EUCLIDEAN GIBBS MEASURES ON LOOP LATTICES: EXISTENCE AND A PRIORI ESTIMATES

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Abstract. We present a new method to prove existence and uniform a priori estimates for Euclidean Gibbs measures corresponding to quantum anharmonic crystals. It is based first on the alternative characterization of Gibbs measures in terms of their logarithmic derivatives through integration by parts formulas, and second on the choice of appropriate Lyapunov functionals.

1. Introduction

This paper is concerned with models of quantum anharmonic lattice systems. In Statistical Physics they are commonly viewed as models for *quantum crystals* (see e.g. [Frö77, DLP79, GJ81, MVZ00, AKKR01b]). A mathematical description of equilibrium properties of quantum systems might be given in terms of their Gibbs states defined on proper algebras of observables (cf. [BrRo81]). However, in the realization of this general concept for the considered quantum models there occur principal difficulties (see e.g. the discussion in [AKKR02a]). In order to overcome these difficulties we shall take the *Euclidean* (or *path space*) *approach,* which is conceptually analogous to the well-known Euclidean strategy in quantum field theory (see e.g. [Si74, Frö77, GJ81]). This analogy was pointed out and first implemented to quantum lattice systems in [AH-K75]; for the recent developments see the review articles [AKRT01, AKKR02a] and the bibliography therein. More precisely, the Euclidean approach transforms the problem of constructing *quantum Gibbs states* G_β as functionals on the algebra of observables into the problem of studying certain *Euclidean Gibbs measures* μ on the *loop lattice* $\Omega_\beta := [C(S_\beta \to \mathbb{R}^d)]^{\mathbb{Z}^d}$ (cf. Subsection 2.1 below for details). Here $\beta := 1/T > 0$ is the inverse (absolute) temperature and $S_\beta \cong [0,\beta]$ is a circle of length β. As a consequence, various probabilistic techniques become available for investigating equilibrium properties of quantum infinite-particle systems. But, as compared with classical lattice systems, the situation with Euclidean Gibbs measures is much more involved, since now the spin (i.e., loop) spaces themselves are *infinite dimensional* and their topological features should be taken into account carefully. Also, as is typical for non-compact spin spaces, we have to restrict ourselves to the set $\mathcal{G}^t_\beta$ of *tempered* Gibbs measures μ, which we specify by some natural support condition (cf. (2.7), (2.15) below).

The aim of this paper is to establish *a new method for proving existence and a priori estimates* for tempered Euclidean Gibbs measures. We obtain improvements and generalizations of essentially all corresponding existence results known so far in the literature (see the discussion in Subsection 2.4). Moreover, this method seems to be quite universal for lattice models and gives structural insight. It has been first implemented for classical lattice systems in [AKRT99,00].

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But the concrete technique suggested in those papers does not apply to loop spaces, so that a proper (highly non-trivial) modification for the quantum case is necessary (see Section 4 below).

One basic idea of our method is to use an *alternative characterization of Gibbs measures via integration by parts* (instead of the usual one in terms of local specifications through the Dobrushin–Lanford–Ruelle equations). Let us note that such alternative descriptions of Gibbs measures via integration by parts formulas or via their Radon–Nikodym derivatives w.r.t. shift transformations of the underlying configuration spaces (*flow characterization*) have long been known for a number of specific models in statistical mechanics and field theory (see, e.g., [CR75, Ro75,76, RoY76, HS76, Frö77, Fri82, M-O85, Iw85, Fö88, Fu91, Ki95]). But for the quantum lattice systems under consideration, a complete characterization of the measures $\mu \in \mathcal{G}^t_\beta$ in terms of their Radon–Nikodym derivatives has been first proved in [AKR97b, Theorem 4.6] (cf. also its extension in Proposition 1 below). Assuming that the interaction potentials are differentiable, we further show that this flow characterization of Gibbs measures is equivalent to their characterization as differentiable measures satisfying integration by parts formulas with prescribed logarithmic derivatives (cf. Proposition 2 below).

Relying on this characterization of $\mu \in \mathcal{G}^t_\beta$, in a *direct analytical way* we then prove the two main results of the paper: the *existence* of tempered Euclidean Gibbs measures (Theorem 1) and *a priori estimates* on their moments in terms of parameters of the interaction (Theorem 2). The essential ingredient of the proofs is that by the characterization of Gibbs measures via integration by parts we can deal with them as solutions of an infinite system of first order PDE's (cf. (3.13) below). This enables us to employ the *Lyapunov function method* (in a similar way as in finite dimensional PDE's) in order to get a priori moment estimates on $\mu \in \mathcal{G}^t_\beta$. The local Gibbs specifications also satisfy the same integration by parts formulas, from which we deduce moment estimates *uniformly* in volume. The latter is crucial for our proof of existence for Euclidean Gibbs measures μ, i.e., that $\mathcal{G}^t_\beta \neq \emptyset$. In addition, from the a priori estimates we obtain precise information on support properties of all tempered Euclidean Gibbs measures.

Some results on the existence of Euclidean Gibbs measures, concerning special classes of anharmonic interactions and based on various other techniques, have been already known before (see Subsection 2.4 for the references and a more detailed discussion). But we emphasize that our tools are completely different and rather elementary, provided one has the integration by parts description of Gibbs measures. To demonstrate our method and present the main ideas, we analyze a class of concrete lattice models given by a system of *d-dimensional* quantum anharmonic oscillators interacting via potentials of *superquadratic* growth (cf. (2.2) below). Trying to keep the exposition more transparent, in the main body of the paper we restrict ourselves further to the case of translation invariant pair interactions between nearest neighbors only. However, our method can be easily extended to *general* (not necessarily translation invariant) *many-particle interactions* of unbounded order and infinite range, not covered at all by any previous work. This extension will be briefly described in Section 5. For a detailed exposition of the latter case we also refer to the forthcoming paper [AKRT02].

Finally, let us notice that the method proposed here can be also modified to apply to the more difficult and less studied case of zero absolute temperature, i.e., $\beta = \infty$, and corresponding Gibbs measures (so-called Euclidean ground states, cf. [MVZ00, MRZ00, LMS02]) on the "path lattice" $\Omega_\infty := C(\mathbb{R})^{\mathbb{Z}^d}$. This case is under present investigation.

The organization of this paper is as follows. Section 2 is devoted to general aspects of the theory of Euclidean Gibbs measures. Here we introduce the models of quantum lattice systems ("anharmonic crystals"). We recall details on the corresponding Gibbsian formalism for Euclidean Gibbs measures μ on the loop lattice Ω_β. The transform from quantum Gibbs states to Euclidean

Gibbs measures, however, is described in more detail in the Appendix. Then we formulate our main Theorems 1 resp. 2 on the existence resp. a priori estimates for tempered Euclidean Gibbs measures $\mu \in \mathcal{G}_\beta^t$ and compare them with previous results obtained by other methods. In Section 3 we discuss the above mentioned alternative description of $\mu \in \mathcal{G}_\beta$ in terms of their shift–Radon–Nikodym derivatives and (its infinitesimal form) in terms of their logarithmic derivatives via the integration by parts formulas. In Section 4 we give complete proofs of our main Theorems 1 and 2, which we divide into several sequential steps formulated as Lemmas 2 to 5. In Section 5 we outline some possible generalizations of our method.

Finally we mention that the results of this paper have been announced in [AKPR01a,b] and presented in various talks since December 2000 during seminars or conferences, e.g., in Berlin, Kiev, Moscow, Oberwolfach, and Pisa.

2. Euclidean Gibbs measures on loop spaces

We begin this section with the description of a model of interacting multidimensional quantum anharmonic oscillators on a lattice (so-called "quantum crystals"). For simplicity of the exposition, we concentrate on a specific case of *translation invariant* systems with pair interactions of *nearest neighbor* type and with *isotropic* self-interaction. Then we give a rigorous definition of the corresponding Euclidean Gibbs measures as classical Gibbs measures but with *infinite dimensional* single spin (i.e., *loop*) spaces. We close the section with the formulation of our main results on the existence and a priori estimates for tempered Gibbs measures on loop lattices.

2.1. **A model of quantum anharmonic crystals.** Consider the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, with distance $|\cdot| = (\cdot,\cdot)^{\frac{1}{2}}$ and basis $(e_\alpha)_{\alpha=1}^d$, and let $\mathbb{Z}^d$ be the integer lattice in $\mathbb{R}^d$. We study a translation invariant system of interacting quantum particles performing d-dimensional oscillations around their equilibrium positions at points $k \in \mathbb{Z}^d$. Each particle individually is described by the quantum mechanical Hamiltonian

$$H_k := -\frac{1}{2\mathfrak{m}}\Delta + \frac{a^2}{2}|q_k|^2 + V(q_k) \tag{2.1}$$

acting in the (physical) Hilbert state space $\mathcal{H}_k := L^2(\mathbb{R}^d, dq_k)$. Here Δ is the usual Laplacian on $\mathbb{R}^d$, $\mathfrak{m} > 0$ is the (reduced) particle mass, $a^2 > 0$ is their rigidity w.r.t. the harmonic oscillations, and $V : \mathbb{R}^d \to \mathbb{R}$ is an anharmonic self-interaction potential. Next, we add a nearest neighbor interaction between the particles, given by a symmetric potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and taken over all (unordered) pairs $\langle k, k'\rangle \subset \mathbb{Z}^d$ such that $|k - k'| = 1$. The whole system is then described by the heuristic Hamiltonian of the form

$$H := \sum_{k\in\mathbb{Z}^d} H_k + \sum_{\langle k,k'\rangle\subset\mathbb{Z}^d} W(q_k, q_{k'}). \tag{2.2}$$

The infinite-volume Hamiltonian (2.2) cannot be defined directly as a mathematical object and is represented by the local (i.e., indexed by finite volumes $\Lambda \subset \mathbb{Z}^d$) Hamiltonians

$$H_\Lambda := \sum_{k\in\Lambda} H_k + \sum_{\langle k,k'\rangle\subset\Lambda} W(q_k, q_{k'}) \tag{2.3}$$

acting in the Hilbert spaces $\mathcal{H}_\Lambda := \otimes_{k\in\Lambda}\mathcal{H}_k$.

Concerning the interaction potentials, we shall suppose that they are twice continuously differentiable, i.e.,

$$V \in C^2(\mathbb{R}^d \to \mathbb{R}),\ W \in C^2(\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}).$$

By $\nabla^{(l)}V(q)$, $\nabla_q^{(l)}W(q,q') \in \mathbb{R}^{d^l}$ we shall denote their l-th derivative ($l = 0,1,2$) w.r.t. the coordinate $q \in \mathbb{R}^d$. Moreover, we impose the following bounds on the *asymptotic behavior* of the interaction potentials at the infinity:

Assumption. There exist some constants $P > R \geq 2$ and $C_V, C_W > 0$ such that:

(V): $V(q) = v(|q|)$, $q \in \mathbb{R}^d$, and a function $v : \mathbb{R}_+ \to \mathbb{R}$ satisfies for all $s \geq s_0$

$$C_V s^{P-l}/2 \leq v^{(l)}(s) \leq C_V s^{P-l}, \quad l = 0,1,2.$$

(W): For all $q, q' \in \mathbb{R}^d$, $\min\{|q|,|q'|\} > s_0$, holds

$$|\nabla_q^{(l)} W(q,q')| \leq C_W \left(|q|^{R-l} + |q'|^{R-l}\right), \quad l = 0,1,2.$$

Example. Typical potentials satisfying Assumptions (**V**) and (**W**) are the polynomials

$$V(q) := \sum\nolimits_{n=0}^{p} b_{2n}|q|^{2n}, \quad W(q,q') := \left(S(q-q'), q-q'\right)^r, \tag{2.4}$$

where S is a symmetric $d \times d$ matrix, $p, r \in \mathbb{N}$, $p > r$, and $b_{2p} > 0$.

Notation. For a set $\Lambda \subset \mathbb{Z}^d$, by $|\Lambda|$ we denote its *cardinality*, by $\Lambda^c := \mathbb{Z}^d \backslash \Lambda$ its *complement*, and by $\partial\Lambda := \{k' \in \Lambda^c \mid \exists k \in \Lambda,\ |k-k'| = 1\}$ its *boundary*. In particular, $\partial k := \left\{k' \in \mathbb{Z}^d \mid |k-k'| = 1\right\}$ is the set of all neighbors of k consisting of $2d$ points. We write $\Lambda \Subset \mathbb{Z}^d$ whenever $1 \leq |\Lambda| < \infty$. As usual, $\Lambda \nearrow \mathbb{Z}^d$ means the limit as $N \to \infty$ along any increasing sequence of volumes $\Lambda^{(N)} \subset \Lambda^{(N+1)} \Subset \mathbb{Z}^d$ such that ${}_{N\in\mathbb{N}}\Lambda^{(N)} = \mathbb{Z}^d$.

As was already mentioned in the Introduction, we take the *Euclidean approach* to study the quantum lattice system (2.2). Such approach involves intricate relations between quantum statistical mechanics and stochastic processes, which for convenience of the non-expert reader we shall briefly discuss in the Appendix. In Subsections 2.2 and 2.3 below we proceed with the rigorous description of the corresponding Gibbsian Euclidean formalism.

2.2. **Loop spaces.** Let us fix some $\beta > 0$ having the meaning of inverse (absolute) temperature. Let $S_\beta \cong [0,\beta]$ be a circle of length β (considered as a compact Riemannian manifold with Lebesgue measure $d\tau$ as a volume element and distance $\rho(\tau,\tau') := \min(|\tau-\tau'|,\ \beta - |\tau-\tau'|)$, $\tau,\tau' \in S_\beta$). For each $k \in \mathbb{Z}^d$, denote by

$$\mathbb{L}_\beta^r := L^r(S_\beta \to \mathbb{R}^d, d\tau), \quad r \geq 1,$$
$$\mathbb{C}_\beta^{n+\eta} := C^{n+\eta}(S_\beta \to \mathbb{R}^d), \quad n \in \mathbb{N}\cup\{0\},\ \eta \in (0,1),$$

the Banach spaces of all integrable resp. (Hölder) continuous functions (i.e., loops) $\omega_k = (\omega_{k,\alpha})_{\alpha=1}^d : S_\beta \to \mathbb{R}^d$ with the norms

$$\begin{gathered} |\omega_k|_{\mathbb{L}_\beta^r} := \left[\int_{S_\beta} |\omega_k(\tau)|^r d\tau\right]^{1/r}, \\ |\omega_k|_{\mathbb{C}_\beta^{n+\eta}} := \sum\nolimits_{l=0}^{n} \sup_{\tau\in S_\beta} |\omega_k^{(l)}|_{\mathbb{C}_\beta} + \sup_{\tau,\tau'\in S_\beta,\ \tau\neq\tau'} \frac{|\omega_k^{(n)}(\tau) - \omega_k^{(n)}(\tau')|}{\rho^\eta(\tau,\tau')}. \end{gathered} \tag{2.5}$$

If $d = 1$, we simply write $L_\beta^r := \mathbb{L}_\beta^r$ and $C_\beta^{n+\eta} := \mathbb{C}_\beta^{n+\eta}$. In particular, $\mathbb{C}_\beta$ with the sup-norm $|\cdot|_{\mathbb{C}_\beta}$ will be treated as the *single spin space*, whereas $\mathbb{L}_\beta^2$ with the inner product $(\cdot,\cdot)_{\mathbb{L}_\beta^2} := |\cdot|_{\mathbb{L}_\beta^2}^2$ as the Hilbert space tangent to $\mathbb{C}_\beta$.

As the *configuration space* for the infinite volume system we define the space of all loop sequences over $\mathbb{Z}^d$

$$\Omega_\beta := [\mathbb{C}_\beta]^{\mathbb{Z}^d} = \left\{\omega = (\omega_k)_{k\in\mathbb{Z}^d} \,\middle|\, \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d},\ \omega_k \in \mathbb{C}_\beta\right\}. \tag{2.6}$$

We endow Ω_β with the *product topology* (i.e., the weakest topology on Ω_β such that all finite volume projections

$$\Omega_\beta \ni \omega \mapsto \mathbb{P}_\Lambda \omega := \omega_\Lambda := (\omega_k)_{k\in\Lambda} \in [\mathbb{C}_\beta]^\Lambda =: \Omega_{\beta,\Lambda}, \quad \Lambda \Subset \mathbb{Z}^d,$$

are continuous) and with the corresponding *Borel σ-algebra* $\mathcal{B}(\Omega_\beta)$ (which coincides with the σ-algebra generated by all cylinder sets

$$\{\omega \in \Omega_\beta \mid \omega_\Lambda \in \Delta_\Lambda\}, \quad \Delta_\Lambda \in \mathcal{B}(\Omega_{\beta,\Lambda}),\ \Lambda \Subset \mathbb{Z}^d\ ).$$

Let $\mathcal{M}(\Omega_\beta)$ denote the set of all probability measures on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$. Next, we define the subset of *(exponentially) tempered configurations*

$$\Omega_\beta^t := \left\{ \omega \in \Omega_\beta \;\middle|\; \forall \delta \in (0,1):\ ||\omega||_\delta := \left[\sum\nolimits_{k\in\mathbb{Z}^d} e^{-\delta|k|}|\omega_k|^R_{\mathbb{L}^R_\beta}\right]^{\frac{1}{R}} < \infty \right\}, \tag{2.7}$$

where the parameter $R \geq 2$ describes a possible order of polynomial growth of the pair potential $W(q,q')$ (cf. Assumption (**W**)). In the context below, $\Omega_\beta^t \in \mathcal{B}(\Omega_\beta)$ will be always viewed as a *locally convex Polish space* with the topology induced by the system of norms $\big(||\omega||_\delta, |\omega_k|_{\mathbb{C}_\beta}\big)_{\delta>0,\ k\in\mathbb{Z}^d}$. Correspondingly, we specify the subset of *tempered measures* as those supported by Ω_β^t, i.e.,

$$\mathcal{M}_\beta^t := \left\{\mu \in \mathcal{M}(\Omega_\beta) \;\middle|\; \mu\left(\Omega_\beta^t\right) = 1\right\}. \tag{2.8}$$

Remark 1. Our definition of temperedness (as well as its modification to the classical case with $|q_k|$ substituting $|\omega_k|_{\mathbb{L}^R_\beta}$) is *more extended* (and simpler) as those usually used in the literature (for comparison, see e.g. [COPP78, BH-K82]). So, Ω_β^t contains all (slowly increasing) configurations $\omega \in \Omega_\beta$ for which

$$\exists p = p(\omega) > 0:\ \sum\nolimits_{k\in\mathbb{Z}^d}(1+|k|)^{-2p}|\omega_k|_{\mathbb{L}^R_\beta} < \infty.$$

Moreover, $\mathcal{M}_\beta^t$ contains all measures $\mu \in \mathcal{M}(\Omega_\beta)$ satisfying the following condition in terms of their moment sequence:

$$\exists p = p(\mu) > 0:\ \sum\nolimits_{k\in\mathbb{Z}^d}(1+|k|)^{-2p}E_\mu|\omega_k|_{\mathbb{L}^R_\beta} < \infty,$$

in particular, those having the so-called Ruelle support (see Sect. 5 below). Here and further on we write $E_\mu f := \int f d\mu$ for any μ-integrable f.

2.3. **Definition of Euclidean Gibbs measures.** Heuristically, the Euclidean Gibbs measures μ we are interested in have the following representation

$$d\mu(\omega) := Z^{-1}\exp\{-\mathcal{E}(\omega)\}\prod_{k\in\mathbb{Z}^d} d\gamma(\omega_k). \tag{2.9}$$

Here Z is the normalization factor and the map

$$\Omega_\beta \ni \omega \longmapsto \mathcal{E}(\omega) := \int_{S_\beta}\left[\sum_{k\in\mathbb{Z}^d} V(\omega_k) + \sum_{<k,k'>\subset\mathbb{Z}^d} W(\omega_k,\omega_{k'})\right] d\tau$$

might be viewed as a potential energy functional describing an *interacting system of loops* $\omega_k \in \mathbb{C}_\beta$ indexed by $k \in \mathbb{Z}^d$, whereas every single spin space $\mathbb{C}_\beta$ is equipped with the Gaussian measure γ canonically generated by the oscillator bridge process of length β. A rigorous meaning to μ as probability measures on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ can be given by the *Dobrushin–Lanford–Ruelle (DLR) formalism* (cf. [Do70, Pr76, Ge88]). Namely, we define μ as random fields on $\mathbb{Z}^d$ with a prescribed family of local specifications $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ as follows:

We first need to construct a measure $\times_{k\in\mathbb{Z}^d} d\gamma(\omega_k)$ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ corresponding to $V = W = 0$ and the harmonic system (2.2) without interaction. With this aim, in the Hilbert space $L^2_\beta := L^2(S_\beta \to \mathbb{R})$ we consider the shifted Laplace–Beltrami operator $A := -\mathfrak{m}d^2/d\tau^2 + a^2\mathbf{1}$ on its maximal domain $\mathcal{D}(A)$ (which is the closure of $C^2_\beta := C^2(S_\beta \to \mathbb{R})$ for the norm $|\varphi|_{W^{2,2}_\beta} := |A\varphi|_{L^2_\beta}$). It is well known that the operator A is self-adjoint with the resolvent of trace class, i.e., $Tr_{L^2_\beta} A^{-1} < \infty$. Respectively, in the Hilbert space $\mathbb{L}^2_\beta := L^2(S_\beta \to \mathbb{R}^d)$ we consider the operator $\mathbb{A} := A \otimes \mathbb{I}_d$, where $\mathbb{I}_d$ is the identity matrix in $\mathbb{R}^d$. Let now γ be a Gaussian measure on $(\mathbb{C}_\beta, \mathcal{B}(\mathbb{C}_\beta))$ with correlation operator $\mathbb{A}^{-1}$, which is uniquely determined by its Fourier transform

$$\int_{\mathbb{L}^2_\beta} \exp i(\phi, \omega_k)_{\mathbb{L}^2_\beta} d\gamma(\omega_k) = \exp\left\{-\frac{1}{2}(\mathbb{A}^{-1}\phi, \phi)_{\mathbb{L}^2_\beta}\right\}, \quad \phi \in \mathbb{L}^2_\beta. \tag{2.10}$$

Actually, γ is supported by the Hölder continuous loops, i.e., $\gamma(\mathbb{C}^\eta_\beta) = 1$, $\forall \eta \in [0, \frac{1}{2})$, and has finite moments $E_\gamma |\omega_k|^Q_{\mathbb{C}^\eta_\beta} < \infty$, $\forall Q \geq 1$ (see e.g. [Si79]).

For every $\Lambda \Subset \mathbb{Z}^d$, we then define a probability kernel

$$\mathcal{B}(\Omega_\beta) \times \Omega_\beta \ni (\Delta, \xi) \to \pi_\Lambda(\Delta|\xi) \in [0, 1]$$

by

$$\pi_\Lambda(\Delta|\xi) := Z^{-1}_\Lambda(\xi) \int_{\Omega_{\beta,\Lambda}} \exp\left\{-\mathcal{E}_\Lambda(\omega|\xi)\right\} \mathbf{1}_\Delta(\omega_\Lambda, \xi_{\Lambda^c})_{k\in\Lambda} d\gamma(\omega_k) \tag{2.11}$$

(where $\mathbf{1}_\Delta$ denotes the indicator on Δ). Here $Z_\Lambda(\xi)$ is the normalization factor, and

$$\mathcal{E}_\Lambda(\omega|\xi) := \int_{S_\beta} \left[\sum_{k\in\Lambda} V(\omega_k) + \sum_{<k,k'>\subset\Lambda} W(\omega_k, \omega_{k'}) + \sum_{k\in\Lambda,\ k'\in\Lambda^c} W(\omega_k, \xi_{k'})\right] d\tau \tag{2.12}$$

is the interaction in the volume Λ under the boundary condition $\xi_{\Lambda^c} := (\xi_{k'})_{k'\in\Lambda^c}$. Due to Assumptions (**V**), (**W**)

$$\inf_{\omega\in\Omega_\beta} \mathcal{E}_\Lambda(\omega|\xi) > -\infty, \quad \forall \Lambda \Subset \mathbb{Z}^d,\ \forall \xi \in \Omega_\beta, \tag{2.13}$$

and thus the RHS in (2.12) makes sense. Moreover, the additive structure of the functional $\mathcal{E}_\Lambda(\omega|\xi)$ yields the *consistency* property for $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$: for all $\Lambda \subset \Lambda' \Subset \mathbb{Z}^d$, $\xi \in \Omega_\beta$ and $\Delta \in \mathcal{B}(\Omega_\beta)$

$$(\pi_{\Lambda'}\pi_\Lambda)(\Delta|\xi) := \int_{\Omega_\beta} \pi_{\Lambda'}(d\omega|\xi)\pi_\Lambda(\Delta|\omega) = \pi_{\Lambda'}(\Delta|\xi).$$

Definition 1. *A probability measure μ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ is called Euclidean Gibbs measure for the specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{Z}^d}$ (corresponding to the quantum lattice system (2.2) at inverse temperature $\beta > 0$) if it satisfies the DLR equilibrium equations: for all $\Lambda \Subset \mathbb{Z}^d$ and $\Delta \in \mathcal{B}(\Omega_\beta)$*

$$\mu\pi_\Lambda(\Delta) := \int_{\Omega_\beta} \mu(d\omega)\pi_\Lambda(\Delta|\omega) = \mu(\Delta). \tag{2.14}$$

Fixing $\beta > 0$, let $\mathcal{G}_\beta$ denote the set of all such measures μ. We shall mostly be concerned with the subset $\mathcal{G}^t_\beta$ of *tempered Gibbs measures* supported by Ω^t_β, i.e.,

$$\mathcal{G}^t_\beta := \mathcal{G}_\beta \cap \mathcal{M}^t_\beta = \left\{\mu \in \mathcal{G}_\beta \;\middle|\; \mu\left(\Omega^t_\beta\right) = 1\right\}. \tag{2.15}$$

2.4. **Formulation of the main results: Theorems 1 and 2.** Now we present our results on the existence and a priori estimates for Euclidean Gibbs states:

Theorem 1. (Existence of Tempered Gibbs States) *Let Assumptions* (**V**) *and* (**W**) *on the potentials* V *and* W *be fulfilled. Then for all values of the mass* $\mathfrak{m} > 0$ *and the inverse temperature* $\beta > 0$:

$$\mathcal{G}^t_\beta \neq \emptyset.$$

Theorem 2. (A Priori Estimates on Tempered Gibbs States) *Under the assumptions of Theorem 1, every* $\mu \in \mathcal{G}^t_\beta$ *is supported by the set of Hölder loops* $\bigcap_{0 \le \eta < \frac{1}{2}} [\mathbb{C}^\eta_\beta]^{\mathbb{Z}^d}$. *Moreover, for all* $Q \geq 1$ *and* $\eta \in [0, \frac{1}{2})$

$$\sup_{\mu \in \mathcal{G}^t_\beta} \sup_{k \in \mathbb{Z}^d} \int_{\Omega_\beta} |\omega_k|^Q_{\mathbb{C}^\eta_\beta} d\mu(\omega) < \infty. \tag{2.16}$$

Corollary 1. *Under the assumptions of Theorems 1 and 2 above, the set* $\mathcal{G}^t_\beta$ *is compact w.r.t. the topology of weak convergence of measures on the Polish space* Ω^t_β.

Let us make some **comments on Theorems 1 and 2**. As is typical for systems with noncompact (in our case, even infinite dimensional) spin spaces, the *existence* of $\mu \in \mathcal{G}^t_\beta$ stated by Theorem 1 is not evident at all. Depending on the specific class of quantum lattice models one considers, this problem has been examined in the literature by the following main methods:

(i) General Dobrushin's criterion for existence of Gibbs distributions: So far, no technical means are available to verify the sufficient conditions of the Dobrushin existence theorem [Do70] in the case of Euclidean Gibbs measures with single spin spaces $\mathbb{L}^r_\beta$ or $\mathbb{C}_\beta$ (in contrast with classical lattice systems with spins in $\mathbb{R}^d$ successfully dealt with, e.g., in [COPP78, BH-K82]).

(ii) Ruelle's technique of superstability estimates: This technique (cf. the original papers [Ru69, LP76] and resp. [PY94] for its modification to the quantum case) otherwise requires that the interaction is *translation invariant* and the many-particle potentials are of *at most quadratic growth* (i.e., (**W**) holds with $R = 2$).

(iii) Cluster expansions is one of the most powerful methods to study $\mu \in \mathcal{G}^t_\beta$ in a *perturbative regime*, i.e., when an effective parameter of the interaction is small (see e.g. [PY95, AKMR98, FM99, MVZ00, MRZ00] and references therein).

(iv) Method of correlation inequalities involves more detailed information about the structure of the interaction potentials (for instance, whether they are ferromagnetic or convex). A number of correlation inequalities (such as *FKG, GKS, Lebowitz, Brascamp-Lieb* etc.) commonly known for classical lattice systems can be extended also to the quantum case (see e.g. [AH-K75, GK90, OS99, AKKR01,02a,b]).

(v) Method of reflection positivity (as a part of (iv)) applies to *translation invariant* systems of type (2.2) with *nearest-neighbor* pair interactions and gives the existence of so-called *periodic* Gibbs states at least under the assumptions on the potentials V, W imposed in Subsect. 2.1 (cf. [BK91]).

Theorem 2 contributes to the fundamental problem of getting **uniform estimates** on correlation functionals of Gibbs measures $\mu \in \mathcal{G}^t_\beta$ in terms of parameters of the interaction. This problem was initially posed for classical lattice systems in [COPP78, BH-K82] and is closely related with the compactness of the set of tempered Gibbs states (cf. Corollary 1). To the best of our knowledge, all previous results on a priori integrability of Gibbs measures on path spaces were based on the **method of stochastic dynamics** (also referred to in mathematical physics as **"stochastic quantization"**); see e.g. [Fu91, DPZ96, AKRT01] and the related bibliography therein. Since in this method the Gibbs measures are treated as invariant distributions for the so-called Langevin stochastic dynamics, it requires *additional restrictions* on the interaction (among them at most quadratic growth of the pair potential W) to ensure the solvability of the corresponding stochastic equations in infinite dimensions (not to mention the extremely difficult ergodicity problem for them). Besides, a priori information about the finiteness of the moments of the measures $\mu \in \mathcal{G}_\beta^t$ is also needed for the study of Gibbs measures by means of the associated Dirichlet operators in the spaces $L^p(\mu),\ p \geq 1$, (this is known as the Holley–Stroock approach [HS76, AKR97a,b]).

As already mentioned in the Introduction, in order to prove Theorems 1 and 2 we shall propose a new technique, which completely differs from those listed under (i)–(v) and relies on the alternative description of $\mu \in \mathcal{G}_\beta$ via integration by parts. Moreover, this technique obviously extends (cf. Subsect. 5) to *general many-particle interactions* (not necessarily translation invariant and possibly having superquadratic growth, unbounded order and infinite range), which were not covered at all by any previous work. On the other hand, our approach is conceptually more straightforward and technically easier in comparison to the stochastic dynamics method mentioned above. This alternative approach has been first realized in [AKRT99,00], however in the much simpler situation of classical lattice systems with finite dimensional spins. In contrast with those papers, in the quantum case we have to do not only a "*lattice analysis*" (depending on the properties of the interaction potentials V, W), but also an additional "*single spin space analysis*" (taking into account the spectral properties of the elliptic operator $\mathbb{A}$). It should also be mentioned, that in the recent preprint [Ha01] some (deterministic) version of integration by parts for local specifications has been used to prove existence of Gibbs measures relative to Brownian motion on the path space $C(\mathbb{R} \to \mathbb{R}^d)$. The study of such Gibbsian (in general non Markovian) processes has been initiated in [OS99]. As a special case they include the so-called $P(\varphi)_1$-processes as Gibbs distributions corresponding to a single quantum particle at zero temperature, i.e., $\beta = \infty$ (see e.g. [Iw85, BL00]).

Finally, let us note that sufficient conditions implying uniqueness of tempered Euclidean Gibbs measures for quantum lattice systems like (2.2) have been proved in [AKRT97a,b, AKKR01,02a,b], whereas the possibility of phase transitions in such models has been discussed in [DLP79, BK91, AKR98, He98].

3. Flow and (IbP)-characterization of Euclidean Gibbs measures

In this section we give an alternative description of Euclidean Gibbs measures (cf. Propositions 1 and 2 below). These are: first, the *flow characterization* of $\mu \in \mathcal{G}_\beta$ in terms of their *Radon–Nikodym derivatives* w.r.t. shift transformations of the configuration space Ω_β; second, the characterization (resulting from the previous one) in terms of their *logarithmic derivatives* via corresponding *integration by parts* (for short, *IbP*) *formulas.* If the interaction potentials are differentiable (as they are in our case), both characterizations are equivalent. Also we observe that the local Gibbs specifications π_Λ, $\Lambda \Subset \mathbb{Z}^d$, also satisfy the same flow and integration by parts descriptions, which later will be crucial for our proof of the existence of $\mu \in \mathcal{G}_\beta^t$.

3.1. **Flow description of Euclidean Gibbs measures.**

We start with the flow description of $\mu \in \mathcal{G}_\beta$ in terms of their *"shift"–Radon–Nikodym derivatives* $a_{\theta h_i}$, $\theta \in \mathbb{R}$, along some set of admissible directions h_i, $i \in \mathcal{I}$, whose linear span is dense in Ω_β.

With this aim, we shall consider the Hilbert space

$$\mathcal{H}_\beta := \left\{ \omega = (\omega_k)_{k \in \mathbb{Z}^d} \in [\mathbb{L}^2_\beta]^{\mathbb{Z}^d} \;\Big|\; ||\omega||^2_\beta := \sum\nolimits_{k \in \mathbb{Z}^d} |\omega_k|^2_{\mathbb{L}^2_\beta} < \infty \right\} \tag{3.1}$$

with the inner product $<\omega,\omega>_\beta := ||\omega||^2_\beta$ as *tangent space* to Ω_β. For the remainder of this paper, we fix an *orthonormal basis* in $\mathcal{H}_\beta$ consisting of the vectors

$$\begin{array}{c} h_i := (\delta_{k-k'}\delta_{\alpha-\alpha'}\varphi_n)_{k'\in\mathbb{Z}^d,\ 1\le\alpha'\le d} \\ \text{indexed by } \ i=(k,n,\alpha)\ \in\ \mathbb{Z}^d\times\mathbb{Z}\times\{1,...,d\} =: \mathcal{I}, \end{array} \tag{3.2}$$

where φ_n are the eigenvectors of the operator A in L^2_β, i.e., $A\varphi_n = \lambda_n\varphi_n$. Recall that the operator A has discrete spectrum

$$\lambda_n := \left(\frac{2\pi}{\beta}n\right)^2 \mathfrak{m} + a^2, \quad n \in \mathbb{Z}, \tag{3.3}$$

and a complete orthonormal system of trigonometric functions

$$\varphi_n(\tau) := \begin{cases} \sqrt{\frac{1}{\beta}}, & n=0, \\ \sqrt{\frac{2}{\beta}}\cos\frac{2\pi}{\beta}n\tau, & n=1,2,...\ , \\ -\sqrt{\frac{2}{\beta}}\sin\frac{2\pi}{\beta}n\tau, & n=-1,-2,...\ . \end{cases} \tag{3.4}$$

Moreover, the set of all trigonometric polynomials

$$\mathcal{T}_\beta := lin.span\{\varphi_n\}_{n\in\mathbb{Z}}$$

is a domain of essential self-adjointness for A. Respectively $\phi_{(n,\alpha)} := (\delta_{\alpha-\alpha'}\varphi_n)^d_{\alpha'=1}$ are the eigenvectors of the operator $\mathbb{A}$ in $\mathbb{L}^2_\beta$, i.e., $\mathbb{A}\phi_{(n,\alpha)} = \lambda_n\phi_{(n,\alpha)}$.

Proposition 1. (Flow Description of $\mu\in\mathcal{G}_\beta$) *For a given direction h_i, $i\in\mathcal{I}$, let $\mathcal{M}_{a,h_i}$ denote the set of all probability measures μ on $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ which are quasi-invariant w.r.t. the shifts $\omega\mapsto\omega+\theta h_i$, $\theta\in\mathbb{R}$, with Radon–Nikodym derivatives*

$$\begin{aligned} a_{\theta h_i}(\omega) &:= \frac{d\mu(\omega+\theta h_i)}{d\mu(\omega)} \\ &= \exp\left\{-\theta(\mathbb{A}\phi_{(n,\alpha)},\omega_k)_{\mathbb{L}^2_\beta} - \frac{\theta^2}{2}(\mathbb{A}\phi_{(n,\alpha)},\phi_{(n,\alpha)})_{\mathbb{L}^2_\beta}\right\} \\ &\quad\times\exp\int_{S_\beta}\left\{V(\omega_k)-V(\omega_k+\theta\phi_{(n,\alpha)}) + \sum_{k'\in\partial k}\left[W(\omega_k,\omega_{k'})-W(\omega_k+\theta\phi_{(n,\alpha)},\omega_{k'})\right]\right\}d\tau. \end{aligned} \tag{3.5}$$

Then

$$\mathcal{G}_\beta =_{i\in\mathcal{I}} \mathcal{M}_{a,h_i} =: \mathcal{M}_a. \tag{3.6}$$

Proof. Proposition 1 extends the analogous result of Theorem 4.6 in [AKR97b], where (for simplicity only) the particular case of harmonic pair interactions was treated. Thus, here we sketch the proof very roughly; for missing details as well as possible generalizations we also refer to [AKPR02].

(i) $\mathcal{G}_\beta\subseteq\mathcal{M}_a$: This inclusion obviously follows from the DLR equations (2.14) and the quasi-invariance of the probability kernels $\pi_\Lambda(d\omega|\xi)$.

(ii) $\mathcal{M}_a\subseteq\mathcal{G}_\beta$: Keeping the notation of Sect. 2, for every $k\in\mathbb{Z}^d$ define $\Lambda_k := \{k\}$, $\Lambda_k^c := \mathbb{Z}^d\backslash\{k\}$ and $\omega_{\Lambda_k^c} := \mathbb{P}_{\Lambda_k^c}\omega\in\Omega_{\beta,\Lambda_k^c}$. Let us disintegrate any $\mu\in\mathcal{M}_a$ w.r.t. its projection $\mu_{\Lambda_k^c} := \mu\mathbb{P}^{-1}_{\Lambda_k^c}$ onto $\Omega_{\beta,\Lambda_k^c}$:

$$\mu(d\omega_k, d\omega_{\Lambda_k^c}) = \nu_{\omega_{\Lambda_k^c}}(d\omega_k)\mu_{\Lambda_k^c}(d\omega_{\Lambda_k^c}),$$

Here $\nu_{\omega_{\Lambda_k^c}}(d\omega_k)$ are some probability measures (= regular conditional distributions given $\omega_{\Lambda_k^c}$) on $(\mathbb{C}_\beta, \mathcal{B}(\mathbb{C}_\beta))$. Moreover, one can verify (cf. [Ro75, Proposition 3]) the quasi-invariance of the measures $\nu_{\omega_{\Lambda_k^c}}$ in the following sense: There exists a Borel subset $\Delta_k \subseteq \Omega_{\beta,\Lambda_k^c}$ such that $\mu_{\Lambda_k^c}(\Delta_k) = 1$ and for every $\omega_{\Lambda_k^c} \in \Delta_k$, $h_i := (\delta_{k-k'}\phi_{(n,\alpha)})_{k'\in\mathbb{Z}^d}$ and $\theta \in \mathbb{R}$:

$$\frac{d\nu_{\omega_{\Lambda_k^c}}(\omega_k + \theta\phi_{(n,\alpha)})}{d\nu_{\omega_{\Lambda_k^c}}(d\omega_k)} = a_{\theta h_i}(\omega_k, \omega_{\Lambda_k^c}), \quad \omega_k \in \mathbb{C}_\beta \ (mod \nu_{\omega_{\Lambda_k^c}}). \tag{3.7}$$

¿From now on fix any $\xi_{\Lambda_k^c} \in \Delta_k$, and let us show that (3.7) implies

$$\nu_{\xi_{\Lambda_k^c}}(d\omega_k) = Z^{-1} \exp\left\{ -\int_{S_\beta} \left[V(\omega_k) + \sum_{k'\in\partial k} W(\omega_k, \xi_{k'}) \right] d\tau \right\} \gamma(d\omega_k). \tag{3.8}$$

However, $\nu := \nu_{\xi_{\Lambda_k^c}}(d\omega_k)$ is the unique probability measure on $(\mathbb{C}_\beta, \mathcal{B}(\mathbb{C}_\beta))$ which satisfies the flow description

$$\frac{d\nu(\omega_k + \phi)}{d\nu(\omega_k)} = a_\phi(\omega_k), \quad \omega_k \in \mathbb{C}_\beta \ (mod \nu), \tag{3.9}$$

for all $\phi \in lin.span\{\phi_{(n,\alpha)}\}$ with the cocycle

$$\begin{aligned} a_\phi(\omega_k) := {} & \exp\left\{ -(\mathbb{A}\phi, \omega_k)_{\mathbb{L}^2_\beta} - \frac{1}{2}(\mathbb{A}\phi, \phi)_{\mathbb{L}^2_\beta} \right\} \\ & \times \exp \int_{S_\beta} \left\{ V(\omega_k) - V(\omega_k + \phi) + \sum_{k'\in\partial k} [W(\omega_k, \xi_{k'}) - W(\omega_k + \phi, \xi_{k'})] \right\} d\tau. \end{aligned} \tag{3.10}$$

To check this, let us introduce the new measure

$$\sigma(d\omega_k) := \exp\left\{ \int_{S_\beta} \left[V(\omega_k) + \sum_{k'\in\partial k} W(\omega_k, \xi_{k'}) \right] d\tau \right\} \nu(d\omega_k), \tag{3.11}$$

which (due to our assumptions on the potentials V and W) is at least σ-finite on $(\mathbb{C}_\beta, \mathcal{B}(\mathbb{C}_\beta))$. By (3.9)–(3.11) we have that

$$\frac{d\sigma(\omega_k + \phi)}{d\sigma(\omega_k)} = \exp\left\{ -(\mathbb{A}\phi, \omega_k)_{\mathbb{L}^2_\beta} - \frac{1}{2}(\mathbb{A}\phi, \phi)_{\mathbb{L}^2_\beta} \right\}. \tag{3.12}$$

But, as it is well-known and could be straightforwardly verified (see e.g. Ro75, Proposition 4]), (3.12) implies $\sigma(\mathbb{C}_\beta) < \infty$ and $\sigma(d\omega_k) = const \cdot \gamma(d\omega_k)$. Consequently, combining (3.8) and (3.11), we deduce that $\nu(d\omega_k) = const \cdot \nu_{\xi_{\Lambda_k^c}}(d\omega_k)$ and, since both are probability measures, they coincide. And finally, noting that $\mathcal{G}_\beta$ is fully determined by $\{\pi_{\Lambda_k}\}_{k\in\mathbb{Z}^d}$ (cf. [Ge88, Theorem 1.33]), we get the desired inclusion $\mu \in \mathcal{G}_\beta$.
■

Actually, (3.6) is true under minimal assumptions on the potentials V, W, which guarantee, (besides the well-definedness of the local specifications $\pi_{\beta,\Lambda}$) continuity and local boundedness of the functions $\mathbb{R}\times\Omega_\beta \ni (\theta, \omega) \longmapsto a_{\theta h_i}(\omega) \in \mathbb{R}$ for all $i \in \mathcal{I}$. However, in applications it is more convenient to use not the flow characterization (3.6) itself, but its infinitesimal form which we describe in the next subsection.

3.2. **Integration by parts formula for Euclidean Gibbs measures.** We shall show that the above flow characterization of $\mu \in \mathcal{G}_\beta$ is equivalent to their characterization as differentiable measures satisfying the integration by parts (for short, *IbP*) formulas

$$\partial_{h_i}\mu(d\omega) = b_{h_i}(\omega)\mu(d\omega), \quad \forall i \in \mathcal{I}, \tag{3.13}$$

with the given *logarithmic derivatives* b_{h_i} *along basis vectors* h_i. More precisely, for the basis vector $h_i = (\delta_{k-k'}\phi_{(n,\alpha)})_{k'\in\mathbb{Z}^d}$, $i = (k,n,\alpha) \in \mathcal{I}$, we define the function $b_{h_i} : \Omega_\beta \to \mathbb{R}$ by

$$b_{h_i}(\omega) := \frac{\partial}{\partial\theta} a_{\theta h_i}(\omega)\Big|_{\theta=0} = -(\mathbb{A}\phi_{(n,\alpha)}, \omega_k)_{\mathbb{L}^2_\beta} - (F_{k,\alpha}(\omega), \phi_{(n,\alpha)})_{\mathbb{L}^2_\beta}, \tag{3.14}$$

where $F_k = (F_{k,\alpha})_{\alpha=1}^d : \Omega_\beta \to \mathbb{C}_\beta$ is the nonlinear Nemytskii-type operator acting by

$$F_k(\omega) := \nabla V(\omega_k) + \sum\nolimits_{k'\in\partial k} \nabla_q W(q,q')|_{q=\omega_k,\ q'=\omega_{k'}} . \tag{3.15}$$

We stress that the main difficulty in dealing with the (IbP)-formulas (3.13) is that *we do not know in advance* (until Theorem 2 below) *whether* $b_{h_i} \in L^1(\Omega_\beta, \mu)$. Thus, we first have to introduce proper classes of differentiable functions on Ω_β to which we can correctly apply the distributional identity (3.13).

For this purpose it is helpful to recall some facts from convex analysis (cf. e.g. [Dei85]): Let X be a locally convex space, and let $\Phi : X \to \mathbb{R}$. The *partial derivatives on the right* resp. *left in the direction* $h \in X$ of the function Φ at a point $x \in X$ are defined by

$$\partial_h^+\Phi(x) := \lim_{\theta\to+0} \frac{\Phi(x+\theta h) - \Phi(x)}{\theta}, \quad \partial_h^-\Phi(x) := \lim_{\theta\to-0} \frac{\Phi(x+\theta h) - \Phi(x)}{\theta}. \tag{3.16}$$

If the right and left limits in (3.16) coincide, one says that there exists the *derivative* $\partial_h\Phi(x)$ *in the direction* h. By $C^1(X;h)$ (resp. $C^1_{b,loc}(X;h)$ or $C^1_b(X;h)$) we denote the spaces of all functions $\Phi : X \to \mathbb{R}$ which are *continuous* (and, moreover, *locally* or *globally bounded*) together with their partial derivatives $\partial_h\Phi : X \to \mathbb{R}$. Actually, the existence either or both of $\partial_h^+\Phi$ and $\partial_h^-\Phi$ along some total set of $h \in X$ will be quite enough for our applications, so we do not discuss here the more involved notions of Gâteaux or Fréchet differentiability. On the other hand, later we shall also need to consider $X := C_\beta$, for which, as is well known, the norm-function $|\cdot|_{C_\beta}$ is not (Gâteaux) differentiable everywhere on $\mathbb{C}_\beta\backslash\{0\}$: indeed,

$$\begin{gathered}\exists\partial_h|x|_{C_\beta} = \partial_h^\pm|x|_{C_\beta} \ \text{ for } x,h \in C_\beta \ \text{ iff} \\ h(\tau) = \pm h(\tau') \ \text{ for all } \ \tau,\tau' \in S_\beta \ \text{ such that } h(\tau) = \pm h(\tau') = |h|_{C_\beta}.\end{gathered} \tag{3.17}$$

In general, for any Banach space $(X, |\cdot|_X)$ and all $x,h \in X$, there exist both $\partial_h^-|x|_X$ and $\partial_h^+|x|_X$, which are uniformly bounded by

$$-|h|_X \le \partial_h^-|x|_X \le \partial_h^+|x|_X \le |h|_X \tag{3.18}$$

and, which for fixed $h \in X$, are semicontinuous (above resp. below) functions of $x \in X$.

Remark 2. Under the assumptions on the potentials V, W imposed in Subsect. 2.1, we have that $a_{\theta h_i}, b_{h_i} \in C^1(\Omega_\beta; h_i)$ for all $\theta \in \mathbb{R}$ and $i = (k,n,\alpha) \in \mathcal{I}$. Moreover, the functions $a_{\theta h_i}, b_{h_i}$, as well as their partial derivatives $\partial_{h_i}a_{\theta h_i}$, $\partial_{h_i}b_{h_i}$, are bounded on the cylinder sets

$$B_{\Lambda,\rho} := \big\{\omega \in \Omega_\beta \mid |\omega_{k'}|_{\mathbb{C}_\beta} \le \rho, \ \ \forall k' \in \Lambda\big\} \ \text{ where } \Lambda \supseteq \{k\} \cup \partial k, \ \rho \in (0,\infty).$$

Further, by a straightforward calculation,

$$\partial_{h_i}a_{\theta h_i}(\omega) = a_{\theta h_i}(\omega)\left[b_{h_i}(\omega+\theta h_i) - b_{h_i}(\omega)\right], \quad \frac{\partial}{\partial\theta}a_{\theta h_i}(\omega) = a_{\theta h_i}(\omega)b_{h_i}(\omega+\theta h_i), \tag{3.19}$$

and thus one can recover $a_{\theta h_i}$ from b_{h_i} by

$$a_{\theta h_i}(\omega) = \exp \int_0^\theta b_{h_i}(\omega + \vartheta h_i) d\vartheta. \tag{3.20}$$

Definition 2. *Fixing a basis vector h_i, $i = (k, n, \alpha) \in \mathcal{I}$, by $C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$ we denote the set of all functions $f \in C^1_b(\Omega_\beta; h_i)$, which satisfy the extra decay condition*

$$\sup_{\omega \in \Omega_\beta} \left| f(\omega) \left(1 + |\omega_k|_{\mathbb{L}^1_\beta} + |F_k(\omega)|_{\mathbb{L}^1_\beta} \right) \right| < \infty. \tag{3.21}$$

Below we shall need a simple approximation lemma (true even in a more general setting on locally convex spaces) in order to justify the (IbP)-formula (3.13) for all $f \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$.

Lemma 1. *(i) For any given $\Lambda \Subset \mathbb{Z}^d$ and $h \in \Omega_\beta$, there exists a sequence $\{\psi^{(N)}\}_{N \in \mathbb{N}} \subset C^1_b(\Omega_\beta; h)$ approximating $\psi \equiv 1$ in the following sense:*

$$\begin{gathered} 0 \le \psi^{(N)}(\omega) := \psi^{(N)}(\omega_\Lambda) \le 1, \quad \operatorname{supp} \psi^{(N)} \subset B_{\Lambda, \rho^{(N)}}, \quad \rho^{(N)} \in (0, \infty), \\ \psi^{(N)} \to 1, \quad \partial_h \psi^{(N)} \to 0 \ \text{ as } N \to \infty, \quad \text{and } \sup\nolimits_{N \in \mathbb{N}} ||\partial_h \psi^{(N)}||_{C_b(\Omega_\beta)} < \infty. \end{gathered} \tag{3.22}$$

(ii) For each $\mu \in \mathcal{M}(\Omega_\beta)$ and h_i, $i \in \mathcal{I}$, the set $C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$ is dense in all spaces $L^r(\Omega_\beta, \mu)$, $1 \le r < \infty$.

Proof. *(i)* Let us first take any cut-off sequence $\{\chi_N\}_{N \in \mathbb{N}} \subset C^1_0(\mathbb{R}_+)$ with the properties:

$$\begin{gathered} \chi_N(s) = 1 \ \text{ for } s \in [0, N], \quad \chi_N(s) = 0 \ \text{ for } s \in [N+1, \infty), \\ 0 \le \chi_{N+1}(s+1) = \chi_N(s) \le 1 \ \text{ for every } s \ge 0. \end{gathered}$$

Next, for each $k \in \mathbb{Z}^d$ define the boundedly supported functions on $\mathbb{C}_\beta$

$$\psi_k^{(N)}(\omega_k) := N \int_0^{\frac{1}{N}} \chi_N \left(|(\omega + \theta h)_k|_{\mathbb{C}_\beta} \right) d\theta$$

and calculate their derivatives

$$\partial_{h_k} \psi^{(N)}(\omega_k) = N \left[\chi_N \left(|(\omega + \frac{1}{N} h)_k|_{\mathbb{C}_\beta} \right) - \chi_N \left(|\omega_k|_{\mathbb{C}_\beta} \right) \right].$$

Then it is easy to check that $\psi^{(N)} := \prod_{k \in \Lambda} \psi_k^{(N)} \to 1$, as $N \to \infty$, in the sense of (3.22).

(ii) By the regularity property of any $\mu \in \mathcal{M}(\Omega_\beta)$ (considered as a Borel measure on the product space $[\mathbb{L}^2_\beta]^{\mathbb{Z}^d}$), the set $C^1_b(\Omega_\beta; h_i)$ (as such containing smooth cylinder functions w.r.t. the base $\{h_i\}_{i \in \mathcal{I}}$) is dense in every $L^r(\Omega_\beta, \mu)$, $1 \le r < \infty$. Now let us fix $h := h_i$, $\Lambda \supseteq \{k\} \cup \partial k$, and let $\psi^{(N)} \to 1$, $N \to \infty$, as in (3.22) above. Then, by Lebesgue's dominated convergence theorem, each $f \in C^1_b(\Omega_\beta; h_i)$ can be approximated in $L^r(\Omega_\beta, \mu)$ by the corresponding sequence $f^{(N)} := f \psi^{(N)} \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$, $N \in \mathbb{N}$. ■

Proposition 2. ((IbP)-Description of $\mu \in \mathcal{G}_\beta$) *Given any direction h_i, $i \in \mathcal{I}$, let $\mathcal{M}_{b,h_i}$ denote the set of all probability measures μ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ which satisfy the (IbP)-formula*

$$\int_{\Omega_\beta} \partial_{h_i} f(\omega) d\mu(\omega) = - \int_{\Omega_\beta} f(\omega) b_{h_i}(\omega) d\mu(\omega) \tag{3.23}$$

for all functions $f \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$. Then

$$\begin{gathered} \forall i \in \mathcal{I}: \quad \mathcal{M}_{b,h_i} = \mathcal{M}_{a,h_i}, \\ \text{and thus} \quad \mathcal{G}_\beta = \mathcal{M}_a =_{i \in \mathcal{I}} \mathcal{M}_{b,h_i} =: \mathcal{M}_b. \end{gathered} \tag{3.24}$$

Proof. The line of reasoning is close to the proof in [Be85, DS88] of the well known fact that every probability measure μ on a vector space X, which is differentiable along some direction $h \in X$ with corresponding logarithmic derivative $b_h \in L^1(\mu)$, is for sure also quasi-invariant w.r.t. all shifts $x \mapsto x + \theta h$. The new difficulty and the principal difference compared with the above mentioned papers is that no assumptions on the global integrability of the logarithmic derivatives b_{h_i} are imposed here. Instead, we shall crucially use the approximation procedure given by Lemma 1 and the observation that $a_{\theta h_i}$, b_{h_i} are continuous locally bounded functions on Ω_β (and hence by Remark 2 for all $\theta \in \mathbb{R}$ there exist $\partial_\theta a_{\theta h_i}$, $\partial_{h_i} a_{\theta h_i} \in C_{b,loc}(\Omega_\beta)$).

(i) $\mathcal{M}_{a,h_i} \subset \mathcal{M}_{b,h_i}$: By Proposition 1, for each $\mu \in \mathcal{M}_{a,h_i}$ and $f \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$

$$\int_{\Omega_\beta} f(\omega) a_{\theta h_i}(\omega) d\mu(\omega) = \int_{\Omega_\beta} f(\omega - \theta h_i) d\mu(\omega), \tag{3.25}$$

and thus for all $\theta \neq 0$

$$\int_{\Omega_\beta} f(\omega) \frac{a_{\theta h_i}(\omega) - 1}{\theta} d\mu(\omega) = \int_{\Omega_\beta} \frac{f(\omega - \theta h_i) - f(\omega)}{\theta} d\mu(\omega). \tag{3.26}$$

Let again $f^{(N)} := f\psi^{(N)}$ and $\{\psi^{(N)}\}_{N\in\mathbb{N}}$ be the cut-off sequence constructed in Lemma 1 for $h := h_i$ and $\Lambda \supseteq \{k\} \cup \partial k$. Hence, combining (3.19) and (3.22), for all $N \in \mathbb{N}$:

$$\sup_{0<|\theta|\le 1,\ \omega\in\Omega_\beta} \left| \frac{f^{(N)}(\omega + \theta h_i) - f^{(N)}(\omega)}{\theta} \right| \le \sup_{\omega\in\Omega_\beta} |\partial_{h_i} f^{(N)}(\omega)| < \infty,$$

$$\sup_{0<|\theta|\le 1,\ \omega\in\Omega_\beta} \left| f^{(N)}(\omega) \frac{a_{\theta h_i}(\omega) - 1}{\theta} \right| \le ||f||_{C_b(\Omega_\beta)} \sup_{0<|\theta|\le 1,\ \omega\in B_{\Lambda,\rho(N)}} \left| \frac{\partial}{\partial\theta} a_{\theta h_i}(\omega_\Lambda) \right| < \infty.$$

Thus, in order to get the (IbP)-formula (3.23), by Lebesgue's dominated convergence theorem one can pass to the limit, first $\theta \to \pm 0$ and thereafter $N \to \infty$, in both sides of (3.26) with $f^{(N)}$ replacing f.

(ii) $\mathcal{M}_{a,h_i} \supset \mathcal{M}_{b,h_i}$: We claim that each $\mu \in \mathcal{M}_{b,h_i}$ is quasi-invariant w.r.t. the shifts $\omega \mapsto \omega + \theta h_i$, $\theta \in \mathbb{R}$, with the Radon–Nikodym derivatives

$$\frac{d\mu(\omega + \theta h_i)}{d\mu(\omega)} = \exp \int_0^\theta b_i(\omega + \vartheta h_i) d\vartheta. \tag{3.27}$$

By Remark 2 this readily implies that $\mu \in \mathcal{M}_{a,h_i}$.

So, given as before any $f \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i)$ and its approximations $f^{(N)} := f\psi^{(N)}$, $N \in \mathbb{N}$, let us define a family of functions indexed by $\theta \in \mathbb{R}$:

$$f^{(N)}(\theta, \cdot) \in C^1_{\mathsf{dec}}(\Omega_\beta; h_i), \quad f^{(N)}(\theta, \omega) := f^{(N)}(\omega + \theta h_i) a_{\theta h_i}(\omega). \tag{3.28}$$

Moreover, one can check by a direct calculation that

$$\begin{aligned} &\frac{d}{d\theta} \int_{\Omega_\beta} f^{(N)}(\theta, \omega) d\mu(\omega) \\ &= \int_{\Omega_\beta} \left[\partial_{h_i} f^{(N)}(\omega + \theta h_i) + f^{(N)}(\omega + \theta h_i) b_{h_i}(\omega + \theta h_i) \right] a_{\theta h_i}(\omega) d\mu(\omega). \end{aligned} \tag{3.29}$$

Substituting the exact expression (3.19) for $\partial_{h_i} a_{\theta h_i}(\omega)$ in (3.29) and then applying the (IbP)-formula (3.23) to $f^{(N)}(\theta, \cdot)$, we find that

$$\frac{d}{d\theta} \int_{\Omega_\beta} f^{(N)}(\theta, \omega) d\mu(\omega) = 0, \quad \forall \theta \in \mathbb{R}.$$

Due to the continuity of $\theta \mapsto E_\mu f^{(N)}(\theta,\cdot)$, the latter yields

$$\int_{\Omega_\beta} f^{(N)}(\omega+\theta h_i)a_{\theta h_i}(\omega)d\mu(\omega)=\int_{\Omega_\beta} f^{(N)}(\omega)d\mu(\omega), \quad \forall\theta\in\mathbb{R}.$$

Herefrom, letting $N\to\infty$, by Fatou's lemma and Lemma 1 we conclude that the description via "shift"-Radon–Nikodym derivatives (3.27) holds for all $f\in L^1(\Omega_\beta,\mu)$. ■.

Remark 3. We briefly discuss here some useful modifications and corollaries of Proposition 2:

(i) Denote by $C^{1,\pm}_{\mathtt{dec}}(\Omega_\beta;h_i)$ the set of all continuous functions $f\in C_b(\Omega_\beta)$ which satisfy the decay condition (3.21) and have globally bounded (but not necessarily continuous) *right* and *left derivatives* $\partial^+_{h_i}f$ and $\partial^-_{h_i}f$ along the direction h_i. If $\mu\in\mathcal{M}_{b,h_i}$, then the (IbP)-formula (3.23) extends to all $f\in C^{1,\pm}_{\mathtt{dec}}(\Omega_\beta;h_i)$ by

$$\int_{\Omega_\beta}\partial^\pm_{h_i}f(\omega)d\mu(\omega)=-\int_{\Omega_\beta}f(\omega)b_{h_i}(\omega)d\mu(\omega). \tag{3.30}$$

To this end, it suffices to repeat the proof of part (i) of Proposition 2, recalling definition (3.16) of $\partial^\pm_{h_i}f$ and using the fact that $\partial^+_{h_i}a_{\theta h_i}=\partial^-_{h_i}a_{\theta h_i}$. Respectively, if μ is tempered (i.e., supported by Ω^t_β), then (3.30) also holds for all $f\in C^{1,\pm}_{\mathtt{dec}}(\Omega^t_\beta;h_i)$, where the set $C^{1,\pm}_{\mathtt{dec}}(\Omega^t_\beta;h_i)$ is defined just as above, but with Ω^t_β instead of Ω_β. Namely, in such *extended form* (3.30) the integration by parts formula will be applied to proper test functions (among others, depending on $|\omega_k|_{\mathbb{C}_\beta}$) in the proofs of Lemmas 2–5 below.

(ii) Fix $i=(k,n,\alpha)\in\mathcal{I}$, and let $\mu\in\mathcal{M}_{b,h_i}$. If in the (IbP)-formula (3.30) we make the special choice of $f(\omega):=g(\omega)|\omega_{k,\alpha}|_{C_\beta}$ with arbitrary $g\in C^1_{\mathtt{dec}}(\Omega_\beta;h_i)$ such that $\operatorname{supp} g\subset B_{\Lambda,\rho}$ for some $\Lambda\supseteq\{k\}\cup\partial k$ and $\rho\in(0,\infty)$, we get that $E_\mu(g\partial^+_{h_i}|\omega_{k,\alpha}|_{C_\beta})=E_\mu(g\partial^-_{h_i}|\omega_{k,\alpha}|_{C_\beta})$. Thus by Lemma $1(ii)$

$$\mu\left(\omega\in\Omega_\beta\;\middle|\;\partial^+_{\varphi_n}|\omega_{k,\alpha}|_{C_\beta}=\partial^-_{\varphi_n}|\omega_{k,\alpha}|_{C_\beta}\right)=1. \tag{3.31}$$

The last identity might also be derived from a result in [Ku82, Lemma 1.3] on the so-called stochastic Gâteaux differentiability of Lipshitz continuous functions on abstract Wiener spaces. Moreover, if $\mu\in\mathcal{M}_b$, then one can conclude from (3.31) (using the description of $\partial_\varphi|\omega_{k,\alpha}|_{C_\beta}$ by (3.17) and taking into account that the set of trigonometric polynomials $\mathcal{T}_\beta$ is dense in C_β) the uniqueness of the global extrema for loops $\omega_{k,\alpha}\in C_\beta$:

$$\mu\left(\omega\in\Omega_\beta\;\middle|\;\exists\ \textit{unique}\ \tau\in S_\beta:|\omega_{k,\alpha}(\tau)|=|\omega_{k,\alpha}|_{C_\beta}\right)=1. \tag{3.32}$$

This generalizes the well-known property of the oscillator bridge process γ, see e.g. [ReY91].

So, based on Propositions 2, instead of Euclidean Gibbs measures $\mu\in\mathcal{G}_\beta$ initially defined as random fields on the lattice $\mathbb{Z}^d$, we can just study probability measures on Ω_β satisfying the (IbP)-formula (3.23) with the prescribed logarithmic derivatives $b_{h_i},\ i\in\mathcal{I}$. Let us stress that the b_{h_i} only depend on the given potentials V and W and hence are the same for all $\mu\in\mathcal{G}_\beta$ associated with the heuristic Hamiltonian (2.2). Solutions $\mu\in\mathcal{M}_b$ to the (IbP)-formula (3.23) will also be called *symmetrizing* measures. For further connections to reversible diffusion processes and Dirichlet operators in infinite dimensions we refer e.g. to [AKR97a,b, AKRT01, BR01, BRW01].

3.3. **Integration by parts formula for the probability kernels of the local specification.** As is immediately to see from definitions (2.11), (2.12) and for classical lattice systems has been already mentioned in [Roy77], the following observation is true:

Measures $\pi_\Lambda(d\omega|\xi)$ are quasi-invariant w.r.t. the shifts $\omega \mapsto \omega + \theta h_i$ with the same Radon–Nikodym derivatives as those for the corresponding Gibbs measures $\mu \in \mathcal{G}_\beta$. More precisely, for every $\theta \in \mathbb{R}$, $\xi \in \Omega_\beta$, $\Lambda \Subset \mathbb{Z}^d$ and $i = (k, n, \alpha) \in \mathcal{I}$ with $k \in \Lambda$,

$$\frac{d\pi_\Lambda(\omega + \theta h_i|\xi)}{d\pi_\Lambda(\omega|\xi)} = a_{\theta h_i}(\omega), \quad \forall \omega \in \Omega_\beta \quad (\pi_\Lambda(d\omega|\xi) - a.e.), \tag{3.33}$$

or, equivalently, for all $f \in L^1(\Omega_\beta, \pi_\Lambda(\cdot|\xi))$

$$\int_{\Omega_\beta} f(\omega) a_{\theta h_i}(\omega) \pi_\Lambda(d\omega|\xi) = \int_{\Omega_\beta} f(\omega - \theta h_i) \pi_\Lambda(d\omega|\xi). \tag{3.34}$$

A reasoning similar to that used in the proof of Proposition 2 then shows that, for every $\xi \in \Omega_\beta$, $\Lambda \Subset \mathbb{Z}^d$ and $i = (k, n, \alpha) \in \mathcal{I}$ with $k \in \Lambda$, the (IbP)-formula

$$\int_{\Omega_\beta} \partial_{h_i} f(\omega) \pi_\Lambda(d\omega|\xi) = -\int_{\Omega_\beta} f(\omega) b_{h_i}(\omega) \pi_\Lambda(d\omega|\xi), \tag{3.35}$$

holds for all functions $f \in C^1_{\mathrm{dec}}(\Omega_\beta; h_i)$.

Suppose now that a sequence $\pi_{\Lambda^{(N)}}(d\omega|\xi^{(N)})$, $N \in \mathbb{N}$, where $\xi^{(N)} \in \Omega_\beta$ and $\Lambda^{(N)} \nearrow \mathbb{Z}^d$ as $N \to \infty$, weakly converges on the metric space Ω_β to some probability measure $\mu_* \in \mathcal{M}(\Omega_\beta)$. Taking into account that $a_{\theta h_i}$, $b_{h_i} \in C_{b,loc}(\Omega_\beta)$ and using the same approximation for $f \in C^1_{\mathrm{dec}}(\Omega_\beta; h_i)$ as in the proof of Proposition 2, one can also pass to the limit in both sides of (3.32) and (3.33). So, for any $\theta \in \mathbb{R}$ and any direction h_i, $i = (k, n, \alpha) \in \mathcal{I}$, we again have the flow description (3.25) and the (IbP)-formula (3.23), which hold for $\mu := \mu_*$ and all $f \in C^1_{\mathrm{dec}}(\Omega_\beta; h_i)$. Combining these properties of μ_* with Propositions 1 and 2, we have thus proved the following:

Proposition 3. (Thermodynamic Limit Points are Gibbs) *Consider any sequence of measures $\pi_{\Lambda^{(N)}}(d\omega|\xi^{(N)})$, $N \in \mathbb{N}$, where $\xi^{(N)} \in \Omega_\beta$ and $\Lambda^{(N)} \nearrow \mathbb{Z}^d$ as $N \to \infty$. Then each of its accumulation points $\mu^* \in \mathcal{M}(\Omega_\beta)$ (w.r.t. the topology of weak convergence of measures on the Polish space Ω_β), provided such exist, is Gibbs.*

In this way, the alternative characterization of Euclidean Gibbs measures enables us to study the existence problem for $\mu \in \mathcal{G}_\beta$ just by showing the tightness of the family of their probability kernels.

4. Proof of Theorems 1 and 2

Assumptions (**V**), (**W**) on the asymptotic behavior of the potentials $V \in C^2(\mathbb{R}^d)$, $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ imposed in Section 2 obviously imply the following global bounds:

(**W***)**:** There exist $R \geq 2$ and $J, I \geq 0$ such that for all $q, q' \in \mathbb{R}^d$ holds

$$|\nabla_q^{(l)} W(q, q')| \left(1 + |q|^l + |q'|^l\right) \leq J\left(|q|^R + |q'|^R\right) + I, \quad l = 0, 1, 2,$$

(**V***)**:** The function V and its derivatives are polynomially bounded (i.e., there exist $R' \geq R$ such that

$$\sup_{q \in \mathbb{R}^d} \left\{ |\nabla^{(l)} V(q)| \left(1 + |q|^l\right)^{-R'} \right\} < \infty, \quad l = 0, 1, 2)$$

and, moreover, satisfy the *coercivity* estimate

$$(\nabla V(q), q) \geq K^{-1}|q|^R + L^{-1} \sum_{l=1,2} |\nabla^{(l)} V(q)| \left(1 + |q|^l\right) - M \tag{i}$$

with some $K, L, M > 0$, uniformly for all $q \in \mathbb{R}^d$. Additionally, the following relation between the parameters holds:

(ii) $$\Theta := 2dJKR < 1 \quad \text{and hence also} \quad \Theta_0 := 4dJK < 1.$$

Below we shall only use $(\mathbf{V}^*)$ and $(\mathbf{W}^*)$. So, Theorems 1 and 2 in fact hold under these much weaker conditions.

In order to control the properties of the logarithmic derivatives b_{h_i}, we introduce (*conventionally* in this paper) the following characteristic of the vector field ∇V.

Definition 3. *The functional*

$$\mathbb{C}_\beta \ni \phi \mapsto \Phi(\phi) := (\nabla V(\phi), \phi)_{\mathbb{L}^2_\beta} \in \mathbb{R}, \tag{4.1}$$

is called the coercivity functional corresponding to the vector field $\mathbb{C}_\beta \ni \varphi \mapsto \nabla V(\varphi) \in \mathbb{C}_\beta$ w.r.t. the tangent Hilbert space $\mathbb{L}^2_\beta$.

Assumption $(\mathbf{V}^*\mathbf{ii})$ obviously implies the uniform lower boundedness of Φ, i.e., that

$$\hat{\Phi}(\phi) := \Phi(\phi) + \beta M \geq 0, \quad \forall \phi \in \mathbb{C}_\beta. \tag{4.2}$$

Because of the identity $\mathcal{G}_\beta = \mathcal{M}_b$, it is equivalent to prove the statements of our main Theorems 1 and 2 for tempered measures $\mu \in \mathcal{M}_b^t := \mathcal{M}_b \cap \mathcal{M}_\beta^t$. The proof of Theorems 1 and 2 will be based on Lemmas 2–5 below.

4.1. **Integrability of $\mathbb{L}^R_\beta-$norms.** As a first preliminary result we prove a priori estimates in the spaces $\mathbb{L}^R_\beta$.

Lemma 2. *Suppose that the parameters in assumptions $(\mathbf{V}^*)$ and $(\mathbf{W}^*)$ satisfy the following relation:*

$$\Theta_0 := 4dJK < 1 \tag{4.3}$$

Then for every $Q \geq 1$ there exists $\mathcal{C}_Q \in (0, \infty)$ such that a priori for all $\mu \in \mathcal{M}_b^t$

$$\sup_{k \in \mathbb{Z}^d} \int_{\Omega_\beta} |\omega_k|^{RQ}_{\mathbb{L}^R_\beta} d\mu(\omega) \leq \mathcal{C}_Q. \tag{4.4}$$

Proof. We perform induction on $Q \in \mathbb{N}$ and respectively divide the proof into several steps.

Step 1. Fix an arbitrary, but small enough $\delta \in (0, \frac{1}{2})$ such that

$$\Theta_\delta := 2dJK_0\left(1 + e^\delta\right) + \kappa d\delta\left[1 + 2dK\left(1 + e^\delta\right)\right] < 1 - 2\delta. \tag{4.5}$$

In view of the definition (2.7) of Ω^t_β, we introduce a sequence of weights $\gamma_{\delta,k}$, $k \in \mathbb{Z}^d$:

$$\gamma_{\delta,k} := \exp(-\delta|k|), \quad |\gamma_\delta|_{l^1} := \sum\nolimits_{k \in \mathbb{Z}^d} \gamma_{\delta,k} < \infty.$$

To get the required estimate (4.4) uniformly in $k \in \mathbb{Z}^d$, we endow the space Ω^t_β with the system of (mutually equivalent) norms $||\cdot||_{\delta,k_0}$, $k_0 \in \mathbb{Z}^d$:

$$||\omega||_{\delta,k_0} := \left[\sum\nolimits_{k \in \mathbb{Z}^d} \gamma_{\delta,k-k_0} |\omega_k|^R_{\mathbb{L}^R_\beta}\right]^{\frac{1}{R}} \leq (\gamma_{\delta,k_0})^{-\frac{1}{R}} ||\omega||_\delta < \infty. \tag{4.6}$$

Now let us take any measure $\mu \in \mathcal{M}_b^t$. For given $Q \geq 1$, $k \in \mathbb{Z}^d$ and $\alpha \in \{1, ..., d\}$, consider the following family of test functions on Ω_β^t:

$$f(\omega) := f_{\tau,\sigma,\varepsilon}(\omega) := (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau)) Z_{\sigma,\varepsilon}^{-1}(\omega), \quad \tau \in S_\beta, \quad \sigma, \varepsilon > 0, \tag{4.7}$$

where the mapping $F_{k,\alpha} : \Omega_\beta \to C_\beta$ is defined by (3.15). Here we set

$$Z(\omega) := Z_{\sigma,\varepsilon}(\omega) := 1 + \sigma ||\omega||_{\delta,k_0}^{2RQ} + \varepsilon |\omega_k|_{\mathbb{C}_\beta}^{2R'Q} \tag{4.8}$$

with a large enough $R' \geq R$ from Assumption ($\mathbf{V}^*$). Then $f \in C^{1,\pm}_{\mathsf{dec}}(\Omega_\beta^t; h_i)$ for each $i = (k, n, \alpha) \in \mathcal{I}$. Hence according to Remark 3 one can correctly apply to such f and μ the extended version (3.30) of integration by parts in all (e.g., not necessarily basis) directions $h \in \Omega_\beta$ of the form (3.2) with any $0 \neq \varphi \in \mathcal{T}_\beta$ instead of φ_n. By the chain rule for all $\omega \in \Omega_\beta^t$

$$\begin{aligned} \partial_h^{\pm} f(\omega) &= (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} \left[\partial_h F_{k,\alpha}(\omega(\tau))\right] Z^{-1}(\omega) \\ &+ R(Q-1)(||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)-1} \left[\partial_h^{\pm} ||\omega||_{\delta,k_0}\right] F_{k,\alpha}(\omega(\tau)) Z^{-1}(\omega) \\ &- (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau)) \left[\partial_h^{\pm} Z(\omega)\right] Z^{-2}(\omega). \end{aligned} \tag{4.9}$$

Due to Young's inequality and assumptions ($\mathbf{V}^*$), ($\mathbf{W}^*$), one has the following bounds in the RHS of (4.9):

$$\begin{aligned} |\varphi|_{C_\beta}^{-1} |\partial_h^{\pm} Z(\omega)| Z^{-1}(\omega) &\leq 2QR' \left(\sigma \beta^{\frac{1}{R}} ||\omega||_{\delta,k_0}^{2RQ-1} + \varepsilon |\omega_k|_{\mathbb{C}_\beta}^{2R'Q-1}\right) Z^{-1}(\omega) \\ &\leq 2QR' \left(\sigma^{\frac{1}{2RQ}} \beta^{\frac{1}{R}} + \varepsilon^{\frac{1}{2R'Q}}\right) =: \mathcal{Z}_{\sigma,\varepsilon} =: \mathcal{Z} \to 0, \quad \sigma, \varepsilon \to +0, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &\max \left\{ |F_{k,\alpha}(\omega)|, \ |\varphi|_{C_\beta}^{-1} |\partial_h F_{k,\alpha}(\omega)| \right\} \\ &\leq \delta \left[\sum\nolimits_{k' \in \partial k} \left(|\omega_k|^R + |\omega_{k'}|^R\right) + (\nabla V(\omega_k), \omega_k) + M\right] + I_\delta =: \hat{F}_{\delta,k}(\omega) \end{aligned} \tag{4.11}$$

with some absolute constant $I_\delta > 0$ (which could be calculated explicitly). Substituting (4.9)–(4.11) into (3.30), we get that

$$\begin{aligned} &\int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau)) (A\varphi, \omega_{k,\alpha})_{L^2_\beta} Z^{-1} d\mu \\ &\leq |\varphi|_{C_\beta} \int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)-1} \left[(1 + \mathcal{Z}) ||\omega||_{\delta,k_0} + \beta^{\frac{1}{R}} R(Q-1)\right] \hat{F}_{\delta,k}(\omega(\tau)) Z^{-1} d\mu \\ &- \int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau)) (F_{k,\alpha}(\omega), \varphi)_{L^2_\beta} Z^{-1} d\mu. \end{aligned} \tag{4.12}$$

The last inequality obviously extends by continuity to arbitrary $\varphi \in \mathcal{D}(A)$.

Step 2. Now we would like to replace φ in (4.13) by the Green function $\mathfrak{G}_\tau := A^{-1}\delta_\tau$, $\tau \in S_\beta$, which is given by the well-known representation

$$\begin{aligned} \mathfrak{G}_\tau(\tau') := (A^{-1}\delta_\tau)(\tau') &= \sum_{n \in \mathbb{Z}} \lambda_n^{-1} \varphi_n(\tau) \varphi_n(\tau') \\ &= \frac{\kappa}{2} \left(e^{-\frac{a}{\sqrt{\mathfrak{m}}}(\beta - \rho(\tau,\tau'))} + e^{-\frac{a}{\sqrt{\mathfrak{m}}}\rho(\tau,\tau')}\right), \quad \tau, \tau' \in S_\beta. \end{aligned} \tag{4.13}$$

Here, for the sake of convenience, we introduce a parameter

$$\kappa := \left[2a\sqrt{\mathfrak{m}} \left(1 - e^{-\frac{a}{\sqrt{\mathfrak{m}}}\beta}\right)\right]^{-1}. \tag{4.14}$$

Moreover, from (4.13) one easily gets the following regularity properties of $\mathfrak{G}_\tau \in \mathcal{D}(A^{\frac{1}{2}}) := W^{2,1}_\beta$ to be crucially used below in the proof of Lemmas 2–5:

$$|\mathfrak{G}_\tau|_{C_\beta} \le \kappa, \quad |\mathfrak{G}_\tau - \mathfrak{G}_{\tau'}|_{C_\beta} \le \kappa \frac{a}{\sqrt{\mathfrak{m}}}\rho(\tau,\tau'), \quad \forall \tau,\tau' \in S_\beta. \tag{4.15}$$

To this end, we construct the (so-called Yosida) approximation of $\varphi_\tau := \mathfrak{G}_\tau$ by $\varphi_\tau^{(N)} \in \mathcal{D}(A)$, $N \in \mathbb{N}$,

$$\varphi_\tau^{(N)} := \left(1 + N^{-1}A\right)^{-1} \mathfrak{G}_\tau, \quad \lim_{N\to\infty} |\varphi_\tau^{(N)} - \mathfrak{G}_\tau|_{C_\beta} = 0. \tag{4.16}$$

Using the fact that A generates a contractive semigroup on C_β, one can easily check that for all $\omega \in \Omega_\beta$

$$\lim_{N\to\infty} (A\varphi_\tau^{(N)}, \omega_{k,\alpha})_{L^2_\beta} = \omega_{k,\alpha}(\tau), \quad \lim_{N\to\infty} (F_{k,\alpha}(\omega), \varphi_\tau^{(N)})_{L^2_\beta} = (A^{-1}F_{k,\alpha}(\omega))(\tau), \tag{4.17}$$

with the uniform bounds

$$|(A\varphi_\tau^{(N)}, \omega_{k,\alpha})_{L^2_\beta}| \le |\omega_k|_{\mathbb{C}_\beta}, \quad |(F_{k,\alpha}(\omega), \varphi_\tau^{(N)})_{L^2_\beta}| \le \kappa |F_k(\omega)|_{\mathbb{L}^1_\beta}, \tag{4.18}$$

Letting $N \to \infty$ in (4.12) with $\varphi_\tau^{(N)}$ replacing φ, by (4.16)–(4.18) and Lebesgue's dominated convergence theorem we get that

$$\begin{aligned} &\int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau))\omega_{k,\alpha}(\tau) Z^{-1} d\mu \\ &\le \kappa \int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)-1} \left[(1+\mathcal{Z})||\omega||_{\delta,k_0} + \beta^{\frac{1}{R}} R(Q-1)\right] \hat{F}_{\delta,k}(\omega(\tau)) Z^{-1} d\mu \\ &- \int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} F_{k,\alpha}(\omega(\tau))(A^{-1}F_{k,\alpha}(\omega))(\tau) Z^{-1} d\mu. \end{aligned} \tag{4.19}$$

Next, we take the sum of (4.19) over all α and integrate them over $\tau \in S_\beta$. Simply dropping the non-negative term with $(\mathbb{A}^{-1}F_k(\omega), F_k(\omega))_{\mathbb{L}^2_\beta}$, we obtain the estimate

$$\begin{aligned} &\int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)} (F_k(\omega), \omega_k)_{L^2_\beta} Z^{-1} d\mu \\ &\le \kappa d \int_{\Omega_\beta} (||\omega||_{\delta,k_0} + \sigma)^{R(Q-1)-1} \left[(1+\mathcal{Z})||\omega||_{\delta,k_0} + \beta^{\frac{1}{R}} R(Q-1)\right] |\hat{F}_{\delta,k}(\omega)|_{\mathbb{L}^1_\beta} Z^{-1} d\mu. \end{aligned} \tag{4.20}$$

We note that, because of (4.5) and (4.10), $\sigma, \varepsilon > 0$ in (4.20) can be chosen so small that the following relation holds:

$$\begin{aligned} \Xi &:= \Xi_{\delta,\sigma,\varepsilon} := \kappa d\delta\,(1 + \mathcal{Z}_{\sigma,\varepsilon}) \\ &< \Xi_{\delta,\sigma,\varepsilon}\left[1 + 2dK_0\left(1 + e^R\right)\right] + 2dJK_0\left(1 + e^R\right) =: \Theta_{\delta,\sigma,\varepsilon} < 1 - 2\delta. \end{aligned} \tag{4.21}$$

Step 3. In particular, for $Q = 1$ we have from (4.20), (4.21) and assumptions $(\mathbf{V}^*)$ and $(\mathbf{W}^*)$ that

$$\begin{aligned} \int_{\Omega_\beta} |\omega_k|^R_{\mathbb{L}^R_\beta} Z^{-1} d\mu &\le K_0 \int_{\Omega_\beta} \hat{\Phi}(\omega_k) Z^{-1} d\mu \\ &\le K_0 (J + \Xi)(1-\Xi)^{-1} \int_{\Omega_\beta} \sum\nolimits_{k' \in \partial k} \left(|\omega_k|^R_{\mathbb{L}^R_\beta} + |\omega_{k'}|^R_{\mathbb{L}^R_\beta}\right) Z^{-1} d\mu \\ &+ \beta K_0 \left(M + 2dI + \delta^{-1} I_\delta \Xi\right)(1-\Xi)^{-1}. \end{aligned} \tag{4.22}$$

Letting first $\varepsilon \to +0$ in (4.22) and then summing with the weights $\gamma_{\delta,k-k_0}$ over $k \in \mathbb{Z}^d$, by Lebesgue's dominated convergence theorem we find that

$$\int_{\Omega_\beta} ||\omega||^R_{\delta,k_0} Z^{-1}_{\sigma,0} d\mu \le \beta K_0 (2\delta)^{-1} |\gamma_\delta|_{l^1} \left(M + 2dI + \delta^{-1} I_\delta + 1\right) =: \mathcal{C}_1. \tag{4.23}$$

Finally, letting $\sigma \to +0$ in (4.23), by Fatou's lemma we conclude that

$$\sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} |\omega_{k_0}|^R_{\mathbb{L}^R_\beta} d\mu(\omega) \le \sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} ||\omega||^R_{\delta,k_0} d\mu(\omega) \le \mathcal{C}_1 \tag{4.24}$$

and thus, getting back to (4.22),

$$\sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} \hat{\Phi}(\omega_{k_0}) d\mu(\omega) \le \mathcal{C}_1 (\delta^{-1} K_0^{-1} + 1) =: \mathcal{C}_1(\hat{\Phi}). \tag{4.25}$$

Step 4. Let us consider the general case of $Q \ge 2$ provided it is already known that

$$\sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} ||\omega||^{R(Q-1)}_{\delta,k_0} d\mu(\omega) \le \mathcal{C}_{Q-1} < \infty \tag{4.26}$$

(as proved by (4.24) for $Q = 2$). Analogously to deriving estimates (4.22)–(4.25), we reduce (4.20) to

$$\begin{aligned} &\int_{\Omega_\beta} ||\omega||^{R(Q-1)}_{\delta,k_0} |\omega_k|^R_{\mathbb{L}^R_\beta} Z^{-1} d\mu \\ &\le \int_{\Omega_\beta} K_0 ||\omega||^{R(Q-1)}_{\delta,k_0} \hat{\Phi}(\omega_k) Z^{-1} d\mu \\ &\le K_0 (J + \Xi)(1 - \Xi - \delta)^{-1} \int_{\Omega_\beta} ||\omega||^{R(Q-1)}_{\delta,k_0} \sum\nolimits_{k' \in \partial k} \left(|\omega_k|^R_{\mathbb{L}^R_\beta} + |\omega_{k'}|^R_{\mathbb{L}^R_\beta}\right) Z^{-1} d\mu \\ &+ I_{Q,\delta} \int_{\Omega_\beta} \left[||\omega||^{R(Q-1)}_{\delta,k_0} + \sum\nolimits_{k' \in \partial k} \left(|\omega_k|^R_{\mathbb{L}^R_\beta} + |\omega_{k'}|^R_{\mathbb{L}^R_\beta}\right) + \hat{\Phi}(\omega_k) + 1\right] d\mu \end{aligned} \tag{4.27}$$

with some constant $I_{Q,\delta} > 0$ which is independent of $k, k_0 \in \mathbb{Z}^d$. Again, letting $\varepsilon \to +0$ in (4.27) and then summing with the weights $\gamma_{\delta,k-k_0}$ over $k \in \mathbb{Z}^d$, by (4.24)–(4.26) and Lebesgue's dominated convergence theorem we get that

$$\int_{\Omega_\beta} ||\omega||^{RQ}_{\delta,k_0} Z^{-1}_{\sigma,0} d\mu(\omega) \le \delta^{-1} I_{Q,\delta} |\gamma_\delta|_{l^1} \left[\mathcal{C}_{Q-1} + 4d\mathcal{C}_1 + \mathcal{C}_1(\hat{\Phi}) + 1\right] =: \mathcal{C}_{Q,\delta} =: \mathcal{C}_Q. \tag{4.28}$$

Thereafter, letting $\varepsilon \to +0$ in (4.28), by Fatou's lemma we readily obtain that

$$\sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} |\omega_{k_0}|^{RQ}_{\mathbb{L}^R_\beta} d\mu(\omega) \le \sup_{k_0 \in \mathbb{Z}^d} \int_{\Omega_\beta} ||\omega||^{RQ}_{\delta,k_0} d\mu(\omega) \le \mathcal{C}_Q. \tag{4.29}$$

Hence by induction the required estimate (4.4) is valid for all $Q \ge 1$. ■

Analogous a priori estimates hold also for the corresponding local Gibbs specification.

Lemma 3. *Fix any boundary condition*

$$\xi \in \Omega^t_\beta \quad \textit{with} \quad \sup_{k \in \mathbb{Z}^d} |\xi_k|_{\mathbb{L}^R_\beta} =: c_\xi < \infty. \tag{4.30}$$

Then, under the assumptions of Lemma 2, for every $Q \ge 1$ there exists $\mathcal{C}_{Q,\xi} \in (0, \infty)$ such that uniformly for all $\Lambda \Subset \mathbb{Z}^d$:

$$\sup_{k \in \mathbb{Z}^d} \int_{\Omega_\beta} |\omega_k|^{RQ}_{\mathbb{L}^R_\beta} \pi_\Lambda(d\omega|\xi) \le \mathcal{C}_{Q,\xi}. \tag{4.31}$$

Proof. Setting $\mu(d\omega) := \pi_\Lambda(d\omega|\xi)$ with arbitrary $\Lambda \Subset \mathbb{Z}^d$, let us go step by step through the proof of Lemma 2. Since $\omega_{\Lambda^c} = \xi_{\Lambda^c}$ ($\pi_\Lambda(d\omega|\xi) - a.e.$) and $\pi_\Lambda(d\omega|\xi) \in \mathcal{M}^t_{b,h_i}$ provided $k \in \Lambda$, all the above formulas (4.19)–(4.21) are still valid for such k. Now we take in (4.22) resp. (4.27) the weighted sum over all $k \in \Lambda$ and add the term $E_\mu\left(||\omega||^{R(Q-1)}_{\delta,k_0}|\xi_{\Lambda^c}|_{\delta,k_0}\right)$ to both sides of the resulting inequality. If $Q = 1$, in a straightforward way one gets that

$$\sup_{k_0\in\mathbb{Z}^d}\int_{\Omega_\beta}|\omega_{k_0}|^R_{\mathbb{L}^R_\beta}\pi_\Lambda(d\omega|\xi) \le \sup_{k_0\in\mathbb{Z}^d}\int_{\Omega_\beta}||\omega||^R_{\delta,k_0}\pi_\Lambda(d\omega|\xi)$$
$$\le (2\delta)^{-1}|\gamma_\delta|_{l^1}\left[\beta K_0\left(M + 2dI + \delta^{-1}I_\delta + 1\right) + c_\xi\right] =: \mathcal{C}_{1,\xi} \tag{4.32}$$

and, thus,

$$\sup_{k_0\in\Lambda}\int_{\Omega_\beta}\hat{\Phi}(\omega_{k_0})\pi_\Lambda(d\omega|\xi) \le \mathcal{C}_{1,\xi}(\delta^{-1}K_0^{-1} + 1) =: \mathcal{C}_{1,\xi}(\hat{\Phi}) \tag{4.33}$$

uniformly for all $\Lambda \Subset \mathbb{Z}^d$. Thereafter, by induction over $Q \in \mathbb{N}$, we conclude that

$$\sup_{k_0\in\mathbb{Z}^d}\int_{\Omega_\beta}|\omega_{k_0}|^{RQ}_{\mathbb{L}^R_\beta}\pi_\Lambda(d\omega|\xi) \le \sup_{k_0\in\mathbb{Z}^d}\int_{\Omega_\beta}||\omega||^{RQ}_{\delta,k_0}\pi_\Lambda(d\omega|\xi)$$
$$\le \delta^{-1}I_{Q,\delta}|\gamma^R_\delta|_{l^1}\left[\mathcal{C}_{Q-1,\xi} + 4d\mathcal{C}_{1,\xi} + \mathcal{C}_{1,\xi}(\hat{\Phi}) + 1 + c_\xi\mathcal{C}_{Q-1,\xi}I^{-1}_{Q,\delta}\right] =: \mathcal{C}_{Q,\xi}, \tag{4.34}$$

as was required. ■

4.2. **Integrability of the coercivity functional.** Next, we strengthen the assertions of Lemmas 1 and 2 by proving a priori integrability estimates for the coercivity functional Φ.

Lemma 4. *For fixed $k \in \mathbb{Z}^d$, suppose that μ is a measure satisfying the (IbP)-formula (3.30) along all directions h_i, $i = (k, n, \alpha)$ with $n \in \mathbb{Z}$ and $1 \le \alpha \le d$, i.e.,*

$$\mu \in \bigcap_{n\in\mathbb{Z},\ 1\le\alpha\le d}\mathcal{M}_{b,h_i},$$

and, moreover, obeys the moment estimate

$$\sup_{k'\in\{k\}\cup\partial k}\int_{\Omega_\beta}|\omega_{k'}|^{RQ}_{\mathbb{L}^R_\beta}d\mu(\omega) \le \mathcal{C}_Q \tag{4.35}$$

with some given $Q \ge 1$ and $\mathcal{C}_Q \in (0,\infty)$. Under assumptions $(\mathbf{V}^*)$ and $(\mathbf{W}^*)$, (4.32) then implies the (even stronger) integrability property

$$\int_{\Omega_\beta}|\Phi(\omega_k)|^Q d\mu(\omega) \le \mathcal{C}_Q(\Phi) \tag{4.36}$$

with a uniform (i.e., independent of μ and k) constant $\mathcal{C}_Q(\Phi) \in (0,\infty)$.

Proof. We proceed in much the same way as in the proof of Lemma 2 and keep all the notation used there. For $1 \le \alpha \le d$ and $0 \ne \varphi \in \mathcal{T}_\beta$, let us perform integration by parts w.r.t. μ along the corresponding direction $h \in \Omega_\beta$, but for a suitable family of test functions g on Ω^t_β, namely those of type:

$$g(\omega) := g_{\tau,\varepsilon}(\omega) := (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1}F_{k,\alpha}(\omega(\tau))Z_\varepsilon^{-1}(\omega),$$
$$\tau \in S_\beta, \quad \varepsilon > 0. \tag{4.37}$$

Since $|\omega_k|^{QR}_{\mathbb{L}^R_\beta} \in L^1(\mu)$, in definition (4.8) one can already set $\sigma = 0$ so that

$$Z(\omega_k) := Z_\varepsilon(\omega_k) := 1 + \varepsilon|\omega_k|^{2R'Q}_{\mathbb{C}_\beta},$$
$$\text{and}\ \ |\varphi|^{-1}_{C_\beta}|\partial^\pm_h Z(\omega_k)|Z^{-1}(\omega_k) \le 2\varepsilon^{\frac{1}{2R'Q}}R'Q =: \mathcal{Z}_\varepsilon := \mathcal{Z} \to 0, \quad \varepsilon \to +0. \tag{4.38}$$

Next, we choose small enough $\delta, \varepsilon \in (0,1)$ so that

$$\Xi := \Xi_{\delta,\varepsilon} := \kappa d\delta\,[1 + L(Q-1) + \mathcal{Z}_\varepsilon] < 1 - \delta. \tag{4.39}$$

The corresponding derivatives

$$\begin{aligned}\partial_h^{\pm} g(\omega) &= (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1}\,[\partial_h F_{k,\alpha}(\omega(\tau))]\, Z^{-1}(\omega_k)\\ &\quad + (Q-1)(\hat{\Phi}(\omega_k) + \varepsilon)^{Q-2}\left[\partial_h \hat{\Phi}(\omega_k)\right] F_{k,\alpha}(\omega(\tau)) Z^{-1}(\omega_k)\\ &\quad - (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} F_{k,\alpha}(\omega(\tau))\left[\partial_h^{\pm} Z(\omega_k)\right] Z^{-2}(\omega_k)\end{aligned} \tag{4.40}$$

can be obviously estimated by means of (4.11), (4.38) and the following bound in the RHS of (4.40)

$$|\varphi|^{-1}_{C_\beta} |\partial_h \hat{\Phi}(\omega_k)| \le |\nabla V(\omega_k)|_{\mathbb{L}^1_\beta} + |(\nabla^{(2)} V(\omega_k))\omega_k|_{\mathbb{L}^1_\beta} \le L\hat{\Phi}(\omega_k).$$

Hence the (IbP)-formula (3.30) implies that

$$\begin{aligned}&\int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} F_{k,\alpha}(\omega(\tau))(A\varphi, \omega_{k,\alpha})_{L^2_\beta} Z^{-1} d\mu\\ &\qquad \le |\varphi|_{C_\beta}[1 + L(Q-1) + \mathcal{Z}] \int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} \hat{F}_{\delta,k}(\omega(\tau)) Z^{-1} d\mu\\ &\qquad\qquad - \int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} F_{k,\alpha}(\omega(\tau))(F_{k,\alpha}(\omega), \varphi)_{L^2_\beta} Z^{-1} d\mu.\end{aligned}$$

Taking for φ the Green function $\mathfrak{G}_\tau := A^{-1}\delta_\tau$ and integrating over $\tau \in S_\beta$ (cf. the arguments (4.16)–(4.20) above), we arrive at the estimate

$$\begin{aligned}&(1 - \Xi) \int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^Q Z^{-1} d\mu\\ &\qquad \le (J + \Xi) \int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} \sum_{k' \in \partial k} \left(|\omega_k|^R_{\mathbb{L}^R_\beta} + |\omega_{k'}|^R_{\mathbb{L}^R_\beta}\right) Z^{-1} d\mu\\ &\qquad\qquad + \beta(M + 2dI + \delta^{-1} I_\delta \Xi + \varepsilon) \int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^{Q-1} Z^{-1} d\mu.\end{aligned}$$

Then by Hölder's inequality it immediately follows that

$$\begin{aligned}&(1 - \Xi) \left\{\int_{\Omega_\beta} (\hat{\Phi}(\omega_k) + \varepsilon)^Q Z^{-1} d\mu\right\}^{\frac{1}{Q}}\\ &\qquad \le (J + \Xi) \left\{\int_{\Omega_\beta} \left[\sum_{k' \in \partial k} \left(|\omega_k|^R_{\mathbb{L}^R_\beta} + |\omega_{k'}|^R_{\mathbb{L}^R_\beta}\right)\right]^Q d\mu\right\}^{\frac{1}{Q}} + \beta(M + 2dI + \delta^{-1} I_\delta \Xi + \varepsilon).\end{aligned}$$

Letting $\varepsilon \to +0$, we conclude by (4.39) and Fatou's lemma that

$$\sup_{k \in \mathbb{Z}^d} \int_{\Omega_\beta} [\hat{\Phi}(\omega_k)]^Q d\mu(\omega) \le \delta^{-Q} \left[4d\mathcal{C}_Q^{\frac{1}{Q}}(J + 1) + \beta(M + 2dI + \delta^{-1} I_\delta)\right]^Q =: \mathcal{C}_Q(\hat{\Phi}), \tag{4.41}$$

which in turn yields the required estimate (4.33). ■

Corollary 2. *Under the assumptions of Lemma 4*

$$\int_{\Omega_\beta} |F_k(\omega_k)|^Q_{\mathbb{L}^1_\beta} d\mu(\omega) \le \mathcal{C}_Q(F) < \infty. \tag{4.42}$$

Proof. The statement follows immediately from (4.33) and Assumptions ($\mathbf{V}^*$) and ($\mathbf{W}^*$). ■

4.3. **Kolmogorov type moment estimates.** This is the crucial step towards the proof of Theorems 1 and 2.

Lemma 5. *For given $k \in \mathbb{Z}^d$, let the measure*

$$\mu \in \bigcap_{n\in\mathbb{N},\ 1\le\alpha\le d} \mathcal{M}_{b,h_i}\,, \quad i = (k,n,\alpha) \in \mathcal{I},$$

be such that

$$\int_{\Omega_\beta} |F_k(\omega_k)|^{2Q}_{\mathbb{L}^1_\beta} d\mu(\omega) \le \mathcal{C}_{2Q}(F) \tag{4.43}$$

with some fixed $Q \in \mathbb{N}$ and $\mathcal{C}_{2Q}(F) \in (0,\infty)$. Then (4.43) implies the moment estimates :

$$\int_{\Omega_\beta} |\omega_k(\tau) - \omega_k(\tau')|^{2Q} d\mu(\omega) \le \Delta\mathcal{K}_{2Q} \cdot \rho^Q(\tau,\tau'), \tag{4.44}$$

$$\int_{\Omega_\beta} |\omega_k(\tau)|^{2Q} d\mu(\omega) \le \mathcal{K}_{2Q}, \tag{4.45}$$

for all $\tau,\tau' \in S_\beta$ and with uniform (i.e., independent of μ and k) constants $\Delta\mathcal{K}_{2Q}, \mathcal{K}_{2Q} \in (0,\infty)$.

Proof. The proof of Lemma 5 follows the same pattern as in Lemmas 2–4 above. In order to prove (4.44), let us consider the following family of test functions on Ω^t_β :

$$\begin{gathered} f(\omega) := f_{\tau,\tau',\varepsilon}(\omega) := [\Delta\omega_{k,\alpha}]^{2Q-1} Z^{-1}(\omega_k), \\ \tau,\tau' \in S_\beta, \quad \varepsilon > 0, \end{gathered} \tag{4.46}$$

where we set

$$\begin{gathered} \Delta\omega_k := \Delta\omega_k(\tau,\tau') := \omega_k(\tau) - \omega_k(\tau'), \\ Z(\omega_k) := Z_\varepsilon(\omega_k) := 1 + \varepsilon|\omega_k|^{2Q}_{\mathbb{C}_\beta}. \end{gathered}$$

Since we already know that $|F_k|_{\mathbb{L}^1_\beta} \in L^1(\mu)$, one can apply the (IbP)-formula (3.30) to such f in all directions $h \in \Omega_\beta$ of the form (3.2) with any $\varphi \in \mathcal{T}_\beta$ instead of the basis vectors φ_n. The partial derivatives

$$\begin{aligned} \partial_h^{\pm} f(\omega) = (2Q-1)|\Delta\omega_{k,\alpha}|^{2Q-2}\,[\varphi(\tau)-\varphi(\tau')]\,Z^{-1}(\omega_k) \\ - 2\varepsilon Q\,[\Delta\omega_{k,\alpha}]^{2Q-1}\,|\omega_k|^{2Q-1}_{\mathbb{C}_\beta}[\partial^{\pm}_{\varphi}|\omega_{k,\alpha}|_{C_\beta}]Z^{-2}(\omega_k), \end{aligned}$$

can be uniformly estimated for all $\varepsilon > 0$ by

$$|\partial_h^{\pm} f(\omega)| \le 8Q|\varphi|_{C_\beta}|\Delta\omega_{k,\alpha}|^{2Q-2} Z^{-1}(\omega_k). \tag{4.47}$$

Thus, combining (3.18), (4.46) and (4.47), we get that

$$\begin{aligned} &\int_{\Omega_\beta} [\Delta\omega_{k,\alpha}]^{2Q-1}\,(A\varphi,\omega_{k,\alpha})_{L^2_\beta} Z^{-1} d\mu \\ &\le |\varphi|_{C_\beta} \int_{\Omega_\beta} \left[8Q|\Delta\omega_{k,\alpha}|^{2Q-2} + |\Delta\omega_{k,\alpha}|^{2Q-1}|F_k(\omega)|_{\mathbb{L}^1_\beta}\right] Z^{-1} d\mu. \end{aligned} \tag{4.48}$$

Substituting $\Delta\varphi_{\tau,\tau'} := \mathfrak{G}_\tau - \mathfrak{G}_{\tau'}$ for φ in (4.48), we find that

$$\int_{\Omega_\beta} |\Delta\omega_{k,\alpha}|^{2Q} Z^{-1} d\mu \le |\mathfrak{G}_\tau - \mathfrak{G}_{\tau'}|_{C_\beta} \int_{\Omega_\beta} \left[8Q|\Delta\omega_{k,\alpha}|^{2Q-2} + |\Delta\omega_{k,\alpha}|^{2Q-1} |F_k(\omega)|_{\mathbb{L}^1_\beta} \right] Z^{-1} d\mu.$$

Therefore by Hölder's inequality

$$\left(\int_{\Omega_\beta} |\Delta\omega_{k,\alpha}|^{2Q} Z^{-1} d\mu \right)^{\frac{1}{Q}} \le |\mathfrak{G}_\tau - \mathfrak{G}_{\tau'}|_{C_\beta} \left[8Q + \mathcal{C}_{2Q}^{\frac{1}{Q}}(F) + \left(\int_{\Omega_\beta} |\Delta\omega_{k,\alpha}|^{2Q} Z^{-1} d\mu \right)^{\frac{1}{2Q}} \right],$$

which obviously implies

$$\left(\int_{\Omega_\beta} |\Delta\omega_k|^{2Q} Z^{-1} d\mu \right)^{\frac{1}{2Q}} \le d|\mathfrak{G}_\tau - \mathfrak{G}_{\tau'}|^{\frac{1}{2}}_{C_\beta} \left[(8Q)^{\frac{1}{2}} + |\mathfrak{G}_\tau - \mathfrak{G}_{\tau'}|^{\frac{1}{2}}_{C_\beta} \left[\mathcal{C}_{2Q}(F)\right]^{\frac{1}{2Q}} \right].$$

Finally, letting $\varepsilon \to +0$ and using the Lipschitz-continuity of the Green function $\mathfrak{G}_\tau$ (cf. (4.15)), we obtain the required estimate

$$\int_{\Omega_\beta} |\Delta\omega_k(\tau,\tau')|^{2Q} d\mu \le \Delta\mathcal{K}_{2Q} \cdot \rho^Q(\tau,\tau')$$

with constant

$$\Delta\mathcal{K}_{2Q} := (4d^2 k \frac{a}{\sqrt{\mathfrak{m}}})^Q \left[(8Q)^Q + (\beta\kappa \frac{a}{\sqrt{\mathfrak{m}}})^Q \mathcal{C}_{2Q}(F) \right]. \tag{4.49}$$

The proof of estimate (4.42) is analogous except that one should start from the test functions

$$\begin{gathered} g(\omega) := g_{\tau,\varepsilon}(\omega) := [\omega_{k,\alpha}(\tau)]^{2Q-1} Z^{-1}(\omega_k), \\ \tau \in S_\beta, \quad \varepsilon > 0. \end{gathered} \tag{4.50}$$

After integrating by parts with these test functions and substituting $\mathfrak{G}_\tau$ for φ, we get that

$$\int_{\Omega_\beta} |\omega_{k,\alpha}(\tau)|^{2Q} Z^{-1} d\mu \le |\mathfrak{G}_\tau|_{C_\beta} \int_{\Omega_\beta} \left[4Q|\omega_{k,\alpha}(\tau)|^{2Q-2} + |F_k(\omega)|_{\mathbb{L}^1_\beta} |\omega_k^\alpha(\tau)|^{2Q-1} \right] Z^{-1} d\mu.$$

Letting $\varepsilon \to +0$ by Fatou's lemma the latter implies the required estimate, i.e.,

$$\int_{\Omega_\beta} |\omega_k(\tau)|^{2Q} d\mu \le (4d^2\kappa)^Q \left[(4Q)^Q + \kappa^Q \mathcal{C}_{2Q}(F) \right] =: \mathcal{K}_{2Q}. \tag{4.51}$$

■

Remark 4. Contrary to the previous Lemmas 2–4, the coercivity property of ∇V is no more needed for the proof of Lemma 5. In fact, the result only depends on the regularity properties of the Green function $\mathfrak{G}_\tau$ of the elliptic operator A.

4.4. **Proof of Theorems 1 and 2.** Having shown for $\mu \in \mathcal{M}_b^t$ the *a priori* estimates from the lemmas above, we are able to prove immediately the main Theorems 1, 2.

Proof of Theorem 2. Consecutively applying Lemmas 2, 4 and 5 to $\mu \in \mathcal{G}_\beta^t = \mathcal{M}_b^t$, we get the following uniform estimates with finite $\Delta\mathcal{K}_{2Q}$, $\mathcal{K}_{2Q} > 0$:

$$\int_{\Omega_\beta} |\omega_k(\tau) - \omega_k(\tau')|^{2Q} d\mu(\omega) \le \Delta\mathcal{K}_{2Q} |\tau - \tau'|^Q, \tag{4.52}$$

$$\int_{\Omega_\beta} |\omega_k(\tau)|^{2Q} d\mu(\omega) \le \mathcal{K}_{2Q}. \tag{4.53}$$

Now we employ a standard argument related to Kolmogorov's continuity criterion. More precisely, using the Garsia–Rodemich–Rumsey lemma (see e.g. in [BY82, Sect. 3]), one can deduce from (4.52) that

$$\sup_{\mu\in\mathcal{G}^t_\beta}\sup_{k\in\mathbb{Z}^d}\int_{\Omega_\beta}\sup_{\tau\neq\tau'}\left[\frac{|\omega_k(\tau)-\omega_k(\tau')|}{\rho^\eta(\tau,\tau')}\right]^Q d\mu(\omega)<\infty \tag{4.54}$$

for all $Q>2$ and $\eta\in[0,\frac{1}{2}-\frac{1}{Q})$. When $Q\to\infty$, both (4.53) and (4.54) give us the required regularity of $\mu\in\mathcal{G}^t_\beta$, namely that

$$\mu\left([\mathbb{C}^\eta_\beta]^{\mathbb{Z}^d}\right)=1 \quad \text{and} \quad \sup_{\mu\in\mathcal{G}^t_\beta}\sup_{k\in\mathbb{Z}^d}\int_{\Omega_\beta}|\omega_k|^Q_{\mathbb{C}^\eta_\beta}d\mu(\omega)<\infty \tag{4.55}$$

for all $Q\geq 1$ and $\eta\in[0,\frac{1}{2})$. ∎

Proof of Theorem 1. Let $Q>2$ and $\eta\in(0,\frac{1}{2}-\frac{1}{Q})$, and fix any boundary condition $\xi\in\Omega^t_\beta$ with $\sup_{k\in\mathbb{Z}^d}|\xi_k|_{\mathbb{C}^\eta_\beta}<\infty$ (for instance, one can take $\xi=0$). Applying Lemmas 3–5 to the probability kernels $\pi_\Lambda(d\omega|\xi)\in\mathcal{M}^t_{b,h_i}$, provided $i=(k,n,\alpha)\in\mathcal{I}$ and $k\in\Lambda$, we get the following moment estimates with finite $\Delta\mathcal{K}_{2Q,\xi}$, $\mathcal{K}_{2Q,\xi}>0$

$$\int_{\Omega_\beta}|\omega_k(\tau)-\omega_k(\tau')|^{2Q}\pi_\Lambda(d\omega|\xi)\leq\Delta\mathcal{K}_{2Q,\xi}|\tau-\tau'|^Q, \tag{4.56}$$

$$\int_{\Omega_\beta}|\omega_k(\tau)|^{2Q}\pi_\Lambda(d\omega|\xi)\leq\mathcal{K}_{2Q,\xi}, \tag{4.57}$$

uniformly for all $\Lambda\Subset\mathbb{Z}^d$. By the same arguments as those used in the proof of Theorem 2, (4.56) and (4.57) together imply that

$$\sup_{\Lambda\Subset\mathbb{Z}^d}\sup_{k\in\mathbb{Z}^d}\int_{\Omega_\beta}|\omega_k|^Q_{\mathbb{C}^\eta_\beta}\pi_\Lambda(d\omega|\xi)<\infty. \tag{4.58}$$

¿From (4.58) we conclude, using Prokhorov's criterion and the compactness of the embedding of $\mathbb{C}^\eta_\beta$ into $\mathbb{C}^{\eta'}_\beta$ when $0\leq\eta'<\eta$, that the family of distributions $\{\pi_\Lambda(d\omega|\xi\}_{\Lambda\Subset\mathbb{Z}^d}$ is tight on the Polish space Ω_β. So, there exists a sequence $\pi_{\Lambda^{(N)}}(d\omega|\xi)$, $N\in\mathbb{N}$, which converges weakly on Ω_β, as $\Lambda^{(N)}\nearrow\mathbb{Z}^d$, to some probability measure $\mu^*\in\mathcal{M}(\Omega_\beta)$. This means by Proposition 3 that $\mu^*\in\mathcal{M}_b=\mathcal{G}_\beta$. But in fact by (4.58) $\sup_{k\in\mathbb{Z}^d}E_{\mu^*}|\omega_k|^Q_{\mathbb{C}_\beta}<\infty$, and thus by Remark 2 $\mu^*\in\mathcal{G}^t_\beta$. ∎

Proof of Corollary 1. The assertion follows from estimates (4.56), (4.57) similarly as in the proof of Theorem 1. ∎

Remark 4. **(i)** Coercivity assumptions on potentials like ($\mathbf{V}^*$i) are standardly used in mathematical physics, especially when one studies stability properties of dynamical systems (for more concrete applications to the infinite dimensional SDE's see, e.g., [DPZ96, Ce01]). If the initial estimate ($\mathbf{V}$) holds with some $P>R$, then so does ($\mathbf{V}^*$i) with *arbitrary* small $K>0$. Moreover, it is easy to show that ($\mathbf{V}^*$i) implies that the potential V grows strongly enough: for any $K_1>K$ there exists $M_1:=M_1(K_1)>0$ such that for all $q\in\mathbb{R}^d$

$$V(q)\geq K_1^{-1}R^{-1}|q|^R-M_1. \tag{4.59}$$

(ii) Roughly speaking, our assumptions mean that the pair interaction is dominated by the single-particle one (so-called *lattice stabilization*). At first, $\Theta<1$ guarantees by (4.59) the semiboundedness from below of the interaction in all finite volumes $\Lambda\Subset\mathbb{Z}^d$, and thus the well-definedness of the corresponding Gibbs specifications π_Λ (cf. Subsect. 2.3). Secondly, ($\mathbf{V}^*$ii) ensures that also $\Theta_0 < 1$, which will be crucial for proving a priori estimates for symmetrizing measures on loop spaces. In fact, from the potential V one can always extract the quadratic term $\frac{a^2}{2}|q_k|^2$ with a small $a^2 > 0$, so that ($\mathbf{V}^*$ii) is still true.

5. Possible generalizations and concluding remarks

(i) Existence of superstable Gibbs states: According to its definition (2.15), $\mathcal{G}_\beta^t$ contains a class $\mathcal{G}_\beta^{sst}$ of so-called Ruelle type "*superstable*" Gibbs measures, which (for the particular case $R = 2$) has been introduced in [PY94] as those measures satisfying the following support condition

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum\nolimits_{|k|\le N} |\omega_k|^2_{\mathbb{L}^2_\beta} \right\} \le C(\omega) < \infty,\ \forall\omega\in\Omega_\beta\ (\mu-\text{a.e.}). \tag{5.1}$$

On the other hand, by an obvious modification of the arguments used in the proof of Theorem 1, one can also construct *periodic* Euclidean Gibbs measures $\mu^{per} \in \mathcal{G}_\beta^t$ which are invariant w.r.t. the group of translations of the lattice $\mathbb{Z}^d$. But for any measure $\mu \in \mathcal{M}(\Omega_\beta)$, which is *translation invariant* and satisfies the a priori estimates (2.16), the support condition (even stronger than (5.1)) holds for all $Q \ge 1$ and $\eta \in [0, \frac{1}{2})$, namely:

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum\nolimits_{|k|\le N} |\omega_k|^Q_{\mathbb{C}^\eta_\beta} \right\} \le C_{Q,\eta}(\omega) < \infty, \quad \forall\omega\in\Omega_\beta \ \ (\mu-\text{a.e.}). \tag{5.2}$$

The latter follows from the Birkhoff–Khinchin ergodic theorem applied to the stationary process ω_k, $k \in \mathbb{Z}^d$, on the probability space $(\Omega_\beta, \mathcal{B}(\Omega_\beta), \mu)$. This means that we can refine the statement of Theorem 1, claiming the existence of $\mu \in \mathcal{G}_\beta^{sst}$ satisfying the support condition (5.1).

(ii) Generalization to many-particle interactions with possibly infinite range and infinite order: Our results generalize to quantum lattice systems with (*not necessarily translation-invariant)* many-particle interactions. Such systems are described by the heuristic infinite dimensional Hamiltonian

$$H = \sum_{k\in\mathbb{Z}^d} H_k + \sum_{n=2}^{N} \sum_{\{k_1,\dots,k_n\}\subset\mathbb{Z}^d} W_{\{k_1,\dots,k_n\}}(q_{k_1},\dots,q_{k_n}), \tag{5.3}$$

where the n-particle interaction potentials, taken over all finite sets $\{k_1, ..., k_n\} \subset \mathbb{Z}^d$ with $n = 2, ..., N$ and $N \in \mathbb{N} \cup \{+\infty\}$, are given by twice continuously differentiable *symmetric* functions $W_{\{k_1,...,k_n\}} \in C^2(\mathbb{R}^{dn} \to \mathbb{R})$. Then the statements of Theorems 1 and 2 for corresponding Euclidean Gibbs measures still hold under Assumptions ($\mathbf{V}$) and ($\mathbf{W}^{**}$), where ($\mathbf{W}^{**}$) is the following modification of ($\mathbf{W}^*$):

($\mathbf{W}^*$)**:** There exist $R \ge 2$, $I \ge 0$ and symmetric matrices $\{J_{k_1,...,k_n}\}_{(k_1,...,k_n)\in\mathbb{Z}^{nd}}$, with positive entries, such that for all $n \le N$, $\{k_1, ..., k_n\} \subset \mathbb{Z}^d$ and $q_1, ..., q_n \in \mathbb{R}^d$

$$|\nabla^{(l)}_{q_1} W_{\{k_1,\dots,k_M\}}(q_1,\dots,q_n)|(1+\sum_{m=1}^{n}|q_m|^l) \le J_{k_1,\dots,k_n}\sum_{m=1}^{n}|q_m|^R + I, \quad l=0,1,2. \tag{i}$$

Moreover, the matrices $\{J_{k_1,...,k_n}\}_{(k_1,...,k_n)\in\mathbb{Z}^{dn}}$, $n = 2, ..., N$, are *exponentially fastly decreasing*, that is for any $\delta > 0$

$$||J||_\delta := \sum_{n=2}^{N} n^R \sup_{k_1\in\mathbb{Z}^d} \left\{ \sum_{\{k_2,\dots,k_n\}\subset\mathbb{Z}^d} J_{k_1,\dots,k_n} e^{\delta\sum_{m=1}^{n}|k_1-k_m|} \right\} < \infty. \tag{ii}$$

The proofs are as before (at least in spirit). For details we refer to the forthcoming paper [AKPR02].

6. Appendix

Here we briefly illustrate the connection between quantum states and measures on loop spaces following the initial paper [AH-K75]; for a more detailed discussion we refer e.g. [KL81, GK90, MM95, AKKR02a, AKPR02]). Let us start with the one-particle case. Due to Assumption (**V**), for each $k \in \mathbb{Z}^d$ the Hamiltonian H_k is a self-adjoint operator with trace class semigroup $e^{-\tau H_k}$, $\tau \geq 0$. On the algebra $\mathcal{A}_k := \mathcal{L}(\mathcal{H}_k)$ of all bounded linear operators in $\mathcal{H}_k$, we may then define the (time-evolution) authomorphism group $\alpha_{\theta,k}$, $\theta \in \mathbb{R}$, and the quantum Gibbs state $G_{\beta,k}$ acting respectively by

$$\alpha_{\theta,k}(B) := e^{i\theta H_k} B e^{-i\theta H_k}, \quad G_{\beta,k}(B) := Tr(Be^{-\beta H_k})\,/Tr(e^{-\beta H_k})\,, \quad B \in \mathcal{A}_k.$$

For any finite set of multiplication operators $(B_i)_{i=1}^n \in L^\infty(\mathbb{R}^d)$ we construct the so-called temperature (or *Euclidean) Green functions*

$$\Gamma_{\beta,k}^{B_1,\dots,B_n}(\tau_1,\dots,\tau_n) := Tr_{\mathcal{H}_k}\left(\prod_{i=1}^n e^{-(\tau_{i+1}-\tau_i)H_k} B_i\right)/Tr(e^{-\beta H_k})\,, \qquad 0 \leq \tau_1 \leq \dots \leq \tau_n \leq \tau_{n+1} := \tau_1 + \beta. \tag{6.1}$$

These functions have analytic continuations to the complex domain

$$\{(z_i := \tau_i + i\theta_i)_{i=1}^n \subset \mathbb{C}^n \mid 0 < \tau_1 < \dots < \tau_n < \beta\}$$

with the boundary values:

$$\Gamma_{\beta,k}^{B_1,\dots,B_n}(-i\theta_1,\dots,-i\theta_n) = G_{\beta,k}\left(\prod_{i=1}^n \alpha_{\theta_i,k}(B_i)\right). \tag{6.2}$$

Since the algebra spanned by the operators $\alpha_{\theta_i,k}(B_i)$ is dense in $\mathcal{A}_k$, (6.2) *fully* determines the Gibbs state $G_{\beta,k}$. A *crucial* observation is that the Green functions (6.1) may be represented (by the Feynman–Kac formula) as the moments

$$\Gamma_{\beta,k}^{B_1,\dots,B_n}(\tau_1,\dots,\tau_n) = E_{\mu_k}\left(\prod_{i=1}^n B_i(\omega_k(\tau_i))\right) \tag{6.3}$$

of a certain probability measure μ_k on the loop space

$$\mathbb{C}_\beta := \left\{\omega_k \in C([0,\beta] \to \mathbb{R}^d) \mid \omega_k(0) = \omega_k(\beta)\right\}. \tag{6.4}$$

More precisely (for simplicity putting here $\mathfrak{m} = 1$ and $a = 0$),

$$d\mu_k(\omega_k) = \frac{1}{Z} E_\beta^{(x,x)}\left\{-\exp\int_0^\beta V(\omega_k(\tau))d\tau\right\}dx, \tag{6.5}$$

where Z is a normalization constant and $E_\beta^{(x,x)}$ is the conditional expectation, given that $\omega_k(0) = \omega_k(\beta) = x$, w.r.t. the Brownian bridge process of length β in $\mathbb{R}^d$ (cf. [Si74]). So, we get a *one-to-one correspondence* between the quantum Gibbs state $G_{\beta,k}$ on the algebra $\mathcal{A}_k$, Euclidean Green functions (2.6) and the measure μ_k on the loop space $\mathbb{C}_\beta$. Moreover, for all local Hamiltonians H_Λ in volumes $\Lambda \Subset \mathbb{Z}^d$, relations similar to (6.1)–(6.3) are valid for the associated Gibbs states $G_{\beta,\Lambda}$ on the algebra $\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda)$ and the measures μ_Λ on the loop space $[\mathbb{C}_\beta]^\Lambda$. This gives a possible way to construct the limiting states when $\Lambda \nearrow \mathbb{Z}^d$, and hence motivates us to consider the set $\mathcal{G}_\beta$ of all Gibbs measures μ on the "*temperature loop lattice*" $\Omega_\beta := [\mathbb{C}_\beta]^{\mathbb{Z}^d}$, as a natural set of states which for sure contains all accumulation points for $\{\mu_\Lambda\}_{\Lambda \Subset \mathbb{Z}^d}$. What is important, is the non-trivial fact that (analogously to the well-known Osterwalder–Schrader reconstruction theorem in Euclidean field theory, see e.g. [Si74, Frö77, GJ81]) from each such Gibbs measure μ it is possible to *reconstruct* (in a certain sense even uniquely) the quantum Gibbs state G_β of the system (2.2) on the algebra of local observables $\mathcal{A}_{loc} := \cup_{\Lambda \Subset \mathbb{Z}^d} \mathcal{A}_\Lambda$. For the above reasons the measures $\mu \in \mathcal{G}_\beta$ are called *Euclidean Gibbs states (in the temperature loop space representation)* for the quantum lattice system (2.2).

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