s}^{(\varepsilon)}\Gamma(P_s^{(\varepsilon)}f,P_s^{(\varepsilon)}f)(x_s)+\frac{1}{h_1(t-s)\int_0^th_1(u)^{-1}\mathrm{d}u}\langle DP_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f)^2(x_s),y-x\rangle}{P_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f+\varepsilon)^2(x_s)}\\&\le\frac{1}{P_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)}f+\varepsilon)^2(x_s)}P_{t-s}^{(\varepsilon)}\Big\{(P_s^{(\varepsilon)}f+\varepsilon)^2\Big[-g+\frac{2\rho(x,y)\sqrt{g}}{h_1(t-s)^{1/2}\int_0^th_1(u)^{-1}\mathrm{d}u}\Big]\Big\}(x_s)\\&\le\frac{\rho(x,y)^2}{h_1(t-s)(\int_0^th_1(u)^{-1}\mathrm{d}u)^2},\quad s\in[0,t],\end{aligned}$$

where

$$g := \frac{\langle RDP_s^{(\varepsilon)} f, DP_s^{(\varepsilon)} f\rangle}{(P_s^{(\varepsilon)} f + \varepsilon)^2}.$$

Therefore, we obtain

$$\phi(t) \le \phi(0) + \frac{\rho(x,y)^2}{\int_0^t h_1(u)^{-1}\mathrm{d}u}.$$

Hence

$$(P_t^{(\varepsilon)} f + \varepsilon)^2(y) \le [P_t^{(\varepsilon)}(f+\varepsilon)^2(x)] \exp\left[\frac{\rho(x,y)^2}{\int_0^t h_1(u)^{-1}\mathrm{d}u}\right].$$

According to [19, Corollary 3.5], one has $P_t^{(\varepsilon)} f \to P_t f$ uniformly as $\varepsilon \to 0$ for any $f \in W$, therefore (1.4) follows by letting $\varepsilon \downarrow 0$.

For general $f \in W$ and any $n \ge 1$, let $h_n \in C^\infty(\mathbb{R})$ such that $0 \le h_n(r) \le |r|, h_n(r) = 0$ for $|r| \le \frac{1}{n}$, and $h_n(r) = |r|$ for $|r| \ge \frac{2}{n}$. Then $h_n(f) \in W$. Applying (1.4) for $h_n(f)$ in place of f then letting $n \to \infty$, we arrive at

$$(P_t|f|)^2(y) \le [P_t f^2(x)] \exp\left[\frac{\rho(x,y)^2}{\int_0^t h_1(u)^{-1}\mathrm{d}u}\right].$$

This implies (1.4) for $f \in W$.

We are now going to prove (1.4) for $f \in C_b(E)$. Let $E_n := \mathrm{spn}\{x_1, \cdots, x_n\}, \pi_n : E \to E_n$ is the orthogonal projection, $n \ge 1$. For $f \in C_b(E)$ and $n \ge 1$, there exists $g_n \in C_b(\mathbb{R}^n)$ such that

$$f_n := f \circ \pi_n = g_n(\langle x_1, \cdot\rangle, \cdots, \langle x_n, \cdot\rangle).$$

Let $\{g_{nm}\}_{m\ge 1} \subset C_0^\infty(\mathbb{R}^n)$ be uniformly bounded such that $g_{nm} \to g_n$ point-wisely as $m \to \infty$. By (1.4) for $f_{nm} := g_{nm}(\langle x_1, \cdot\rangle, \cdots, \langle x_n, \cdot\rangle)$ and using the dominated convergence theorem, we obtain

$$|P_t f_n(x)|^2 = \lim_{m\to\infty} |P_t f_{nm}(x)|^2 \le \left\{\lim_{m\to\infty} P_t|f_{nm}|^2(y)\right\}\Phi(x,y) = [P_t|f_n|^2(y)]\Phi(x,y),$$

where $\Phi(x,y) := \exp\left[\frac{\rho(x,y)^2}{\int_0^t h_1(s)^{-1}\mathrm{d}s}\right]$. Since $f \in C_b(E)$ and $\pi_n x \to x$ as $n \to \infty$ for any $x \in E$, using the dominated convergence theorem once again, we prove (1.4) for $f \in C_b(E)$.

If $M = 0$, then L is a second order differential operator, hence (1.5) follows by repeating the above argument with

$$\phi(s) := \log P_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)} f + \varepsilon)^p(x_s)$$

and using the chain rule. □

Proof of Corollary 1.2. (1) By Theorem 1.1 and the proof of Lemma 2.2, it suffices to prove that P_t is strong Feller. Let μ_t^1 and μ_t^2 be probability measures with Fourier transforms

$$\hat{\mu}_t^1(\xi) := \exp\left[\int_0^t [i\langle T_s^*\xi, a\rangle - \frac{1}{2}\langle T_s^*\xi, RT_s^*\xi\rangle]\mathrm{d}s\right],$$
$$\hat{\mu}_t^2(\xi) := \exp\left[\int_0^t \mathrm{d}s\int_E \left(\mathrm{e}^{i\langle T_s^*\xi, x\rangle} - 1 - \frac{i\langle T_s^*\xi, x\rangle}{1+\|x\|^2}\right)M(\mathrm{d}x)\right].$$

Let P_t^1 be the generalized Mehler semigroup determined by T_t and μ_t^1. If (1.6) holds then according to [27] (see also [11]) P_t^1 is strong Feller. For any $f \in \mathcal{B}_b(E)$, let

$$f_t(x) := \int_E f(x+z)\mu_t^2(\mathrm{d}z), \quad x \in E.$$

One has $f_t \in \mathcal{B}_b(E)$ and

$$P_t f(x) = \int_{E\times E} f(T_t x + y + z)\mu_t^1(\mathrm{d}y)\mu_t^2(\mathrm{d}z) = P_t^1 f_t(x),$$

so, P_t is strong Feller, too. In particular, $P_t(x, \mathrm{d}y)$ has a density $p_t(x,y)$ w.r.t. μ for all $x \in \operatorname{supp}\mu$.

(2) Since $T_t R^{1/2}E$ is dense in E for some $t > 0$, to prove $\operatorname{supp}\mu = E$ by [3, Proposition 2.7] it suffices to check that $\mu\circ\theta_x^{-1}$ is absolutely continuous w.r.t. μ for all $x \in T_t R^{1/2}E$. Let $x = T_t R^{1/2}x'$. It follows from (1.4) that for any $A \in \mathcal{B}(E)$ with $\mu(A) = 0$, one has for $\Phi(x,y) := \exp[\rho(x,y)/\int_0^t h_1(s)^{-1}\mathrm{d}s]$ that

$$\int_E (P_t 1_A)^2(R^{1/2}x' + y)\Phi(R^{1/2}x' + y, y)^{-1}\mu(\mathrm{d}y) \le \mu(P_t 1_A) = \mu(A) = 0.$$

But $\Phi(R^{1/2}x' + y, y)^{-1} > 0$ since $R^{1/2}x' + y - y \in R^{1/2}E$, so $(P_t 1_A)\circ\theta_{R^{1/2}x'} = 0$ μ-a.e. Thus,

$$\mu\circ\theta_x^{-1}(A) = \mu(P_t(1_A\circ\theta_x)) = \mu((P_t 1_A)\circ\theta_{R^{1/2}x'}) = 0.$$

This means that $\mu\circ\theta_x^{-1}$ is absolutely continuous w.r.t. μ. Therefore, $\operatorname{supp}\mu = E$.

Let μ_1 and μ_2 be two invariant probability measures of P_t, then they have full support. Now it is easy to see that they are equivalent. Indeed, if e.g. $\mu_1(B) = 0$ then $\mu_1(P_t 1_B) = 0$. But $P_t 1_B$ is continuous and μ_1 has full support, then $P_t 1_B \equiv 0$ so that $\mu_2(B) = \mu_2(P_t 1_B) = 0$. Let $\mu_2 = \psi\mu_1$ and $\mu := \frac{1}{2}(\mu_1 + \mu_2)$. We have $\mu_2 = \frac{2\psi}{1+\psi}\mu =: \phi\mu$. Then $P_t^*\phi = \phi$, where P_t^* is the adjoint of P_t in $L^2(\mu)$. Since P_t^* is sub-Markovian, for any $c > 0$ one has

$$P_t^*(c-\phi)^+ \ge (c - P_t^*\phi)^+ = (c-\phi)^+.$$

Since $\mu(P_t^*(c-\phi)^+) = \mu((c-\phi)^+)$, it follows that $P_t^*(c-\phi)^+ = (c-\phi)^+$ μ-a.e. Thus, if $\mu((c-\phi)^+) > 0$ then $\frac{(c-\phi)^+}{\mu((c-\phi)^+)}\mu$ is an invariant probability measure for P_t and hence is equivalent to μ as explained above. Therefore $(c-\phi)^+ > 0$ μ-a.e. This means that for any $c > 0$ with $\mu(\phi < c) > 0$, one has $\mu(\phi < c) = 1$. Hence ϕ has to be constant, i.e. $\mu_1 = \mu_2$. □

Proof of Theorem 1.3. Let $P_t^{(\varepsilon)}$ be as in the proof of Theorem 1.1, where $\varepsilon \in (0,1)$. We have $P_t^{(\varepsilon)} W \subset W$ for any $t \geq 0$. Since (H_4) holds, by Lemma 2.1 one has (with $h_1 := 0$ if $R = 0$)

$$\langle RDP_t^{(\varepsilon)} f, DP_t^{(\varepsilon)} f\rangle \leq h_1(t) P_t^{(\varepsilon)} \langle RDf, Df\rangle, \quad f \in W, t \geq 0. \tag{2.6}$$

To obtain the Poincaré inequality, we adopt a trick due to Bakry-Ledoux [6]. For $\phi(s) := P_{t-s}^{(\varepsilon)}(P_s^{(\varepsilon)} f)^2$, it follows from (1.9) and (2.6) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(s) &= -P_{t-s}^{(\varepsilon)}\Gamma(P_s^{(\varepsilon)} f, P_s^{(\varepsilon)} f)\\ &\geq -h_1(s) P_t^{(\varepsilon)}\langle RDf, Df\rangle - P_{t-s}^{(\varepsilon)}\Big\{\int_E [P_s^{(\varepsilon)} f(x) - P_s^{(\varepsilon)} f(x+y)]^2 M_\varepsilon(\mathrm{d}y)\Big\}, \quad s \in [0,t].\end{aligned}$$

Then

$$\begin{aligned}P_t^{(\varepsilon)} f^2 - (P_t^{(\varepsilon)} f)^2 \leq & \Big(\int_0^t h_1(s)\mathrm{d}s\Big) P_t^{(\varepsilon)}\langle RDf, Df\rangle\\ &+ \int_0^t P_{t-s}^{(\varepsilon)}\Big\{\int_E [P_s^{(\varepsilon)} f(x) - P_s^{(\varepsilon)} f(x+y)]^2 M_\varepsilon(\mathrm{d}y)\Big\}\mathrm{d}s, \quad f \in W.\end{aligned}$$

Thus, it follows from (2.2) that

$$P_t^{(\varepsilon)} f^2 - (P_t^{(\varepsilon)} f)^2 \leq C(t) P_t^{(\varepsilon)}\Gamma^{(\varepsilon)}(f,f), \quad f \in W,\ t > 0, \tag{2.7}$$

where

$$\Gamma^{(\varepsilon)}(f,f) := \langle RDf, Df\rangle + \int_E [f(\cdot) - f(\cdot + y)]^2 M_\varepsilon(\mathrm{d}y)$$

Letting $\varepsilon \downarrow 0$ we prove (1.10).

If $M = 0$ then for $f \in W$ and $\varepsilon > 0$, let $\phi(s) := P_{t-s}^{(\varepsilon)}\{(P_s^{(\varepsilon)} f^2 + \varepsilon)\log(P_s^{(\varepsilon)} f^2 + \varepsilon)\}$. One obtains

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(s) = -P_{t-s}^{(\varepsilon)} \frac{\langle RD(P_s^{(\varepsilon)} f^2), D(P_s^{(\varepsilon)} f^2)\rangle}{2(P_s^{(\varepsilon)} f^2 + \varepsilon)}$$
$$\geq -\frac{1}{2} P_{t-s}^{(\varepsilon)} \frac{h_1(s)\big(P_s^{(\varepsilon)}\sqrt{\langle RDf^2, Df^2\rangle}\big)^2}{P_s^{(\varepsilon)} f^2 + \varepsilon} \geq -2h_1(s) P_t^{(\varepsilon)} \Gamma(f,f), \quad s \in [0,t].$$

This implies (1.11) by first integrating over s from 0 to t and then letting $\varepsilon \downarrow 0$. □

Proof of Corollary 1.4. Since the assumptions imply that $P_t f \to \mu(f)$ as $t \to \infty$ for any $f \in C_b(E)$, the Poincaré and log-Sobolev inequalities in (1) immediately follow from Theorem 1.3. Next, it is easy to see that (H_1') holds for M_ε in place of M. Let $a_\infty^{(\varepsilon)}$ and $M_\infty^{(\varepsilon)}$ be given in (H_1') (iii) and (iv) with M replaced by M_ε, and let $\lambda_\infty^{(\varepsilon)}$ be defined by (1.3) for $a_\infty^{(\varepsilon)}, R_\infty$ and $M_\infty^{(\varepsilon)}$ replacing a, R and M respectively. Then $P_t^{(\varepsilon)}$ has a unique invariant probability measure $\mu^{(\varepsilon)}$ with characteristic function $\hat{\mu}^{(\varepsilon)}(\xi) := \mathrm{e}^{-\lambda_\infty^{(\varepsilon)}(\xi)}$. It is clear that $\hat{\mu}^{(\varepsilon)}(\xi) \to \hat{\mu}(\xi)$ for all $\xi \in E$ and hence $\mu^{(\varepsilon)} \to \mu$ weakly as $\varepsilon \to 0$. By letting $t \to \infty$ in (2.7), we obtain

$$\mu^{(\varepsilon)}(f^2) - \mu^{(\varepsilon)}(f)^2 \leq C(\infty)\mu^{(\varepsilon)}(\Gamma^{(\varepsilon)}(f,f)), \qquad f \in W.$$

Since, according to [19], $P_t^{(\varepsilon)} W \subset W \subset \mathcal{D}(L^{(\varepsilon)})$, where $L^{(\varepsilon)}$ is the generator of $P_t^{(\varepsilon)}$ in $L^2(\mu^{(\varepsilon)})$, this Poincaré inequality implies that

$$\mu^{(\varepsilon)}((P_t^{(\varepsilon)} f - \mu^{(\varepsilon)}(f))^2) \leq \mu^{(\varepsilon)}((f - \mu^{(\varepsilon)}(f))^2)\mathrm{e}^{-2t/C(\infty)}$$

for all $t > 0$ and all $f \in W$. Letting $\varepsilon \to 0$ we prove (1.12) for all $f \in W$ and hence for all $f \in L^2(\mu)$ since W is dense in $L^2(\mu)$.

So, let us prove (2). Since $\varepsilon_t < 1$, according to [10, 13] we have $\|P_t\|_{p\to q} \leq 1$ for any $p > 1$ and $q = 1 + (p-1)\varepsilon_t^{-1}$. By (1.11) and Proposition 2.3 below, we arrive at

$$\mu(f^2 \log f^2) \leq \left(\frac{2p}{1-\varepsilon_t}\int_0^t h(s)\mathrm{d}s\right)\mu(\Gamma(f,f)), \quad f \in W, \mu(f^2) = 1.$$

The proof is then finished by letting $p \to 1$.

Finally, by Corollary 1.2 if (1.6) holds then $P_t(x, \mathrm{d}y)$ has a density w.r.t. μ for any $t > 0, x \in \operatorname{supp}\mu$. Therefore, P_t is compact in $L^p(\mu)$ for any $t > 0$ according to [26, Theorem 2.3] (see also [15] and [25]). Hence the essential spectrum of L in $L^p(\mu)$ is empty, see e.g. [18, Theorem 6.29]. □

Proposition 2.3. *If there exist $t > 0, 1 < p < q < \infty$ and $C_1, C_2 > 0$ such that $\|P_t\|_{p\to q} \leq C_1$ and*

$$P_t(f^2 \log f^2) \leq C_2 P_t \Gamma(f,f) + P_t f^2 \log P_t f^2, \quad f \in W, \tag{2.8}$$

then

$$\mu(f^2\log f^2) \le \frac{C_2 p(q-1)}{q-p}\mu(\Gamma(f,f)) + \frac{pq}{q-p}\log C_1, \quad f\in W,\ \mu(f^2)=1. \tag{2.9}$$

Proof. The proof is essentially taken from [22]. Since μ is an invariant measure, (2.8) yields

$$\mu(f^2\log f^2) \le C_2\mu(\Gamma(f,f)) + \mu(P_t f^2 \log P_t f^2). \tag{2.10}$$

For any $s\in(0,(p-1)/p)$, let $r := \frac{ps}{p-1}, p_s := \frac{1}{1-s}, q_s := \frac{1}{1-\delta s}, \delta := \frac{p(q-1)}{(p-1)q}$. Then $r\in(0,1)$ and

$$\frac{1}{p_s} = 1-r+\frac{r}{p},\quad \frac{1}{q_s} = 1-r+\frac{r}{q}.$$

So, by the Riesz-Thorin interpolation theorem,

$$\|P_t\|_{p_s\to q_s} \le C_1^r = C_1^{ps/(p-1)}.$$

Therefore, for any $f\ge 0$ with $\mu(f^2)=1$,

$$\int (P_t f^{2(1-s)})^{q_s}\mathrm{d}\mu \le \|P_t\|_{p_s\to q_s}^{q_s} \le C_1^{ps/(p-1)(1-\delta s)}. \tag{2.11}$$

Noting that for $s=0$ all three quantities in (2.11) are equal to 1, we obtain

$$\delta\mu(P_t f^2\log P_t f^2) - \mu(f^2\log f^2) = \frac{\mathrm{d}}{\mathrm{d}s}\mu\Big(\big(P_t f^{2(1-s)}\big)^{q_s}\Big)\Big|_{s=0} \le \frac{p}{p-1}\log C_1.$$

Combining this with (2.10), we prove (2.9). □

Remark. According to an example due to Fuhrman [13] that for the non-symmetric case the hypercontractivity and the log-Sobolev inequality are no longer equivalent. Consider, for instance, $E=\mathbb{R}^n$. Let $M=0$ and R and T_t be such that $\det R=0$ but $\det R_t>0$ for $t>0$, and $\|R_\infty^{-1/2}T_tR_\infty^{1/2}\|<1$ for some $t>0$. According to [10, 13] P_t is hypercontractive. Let $0\ne\xi\in\mathbb{R}^n$ be such that $R\xi=0$. Taking $f(x)=F(\langle\xi,x\rangle), F\in\mathcal{S}(\mathbb{R}^m,\mathbb{R})$, then $\langle R\nabla f,\nabla f\rangle=0$. Since f is not constant, and μ is a normal distribution, there is no Poincaré inequality. Therefore condition (2.8) in Proposition 2.3 can not be dropped.

Proof of Theorem 1.5. For any bounded, nonnegative measurable f, we have

$$
\begin{aligned}
P_t f(x) &= \int f(T_t x + z)\mu_t(\mathrm{d}z) = \int f(T_t y + z)\mu_t \circ \theta^{-1}_{T_t(x-y)}(\mathrm{d}z) \\
&= \int f(T_t y + z)\eta_t(x-y,z)\mu_t(\mathrm{d}z) \\
&\le [P_t f^p(y)]^{1/p}\Phi_{t,p'}(x-y).
\end{aligned}
$$

This proves (1.14).

Next, let f be such that $\mu(|f|^p) = 1$, by (1.14) we have

$$
|P_t f(x)|^p \int \Phi_{t,p'}(x-y)^{-p}\mu(\mathrm{d}y) \le 1.
$$

Then (1.16) follows from (1.15) immediately. □

Proof of Proposition 1.6. By (1.14), for any f with $\|f\|_p = 1$, we have

$$
|P_t f(x)|^p \le \frac{1}{\int \Phi_{t,p'}(x-y)^{-p}\mu(\mathrm{d}y)}.
$$

If $C(t,p',0) < \infty$, then $\mathcal{F} := \{P_t f : \|f\|_p \le 1\}$ is uniformly integrable in $L^p(\mu)$. Moreover, by Lemma 2.2 P_t has density w.r.t. μ, hence as shown in the proof of Corollary 1.4 it follows that P_s is compact in $L^p(\mu)$ for $s > t$. □

3 Examples

We first present an example to illustrate Theorem 1.1 and Corollary 1.4.

Example 3.1. *Let a and R be arbitrary as in (1.3) and let $T_t = \mathrm{e}^{-\beta t} I$ for some $\beta > 0$.*
(1) *If M is such that (H_1') (iii), (iv) hold, i.e.*

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}t \int_E \left[1 \wedge (\|x\|^2 \mathrm{e}^{-2\beta t})\right] M(\mathrm{d}x) < \infty \\
&\int_0^\infty \mathrm{d}t \int_E \mathrm{e}^{-\beta t} x \Big(\frac{1}{1+\mathrm{e}^{-2\beta t}\|x\|^2} - \frac{1}{1+\|x\|^2}\Big) M(\mathrm{d}x) \in E \text{ exits.}
\end{aligned}
$$

Then (1.4) holds for $h_1(t) := \mathrm{e}^{-\beta t}$.
(2) *Let $P(\mathrm{d}\xi)$ be a finite measure on $S_1 := \{\xi \in E : \|\xi\| = 1\}$ and let $r \in (0,1)$. Write $x := (\xi, s)$ if $x = s\xi, s > 0$. Define M by*

$$
M(\mathrm{d}x) := 1_{(0,\infty)}(s) s^{-(1+r)} P(\mathrm{d}\xi)\mathrm{d}s.
$$

Then (H_4) *holds for* $h_2(t) = \mathrm{e}^{-r\beta t}$. *Hence* (H_1') *is satisfied and the following Poincaré inequality holds*

$$\mu(f^2) - \mu(f)^2 \le \frac{1}{r\beta}\mu(\Gamma(f,f)), \quad f \in W. \tag{3.1}$$

(3) *Let* $M = 0$, *then*

$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f,f)), \quad f \in W, \mu(f^2) = 1. \tag{3.2}$$

holds for $C = 2/\beta$.

Proof. Since $T_t x \to 0$ as $t \to \infty$, (H_1') holds. (H_2) is trivial since E is separable and each $\xi \in E$ is an eigenvector of T_t. Therefore, Theorem 1.1 applies and hence (1) follows. Next, (3) follows directly from Corollary 1.4. Finally, it is easy to see that M given in (2) is a Levy measure satisfying (H_1') (iii), (iv), and for any $A \in \mathcal{B}(E)$

$$\begin{aligned} M \circ T_t^{-1}(A) &= \int_{S_1} P(\mathrm{d}\xi) \int_0^\infty 1_{\mathrm{e}^{\beta t}A}(s\xi) s^{-(1+r)} \mathrm{d}s \\ &= \int_{S_1} P(\mathrm{d}\xi) \int_0^\infty \mathrm{e}^{-r\beta t} 1_A(s\xi) s^{-(1+r)} \mathrm{d}s = \mathrm{e}^{-r\beta t} M(A). \end{aligned}$$

Hence (H_4) holds for $h_2(t) = \mathrm{e}^{-r\beta t}$. Thus, the Poincaré inequality follows from Corollary 1.4. □

In the next example P_t is given by a Gaussian part and an α-stable jump part. We show that this semigroup possesses the Poincaré inequality but is not hyperbounded.

Example 3.2. *Let* $E = \mathbb{R}^n, T_t x = \mathrm{e}^{-t\beta} x$ *and* $\lambda(\xi) = \|\xi\|^\alpha + \delta\|\xi\|^2$, *where* $\beta > 0, \delta \ge 0, \alpha \in (0,2)$ *are constants. Then:*

(1) *For any* $t > 0$, *there exists* $C(t) > 0$ *such that*

$$|P_s f(x)| \le P_s|f|(y) C(t)(1 + \mathrm{e}^{-(n+\alpha)s}\|x-y\|^{n+\alpha}), \quad s > t.$$

(2) *For any* $t > 0$ *and any* $1 < p < q < \infty$, $\|P_t\|_{p \to q} = \infty$.

(3) *The Poincaré inequality* (3.1) *holds with* r *replaced by* $1 \wedge \alpha$. *If* $\delta = 0$ *then it holds with* r *replaced by* α.

Proof. Obviously,

$$\hat{\mu}_t(\xi) = \exp\left[-\frac{1}{\alpha\beta}(1 - \mathrm{e}^{-\alpha\beta t})\|\xi\|^\alpha - \frac{\delta}{2\beta}(1 - \mathrm{e}^{-2\beta t})\|\xi\|^2 \right].$$

By Theorem 3.1 in [14], the unique invariant measure μ is determined by

$$\hat{\mu}(\xi) = \exp\Big[-\frac{1}{\alpha\beta}\|\xi\|^\alpha - \frac{\delta}{2\beta}\|\xi\|^2\Big].$$

Let $t > 0$ be fixed and ν_1, ν_2 two probability measures with

$$\hat{\nu}_1(\xi) = \exp\Big[-\frac{\|\xi\|^\alpha}{\alpha\beta}(1-\mathrm{e}^{-\alpha\beta t})\Big], \quad \hat{\nu}_2(\xi) = \exp\Big[-\frac{\delta}{2\beta}\|\xi\|^2(1-\mathrm{e}^{-2\beta t})\Big].$$

Then $\mu_t = \nu_1 * \nu_2$. Let $\rho_1(x) = \frac{\mathrm{d}\nu_1}{\mathrm{d}x}$. Since ν_1 is α-stable, we have (see [7])

$$\frac{1}{C_1(1+\|x\|^{n+\alpha})} \le \rho_1(x) \le \frac{C_1}{1+\|x\|^{n+\alpha}}$$

for some $C_1 > 1$ and any $x \in \mathbb{R}^n$. Since ν_2 is a normal distribution for $\delta > 0$ and a Dirac measure for $\delta = 0$, it is easy to check that

$$\frac{1}{C_2(1+\|x\|^{n+\alpha})} \le \eta_t(x) := \frac{\mathrm{d}\mu_t}{\mathrm{d}x} \le \frac{C_2}{1+\|x\|^{n+\alpha}} \tag{3.3}$$

for all x and some constant $C_2 > 1$. Therefore, there exists $C_3 > 1$ such that

$$\sup_z |\eta_t(x,z)| \le C_3(1+\|x\|^{n+\alpha}).$$

Obviously, $C_3 = C_3(t)$ can be taken such that $C(t) := \sup_{s\ge t} C_3(s) < \infty$. Hence the first assertion follows from Theorem 1.5.

Next, it is well-known that for the present λ one has $M(\mathrm{d}x) = c(\alpha,n)\|x\|^{-(n+\alpha)}\mathrm{d}x$ for some constant $c(\alpha,n) > 0$. Then (H_4) holds for $h_2(t) = \mathrm{e}^{-\beta\alpha t}$, and hence the desired Poincaré inequality follows from Corollary 1.4.

Finally, let us fix $t > 0$ and $1 < p < q < \infty$. Take $\varepsilon \in (\alpha/q, \alpha/p)$ and put $f(x) = \|x\|^\varepsilon$. It is easy to see that $\|f\|_p < \infty$ since the same type of estimate in (3.3) holds for $\eta(x) := \frac{\mathrm{d}\mu}{\mathrm{d}x}$. But by (3.3),

$$\begin{aligned} P_t f(x) &= \int \|e^{-t\beta}x + y\|^\varepsilon \eta_t(y)\mathrm{d}y \ge \frac{1}{C_2}\int \frac{\|\mathrm{e}^{-t\beta}x+y\|^\varepsilon}{1+\|y\|^{n+\alpha}}\mathrm{d}y \\ &\ge \frac{1}{2C_2}\int_{\{\|y\|\le 1\}} |\mathrm{e}^{-t\beta}\|x\| - \|y\||^\varepsilon \mathrm{d}y \ge C_4\|x\|^\varepsilon - C_5 \end{aligned}$$

for some $C_4, C_5 > 0$ and all $x \in \mathbb{R}^n$. Therefore, there exists $C_6 > 0$ such that

$$\mu\big([P_t f]^q\big) \ge \int_{\{\|x\|^\varepsilon > C_5/C_4\}} \frac{C_6(C_4\|x\|^\varepsilon - C_5)^q}{1+\|x\|^{n+\alpha}}\mathrm{d}x = \infty$$

since $\varepsilon q > \alpha$. This means that $\|P_t\|_{p\to q} = \infty$. □

We now intend to construct a non-Gaussian generalized Mehler semigroup which is hyperbounded in L^1 and hence in L^p for all $p \geq 1$. Although the semigroup below looks like a perturbation of a symmetric Gaussian generalized Mehler semigroup, the L^1-hyperboundedness can not be obtained by perturbation arguments. The reason is that the Gaussian generalized Mehler semigroup itself is not L^1-hyperbounded w.r.t. its invariant measure. Indeed, for a symmetric semigroup the L^1-hyperboundedness is equivalent to the ultracontractivity: the boundedness from L^1 to L^∞. But the Guassian generalized Mehler semigroup is not ultracontractive.

Example 3.3. *Let $E = \mathbb{R}^n$. Consider $T_t x = \mathrm{e}^{-t\beta} x$ and $\lambda(\xi) = \delta\|\xi\|^2 + 2\beta \sum_{i=1}^n \frac{\xi_i^2}{1+\xi_i^2}, \beta, \delta > 0$. Then (T_t, λ) determines a generalized Mehler semigroup P_t satisfying*

$$|P_t f(x)| \leq P_t|f|(y)(1-\mathrm{e}^{-2\beta t})^{-(n+1)/2} \exp\left[C + 2(n-1)\beta t + \mathrm{e}^{-t\beta}|x-y|_1\right] \tag{3.4}$$

for all $f \in \mathcal{B}_b(\mathbb{R}^n)$ and some $C = C(n, \delta, \beta) > 0$, where $|x|_1 := \sum_i |x_i|$. Hence

$$\|p_t(x, \cdot)\|_\infty \leq (1-\mathrm{e}^{-2\beta t})^{-n/2} \exp[C + \mathrm{e}^{-\beta t}|x|_1], \quad t > 0, \ x \in \mathbb{R}^n. \tag{3.5}$$

Consequently, $\|P_t\|_{p \to p(1+\varepsilon)} < \infty$ for any $p \geq 1, t > 0$ and $\varepsilon \in (0, \mathrm{e}^{t\beta} - 1)$. Therefore, P_t is compact in $L^p(\mu)$ and the L^p-essential spectrum of L is empty for any $t > 0$ and $p \geq 1$.

Proof. Obviously, $f_i(\xi) := \frac{1}{1+\xi_i^2}$ is positive definite since it is the characteristic function of $(\frac{1}{2}\mathrm{e}^{-|x_i|}\mathrm{d}x_i) \times (\delta_0)^{n-1}$, where δ_0 denotes the Dirac measure at 0 on $\mathbb{R}$. Then $\frac{\xi_i^2}{1+\xi_i^2} = 1 - f_i(\xi)$ is negative definite for all i, and hence so is $\lambda(\xi)$. We have

$$\begin{aligned}
\hat{\mu}_t(\xi) &:= \exp\left[-\int_0^t \lambda(\mathrm{e}^{-\beta s}\xi)\mathrm{d}s\right] = \exp\left[-\frac{(1-\mathrm{e}^{-2\beta t})\delta\|\xi\|^2}{2\beta}\right] \prod_{i=1}^n \frac{1+\mathrm{e}^{-2\beta t}\xi_i^2}{1+\xi_i^2} \\
&= \exp\left[-\frac{(1-\mathrm{e}^{-2\beta t})\delta\|\xi\|^2}{2\beta}\right] \prod_{i=1}^n \left(\mathrm{e}^{-2\beta t} + \frac{1-\mathrm{e}^{-2\beta t}}{1+\xi_i^2}\right) \\
&= \exp\left[-\frac{(1-\mathrm{e}^{-2\beta 2\beta t})\delta\|\xi\|^2}{2\beta}\right] \left\{\mathrm{e}^{-2n\beta t} + \sum_{k=1}^n \sum_{i_1, \cdots, i_k = 1}^n \frac{\mathrm{e}^{-2(n-k)\beta t}(1-\mathrm{e}^{-2\beta t})^k}{(1+\xi_{i_1}^2)\cdots(1+\xi_{i_k}^2)}\right\}.
\end{aligned}$$

Let

$$\hat{\mu}(\xi) := \lim_{t \to \infty} \hat{\mu}_t(\xi) = \frac{\exp[-\delta\|\xi\|^2/2\beta]}{(1+\xi_1^2)\cdots(1+\xi_n^2)},$$

then μ is the unique invariant measure of P_t. Let

$$\psi_1(r) = \frac{\sqrt{\beta}}{\sqrt{2\pi\delta(1-\mathrm{e}^{-2\beta t})}} \exp\left[-\frac{\beta r^2}{2\delta(1-\mathrm{e}^{-2\beta t})}\right], \quad \psi_2(r) = \frac{1}{2}\mathrm{e}^{-|r|}, \quad r \in \mathbb{R}.$$

We have

$$\eta_t(x) := \frac{\mathrm{d}\mu_t}{\mathrm{d}x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle}\hat{\mu}_t(\xi)\mathrm{d}\xi = \mathrm{e}^{-2n\beta t}\prod_{i=1}^n \psi_1(x_i)$$
$$+\sum_{k=1}^n \mathrm{e}^{-2(n-k)\beta t}(1-\mathrm{e}^{-2\beta t})^k \sum_{i_1,\cdots,i_k=1}^n \Big[\prod_{j=1}^k (\psi_1 * \psi_2)(x_{i_j})\Big] \prod_{i\neq i_1,\cdots,i_k} \psi_1(x_i).$$

Noting that

$$-\frac{\beta r^2}{2\delta(1-\mathrm{e}^{-2\beta t})} \le -r + \frac{\delta(1-\mathrm{e}^{-2\beta t})}{2\beta} \le -r + \frac{\delta}{2\beta},$$

we conclude that there exists $C_1 = C_1(n,\delta,\beta) > 0$ such that

$$\mathrm{e}^{-2(n-1)\beta t}\sqrt{1-\mathrm{e}^{-2\beta t}}\mathrm{e}^{-|x|_1 - C_1} \le \eta_t(x) \le (1-\mathrm{e}^{-2\beta t})^{-n/2}\mathrm{e}^{-|x|_1 + C_1}. \tag{3.6}$$

Moreover, there is $C = C(n,\delta,\beta) > 0$ such that $\mathrm{e}^{-|x|_1 - C} \le \eta(x) := \frac{\mathrm{d}\mu}{\mathrm{d}x} \le \mathrm{e}^{-|x|_1 + C}$. Thus, (3.5) follows. Furthermore, (3.6) yields that

$$\eta_t(x,z) := \frac{\eta_t(z - T_t x)}{\eta_t(z)} \le (1-\mathrm{e}^{-2\beta t})^{-(n+1)/2}\exp[\mathrm{e}^{-\beta t}|x|_1 + 2C_1 + 2(n-1)\beta t], \quad x,z\in\mathbb{R}^n.$$

This implies (3.4) by Theorem 1.5.

Next, there exist $C_2, C_3, C_4 > 0$ depending also on t such that

$$\int_{\mathbb{R}^n} \frac{\mu(\mathrm{d}y)}{\Phi_{t,\infty}(x-y)} \ge C_2\mathrm{e}^{-2C_1}\int \exp\big[-\mathrm{e}^{-t\beta}|x-y|_1\big]\mu(\mathrm{d}y)$$
$$\ge C_3\int_{\mathbb{R}^n}\exp\big[-\mathrm{e}^{-t\beta}|x-y|_1 - |y|_1\big]\mathrm{d}y \ge C_4\exp\big[-\mathrm{e}^{-t\beta}|x|_1\big].$$

Therefore,

$$C(t,\infty,\varepsilon) \le C_4^{-(1+\varepsilon)}\int_{\mathbb{R}^n}\exp\big[(1+\varepsilon)\mathrm{e}^{-t\beta}|x|_1\big]\mu(\mathrm{d}x) < \infty$$

whenever $(1+\varepsilon)\mathrm{e}^{-t\beta} < 1$. Then it follows from Theorem 1.5 that

$$\|P_t\|_{1\to 1+\varepsilon} < \infty, \quad \varepsilon < \mathrm{e}^{t\beta} - 1. \tag{3.7}$$

Finally, for any $p > 1$, let $q = (1+\varepsilon)p$ for $\varepsilon < \mathrm{e}^{t\beta} - 1$. Let $r = 1/p, p_0 = q_0 = \infty, p_1 = 1, q_1 = 1+\varepsilon$. Then

$$\frac{1}{p} = \frac{r}{p_1} + \frac{1-r}{p_0}, \quad \frac{1}{q} = \frac{r}{q_1} + \frac{1-r}{q_0}.$$

The proof is then finished by (3.7), the Riesz-Thorin interpolation theorem and Lemma 2.2. □

Finally, to see that the log-Sobolev constants given in Corollary 1.4 can be sharp, let $T_t x = \mathrm{e}^{-t/2}x$. We have $h_1(t) = \varepsilon_t = \mathrm{e}^{-t}$. Then each of (1) and (2) in Corollary 1.4 implies (3.2) for $C = 2$. This constant is sharp as is well known for the standard Ornstein-Uhlenbeck process.

We remark that one may also prove the log-Sobolev inequality by using a curvature condition. Let A be the generator of T_t, then P_t is generated by

$$Lf(x) := \frac{1}{2}\mathrm{Tr}(QD^2 f(x)) + \langle Df(x), Ax\rangle.$$

One defines the curvature operator by

$$\Gamma_2(f,f) := \frac{1}{2}L\Gamma(f,f) - \Gamma(f, Lf),$$

where $\Gamma(f,g) := \langle RDf, Df\rangle$. We say that the curvature of L has lower bound $K \in \mathbb{R}$, if

$$\Gamma_2(f,f) \geq K\Gamma(f,f) \tag{3.8}$$

for all $f \in \mathcal{A}$, where $\mathcal{A}$ is a core of L, stable by L and P_t and by the action of composition with C^∞ real functions which are zero at zero. According to [6], (3.8) is equivalent to (2.1) with $h_1(t)$ replaced by e^{-2Kt}, therefore we prove (3.2) for

$$C = 2\int_0^\infty \mathrm{e}^{-2Kt}\mathrm{d}t = \frac{1}{K \vee 0}. \tag{3.9}$$

Intuitively, if $h_1(t)$ is not an exponential function, then Corollary 1.4 could provide a sharper constant than (3.9). To see this, we present a simple example below.

Example 3.4. *Consider* $E = \mathbb{R}^2, R = I, T_t x = (\mathrm{e}^{-t}x_1 + ct\mathrm{e}^{-t}x_2, \mathrm{e}^{-t}x_2), c \geq 0$. *Then Corollary 1.4 (1) implies* (3.2) *for* $C = 1 + \frac{c}{2}$, *while* (3.9) *gives* $C = \frac{2}{(2-c)^+}$ *which is larger than* $1 + \frac{c}{2}$ *for* $c > 0$.

Proof. We note that T_t is a semigroup since

$$T_t T_s x = (\mathrm{e}^{-t}(\mathrm{e}^{-s}x_1 + cs\mathrm{e}^{-s}x_2) + ct\mathrm{e}^{-t}\mathrm{e}^{-s}x_2, \mathrm{e}^{-(s+t)}x_2) = T_{s+t}x.$$

Obviously,

$$\|T_t x\|^2 = \mathrm{e}^{-2t}[(x_1 + ctx_2)^2 + x_2^2] \le \mathrm{e}^{-2t}(1+ct)\|x\|^2.$$

Then Corollary 1.4 (1) implies (3.2) for

$$C = 2\int_0^\infty \mathrm{e}^{-2t}(1+ct)\mathrm{d}t = 1 + \frac{c}{2}.$$

On the other hand, we have

$$L = \frac{1}{2}\Delta - (x_1 - cx_2)\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} := \frac{1}{2}\Delta + X.$$

By the Bochner formula, (3.8) is equivalent to

$$-\langle \nabla_Y X, Y\rangle \ge K\|Y\|^2, \quad Y := y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2}, \; y \in \mathbb{R}^2.$$

Noting that $-\langle \nabla_Y X, Y\rangle = (y_1 - cy_2)y_1 + y_2^2$, we see that the best choice of K is $1 - \frac{c}{2}$. So, the proof is completed. □

4 A generalized Mehler semigroup associated with the Dirichlet heat semigroup

We consider the following model discussed in [20] stimulated by the study of stochastic heat equations. Consider $E := L^2((0,1);\mathrm{d}x)$ and let T_t be the Dirichlet heat semigroup of Δ on $(0,1)$. Let

$$\lambda(\xi) := \|\xi\|^2 + \|\xi\|^\alpha, \quad \xi \in E,$$

where $\alpha \in (0,2)$ is fixed. Let μ_t be defined through its Fourier transform

$$\hat{\mu}_t(\xi) := \exp\left[-\int_0^t (\|T_s\xi\|^2 + \|T_s\xi\|^\alpha)\mathrm{d}s\right]. \tag{4.1}$$

It was shown in [20] that $\hat{\mu}_t$ is Sazonov continuous for all $t > 0$, so μ_t indeed exists as a probability measure on $(E, \mathcal{B}(E))$. Furthermore, it was shown in [20] that the generalized Mehler semigroup P_t determined by λ and T_t has an invariant probability measure $\mu := \mu_\infty$. Let $\{\xi_n\}$ be the set of unit eigenfunctions of the Dirichlet Laplacian on $(0,1)$, i.e. we have $\Delta\xi_n = -n\pi^2\xi_n, \; n \ge 1$.

Observing that the above λ is indeed not Sazonov continuous, we use finite-dimensional approximations to apply our results. For $n \ge 1$, let $E_n := \mathrm{spn}\{\xi_1, \cdots, \xi_n\}$. Let $\pi_n : E \to E_n$ be the natural projection.

Theorem 4.1. *Let P_t be determined by λ and T_t above. Then*

$$[P_t f(x)]^2 \le [P_t f^2(y)] \exp\left[\frac{\pi^2\|x-y\|^2}{e^{\pi^2 t}-1}\right], \quad f \in C_b(E), t>0, x,y\in E. \tag{4.2}$$

Next, for any $t>0$ and any $1<p<q<\infty$, one has $\|P_t\|_{p\to q}=\infty$.

Proof. Let μ_t^n denote the projection of μ_t on E_n. Since $\{\xi_n\}_{n\ge1}$ are eigenvectors of T_t, one has $T_tE_n=E_n$ for any $t\ge0, n\ge1$. Let $f\in C_b(E)$ with $f=f\circ\pi_n$, one has, for all $x\in E$

$$P_tf(x)=\int_E f(T_tx+y)\mu_t(dy)=\int_{E_n} f(T_t\pi_nx+y)\mu_t^n(dy)=: P_t^{(n)}f(\pi_nx).$$

Since $\|T_t\|\le e^{-\pi^2 t}, t\ge0$, it follows from Theorem 1.1 (with E, P_t replaced by $E_n, P_t^{(n)}$ respectively) that

$$\begin{aligned}[P_tf(x)]^2 = (P_t^{(n)}f(\pi_nx))^2 &\le [P_t^{(n)}f^2(\pi_ny)]\exp\left[\frac{\pi^2\|\pi_n(x-y)\|^2}{e^{\pi^2 t}-1}\right]\\ &\le [P_tf^2(y)]\exp\left[\frac{\pi^2\|x-y\|^2}{e^{\pi^2 t}-1}\right], \quad x,y\in E, t>0.\end{aligned}$$

Thus, (4.2) holds. For general $f\in C_b(E)$, first applying (4.2) to $f_n:=f\circ\pi_n$ then letting $n\to\infty$, we prove (4.2).

To see that P_t is not hyperbounded, let $t>0$ and $1<p<q<\infty$ be fixed. Since for f with $f=f\circ\pi_1$ one has $P_tf=(P_t^{(1)}f)\circ\pi_1$, we have $\|P_t\|_{p\to q}\ge\|P_t^{(1)}\|_{L^p(\mu^{(1)})\to L^q(\mu^{(1)})}$, where $\mu^{(1)}$ stands for the projection of μ on E_1. But according to Example 3.2 (2) with $n=1$, we have $\|P_t^{(1)}\|_{L^p(\mu^{(1)})\to L^q(\mu^{(1)})}=\infty$. □

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