Symmetrizing Measures for Infinite Dimensional Diffusions: An Analytic Approach

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Abstract. The existence problem, characterizations and uniform a priori estimates for symmetrizing measures of lattice stochastic dynamics are discussed. A constructive criterion for the uniqueness of such measures is proven.

1 Introduction

The study of stochastic differential equations (SDE) for processes taking values in infinite dimensional spaces is not only an interesting generalization of the classical theory of SDE, but also has been strongly motivated by applications in Physics, Biology, Economics and Engineering. The theory of such stochastic processes (considered also as infinite dimensional diffusions) presents many new specific aspects as compared with the finite dimensional case. We refer the reader to the paper [5] for a review which contains more than 150 references to works discussing such stochastic processes. The connections between finite dimensional diffusions and certain aspects of the theory of partial differential equations (PDE) provide, in particular, very powerful analytic techniques in the study of the properties of these processes. However, most of the methods of the finite dimensional theory of PDE do not admit a direct extension to the infinite dimensional case. In this connection, it is very important to develop analytic techniques which are applicable also in the infinite dimensional situation. In this work we describe one of such methods. More precisely, we shall study the problem of existence and uniqueness of symmetrizing measures for infinite dimensional diffusions. In fact, our approach will be entirely analytic. The underlying SDE is only motivation and the reader is not expected to know anything about stochastic equations. The reason is that the measures in question are exactly the symmetrizing measures for the generator of the diffusion which is a second order partial differential operator in infinitely many variables.

The class of processes we consider corresponds to infinite systems of coupled SDE's. To explain the ideas, we deal with the simplest case of linear interactions between components in the drift term and constant diffusion coefficients. The gradient structure of the drift term is also essential for us. Then the symmetrizing measures we are looking for can be characterized by their Radon-Nikodym derivatives w.r.t. local shifts [2], but in applications an infinitesimal form of this characterization via an integration by parts formula is more useful. In Chapter 2 we discuss this characterization (see Proposition 3 below) which describes the symmetrizing measures as the solutions to a first order, but infinite system of PDE's. This gives rise to apply ideas from PDE's to construct these solutions via a priori estimates and tightness criteria, particularly suitable for measures. E.g., one can prove a priori estimates on the moments of symmetrizing measures which we discuss briefly in Section 2(cf. Theorem 5). There exists an alternative constructive description of symmetrizing measures as Gibbs measures. This concept comes from statistical physics (see, e.g., [12]). Together with the mentioned a priori bounds it gives a possibility to show the existence of symmetrizing measures. In [4] this approach is realized in very general cases.

Another practical consequence of the Gibbs characterization is related to the uniqueness problem for symmetrizing measures. This problem is discussed in Section 3. We show there how the uniqueness conditions can be reduced to spectral gap estimates for Dirichlet operators associated with one-point conditional Gibbs measures. As a consequence we prove an easy to check criterion for the uniqueness of symmetrizing measures expressed in terms of parameters of the considered system.

We emphasize that, while Chapter 2 is is a review of our previous results on existence in [4] (formulated for pedagogical reasons in a concrete special case), the results about uniqueness of symmetrizing measures presented in Chapter 3 are new for the class of models considered in this paper.

Finally we would like to point out some important analytic consequences of our results for the study of partial differential operators in infinitely many variables. As mentioned above the generator L of a diffusion of the type studied in this paper is a second order elliptic partial differential operator which can be written as the sum of an infinite dimensional Laplacian and a vector field (called "drift") which is an infinite dimensional (generalized) gradient (cf. (9) below for the precise definition). Furthermore, L with the smooth cylinder functions as its domain is symmetric on $L^2(S', \mu)$, where S' is the infinite dimensional state space of the diffusion process and μ its symmetrizing measure (discussed above). In the analysis of L, μ now takes the role of the Lebesgue measure on $\mathbb{R}^d$ and one can consider corresponding Sobolev spaces $W^{r,p}(S', \mu)$. Though in contrast to the finite dimensional case functions belonging to all such Sobolev spaces are not smooth or not even continuous, in a number of cases regularity of solutions to the generalized heat equation corresponding to L or regularity properties of its resolvent can still be proved on the bases of suitable finite dimensional approximations and making full use of classical regularity theory on $\mathbb{R}^d$. We refer to [4] and for more recent work on this question to [9] and [14]. The latter treat the case where the above mentioned measure does not even have to be symmetrizing for L, but just has to be L-coharmonic, i.e. has to solve the equation $L^*\mu = 0$.

2 Symmetrizing measures for stochastic gradient systems

We shall consider anharmonic systems on the integer lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$, with the Euclidean distance $|k-j|$, $k, j \in \mathbb{Z}^d \subset \mathbb{R}^d$. With any lattice point $k \in \mathbb{Z}^d$ there is associated a particle (oscillator) with the single spin space $\mathbb{R} \ni x_k$. In physical terminology the variable x_k describes the displacement of the particle from the equilibrium position at the point k and only for simplicity of notations it is assumed to be one-dimensional (which corresponds to having polarized oscillations). In fact, all our considerations below can be easily modified to the case of vector spins $x_k \in \mathbb{R}^l$, $l \leq d$. The configuration space of our system $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ contains all real sequences $x = (x_k)_{k\in\mathbb{Z}^d}$ and is equipped with the product topology and with the corresponding Borel σ-algebra $\mathcal{B}(\Omega)$. The subset of finite sequences will be denoted by $\Omega_0 := \mathbb{R}_0^{\mathbb{Z}^d}$. In the following we will need also some topological subspaces of the configuration space Ω. Namely, for any $p \in \mathbb{Z}$ we define the Hilbert space

$$S_p := \left\{ x \in \Omega : \ |x|_p^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|)^{2p} x_k^2 < \infty \right\}$$

and the mutually dual spaces

$$S := S(\mathbb{Z}^d) = \bigcap_{p\geq 1} S_p(\mathbb{Z}^d), \quad S' := S'(\mathbb{Z}^d) = \bigcup_{p\geq 1} S_{-p}(\mathbb{Z}^d)$$

of fastly decreasing resp. slowly increasing sequences over $\mathbb{Z}^d$. This gives a nuclear triplet

$$S(\mathbb{Z}^d) \subset l^2(\mathbb{Z}^d) \subset S'(\mathbb{Z}^d)$$

for the tangent Hilbert space $S_0 = l^2(\mathbb{Z}^d)$ with the scalar product $\langle x, x\rangle := (x,x)_0$, $x \in l^2(\mathbb{Z}^d)$. The duality between S and S' can be expressed in terms of the following scalar product:

$$\langle \varphi, x\rangle = \langle x, \varphi\rangle := \sum_{k\in\mathbb{Z}^d} \varphi_k x_k, \quad \varphi \in S, \ x \in S'. \tag{1}$$

We define $\mathcal{F}C_b^l\left(\mathbb{R}^{\mathbb{Z}^d}\right)$, $l \in \mathbb{N}\cup\{0,\infty\}$, as the set of all cylinder functions $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ which can be represented as

$$f(x) = f_N(x_{k_1}, \ldots, x_{k_N}), \quad x \in \mathbb{R}^{\mathbb{Z}^d}, \tag{2}$$

for some $N \in \mathbb{N}$, $\{k_1, ..., k_N\} \subset \mathbb{Z}^d$ and $f_N \in C_b^l(\mathbb{R}^N)$. Correspondingly, $\mathcal{F}C^l\left(\mathbb{R}^{\mathbb{Z}^d}\right)$ shall denote the space of cylinder functions with $f_N \in C^l(\mathbb{R}^N)$. For $f \in \mathcal{F}C_b^\infty\left(\mathbb{R}^{\mathbb{Z}^d}\right)$ we set $\nabla_k f := \frac{\partial f}{\partial x_k} \in \mathcal{F}C_b^\infty\left(\mathbb{R}^{\mathbb{Z}^d}\right)$. Then the gradient of f in a point $x \in \mathbb{R}^{\mathbb{Z}^d}$is defined as the vector $\nabla f(x) := (\nabla_k f(x))_{k \in \mathbb{Z}^d} \in \mathbb{R}_0^{\mathbb{Z}^d}$.

For $p \in \mathbb{N}$, $C_b^1(S_{-p})$ will denote the set of all bounded Fréchet differentiable functions $f : S_{-p} \to \mathbb{R}$ with bounded continuous derivatives $f' : S_{-p} \to S_{+p}$ given by

$$\langle f'(x), h \rangle := \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}, \qquad x, h \in S_{-p}.$$

As usual, $C_0^1(S_{-p}) \subset C_b^1(S_{-p})$ denotes the subset of functions with bounded support in S_{-p}.

Let us consider a system of anharmonic oscillators which can be described by means of the following heuristic potential energy functional

$$E(x) := \frac{J}{2} \sum_{\langle k,,j \rangle} (x_k - x_j)^2 + \sum_{k \in \mathbb{Z}^d} V(x_k). \tag{3}$$

Here $\sum_{\langle k,,j \rangle}$ is taken over all nearest-neighbor pairs $\langle k, j \rangle$ in $\mathbb{Z}^d$ and gives the potential energy of a quadratic interaction with the constant intensity $J > 0$. The self-interaction potential $V \in C^1(\mathbb{R})$ is assumed to satisfy the polynomial growth condition

$$|V(q)| \leq C(1+|q|)^L, \quad |V'(q)| \leq C(1+|q|)^{L-1}, \tag{4}$$

and the coercivity estimate

$$V'(q) \cdot q \geq A|q|^{2+\sigma} - B \tag{5}$$

(with some constants $A, B, C, \sigma > 0, L \geq 1$). *In all considerations below we assume that these conditions are satisfied without any additional references to them in the formulations of our statements.*

Note that the energy functional is well defined as a function on $\mathbb{R}_0^{\mathbb{Z}^d}$ but it has no meaning as a function on $S'(\mathbb{Z}^d)$. On the other hand, for any $k \in \mathbb{Z}^d$ we have heuristically

$$\nabla_k E(x) = J \sum_{j:|k-j|=1} (x_k - x_j) + V'(x_k), \nabla_k E \in \mathcal{F}C\left(\mathbb{R}^{\mathbb{Z}^d}\right) \tag{6}$$

with its right hand side being well-defined. Taking this as a definition of $\nabla_k E$, we conclude from (4) for the vector field $\nabla E := (\nabla_k E)_{k \in \mathbb{Z}^d}$ that

$$\nabla E \in C(S' \to S').$$

Let us consider the following infinite system of stochastic differential equations:

$$d\xi_k(t) = -\nabla_k E(\xi(t))\,dt + \sqrt{\frac{2}{\beta}}dw_k(t),$$
$$\xi_k(0) = x_k, \quad k \in \mathbb{Z}^d, t \geq 0. \tag{7}$$

Here $\xi(t) = (\xi_k(t))_{k\in\mathbb{Z}^d}$, $w(t) = (w_k(t))_{k\in\mathbb{Z}^d}$ is a family of independent standard one-dimensional Wiener processes, $\beta > 0$ denotes the inverse (absolute) temperature. This system of equations can be written in vector form as

$$d\xi(t) = -\nabla E(\xi(t))\,dt + \sqrt{\frac{2}{\beta}}dw(t), \xi(0) = x \in \Omega, \tag{8}$$

and describes an infinite dimensional dynamical system with gradient type vector field perturbed by Gaussian white noise with intensity depending on the temperature (so-called thermal noise).This system is called a stochastic gradient system, see e.g. [11], or a lattice Brownian motion with drift $b := -\nabla E$. The problem of existence and uniqueness of solutions for the system (7) has been intensively studied, for references and historical comments we refer the reader to [5]. In particular, for any $p \in \mathbb{N}$ and any given initial condition $x \in S_{-p}(\mathbb{Z}^d)$ there exists $p' \in \mathbb{N}$ s.t. (7) has a unique strong solution $\xi(t) = \xi^x(t) \in S_{-p'}(\mathbb{Z}^d), t \geq 0$, which is called the stochastic dynamics associated with our classical anharmonic lattice system.

A direct applications of the Ito's formula to the considered stochastic dynamics shows that the generator of the diffusion process $\xi(t)$ is the following infinite dimensional partial differential operator:

$$L_\beta f := \frac{1}{\beta}\sum_{k\in\mathbb{Z}^d} \varDelta_k f + \sum_{k\in\mathbb{Z}^d} b_k \nabla_k f, \quad f \in \mathcal{F}C_b^\infty\left(\mathbb{R}^{\mathbb{Z}^d}\right), \tag{9}$$

where $\varDelta_k f := \frac{\partial^2 f}{\partial x_k^2}$ and $b_k := -\nabla_k E$. We are interested in the study of symmetrizing measures μ for such stochastic dynamics, i.e., in measures for which the generator L_β become a symmetric operator in the corresponding $L^2(\mu)$-space. Considering such measures as initial distributions for the process we obtain time reversible infinite dimensional diffusions defined for all $t \in \mathbb{R}$. We first restrict the class of admissible measures.

Definition 1. *A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called tempered if*

$$\exists p = p(\mu) > d/2 : \qquad \mu\left(S_{-p}(\mathbb{Z}^d)\right) = 1.$$

We shall denote by $\mathcal{M}_t$ the set of all tempered probability measures on $(\Omega, \mathcal{B}(\Omega))$.

Definition 2. *Let $\mu \in \mathcal{M}_t$, $\mu\left(S_{-p}(\mathbb{Z}^d)\right) = 1$. The measure μ is called a symmetrizing measure for the generator L_β if*

$$\int_{\Omega} g \cdot L_\beta f \; d\mu = \int_{\Omega} f \cdot L_\beta g \; d\mu, \quad f, g \in C_0^2(S_{-p}). \tag{10}$$

Note that for any $f \in C_0^2(S_{-p})$ we have $L_\beta f \in C_0(S_{-p})$ (and, therefore, the integrals in (10) are well defined).

Now we can formulate a characterization result for symmetrizing measures. For the proof (even in the case where much more general interaction potentials in classical lattice systems are considered) we refer to [4].

Proposition 3. *Denote by $\mathcal{M}_t^b$ the set of all tempered probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that the following integration by parts (IbP) formula*

$$\int_{\Omega} \partial_k f(x)\, d\mu(x) = -\beta \int_{\Omega} f(x) b_k(x)\, d\mu(x) \tag{11}$$

holds for all functions $f \in C_0^1(S_{-p})$ and all $k \in \mathbb{Z}^d$. Then $\mathcal{M}_t^b$ coincides with the set of all symmetrizing measures for the generator L_β.

Remark 4. 1. The (IbP)-formula (11) shows that the vector field $\beta b(\cdot)$ plays the role of the vector logarithmic derivative of the measure μ.

2. The relations (11) can be considered as an infinite system of first order PDE for the unknown measures μ.

3. Different versions of Proposition 1 are known essentially from the beginning of the study of gradient dynamics in several particular cases. But in all of them, before the work [4], very restrictive a priori assumptions ensuring the integrability of the components b_k of the logarithmic derivative were imposed on the solutions μ, see [4] for details and references.

The (IbP)-characterization of symmetrizing measures gives a powerful tool for the analysis of such measures. Namely, one can show their existence by solving the first order system of PDE's given by (11). The first step is to prove a priori estimates, e.g. for the moments of the measure μ, as formulated in the following theorem proved in [4]. In fact, the proof of this theorem is based on a direct application of (11) to a proper class of functions $f \in C_0^1(S_{-p})$.

Theorem 5 (a priori estimates). *For any $M \geq 1$ there exists $C_M > 0$ s.t. for all $k \in \mathbb{Z}^d$ and all $\mu \in \mathcal{M}_t^b$*

$$\int |x_k|^M d\mu(x) \leq C_M. \tag{12}$$

As a consequence of the a priori estimates and (4) we have the following: every $\mu \in \mathcal{M}_t^b$ is supported by $\cap_{p>d/2} S_{-p}(\mathbb{Z}^d)$ and for any $M \geq 1$

$$\sup_{\mu \in \mathcal{M}_t^b} \sup_{k \in \mathbb{Z}^d} \int |b_k(x)|^M d\mu(x) < \infty. \tag{13}$$

The latter a priori integrability bound permits to extend the (IbP)-formula (11) to all $f \in \mathcal{F}C_b^{\infty}\left(\mathbb{R}^{\mathbb{Z}^d}\right)$ and to prove the symmetry of the generator L_β on the domain $\mathcal{F}C_b^{\infty}\left(\mathbb{R}^{\mathbb{Z}^d}\right)$ in $L^2(\mu)$ for any symmetrizing measure μ. Moreover, L_β is related with the classical Dirichlet form (see, e.g., [6]) corresponding to the measure μ from the (IbP)-formula it follows that

$$-\left(f, L_\beta g\right)_{L^2(\mu)} = \frac{1}{\beta}\int \langle \nabla f, \nabla g\rangle \, d\mu, \quad f, g \in \mathcal{F}C_b^{\infty}\left(\mathbb{R}^{\mathbb{Z}^d}\right). \tag{14}$$

As a consequence, we have a certain a priori information about symmetrizing measures for the stochastic dynamics we have considered, however, the question of the existence of such measures remains open. To solve this problem, we need a more constructive description of these measures.

For any finite subset $\Lambda \subset \mathbb{Z}^d$ (finite volume) we define a family of local specifications via stochastic kernels $\mu_\Lambda : \mathcal{B}(\Omega) \times \Omega \to [0,1]$ by the following formula, $\forall \Delta \in \mathcal{B}(\Omega) \quad \forall y \in \Omega$,

$$\mu_\Lambda(\Delta|y) := \frac{1}{Z_\Lambda(y)} \int\limits_{\mathbb{R}^\Lambda} \exp\left\{-\beta E_\Lambda(x_\Lambda \times y_{\Lambda^c})\right\} \mathbf{1}_\Delta(x_\Lambda \times y_{\Lambda^c}) \times_{k\in\Lambda} dx_k, \tag{15}$$

Here $x_\Lambda := (x_k)_{k\in\Lambda}$, $y_{\Lambda^c} := (y_j)_{j\in\Lambda^c}$, $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$,

$$E_\Lambda(x_\Lambda \times y_{\Lambda^c}) := \frac{J}{2}\sum_{\{k,j\}\subset\Lambda}(x_k - x_j)^2 + \frac{J}{2}\sum_{k\in\Lambda, j\in\Lambda^c}(x_k - y_j)^2 + \sum_{k\in\Lambda} V_k(x_k) \tag{16}$$

is the energy in the volume Λ corresponding to the boundary condition y, and

$$Z_\Lambda(y) := \int\limits_{\mathbb{R}^\Lambda} \exp\left\{-\beta E_\Lambda(x_\Lambda \times y_{\Lambda^c})\right\} \times_{k\in\Lambda} dx_k \tag{17}$$

is the normalization factor (which is called partition function in the terminology of Statistical Physics). Because of our assumptions on the potentials, (17) is indeed well defined. Moreover, the family of stochastic kernels $\mu_\Lambda(\cdot|y)$ satisfies the consistency condition (cf. [12]):

$$\mu_{\Lambda'}\ \mu_\Lambda = \mu_{\Lambda'}, \quad \Lambda \subset \Lambda'. \tag{18}$$

Definition 6. *A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called a Gibbs measure for the local specification $\{\mu_\Lambda,\ \Lambda \subset \mathbb{Z}^d\}$ iff it satisfies the equation*

$$\forall \Lambda \subset \mathbb{Z}^d, \forall \Delta \in \mathcal{B}(\Omega) \quad \int\limits_\Omega \mu(dx)\mu_\Lambda(\Delta|x) = \mu(\Delta). \tag{19}$$

Equation (18) is called the Dobrushin-Lanford-Ruelle (DLR) equilibrium equation.

Let $\mathcal{G} = \mathcal{G}(J, V, \beta)$ denote the set of all Gibbs measures for our system. As before, we restrict our considerations to the set $\mathcal{G}_t = \mathcal{M}_t \cap \mathcal{G}$ of tempered Gibbs measures. The following proposition is a particular version of results from [1, 2], see also [4] for more general statements and historical comments.

Proposition 7. *We have*

$$\mathcal{M}_t^b = \mathcal{G}_t.$$

Proposition 7 asserts that the symmetrizing measures for the gradient stochastic dynamics can be described constructively as Gibbs measures. Combining this fact with the a priori bounds from Theorem 5 (which can be established in the same way for any measure $\mu_\Lambda(\cdot|x)$ of the specification) we can prove the following existence result, see [4].

Theorem 6. *There exists at least one symmetrizing measure, i.e.,*

$$\mathcal{M}_t^b = \mathcal{G}_t \neq \emptyset.$$

Previous proofs of existence of Gibbs measures for the case of unbounded spin systems have been based on a quite delicate analysis of the specifications for the given particular type of interactions, see, e.g. [7], [8]. The (IbP)-characterization leads to a (comparing with other methods) relatively easily proof for existence of Gibbs measures for a large class of interaction potentials, including the case of multiparticle interactions, see [4].

3 Uniqueness problem

One of the specifically infinite dimensional effects in the considered stochastic dynamics is the possibility of having several symmetrizing measures. Due to Proposition 7 this phenomenon corresponds to what is called phase transition in Statistical Physics. In particular, if the lattice dimension $d \geq 3$ and the one-particle potential V has a symmetric two-wells shape, then for big enough β (low temperature regime) or J (strong coupling regime) our system has phase transitions, i.e., $|\mathcal{G}_t| > 1$. The latter follows by an application of the reflection positivity method and related infrared bounds to the periodic Gibbs state of the discussed model, see, e.g., [15] .

On the other hand, if the potential V is uniformly convex, then for all $J, \beta > 0$ we have $|\mathcal{G}_t| = 1$, see, e.g., [5]. Note that even for $V \equiv 0$ in the case where $d \geq 3$ one has $|\mathcal{G}_t| = \infty$ for all $J, \beta > 0$. Therefore, we should expect that the uniqueness conditions can be expressed in terms of the deviation of the one-particle potential from a convex one. To realize this observation, assume that the potential V has the form

$$V(q) = V_0(q) + W(q), \tag{20}$$

where V_0 is a uniformly convex function:

$$V_0 \in C^2(\mathbb{R}), \quad V''(q) \geq a^2 > 0, q \in \mathbb{R}, \tag{21}$$

which is together with its derivatives up to the second order polynomially bounded. The perturbation W is given by a bounded function

$$W \in C_b(\mathbb{R}). \tag{22}$$

We define the deviation $\delta(W)$ of the function W by

$$\delta(W) = \sup_{\mathbb{R}} W - \inf_{\mathbb{R}} W. \tag{23}$$

Theorem 9. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{e^{\beta\delta(W)}}{2d + J^{-1}a^2} < \frac{1}{2d}. \tag{24}$$

Then there exists exactly one symmetrizing measure, i.e.,

$$|\mathcal{M}_t^b| = |\mathcal{G}_t| = 1.$$

Proof. For the proof of this fact, we shall use a modification of an approach developed in [3] for the derivation of a corresponding result for quantum lattice systems. The existence of a symmetrizing measure was already established in Section 1. To show uniqueness we will a well-known general uniqueness criterion by R.L. Dobrushin, see, e.g.,[10], [12] for a detailed discussion of this approach and related notions which we use below without giving additional references.

Let us consider the one-dimensional Gibbs distributions which correspond to the specification (15). Namely, for any $k \in \mathbb{Z}^d$ and any boundary condition $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ we define a probability measure on $\mathcal{B}(\mathbb{R})$ as follows:

$$\nu_k(B|\xi) = \mu_{\{k\}}\left(\left\{x \in \mathbb{R}^{\mathbb{Z}^d} : x_k \in B\right\}|\xi\right), \quad B \in \mathcal{B}(\mathbb{R}).$$

Using (15) and (16) we have

$$\nu_k(dx_k|\xi) = \frac{1}{Z_k(\xi)} \exp\left\{-dJ\beta x_k^2 - \beta V(x_k) + J\beta x_k \sum_{|j-k|=1} \xi_j\right\} dx_k, \tag{25}$$

where as before $Z_k(\xi)$ is a normalizing factor. The Dobrushin approach is based on a comparision of these measures for different boundary conditions in the Kantorovich(–Ornstein–Rubinstein–Vasserstein) metric which is defined for two probability measures μ_1, μ_2 on $\mathcal{B}(\mathbb{R})$ by

$$R(\mu_1, \mu_2) := \sup_{f \in \mathrm{Lip}_1(\mathbb{R})} \left|\int f d\mu_1 - \int f d\mu_2\right|,$$

where

$$\mathrm{Lip}_1(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \mid \ [f]_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|} \leq 1 \right\}. \tag{26}$$

The Dobrushin matrix $C = (C_{kj})_{k,j \in \mathbb{Z}^d}$ is defined by the relations

$$C_{kj} = \sup_{\substack{\xi,\eta \\ \forall i \neq j : \xi_i = \eta_i}} \left\{ \frac{R(\nu_k(\cdot|\xi), \nu_k(\cdot|\eta))}{|\xi_j - \eta_j|} \right\}. \tag{27}$$

In the considered case of translation invariant nearest-neighbors interactions on the lattice $\mathbb{Z}^d$, (25) implies $C_{kj} = 0$ if $|k-j| > 1$. Due to the Dobrushin criterion [12] the uniqueness of tempered Gibbs measures follows from the inequality

$$\sup_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d, j \neq k} C_{kj} < 1 \tag{28}$$

and an obvious sufficient condition is then

$$C_{kj} < \frac{1}{2d}. \tag{29}$$

Note that due to (25) the measure $\nu_k(dx_k|\xi)$ depends on the boundary condition ξ only through the value $s_k = \sum_{|j-k|=1} \xi_j$.

Let us introduce a family of probability measures on $\mathcal{B}(\mathbb{R})$ parametrized by $s \in \mathbb{R}$ and given by

$$\nu^s(dx) := \frac{1}{Z_s} \exp\left\{ -J\beta d x^2 - \beta V(x) + J\beta x s \right\} dx. \tag{30}$$

Then for all $k, j \in \mathbb{Z}^d$, $|k-j| = 1$ one has

$$C_{kj} = \sup_{\substack{s.t \\ s \neq t}} \left\{ \frac{R(\nu^s, \nu^t)}{|s-t|} \right\}. \tag{31}$$

Consider the mapping

$$\mathbb{R} \ni s \longmapsto \langle f \rangle_s := \int f(x)\, d\nu^s(x) \in \mathbb{R}$$

with fixed $f \in \mathrm{Lip}_1(\mathbb{R})$. Then

$$\frac{d}{ds} \langle f \rangle_s = J\beta \left(\langle f(\cdot) \cdot \rangle_s - \langle f(\cdot) \rangle_s \langle \cdot \rangle_s \right) = J\beta Cov_{\nu^s}(f(\cdot), \cdot), \tag{32}$$

and, therefore,

$$\left| \frac{d}{ds} \langle f \rangle_s \right| \leq J\beta \left(\mathrm{Var}_{\nu^s} f(\cdot) \right)^{\frac{1}{2}} \left(\mathrm{Var}_{\nu^s} \cdot \right)^{\frac{1}{2}}, \tag{33}$$

where

$$\mathrm{Var}_{\nu^s} f = \int \left(f - \langle f \rangle_s\right)^2 d\nu^s.$$

As will be shown separately below, this variance satisfies the following bound:

$$\mathrm{Var}_{\nu^s} f \le C := \frac{e^{\beta\delta(W)}}{2J\beta d + a^2\beta}. \tag{34}$$

Hence

$$\left|\frac{d}{ds} \langle f \rangle_s \right| \le J\beta C \tag{35}$$

and the mean-value theorem gives

$$\left| \int f d\nu^s - \int f d\nu^t \right| \le J\beta C|s-t|, \quad f \in \mathrm{Lip}_1(\mathbb{R}), \tag{36}$$

or

$$R\left(\nu^s, \nu^t\right) \le J\beta C|s-t|. \tag{37}$$

Using (30) and the assumption (24) of the theorem we obtain the estimate (28) and conclude then that $|\mathcal{G}_t| = 1$.

So, only the proof of the estimate (34) remains to be done. Let us consider the classical Dirichlet form corresponding to the probability measure ν^s:

$$\mathcal{E}_{\nu^s}(u,u) := \int |\nabla u|^2 d\nu^s, \quad u \in C_b^\infty(\mathbb{R}) \ . \tag{38}$$

We rewrite the measure ν^s in the form

$$d\nu^s(x) = \frac{1}{N} e^{-\beta W(x)} d\nu_0^s(x), \tag{39}$$

where the probability measure ν_0^s is defined by formula (29) in which V is changed to V_0 :

$$\begin{aligned} \nu_0^s(dx) &:= \frac{1}{Z_s'} \exp\left\{-J\beta dx^2 - \beta V_0(x) + J\beta xs\right\} dx \\ &= \frac{1}{Z_s'} \exp\left\{-U_s(x)\right\} dx. \end{aligned} \tag{40}$$

Because $U_s''(x) \ge 2J\beta d + a^2\beta$ uniformly in $x, s \in \mathbb{R}$, we have the following spectral gap inequality, see, e.g., [13]:

$$\mathcal{E}_{\nu^s}(u,u) \ge \frac{2J\beta d + a^2\beta}{e^{\beta\delta(W)}} \|u\|^2_{L^2(\nu^s)} \tag{41}$$

for all $u \in C_b^\infty(\mathbb{R})$ such that $\int u d\nu^s = 0$. After an obvious approximation the spectral gap inequality can be written as

$$\int (u - \langle u \rangle_{\nu^s})^2 \, d\nu^s = \frac{1}{2} \int \int (u(x) - u(y))^2 \, d\nu^s(x) \, d\nu^s(y)$$
$$\le \frac{e^{\beta\delta(W)}}{2J\beta d + a^2\beta} \int |\nabla u|^2 d\nu^s \tag{42}$$

for all $u \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. Thus

$$\mathrm{Var}_{\nu^s} u \le \frac{e^{\beta\delta(W)}}{2J\beta d + a^2\beta} \sup |\nabla u|^2 = \frac{e^{\beta\delta(W)}}{2J\beta d + a^2\beta} [u]_{\mathrm{Lip}}^2 . \tag{43}$$

Now let us take any $f \in \mathrm{Lip}(\mathbb{R})$. Then there exists a sequence $(u_n)_{n \ge 1} \subset C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ s.t.

$$[u_n]_{\mathrm{Lip}} \le [f]_{\mathrm{Lip}}$$

and $u_n \to f, n \to \infty$, pointwise (to this end we can use a Friedrichs mollifier). Using such an approximation we obtain from (41) the desired inequality

$$\mathrm{Var}_{\nu^s} f = \int (f - \langle f \rangle_s)^2 \, d\nu^s \le \frac{e^{\beta\delta(W)}}{2J\beta d + a^2\beta} [f]_{\mathrm{Lip}}^2 . \tag{44}$$

The proof of the theorem is thus completed. ■

We remark that the uniqueness criterion in Theorem 9 has a simple physical interpretation. Namely, we always can get to the "uniqueness regime" using high temperature or small coupling conditions.

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