Strong Feller Properties for Distorted Brownian Motion and Applications to Finite Particle Systems with Singular Interactions

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ABSTRACT. We prove strong Feller properties for a class of distorted Brownian motions on $\mathbb{R}^d$. We also construct a weak solution to the corresponding stochastic differential equation starting from any point in $\{\varrho \neq 0\}$ and staying in $\{\varrho \neq 0\}$ before possibly going out of any ball in $\mathbb{R}^d$. Here ϱ is the Lebesgue density of the symmetrizing measure μ. Our condition on the logarithmic derivative $\frac{\nabla \varrho}{\varrho}$ is that it should be locally in $L^{d+\varepsilon}$, but only with respect to the symmetrizing measure $\mu = \varrho\, dx$, not necessarily the Lebesgue measure dx. This allows applications to singular situations. In particular, finite particle systems with two body interactions with infinitely strong repulsion can be treated by our results. Among other things it is shown that particles never meet no matter what their starting configuration was. Another application treats diffusions in random media.

Dedicated to Len Gross on the occasion of his 70th birthday.

CONTENTS

1. Introduction

Already in the late seventies and early eighties a number of papers (cf. [**AKS86**], [**AH-KS77**], [**AFKS81**], [**Fu82a, Fu82b**], [**Fu84**]) were devoted to the study

2000 *Mathematics Subject Classification.* primary: 60J60, secondary: 31C25.

of symmetric distorted Brownian motion $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ with singular drift, i.e. $(X_t)_{t\geq 0}$ is the (weak) solution to the stochastic equation

$$dX_t = \sqrt{2}\, dW_t + \frac{\nabla \varrho}{\varrho}(X_t)\, dt\ ,\ X_0 = x\ (\in \mathbb{R}^d), \tag{1.1}$$

with $(W_t)_{t\geq 0}$ = Brownian motion on $\mathbb{R}^d$ and ϱ = Lebesgue density of the symmetrizing measure μ. These papers, as well as their subsequent generalizations to infinite dimensional state spaces (see e.g. [**AR89**], [**AR90**], [**AKR90**], [**AR91**], [**ARZ93**]), were motivated in part by Mathematical Physics, and mostly based on techniques from the theory of symmetric Dirichlet forms. However, these lists of references are far from being complete and the reader should also consult the standard reference on symmetric Dirichlet forms [**FOT94**].

In recent years, the interest in equations of type (1.1) has risen again, since generalizations of distorted Brownian motion to infinite dimensional manifolds, so called "configuration spaces", have been constructed (see e.g. [**Os96**], [**Yo96**], [**AKR96a, AKR96b**], [**AKR98a, AKR98b**], [**Rö98**], [**MR00**], [**GKLR01**]). and identified as describing exactly infinite particle systems in continuum undergoing very singular interactions, which in turn have been in the focus of study in Statistical Mechanics for many years (see e.g. [**La77**], [**Sp86**], [**Fr87**], [**Ta96**], [**Ol94**]).

In this paper we are merely interested in the case of finitely many particles. The motivation is the following. Dirichlet form techniques give weak solutions to (1.1) (respectively their infinite dimensional analogues), which can only start from a set of points in $\mathbb{R}^d$ (respectively in the infinite dimensional state space) whose complement is of zero capacity, but maybe is non-empty. In general it is impossible or at least extremely difficult to give an explicit analytic description of the "allowed" starting points. In this paper we, however, shall prove that under suitable integrability conditions on $\frac{\nabla \varrho}{\varrho}$ with respect to $\mu = \varrho\, dx$ (which still allow applications to the said finite particle systems even if the interactions are as singular as they should be in physically relevant models) (1.1) can be (weakly) solved for any initial condition in $\{\varrho \neq 0\}$. These conditions are as follows: For $\varrho : \mathbb{R}^d \to \mathbb{R}_+$ we shall assume

(H1) $\sqrt{\varrho} \in W^{1,2}_{loc}(\mathbb{R}^d, dx),\ \varrho > 0,\ dx - a.e.$

(H2) $\dfrac{|\nabla \varrho|}{\varrho} = 2\dfrac{|\nabla \sqrt{\varrho}|}{\sqrt{\varrho}} \in L^{d+\varepsilon}_{loc}(\mathbb{R}^d, \mu)$ for some $\varepsilon > 0$.

Here dx denotes Lebesgue measure on $\mathbb{R}^d$, $W^{s,q}_{(loc)}(\mathbb{R}^d, dx),\ s > 0,\ q \geq 1$ the classical (local) Sobolev space of order s in $L^q_{(loc)}(\mathbb{R}^d, dx)$, and $\mu := \varrho\ dx$. $L^q_{(loc)}(\mu) := L^q_{(loc)}(\mathbb{R}^d, \mu),\ q > 0$, denote the corresponding real (local) L^p-spaces. Corresponding norms are denoted by $\|\cdot\|_{L^q(\mathbb{R}^d,\mu)},\ \|\cdot\|_{W^{s,q}(\mathbb{R}^d,dx)}$ etc.

(H1) alone already implies that the symmetric positive definite bilinear form

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} < \nabla u, \nabla v >_{\mathbb{R}^d} d\mu\ ;\ u,v \in C_0^\infty(\mathbb{R}^d) \tag{1.2}$$

is closable on $L^2(\mathbb{R}^d, \mu)$ and that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular local symmetric Dirichlet form (cf. [**Fu84**]).

By the regularity $(\mathcal{E}, D(\mathcal{E}))$ is associated with a Hunt process $\tilde{\mathbb{M}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d_\Delta$:= Alexandrov compactification of

$\mathbb{R}^d_\Delta$, with continuous sample paths and lifetime ζ. Since $(\mathcal{E}, D(\mathcal{E}))$ has no zero order (= killing) part, we may assume that

$$\tilde{\Omega} = \{\omega = (\omega(t))_{t\geq 0} \in C([0,\infty), \mathbb{R}^d_\Delta) \mid \omega(t) = \Delta \; \forall \, t \geq \zeta\} \tag{1.3}$$

(cf. [**FOT94**, Theorem 4.5.3]), and

$$\tilde{X}(t) = \omega(t), \, \omega \in \tilde{\Omega}, \, t \geq 0. \tag{1.4}$$

$(\tilde{X}_t)_{t\geq 0}$ weakly solves (1.1) under $\tilde{\mathbb{P}}_x$ for $x \in \mathbb{R}^d$ outside a set of capacity zero. We refer to [**Fu80**] and [**FOT94**] for details and terminology.

We shall also discuss the case where in addition to (H1), (H2), we also have that $(\mathcal{E}, D(\mathcal{E}))$ is conservative (cf. (H3) in Section 3 below), in which we get slightly more refined results.

We note that (as will be explained in the main body of the paper) (H2) implies that ϱ is continuous (or more precisely has a Hölder-continuous dx-version, cf. Corollary 2.2 below). So, the set $\{\varrho > 0\}$, which we shall identify as the set of allowed starting points, is open. Then our main result can be formulated as follows:

THEOREM 1.1. *Suppose that* (H1) *and* (H2) *with* $p := d + \varepsilon$ *hold. Then there exists a diffusion process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\{\varrho>0\}})$ *(i.e. a strong Markov process with continuous paths) with state space* $\{\varrho > 0\}$ *and cemetery* $\Delta :=$ *Alexandrov point of* $\mathbb{R}^d$*, whose transition semigroup* $(P_t)_{t>0}$ *is* $\mathcal{L}^r(\mu)$*-strong Feller (i.e.* $P_t\mathcal{L}^r(\mu) \subset C(\{\varrho > 0\})$*, see Section 3),* $r \in [p, \infty)$*, and which solves* (1.1) *in the (weak) sense for all initial conditions* $x \in \{\varrho > 0\}$*. If* $(\mathcal{E}, D(\mathcal{E}))$ *is, in addition, conservative, then so is* $\mathbb{M}$*. Furthermore,* $(P_t)_{t>0}$ *is strong Feller in this case (i.e.* $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\{\varrho > 0\})$ *for all* $t > 0$*).*

The process in Theorem 1.1 is constructed below directly and is not derived from the process $\tilde{\mathbb{M}}$ associated to the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. But the mere existence of $\tilde{\mathbb{M}}$ solving (1.1) from μ*−a.e.* $x \in \mathbb{R}^d$ is used in a crucial way. Other main ingredients of the proof are results on elliptic regularity obtained by V.Bogachev, N.Krylov and the last named author of this paper in [**BKR97**] and [**BKR01**], as well as techniques from [**Do02**].

The organization of this paper is as follows: In Section 2 we recall the mentioned elliptic regularity results and present the consequences which are relevant for this paper. In Section 3 we construct the $\mathcal{L}^r(\mu)$-strong Feller semigroup (of kernels) and strong Feller resolvent (of kernels) for our diffusion process on $\{\varrho > 0\}$ which itself is constructed in Section 4. Of course, this process is then associated to $(\mathcal{E}, D(\mathcal{E}))$ given as the closure of the form (1.2). In Section 5 we prove that this process solves (1.1) in the sense of (Markov selections for) martingale problems (see [**SV79**]), hence in the weak sense. The first part of Section 6 is devoted to the said application to finite particle systems with singular interactions. Its second part treats diffusions in random media.

Finally we should mention that this paper was strongly motivated by A.V.Skorohod's paper [**Sk99**], in which he constructs solutions to (1.1) starting from $x \in \{\varrho > 0\}$ in the case of finite particle systems as we do in Subsection 6.1. His method is, however, based on Girsanov's transformation and therefore requires assumptions on $\frac{\nabla \varrho}{\varrho}$ which are much stronger than our assumptions (H1) and (H2) above.

2. An elliptic regularity result and its consequences

Suppose that conditions (H1), (H2) (formulated in the Introduction) hold. Let $(L_2, D(L_2))$ be the generators of $(\mathcal{E}, D(\mathcal{E}))$ (cf. (1.2)) on $L^2(\mathbb{R}^d, \mu)$ (cf. [**FOT94**]), which is negative definite and self-adjoint and defined as the Friedrichs extension of its restriction L to $C_0^\infty(\mathbb{R}^d)$, (due to integration by parts) given by

$$Lu := \Delta u + \langle \frac{\nabla \varrho}{\varrho}, \nabla u \rangle_{\mathbb{R}^d}, \; u \in C_0^\infty(\mathbb{R}^d). \tag{2.1}$$

Let $T_t := e^{tL_2}$, $t > 0$, and $G_\lambda := (\lambda - L_2)^{-1}$, $\lambda > 0$, be the corresponding C_0-semigroup, resolvent on $L^2(\mathbb{R}^d, \mu)$, respectively. Then each T_t is sub Markovian, i.e. $0 \leq f \leq 1 \;\Rightarrow\; 0 \leq T_t f \leq 1$, and so is each λG_λ (see [**FOT94**]). It follows that both all T_t and all λG_λ restricted to $L^1(\mathbb{R}^d, \mu) \cap L^\infty(\mathbb{R}^d, \mu)$ extend to bounded operators on all $L^r(\mathbb{R}^d, \mu)$, $r \in [1, \infty)$, with operator norms less than 1. We denote these extensions by the same symbols. $(T_t)_{t>0}$ is then (because of its symmetry on $L^2(\mathbb{R}^d, \mu)$) an analytic semigroup on every $L^r(\mathbb{R}^d, \mu)$, $r \in (1, \infty)$. We refer e.g. to [**LS96**] for details on all this, in particular, Theorem 2.1 therein for the latter. Let $(L_r, D(L_r))$ denote the corresponding generators on $L^r(\mathbb{R}^d, \mu)$. Then for $r \in [1, \infty)$ it is well-known and easy to see that $C_0^\infty(\mathbb{R}^d) \subset D(L_r)$ and for all $u \in C_0^\infty(\mathbb{R}^d)$

$$L_r u = \Delta u + \langle \frac{\nabla \varrho}{\varrho}, \nabla u \rangle_{\mathbb{R}^d}, \tag{2.2}$$

provided $\frac{|\nabla \varrho|}{\varrho} \in L^r_{loc}(\mathbb{R}^d, \mu)$.

We need to prove regularity properties of $(T_t)_{t>0}$ and $(G_\lambda)_{\lambda>0}$. We shall derive these from special cases of the elliptic regularity results in [**BKR97**] and [**BKR01**]. Let us restate these special cases, relevant for this paper.

PROPOSITION 2.1. *(*[**BKR97**, Theorem 1(iii)(b)], [**BKR01**, Remark 2.15]*) Let Ω be an open set in $\mathbb{R}^d$ and $B = (B^i) : \Omega \to \mathbb{R}^d$, $c : \Omega \to \mathbb{R}$ Borel measurable maps. Suppose μ is a (signed) Radon measure on Ω and $f \in L^1_{loc}(\Omega, dx)$ such that $|B|, c \in L^1_{loc}(\Omega, \mu)$ and*

$$\int Nu(x)\, \mu(dx) = \int u(x) f(x)\, dx \;\; \forall\, u \in C_0^\infty(\Omega),$$

where

$$Nu(x) := \Delta u(x) + \langle B(x), \nabla u(x) \rangle + c(x) u(x).$$

If for some $p > d$, $|B| \in L^p_{loc}(\Omega, \mu)$, $c \in L^{pd/(p+d)}_{loc}(\Omega, \mu)$, and $f \in L^{pd/(p+d)}_{loc}(\Omega, dx)$, then $\mu = \varrho\, dx$ with ϱ continuous and

$$\varrho \in W^{1,p}_{loc}(\Omega, dx) \; (\subset C^{1-d/p}_{loc}(\Omega)),$$

where $C^{1-d/p}_{loc}(\Omega)$ denotes the set of all locally Hölder continuous functions of order $1 - d/p$ on Ω. If $\Omega_0 := \Omega \cap \{\varrho > 0\}$ and moreover $f, c \in L^p_{loc}(\Omega_0)$, then for any open ball $B \subset \overline{B} \subset \Omega_0$ there exists $c_B \in (0, \infty)$ (independent of ϱ and f) such that

$$\|\varrho\|_{W^{1,p}(B,dx)} \leq c_B (\|\varrho\|_{L^1(B,dx)} + \|f\|_{L^p(B,dx)}). \tag{2.3}$$

COROLLARY 2.2. *Let $\varrho : \mathbb{R}^d \to \mathbb{R}$ be as specified in the Introduction, so satisfying* (H1), (H2). *Then $\varrho \in W^{1,d+\varepsilon}_{loc}(\mathbb{R}^d, x)$ (with ε as in* (H2)*) and ϱ has a continuous dx-version in $C^{1-d/(d+\varepsilon)}_{loc}(\mathbb{R}^d)$.*

PROOF. For all $u \in C_0^\infty(\mathbb{R}^d)$ and $\mu := \varrho\, dx$ integrating by parts we obtain from (2.1).

$$\int Lu\, d\mu = 0.$$

Since $\frac{|\nabla \varrho|}{\varrho} \in L^{d+\varepsilon}_{loc}(\mathbb{R}^d, \mu)$, the assertion follows by Proposition 2.1. □

Below we shall always consider the continuous version of ϱ and denote it also by ϱ.

COROLLARY 2.3. *Assume* (H1), (H2), *set $p := (d+\varepsilon) \vee 2$ (with ε as in* (H2)*), and let $\lambda > 0$. Suppose $f \in L^r(\mathbb{R}^d, \mu)$, $r \in [p, \infty)$. Then*

$$\varrho\, G_\lambda f \in W^{1,p}_{loc}(\mathbb{R}^d, dx)$$

and for any open ball $B \subset \overline{B} \subset \{\varrho > 0\}$ there exists $c_{B,\lambda} \in (0,\infty)$, independent of f, such that

$$\|\varrho\, G_\lambda f\|_{W^{1,p}(B,dx)} \le c_{B,\lambda}(\|G_\lambda f\|_{L^1(B,\mu)} + \|f\|_{L^p(B,\mu)}). \tag{2.4}$$

PROOF. Let us first assume that $f \in C_0^\infty(\mathbb{R}^d)$. Then $G_\lambda f \in L^\infty(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu)$. So, by the symmetry of L on $L^2(\mathbb{R}^d, \mu)$ we have that

$$\int (\lambda - L)u\, G_\lambda f \varrho\, dx = \int u f \varrho\, dx \quad \forall\, u \in C_0^\infty(\mathbb{R}^d).$$

Now we may apply Proposition 2.1 with the measure $\nu := \varrho\, G_\lambda f$ taking the role of μ to prove the assertion for $f \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^r(\mathbb{R}^d, \mu)$ with respect to $\|\ \|_{L^r(\mathbb{R}^d,\mu)}$, $r \in [1,\infty)$, the assertion follows by continuity. □

COROLLARY 2.4. *Assume* (H1), (H2), *set $p := (d+\varepsilon) \vee 2$ (with ε, as in* (H2)*) and let $t > 0$, $r \in [p, \infty)$.*

(i) Let $u \in D(L_r)$. Then

$$\varrho\, T_t u \in W^{1,p}_{loc}(\mathbb{R}^d, dx)$$

and for any open ball $B \subset \overline{B} \subset \{\varrho > 0\}$ there exists $c_B \in (0,\infty)$ (independent of u and t) such that

$$\begin{aligned} &\|\varrho\, T_t u\|_{W^{1,p}(B,dx)} \\ &\le c_B \left(\|T_t u\|_{L^1(B,\mu)} + \|T_t(1-L_r)u\|_{L^p(B,\mu)}\right) \\ &\le c_B \left(\mu(B)^{\frac{r-1}{r}} \|u\|_{L^r(\mathbb{R}^d,\mu)} + \mu(B)^{\frac{r-p}{rp}} \|(1-L_r)u\|_{L^r(\mathbb{R}^d,\mu)}\right). \end{aligned} \tag{2.5}$$

(ii) Let $f \in L^r(\mathbb{R}^d, \mu)$. Then the above statements still hold with (2.5) *replaced by*

$$\begin{aligned} &\|\varrho\, T_t f\|_{W^{1,p}(B,dx)} \\ &\le \tilde{c}_B\, t^{-1} \|f\|_{L^r(\mathbb{R}^d,\mu)}, \end{aligned} \tag{2.6}$$

where $\tilde{c}_B \in (0,\infty)$ (independent of f and t).

PROOF. (i) We have that

$$T_t u = G_1 T_t (1 - L_r) u.$$

Since $T_t(1 - L_r)u \in L^r(\mathbb{R}^d, \mu)$, the assertion follows by Corrollary 2.3.

(ii) $(T_t)_{t>0}$ is an analytic semigroup on $L^r(\mathbb{R}^d, \mu)$, and since (2.6) holds by (i) for all $f \in C_0^\infty(\mathbb{R}^d)$ which are dense in $L^r(\mathbb{R}^d, \mu)$, the assertion follows by continuity.

□

REMARK 2.5. We note that since by Sobolev $W^{1,p}(B, dx) \subset C^{1-d/p}(B, dx)$ continuously, it follows by (2.5) for $r \in [p, \infty)$, $R > 0$, that the set

$$\{T_t u \mid t > 0,\ u \in D(L_r),\ \|u\|_{L^r(\mathbb{R}^d,\mu)} + \|L_r u\|_{L^r(\mathbb{R}^d,\mu)} \le R\}$$

is equicontinuous on $\{\varrho > 0\}$.

3. Construction of the semigroup and resolvent of kernels

Assume throughout this section that (H1) and (H2) hold and let $p := (d+\varepsilon) \vee 2$ with ε as in (H2). Below for a topological space X with Borel σ-algebra $\mathcal{B}(X)$ we denote the set of all $\mathcal{B}(X)$-measurable $f : X \to \mathbb{R}$ which are bounded, or nonnegative by $\mathcal{B}_b(X)$, $\mathcal{B}^+(X)$ respectively.

For $r \in [1, \infty)$ let $\mathcal{L}^r(\mu)$ denote the set of all $\mathcal{B}(\mathbb{R}^d)$-measurable functions such that $\int |f|^r \, d\mu < \infty$. For $f \in \mathcal{L}^r(\mu)$ and an operator T on $L^r(\mathbb{R}^d, \mu)$ we write Tf in the sense that T is applied to the corresponding μ-class in $L^r(\mathbb{R}^d, \mu)$ provided this class is in the domain of T.

For $t > 0$ and $f \in \mathcal{L}^p(\mu)$ we know by Corollary 2.4(ii) and Sobolev embedding (cf. Remark 2.5) that $T_t f$ has a (real-valued and) continuous, hence unique μ-version $\widetilde{T_t f}$ on $\{\varrho > 0\}$. Furthermore, we have:

LEMMA 3.1. *Let $t > 0$ and $x \in \{\varrho > 0\}$. Then the map*

$$f \mapsto \widetilde{T_t f}(x)$$

on $\mathcal{L}^p(\mu)$ is a Daniell-Integral, hence there exists a unique positive measure $P_t(x, dy)$ on $\mathcal{B}(\mathbb{R}^d)$ such that

$$\widetilde{T_t f}(x) = \int f(y) \, P_t(x, dy) \ \forall\, f \in \mathcal{L}^p(\mu).$$

PROOF. Since the map $f \mapsto \widetilde{T_t f}$ is positive and linear $\mu - a.e.$ on $\{\varrho > 0\}$, it is positive and linear pointwise on $\{\varrho > 0\}$ by continuity. Furthermore, if $f_n \in \mathcal{L}^p(\mu)$, $n \in \mathbb{N}$, such that $f_n \downarrow 0$, then by (2.6) (applied with $r := p$) and Sobolev embedding (cf. Remark 2.5) $\widetilde{T_t f_n}(x) \downarrow 0$. So, we can apply the Daniell-Stone theorem. □

As usual we define for $t > 0$, $x \in \{\varrho > 0\}$, and $f \in L^1(\mathbb{R}^d; P_t(x, dy)) \cup \mathcal{B}^+(\mathbb{R}^d)$

$$P_t f(x) := \int f(y) \, P_t(x, dy) \tag{3.1}$$

and $P_0 f(x) := f(x)$.

PROPOSITION 3.2. *(i) Let $t > 0$. Then $P_t 1(x) \leq 1$ for all $x \in \{\varrho > 0\}$ and there exists a $\mathcal{B}(\{\varrho > 0\} \times \mathbb{R}^d)$-measurable map $(x, y) \mapsto p_t(x, y)$ such that $P_t(x, dy) = p_t(x, y)\ \mu(dy)$. In particular, $P_t(x, \{\varrho = 0\}) = 0$ for all $x \in \{\varrho > 0\}$, so P_t can be considered as a kernel on $\{\varrho > 0\}$ (denoted below by the same symbol).*

*(ii) $(P_t)_{t>0}$ is a semigroup of kernels on $\{\varrho > 0\}$ which is $\mathcal{L}^r(\mu)$-strong Feller for all $r \in [p, \infty)$, i.e. (cf. **[Do02]**) $P_t f \in C(\{\varrho > 0\})$ for all $t > 0$, $f \in \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$.*

(iii) For all $u \in C_0^\infty(\mathbb{R}^d)$ and all $s \geq 0$

$$\lim_{t\to 0} P_{t+s}u(x) = P_s u(x)\ \forall\, x \in \{\varrho > 0\}.$$

(iv) $(P_t)_{t>0}$ is a measurable semigroup on $\{\varrho > 0\}$, i.e. for $f \in \mathcal{B}^+(\mathbb{R}^d)$ the map $(t, x) \mapsto P_t f(x)$ is $\mathcal{B}([0, \infty) \times \{\varrho > 0\})$-measurable.

PROOF. (i) Let K_n, $n \in \mathbb{N}$, be compact sets in $\mathbb{R}^d$ such that $K_n \uparrow \mathbb{R}^d$. Then because $T_t 1_{K_n} \leq 1\ \mu - a.e.$ for all $n \in \mathbb{N}$, we have for all $x \in \{\varrho > 0\}$ by continuity that $P_t 1_{K_n}(x) \leq 1$ for all $n \in \mathbb{N}$. Hence

$$P_t 1(x) = \lim_{n\to\infty} P_t 1_{K_n}(x) \leq 1\ \forall\, x \in \{\varrho > 0\}.$$

Let $N \in \mathcal{B}(\mathbb{R}^d)$, $\mu(N) = 0$. Then $P_t 1_N = 0\ \mu - a.e.$ on $\{\varrho > 0\}$, so by continuity everywhere on $\{\varrho > 0\}$. So, the existence of $p_t(x, y)$ follows. That it can be chosen measurable in both arguments is standard.

(ii) Let $s, t > 0$. Then for all $u \in C_0^\infty(\mathbb{R}^d)$, since $T_{t+s}u = T_t T_s u\ \ \mu - a.e.$,

$$P_{t+s}u(x) = P_t(P_s u)(x) \text{ for } \mu - a.e.\ x \in \{\varrho > 0\}.$$

Since $u, P_s u \in \mathcal{L}^p(\mu)$, both sides of the equality are continuous, so it holds for all $x \in \{\varrho > 0\}$. Now the first assertion follows by a monotone class argument. To show the second let $t > 0$, $r \in [p, \infty)$ and $f \in \mathcal{L}^r(\mu)$. Since $f = f^+ - f^-$, we may assume that $f \geq 0$. Let $\chi_n \in C_0^\infty(\mathbb{R}^n)$, $\chi_n \geq 0$, for $n \in \mathbb{N}$ such that $\chi_n \uparrow 1$ as $n \to \infty$. Set $f_n := \chi_n f$. Then $P_t f_n \uparrow P_t f$ pointwise on $\{\varrho > 0\}$ as $n \to \infty$. But since $f_n \to f$ in $L^r(\mathbb{R}^d, \mu)$ as $n \to \infty$, by (2.6) and Sobolev embedding we also have that $(P_t f_n)_{n\in\mathbb{N}}$ has a continuous pointwise limit on $\{\varrho > 0\}$. So, $P_t f$ must be this limit.

(iii) Let $u \in C_0^\infty(\mathbb{R}^d)$. Then by Remark 2.5 and the definition of $P_t, t > 0$, the set $\{P_t(P_s u) \mid t > 0\}$ is equicontinuous on $\{\varrho > 0\}$. (Note that $P_s u \in D(L_p)$ since $u \in D(L_p)$). Suppose $x \in \{\varrho > 0\}$ such that for some $\delta > 0$ there exists a sequence $(t_n)_{n\in\mathbb{N}}$ tending to zero such that

$$|P_{t_n}(P_s u)(x) - P_s u(x)| \geq \delta\ \forall\, n \in \mathbb{N}. \tag{3.2}$$

Since $T_{t_n} T_s u \to T_s u$ in $L^p(\mathbb{R}^d, \mu)$ as $n \to \infty$, (selecting another subsequence, if necessary) we may assume that for

$$M := \left\{ y \in \{\varrho > 0\} : \lim_{n\to\infty} P_{t_n}(P_s u)(y) = P_s u(y) \right\}$$

we have $\mu(\{\varrho > 0\} \setminus M) = 0$. So, M is dense in $\{\varrho > 0\}$. Since $\{P_{t_n}(P_s u) \mid n \in \mathbb{N}\}$ is equicontinuous, it follows that

$$\lim_{n\to\infty} P_{t_n}(P_s u)(y) = P_s u(y)\ \ \forall\, y \in \{\varrho > 0\}$$

contradicting (3.2).

(iv) This is a consequence of (iii) by a monotone class argument.

□

REMARK 3.3. We would like to point out that it is not claimed above that $P_t f$ for $t > 0$, is continuous for $f \in \mathcal{B}_b(\mathbb{R}^d)$, but only for $f \in \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$. So, $(P_t)_{t>0}$ is not strong Feller in the classical sense. For this reason we shall have a closer look at the corresponding resolvent for which this is the case. We emphasize that all "strong Feller statements" below by Corollaries 2.3 and 2.4 always state that $P_t f$, $R_\lambda f$ are not only continuous, but even in $W^{1,p}_{loc}(\mathbb{R}^d, dx) \subset C^{1-d/p}_{loc}(\mathbb{R}^d)$ for the respectively specified functions f.

For $\lambda > 0$, $f \in \mathcal{L}^p(\mu)$ we know by Corollary 2.3 that $G_\lambda f$ has a unique (real-valued and) continuous version $\widetilde{G_\lambda f}$ on $\{\varrho > 0\}$. Hence because of (2.4) exactly as in Lemma 3.1 one proves:

LEMMA 3.4. *Let $\lambda > 0$ and $x \in \{\varrho > 0\}$. Then the map*

$$f \mapsto \widetilde{G_\lambda f}(x)$$

on $\mathcal{L}^p(\mu)$ is a Daniell-integral, hence there exists a unique positive measure $R_\lambda(x, dy)$ on $\mathcal{B}(\mathbb{R}^d)$ such that

$$\widetilde{G_\lambda f}(x) = \int f(y)\, R_\lambda(x, dy) \;\; \forall\, f \in \mathcal{L}^p(\mu)$$

As usual we define for $\lambda > 0$, $x \in \{\varrho > 0\}$, and $f \in L^1(\mathbb{R}^d, R_\lambda(x, dy)) \cup \mathcal{B}^+(\mathbb{R}^d)$

$$R_\lambda f(x) := \int f(y)\, R_\lambda(x, dy). \tag{3.3}$$

PROPOSITION 3.5. *(i) Let $\lambda > 0$ and $x \in \{\varrho > 0\}$. Then $\lambda R_\lambda 1(x) \leq 1$ and there exists a $\mathcal{B}(\{\varrho > 0\} \times \mathbb{R}^d)$-measurable map $(x, y) \mapsto r_\lambda(x, y)$ such that $R_\lambda(x, dy) = r_\lambda(x, y)\, \mu(dy)$. In particular, $R_\lambda(x, \{\varrho = 0\}) = 0$, so R_λ can be considered as a kernel on $\{\varrho > 0\}$ (denoted below by the same symbol).*

(ii) $(R_\lambda)_{\lambda>0}$ is a resolvent of kernels on $\{\varrho > 0\}$.

(iii) $(R_\lambda)_{\lambda>0}$ is $\mathcal{L}^r(\mu)$-strong Feller for all $r \in [p, \infty]$, i.e. $R_\lambda f \in C_b(\{\varrho > 0\})$ for all $f \in \mathcal{B}_b(\mathbb{R}^d)$, and $R_\lambda f \in C(\{\varrho > 0\})$ for all $\lambda > 0$ and all $f \in \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$.

(iv) Let $\lambda > 0$. Then for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cup \mathcal{B}^+(\mathbb{R}^d)$ and all $x \in \{\varrho > 0\}$

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt. \tag{3.4}$$

(v) For all $u \in C_0^\infty(\mathbb{R}^d)$

$$\lim_{\lambda\to\infty} \lambda R_\lambda u(x) = u(x) \;\; \forall\, x \in \{\varrho > 0\}.$$

PROOF. The proofs of (i), (ii) are because of (2.4) analogous to the corresponding statements in Proposition 3.2.

(iii) Let $\lambda > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d) \cup \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$. Since $f = f^+ - f^-$, we may assume that $f \geq 0$. Then for f_n, $n \in \mathbb{N}$, defined as in the proof of Proposition 3.2(ii), we have $R_\lambda f_n \uparrow R_\lambda f$ pointwise on $\{\varrho > 0\}$ as $n \to \infty$. But since $R_\lambda f_n \to R_\lambda f$ in $L^1(B, \mu)$ and $f_n \to f$ in $L^p(B, \mu)$ as $n \to \infty$ for every ball $B \subset \overline{B} \subset \{\varrho > 0\}$, by (2.4) and Sobolev embedding we also have that $(R_\lambda f_n)_{n\in\mathbb{N}}$ has a continuous pointwise limit on $\{\varrho > 0\}$. So, $R_\lambda f$ must be this limit, which is bounded by (i), if $f \in \mathcal{B}_b(\mathbb{R}^d)$.

(iv) It suffices to consider $f \in C_0^\infty(\mathbb{R}^d)$. Since $G_\lambda f$ is the Laplace transform of $T_t f$, $t > 0$, in $L^2(\mathbb{R}^d, \mu)$, it follows that (3.4) holds for $\mu - a.e.\ x \in \{\varrho > 0\}$. Since both sides of equality (3.4) are continuous on $\{\varrho > 0\}$, the assertion follows.

(v) Transforming the integral in (3.4), the assertion follows from (iv) by Proposition 3.2(iii), and Lebesgue's dominated convergence theorem.

□

LEMMA 3.6. *(i) Let $t, \lambda > 0$ and $f \in \mathcal{B}^+(\mathbb{R}^d)$. Then $P_t f$ (resp. $R_\lambda f$) is lower-semicontinuous on $\{\varrho > 0\}$. If $g \in \mathcal{B}^+(\mathbb{R}^d)$ such that $f \leq g$ and $P_t g$ (resp. $R_\lambda g$) is (real-valued and) continuous on $\{\varrho > 0\}$, then so is $P_t f$ (resp. $R_\lambda f$).*

(ii) Let $\lambda > 0$, $t > 0$. Then $P_t f$ is continuous on $\{\varrho > 0\}$ for all $\mathcal{B}(\mathbb{R}^d)$-measurable $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f| \leq cR_1 1$ for some $c \in (0, \infty)$.

PROOF. (i) We shall prove the assertion for P_t. The proof for R_λ is exactly the same. There exist $f_n \in \mathcal{L}^p(\mu)$, $f_n \geq 0$, $n \in \mathbb{N}$, such that $f_n \uparrow f$. Hence $P_t f$ is lower semicontinuous on $\{\varrho > 0\}$, since each $P_t f_n$ is continuous there. Hence also $P_t(g - f)$ is lower-semicontinuous, so

$$P_t f = P_t g - P_t(g - f)$$

is continuous on $\{\varrho > 0\}$.

(ii) Since $P_t 1 \leq 1$, it follows by Proposition 3.5(iii) that $R_\lambda P_t 1$ is continuous on $\{\varrho > 0\}$. But clearly by (3.4), $R_\lambda P_t 1 = P_t R_\lambda 1$ on $\{\varrho > 0\}$, hence the assertion follows by (i).

□

Let us now assume conservativity of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mathbb{R}^d, \mu)$, more precisely:

(H3) For the adjoint semigroup $(T_t^*)_{t>0}$ of $(T_t)_{t>0}$ considered as a C_0-semigroup on $L^1(E, \mu)$, i.e. $(T_t^*)_{t>0}$ is a (non-C_0) semigroup on $L^\infty(\mathbb{R}^d, \mu)$, we have $T_t^* 1 = 1$ ($\mu - a.e.$) for all $t > 0$.

REMARK 3.7. (i) There is a complete (analytic) characterization of (H3) in terms of properties of ϱ by W.Stannat in [**St99**] to which we refer for details, in particular [**St99**, Corollary 2.2 and Proposition 1.10]. We would only like to mention here that Stannat, in particular, proved that (H3) is equivalent to the so-called L^1-uniqueness, i.e. the property that $C_0^\infty(\mathbb{R}^d)$ is a core for the generator $(L_1, D(L_1))$ of $(T_t)_{t>0}$ on $L^1(\mathbb{R}^d, \mu)$ (cf. [**St99**, Corollary 2.2 and Remark 2.4]). E.g. if $\mu(\mathbb{R}^d) < \infty$, then (H3) holds.

(ii) For global L^p-estimates on the resolvent under global assumptions (on the coefficients) we refer to [**Fu77**].

PROPOSITION 3.8. *If (in addition to* (H1), (H2) *also)* (H3) *holds, then:*

(i) $\lambda R_\lambda 1(x) = 1$ for all $x \in \{\varrho > 0\}$, $\lambda > 0$.

(ii) $(P_t)_{t>0}$ is strong Feller on $\{\varrho > 0\}$, i.e. $P_t(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\{\varrho > 0\})$ for all $t > 0$.

(iii) $P_t 1(x) = 1$ for all $x \in \{\varrho > 0\}$, $t > 0$.

PROOF. (i) Let $K_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, be compact. Then for $t > 0$ and $g \in L^1(\mathbb{R}^d, \mu) \cap L^\infty(\mathbb{R}^d, \mu)$, by the symmetry of T_t on $L^2(\mathbb{R}^d, \mu)$

$$\begin{aligned}\int g\, d\mu &= \int T_t^* 1\, g\, d\mu = \int T_t g\, d\mu \\ &= \lim_{n\to\infty} \int 1_{K_n}\, T_t g\, d\mu \\ &= \lim_{n\to\infty} \int P_t 1_{K_n}\, g\, d\mu = \int P_t 1\, g\, d\mu.\end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $e^{-\lambda t}$ for $\lambda > 0$, and integrating with respect to dt over $[0, \infty)$, by Fubini's theorem we obtain that $\lambda R_\lambda 1 = 1$ $\mu - a.e.$, hence by continuity we obtain (i).

(ii) Let $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $c := \sup_{x\in\mathbb{R}^d} f(x)$. Then by (i) we have that $f \leq cR_1 1$ and the assertion follows by Lemma 3.6(ii).

(iii) Let $t > 0$. It follows by (3.5) that $P_t 1 = 1$ $\mu - a.e.$ But by (ii), $P_t 1$ is continuous on $\{\varrho > 0\}$. So, (iii) follows.

□

4. Construction of the associated diffusion process

Throughout this section we assume (H1) and (H2) and (as before) denote the Alexandrov compactification of $\mathbb{R}^d$ by $\mathbb{R}^d_\Delta$. We extend our semigroup of kernels $(P_t)_{t>0}$ defined in Section 3 as kernels from $\{\varrho > 0\}$ to $\mathcal{B}(\mathbb{R}^d)$ to a semigroup on $\mathcal{B}(\mathbb{R}^d_\Delta)$ in the following standard way: for $x \in \{\varrho > 0\}$, $t > 0$, extend $P_t(x, dy)$ to a probability measure $P_t^\Delta(x, dy)$ on $\mathcal{B}(\mathbb{R}^d_\Delta)$ $(= \{A \subset \mathbb{R}^d_\Delta \mid A \in \mathcal{B}(\mathbb{R}^d)$ or $A = A_0 \cup \{\Delta\}, A_0 \in \mathcal{B}(\mathbb{R}^d)\})$ by setting

$$P_t^\Delta(x, dy) := (1 - P_t(x, \mathbb{R}^d))\varepsilon_\Delta(dy) + P_t(x, dy) \tag{4.1}$$

and for $x \in \{\varrho = 0\} \cup \{\Delta\}$ define

$$P_t^\Delta(x, dy) := \varepsilon_x(dy), \tag{4.2}$$

where ε_x denotes Dirac measure at x.

It is straightforward to check that $(P_t^\Delta)_{t>0}$ is a semigroup of probability kernels on $\mathcal{B}(\mathbb{R}^d_\Delta)$. We extend μ to $\mathcal{B}(\mathbb{R}^d_\Delta)$ by zero. For $n \in \mathbb{N}$, set

$$S_n := \{k2^{-n} \mid k \in \mathbb{N} \cup \{0\}\} \text{ and } S := \bigcup_{n\in\mathbb{N}} S_n.$$

By Kolmogorov's standard construction scheme there exist probability measures $\mathbb{P}_x$, $x \in \mathbb{R}^d_\Delta$, on $\Omega := (\mathbb{R}^d_\Delta)^S$, equipped with the product σ-field $\mathcal{F}^0$, so that $\mathbb{M}^0 := (\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s\in S}, (X^0_s)_{s\in S}, (\mathbb{P}_x)_{x\in\mathbb{R}^d_\Delta})$ is a normal Markov process on $\mathbb{R}^d_\Delta$ with transition semigroup $(P_s^\Delta)_{s\in S}$. Here $X^0_s : (\mathbb{R}^d_\Delta)^S \to \mathbb{R}^d_\Delta$ are the coordinate maps, and $\mathcal{F}^0_s := \sigma(X^0_r \mid r \in S, r \leq s)$. Define

$$\mathbb{P}_\mu := \int_{\mathbb{R}^d_\Delta} \mathbb{P}_x\, \mu(dx). \tag{4.3}$$

There exists a complete metric on $\mathbb{R}^d_\Delta$ compatible with the Alexandrov topology of $\mathbb{R}^d_\Delta$. Our aim is to find a set $\Omega_0 \in \mathcal{F}^0$ such that $\mathbb{P}_x(\Omega_0) = 1$ for all $x \in \{\varrho > 0\}$ and each $\omega = (\omega(s))_{s\in S}$ has a unique extension to a continuous path in $\{\varrho > 0\} \cup \{\Delta\}$ (equipped with the trace topology induced by the Alexandrov topology on $\mathbb{R}^d_\Delta$) which stays in Δ once it gets there.

Now (with the aim to construct Ω_0 as specified above) we are going to use the theory of symmetric Dirichlet forms, in particular, the Hunt process $\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$, with $\tilde{\Omega}$ as in (1.3), associated to $(\mathcal{E}, D(\mathcal{E}))$, which was introduced in the Introduction. We shall follow the notation and terminology of [**FOT94**] without repeating everything here.

The following result is crucial for our further analysis. It holds even merely under (H1).

LEMMA 4.1. *$\{\varrho = 0\}$ is of capacity zero with respect to $(\mathcal{E}, D(\mathcal{E}))$.*

PROOF. See [**Fu84**]. □

By virtue of e.g. [**MR92**, Chap. IV, Lemma 4.5 and Chap. III, Exercise 2.10] Lemma 4.1 implies that there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of $\{\varrho > 0\}$, so that for the first hitting times $\sigma_{K_n^c}$ of their complements $K_n^c := \mathbb{R}^d \setminus K_n$, i.e., $\sigma_{K_n^c} := \inf\{t > 0 \mid \tilde{X}_t \in K_n^c\}$, $n \in \mathbb{N}$, and

$$\tilde{\Omega}_0 := \left\{\omega \in \tilde{\Omega} \mid \omega(0) \in \{\varrho > 0\},\ \lim_{n\to\infty} \sigma_{K_n^c} \geq \zeta\right\}, \tag{4.4}$$

we have

$$\tilde{\mathbb{P}}_\mu(\tilde{\Omega}_0) = 1, \tag{4.5}$$

where

$$\tilde{\mathbb{P}}_\mu := \int_{\mathbb{R}^d} \tilde{\mathbb{P}}_x\ \mu(dx). \tag{4.6}$$

Consider the one-to-one map $G : \tilde{\Omega} \to \Omega$ defined by

$$G(\omega) := (\omega(s))_{s\in S},\ \omega = (\omega(t))_{t\in[0,\infty)} \in \tilde{\Omega}. \tag{4.7}$$

Note that if $\tilde{\mathcal{F}}^0 := \sigma\{\tilde{X}_s \mid s \in S\}$, then obviously $\tilde{\Omega}_0 \in \tilde{\mathcal{F}}^0$ and G is $\tilde{\mathcal{F}}^0/\mathcal{F}^0$-measurable. Define the image measure of the restriction of $\tilde{\mathbb{P}}_\mu$ to $\tilde{\mathcal{F}}^0$ of $\tilde{\mathbb{P}}_\mu$ (defined in (4.7)) under G by

$$\hat{\mathbb{P}}_\mu := \tilde{\mathbb{P}}_\mu\big|_{\tilde{\mathcal{F}}^0} \circ G^{-1}. \tag{4.8}$$

LEMMA 4.2.

$$\hat{\mathbb{P}}_\mu = \mathbb{P}_\mu.$$

PROOF. Let $N \in \mathbb{N}$, $A_0, \dots, A_N \in \mathcal{B}(\mathbb{R}^d_\Delta)$, $s_1, \dots, s_N \in S$, $0 \leq s_1 \leq \dots \leq s_N$. Let $(\tilde{P}_t)_{t>0}$ be the transition semigroup of $\tilde{\mathbb{M}}$ and define $\tilde{P}_t^\Delta(x, dy)$ for $x \in \mathbb{R}^d$ as in (4.1) and $\tilde{P}_t^\Delta(\Delta, dy) := \varepsilon_\Delta$. Then since $\tilde{\mathbb{M}}$ is associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e. $\tilde{P}_t f$ is a μ-version $T_t f$ for all $f \in \mathcal{L}^2(\mu)$, $t > 0$, and since $\mu(\{\varrho > 0\}) = 0$ we obtain by definition of $\hat{\mathbb{P}}_\mu$ that

$$\begin{aligned}
&\hat{\mathbb{P}}_\mu[X_0^0 \in A_0, X_{s_1}^0 \in A_1, \dots, X_{s_N}^0 \in A_N] \\
&\qquad = \int 1_{A_0}\, \tilde{P}_{s_1}^\Delta \left(1_{A_1}\, \tilde{P}_{s_2-s_1}^\Delta\, 1_{A_2}\, \cdots\, 1_{A_{N-1}}\, \tilde{P}_{s_N}^\Delta\, 1_{A_N}\right)\, d\mu \\
&\qquad = \int 1_{A_0}\, P_{s_1}^\Delta \left(1_{A_1}\, P_{s_2-s_1}^\Delta\, 1_{A_2}\, \cdots\, 1_{A_{N-1}}\, P_{s_N}^\Delta\, 1_{A_N}\right)\, d\mu \\
&\qquad = \mathbb{P}_\mu[X_0^0 \in A_0, X_{s_1}^0 \in A_1, \dots, X_{s_N}^0 \in A_N],
\end{aligned}$$

hence $\hat{\mathbb{P}}_\mu = \mathbb{P}_\mu$ on $\mathcal{F}^0$. □

LEMMA 4.3.

$$G(\tilde{\Omega}_0) \in \mathcal{F}^0 \text{ and } \mathbb{P}_\mu(G(\tilde{\Omega}_0)) = 1.$$

PROOF. Since G is one-to-one and $\tilde{\mathcal{F}}^0/\mathcal{F}^0$-measurable, the first assertion follows, since $\mathcal{F}^0$ is countably generated and $\tilde{\mathcal{F}}^0$ is standard Borel (cf. [**Pa67**, Theorem 2.4, p. 135]). The second statement follows, since by Lemma 4.2, (4.8) and (4.5)

$$\mathbb{P}_\mu(G(\tilde{\Omega}_0)) = \hat{\mathbb{P}}_\mu(G(\tilde{\Omega}_0)) = \tilde{\mathbb{P}}_\mu(\tilde{\Omega}_0) = 1.$$

□

To construct from $G(\tilde{\Omega}_0)$ the desired set Ω_0 of $\mathbb{P}_x$-measure for one for all $x \in \{\varrho > 0\}$, we follow the strategy of [**Do02**] and define

$$\Omega_1 := \bigcap_{\substack{s \in S,\\ s>0}} \theta_s^{-1}(G(\tilde{\Omega}_0)), \tag{4.9}$$

where $\theta_s : \Omega \to \Omega$, $\theta_s(\omega) := \omega(\cdot + s)$, for $s \in S$, is the usual time shift operator.

LEMMA 4.4. *Suppose $A \in \mathcal{F}^0$, such that $\mathbb{P}_\mu(A) = 1$. Then for every $s \in S$, $s > 0$,*

$$\mathbb{P}_x(\theta_s^{-1}(A)) = 1 \; \forall \, x \in \{\varrho > 0\}. \tag{4.10}$$

In particular, $\mathbb{P}_x(\Omega_1) = 1$ for all $x \in \{\varrho > 0\}$.

PROOF. Let $x \in \{\varrho > 0\}$. Then by the Markov property and Proposition 3.2(i)

$$\begin{aligned} \mathbb{P}_x(\Omega \setminus \theta_s^{-1}(A)) &= \mathbb{P}_x(\theta_s^{-1}(\Omega \setminus A)) \\ &= \mathbb{E}_x\left[\mathbb{E}_x(1_{\Omega\setminus A} \circ \theta_s \mid \mathcal{F}_s^0)\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_{X_s^0}(1_{\Omega\setminus A})\right] \\ &= P_s(\mathbb{P}_\cdot(\Omega \setminus A))(x) \\ &= \int \mathbb{P}_y(\Omega \setminus A)\, p_s(x,y)\, \mu(dy) \\ &= 0. \end{aligned} \tag{4.11}$$

Here $\mathbb{E}_x(\,\cdot\,)$, $\mathbb{E}_x(\,\cdot \mid \mathcal{F}_s^0)$ denotes expectation, conditional expectation with respect to $\mathbb{P}_x$ respectively. This proves the first assertion, the second then follows by Lemma 4.3. □

REMARK 4.5. Lemma 4.4 will be a key ingredient of our line of arguments below. In its proof we used the absolute continuity of $P_t(x, dy)$ with respect to μ for $x \in \{\varrho > 0\}$, rather than the strong Feller property. This is in contrast to the corresponding results in [**DPR01**] and [**Do02**]. But it is in accordance with general results in the theory of symmetric Dirichlet forms that a number of statements valid for all x ouside a capacity zero set, turn into statements for all x in the state space, if $P_t(x, dy)$ has a density with respect to the underlying measure (cf. [**FOT94**]).

Obviously, Ω_1 defined in (4.9) consists of paths in Ω which have unique continuous extensions to $(0, \infty)$ which still lie in $\{\varrho > 0\} \cup \{\Delta\}$ and stay in Δ once they have hit Δ. So, we have to handle the limits at $s = 0$. To this end we define

$$\Omega_0 := \left\{\omega \in \Omega_1 \,\middle|\, \lim_{s\downarrow 0} X_s^0(\omega) \text{ exists in } \{\varrho > 0\}\right\}. \tag{4.12}$$

We shall see that Ω_0 is our desired set.

LEMMA 4.6. *Let* $x \in \{\varrho > 0\}$. *Then*

$$\lim_{s\downarrow 0} X_s^0 = x \ \ \mathbb{P}_x - a.s. \tag{4.13}$$

PROOF. Let $u_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, separating the points of $\mathbb{R}^d_\Delta$. Fix $n \in \mathbb{N}$ and define

$$f_{n,1} := ((1-L)u_n)^+, \ f_{n,2} := ((1-L)u_n)^-.$$

Since $(1-L)u_n \in \mathcal{L}^p(\mu)$, it follows by Proposition 3.5(iii) that $R_1(f_{n,1})$, $R_1(f_{n,2})$ are both (real-valued and) continuous on $\{\varrho > 0\}$. Furthermore (as is well known and easily follows from the Markov property), $(e^{-s}R_1 f_{n,i}(X_s^0))_{s\in S}$, $i = 1,2$, are positive supermartingales, so by the martingale convergence theorem $\mathbb{P}_x - a.s.$

$$\lim_{s\downarrow 0} e^{-s} R_1 f_{n,i}(X_s^0) \text{ exists in } \mathbb{R}_+ \text{ for } i = 1,2,$$

so $\mathbb{P}_x - a.s.$ on Ω_1

$$\lim_{s\downarrow 0} u_n(X_s^0) \text{ exists in } \mathbb{R} \ \forall\, n \in \mathbb{N}, \tag{4.14}$$

since $u_n = R_1 f_{n,1} - R_1 f_{n,2}$ on $\{\varrho > 0\}$ for all $n \in \mathbb{N}$.
But by Proposition 3.2(iii) for all $n \in \mathbb{N}$

$$\mathbb{E}_x \left[(u_n(X_s^0) - u_n(x))^2\right] = P_s u_n^2(x) - 2u_n(x) P_s u_n(x) + u_n^2(x) \to 0$$

as $s \downarrow 0$, which together with (4.14) implies that $\mathbb{P}_x - a.s.$ on Ω_1

$$\lim_{s\downarrow 0} u_n(X_s^0) = u_n(x) \ \forall\, n \in \mathbb{N}.$$

Since $u_n, n \in \mathbb{N}$, separate the points of $\mathbb{R}^d_\Delta$, this implies (4.13), since $\mathbb{P}_x(\Omega_1) = 1$ by Lemma 4.4. □

Now we define for $\omega \in \Omega$ and $t \geq 0$

$$X_t(\omega) := \begin{cases} \lim\limits_{\substack{s\downarrow t\\ s\in S}} X_s^0(\omega) & , \text{ if } \omega \in \Omega_0 \\ x_0 & , \text{ if } \omega \in \Omega \setminus \Omega_0, \end{cases} \tag{4.15}$$

where x_0 is a fixed point in $\{\varrho > 0\}$.

Now we can formulate the final result of this section.

THEOREM 4.7. *There exists a diffusion process (i.e. strong Markov with continuous sample paths)* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\{\varrho>0\}})$ *with state space* $\{\varrho > 0\}$ *and cemetery* $\Delta :=$ *the Alexandrov point of* $\mathbb{R}^d$*, having as transition semigroup the strong* $\mathcal{L}^r(\mu)$*-Feller semigroup* $(P_t)_{t>0}$, $r \in [p, \infty)$ *(defined in Lemma 3.1 and* (3.1)*).*

PROOF. Defining X_t, $t \geq 0$, as in (4.15) and $(\mathcal{F}_t)_{t\in[0,\infty]}$ as the corresponding natural filtration (see e.g. [**MR92**, Chap. IV, Definition 1.8, formulae (1.6), (1.7)]), the proof is standard. We only note that for $u \in C_0^\infty(\mathbb{R}^d)$ and $t > 0,\ x \in E$,

$$\begin{aligned}\int \left|u(X_t^0) - u(X_t)\right|^2 \, d\mathbb{P}_x &= \lim_{\substack{s\in S\\ s\downarrow t}} \int \left|u(X_t^0) - u(X_s^0)\right|^2 \, d\mathbb{P}_x \\ &= \lim_{\substack{s\in S\\ s\downarrow t}} (p_t u^2(x) - 2p_t(up_{s-t}u)(x) + p_s u^2(x)) \\ &= 0\end{aligned}$$

by Proposition 3.2(iii). Hence $\mathbb{P}_x[X_t^0 \neq X_t] = 0$. □

REMARK 4.8. If, in addition, (H3) holds, one can drop Δ in the above theorem, and one obtains a conservative diffusion process.

5. Solution to the stochastic equation

Throughout this section we assume (H1), (H2) to hold. Our aim is to prove that the diffusion process $\mathbb{M}$ from Theorem 4.7 gives rise to a weak (= martingale) solution to the stochastic equation (1.1) and is unique, in a sense specified below. So, let $\mathbb{P}_x$, $x \in \{\varrho > 0\}$, be as in Theorem 4.7. As usual below we extend every function $f : \{\varrho > 0\} \to \mathbb{R}$ to $\{\varrho > 0\} \cup \{\Delta\}$ by setting $f(\Delta) = 0$. Thus, we can avoid to explicity mention the explosion time (= life time ζ from the previous section), i.e. the time the process leaves any compact set.

The following lemma is crucial to prove that each $\mathbb{P}_x$ solves the martingale problem for $(L, C_0^\infty(\{\varrho > 0\}))$ with the initial condition $x \in \{\varrho > 0\}$.

LEMMA 5.1. *(i) Let $f \in \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$, $f \geq 0$, then for all $t > 0$, $x \in \{\varrho > 0\}$,*

$$\int_0^t P_r f(x) \, dr < \infty,$$

hence

$$\int\int_0^t f(X_r) \, dr \, d\mathbb{P}_x < \infty$$

(ii) Let $u \in C_0^\infty(\mathbb{R}^d)$, $\lambda > 0$. Then

$$R_\lambda((\lambda - L)u)(x) = u(x) \quad \text{for all} \quad x \in \{\varrho > 0\}. \tag{5.1}$$

(iii) Let $u \in C_0^\infty(\mathbb{R}^d)$, $t > 0$. Then

$$P_t u(x) - u(x) = \int_0^t P_r(Lu)(x) \, dr \quad \text{for all} \quad x \in \{\varrho > 0\}. \tag{5.2}$$

PROOF. (i) Let $t > 0,\ x \in \{\varrho > 0\}$. By (3.4) and monotone convergence

$$R_1 f(x) = \int_0^\infty e^{-r} P_r f(x) \, dr.$$

But $R_1 f$ is continuous, hence finite on $\{\varrho > 0\}$, since $f \in \bigcup_{r\in[p,\infty)} \mathcal{L}^r(\mu)$. So, assertion (i) follows.

(ii) Since $G_\lambda(\lambda - L)u = u$ in $L^p(\mathbb{R}^d, \mu)$, it follows that (5.1) holds for μ-a.e. $x \in \{\varrho > 0\}$. Since $(\lambda - L)u \in \mathcal{L}^p(\mu)$, we have that $R_\lambda((\lambda - L)u)$ is continuous on $\{\varrho > 0\}$ and so assertion (ii) follows.

(iii) Note that since $Lu \in \mathcal{L}^p(\mu)$ the integral on the right hand side of (5.2) exists and is a μ-version of $\int_0^t T_r(Lu)\,dr$, as is the left hand side of $T_t u - u$. Hence (5.2) holds for μ-a.e. $x \in \{\varrho > 0\}$. Since the left hand side of (5.2) is continuous on $\{\varrho > 0\}$, the assertion follows if we can prove that so is the right hand side. But by (ii) and (3.4), which extends to all $f \in \mathcal{L}^p(\mu)$, we have for all $x \in \{\varrho > 0\}$

$$\begin{aligned}
&\int_0^t P_r(Lu)(x)\,dr \\
&= e^r \int_0^r e^{-s} P_s(Lu)(x)\,ds\Big|_{r=0}^{r=t} - \int_0^t e^r \int_0^r e^{-s} P_s(Lu)(x)\,ds\,dr \\
&= e^t \left[R_1(Lu)(x) - \int_t^\infty e^{-s} P_s(Lu)(x)\,ds \right] \\
&\quad - \int_0^t e^r \left[R_1(Lu)(x) - \int_r^\infty e^{-s} P_s(Lu)(x)\,ds \right] dr \\
&= e^t [R_1(Lu)(x) - e^{-t} P_t(R_1(Lu))(x)] \\
&\quad - (e^t - 1) R_1(Lu)(x) - \int_0^t P_r(R_1(Lu))(x)\,dr.
\end{aligned}$$

We see that, since all involved functions are in $\mathcal{L}^p(\mu)$, all functions in the last expression are obviously continuous in $x \in \{\varrho > 0\}$ apart from the last. But by Fubini's theorem for $f := (Lu)^+$ or $(Lu)^-$

$$\int_0^t P_r(R_1 f)(x) dr = R_1\Big(\int_0^t P_r f(\cdot) dr\Big)(x).$$

But $\int_0^t P_r f(\cdot)\,dr \in \mathcal{L}^p(\mu)$ since it is a μ-version of $\int_0^t T_r f\,dr$ which is in $L^p(\mathbb{R}^d, \mu)$. So, also this last term is continuous on $\{\varrho > 0\}$. So, assertion (iii) is proved.

□

THEOREM 5.2. *For every $x \in \{\varrho > 0\}$, $\mathbb{P}_x$ from Theorem 4.7 solves the martingale problem for $(L, C_0^\infty(\{\varrho > 0\}))$ with initial condition x, i.e. under $\mathbb{P}_x$ for all $u \in C_0^\infty(\{\varrho > 0\})$*

$$u(X_t) - u(x) - \int_0^t Lu(X_s)\,ds,\ t \geq 0, \tag{5.3}$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale starting at zero.

PROOF. By Lemma 5.1 the proof is now standard by the Markov property. All integrability issues in the computations are clear from Lemma 5.1. □

Now we turn to uniqueness.

DEFINITION 5.3. *A diffusion process $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in\{\varrho>0\}})$ on $\{\varrho > 0\}$ with lifetime ζ', cemetery Δ, and semigroup $(P'_t)_{t>0}$ is said to satisfy the $L^1(\{\varrho > 0\}, \mu)$-martingale problem for $(L, C_0^\infty(\{\varrho > 0\}))$, if*

(i) For some M', $\varepsilon' \in (0, \infty)$

$$\int_E |P'_t f|\,d\mu \leq M' \int_E |f|\,d\mu \ \ \forall f \in C_b(\{\varrho > 0\}),\ t \in (0, \varepsilon').$$

(ii) For all $u \in C_0^\infty(\{\varrho > 0\})$ *under* $\mathbb{P}'_\mu := \int \mathbb{P}'_x \, \mu(dx)$

$$u(X'_t) - \int_0^t Lu(X'_s)\, ds, \; t \geq 0,$$

is an $(\mathcal{F}'_t)_{t\geq 0}$*-martingale.*

PROPOSITION 5.4. *The diffusion process* $\mathbb{M}$ *from Theorem 4.7 solves the* $L^1(\{\varrho > 0\}, \mu)$*-martingale problem for* $(L, C_0^\infty(\{\varrho > 0\}))$.

PROOF. The proof is completely analogous to that of Proposition 8.2 in [**DPR01**]. □

THEOREM 5.5. *("Uniqueness") Assume (in addition to* (H1), (H2)*) that* (H3) *holds. Let* $\mathbb{M}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in\{\varrho>0\}})$ *be a diffusion process on* $\{\varrho > 0\}$ *with transition semigroup* $(P'_t)_{t>0}$ *such that* $\mathbb{M}'$ *satisfies the* $L^1(\{\varrho > 0\}, \nu)$*-martingale problem for* $(L, C_0^\infty(\{\varrho > 0\}))$*. Then* $\mathbb{P}'_x = \mathbb{P}_x$ *for* $\mu - a.e.\ x \in \{\varrho > 0\}$*, where* $\mathbb{P}_x$*,* $x \in \{\varrho > 0\}$*, is the probability measure of* $\mathbb{M}$ *in Theorem 4.7. If, in addition,* $P'_t(C_0^\infty(\{\varrho > 0\})) \subset C(\{\varrho > 0\})$ *for all* $t > 0$*, then* $\mathbb{P}'_x = \mathbb{P}_x$ *for every* $x \in \{\varrho > 0\}$*.*

PROOF. As mentioned in Remark 3.7, by [**St99**] the conservativity (H3) implies L^1-uniqueness of $(L, C_0^\infty(\{\varrho > 0\}))$. Hence the proof of the assertion is completely analogous to that of Theorem 8.3 in [**DPR01**]. □

REMARK 5.6. In a standard way one derives from Theorem 5.2 that $(X_t)_{t\geq 0}$ under $\mathbb{P}_x$ weakly solves the stochastic equation (1.1) for all starting points $x \in \{\varrho > 0\}$, up to the explosion time (= lifetime ζ), i.e. up to the time the process leaves any compact set. The only thing to do is to calculate the quadratic variation of the martingale in (5.3) to be equal to

$$\int_0^t |\nabla u|^2(X_s)\, ds, \; t \geq 0,$$

under $\mathbb{P}_x$ for all $x \in \{\varrho > 0\}$. This can be done directly by a little lengthy calculation with several integrability issues to be solved on the basis of Lemma 5.1 and the arguments in its proof (cf. [**RS02**]). By P.Levy's characterization theorem and a localization argument one then obtains (1.1). We would like to stress that one cannot use the Fukushima decomposition to get (1.1) for $\mathbb{M}$, as is usually done (cf. [**Fu82a, Fu82b**], [**AR91**] and also [**FOT94**]), because this would give (1.1) only for $x \in E$ outside a capacity zero set. More refined results from [**FOT94**] to give (1.1) for every x do not seem to apply directly, since our transition semigroup is only defined for $x \in \{\varrho > 0\}$ and has only a density $(x, y) \mapsto p_t(x, y)$ with respect to μ for such x. But we are convinced that also by these refined results, properly modified, we can derive (1.1) for our $\mathbb{M}$ and all $x \in \{\varrho > 0\}$.

6. Applications to stochastic dynamics

In this chapter we describe two classes of models from Mathematical Physics in which the stochastic equations of the form (1.1) with singular drifts $\frac{\nabla\varrho}{\varrho}$ appear naturally. In our considerations we do not try to analyze these models under most the general assumptions. We rather restrict ourselves to some particular situations which are reasonable from the physical point of view and give an illustration of the effectiveness of our general results in applications. More detailed and extended analysis of these models will be a subject of the forthcoming paper [**AKR02**].

6.1. Stochastic gradient dynamics of N-particle systems. Let us consider a system of N particles in the Euclidean space $\mathbb{R}^d, d \geq 2$, which have positions $x_k \in \mathbb{R}^d, 1 \leq k \leq N$. The particles interact via a pair potential $V : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$.

The stochastic motion of the particles is described by the system of stochastic differential equations (SDE)

$$\begin{aligned} dx_k(t) &= -\sum_{j=1, j\neq k}^{N} \nabla V(x_k(t) - x_j(t))\, dt + \sqrt{2} dw_k(t) \\ x_k(0) &= x_k \in \mathbb{R}^d, 1 \leq k \leq N, \end{aligned} \tag{6.1}$$

where $\{x_k, 1 \leq k \leq N\}$ are different points in $\mathbb{R}^d$ and $\{w_k\ ,\ 1 \leq k \leq N\}$ are independent standard Wiener processes in $\mathbb{R}^d$. Following [**Sk96**] (see also [**Sk99**]) we will call the system regular if in the stochastic motion particles never can hit each other.

We assume that the potential V has the form $V(x) = \phi(|x|), x \in \mathbb{R}^d$, where $\phi \in C^2(\mathbb{R}_+ \setminus \{0\})$ and this function and its first and second derivatives $\phi^{(1)}$, $\phi^{(2)}$ have the following asymptotic properties:

$$\phi^{(2)}(r) \sim \frac{A_1}{r^{\alpha_1+2}}, \quad \alpha_1, A_1 > 0,\ r \to 0+, \tag{6.2}$$

$$|\phi^{(s)}(r)| \leq \frac{A_2}{r^{\alpha_2+s}}, \quad \alpha_2 > d,\ A_2 > 0,\ s = 0,1,2,\ r \geq r_0 > 0. \tag{6.3}$$

We should (with $\phi^0 \equiv \phi$) admit a singularity of ϕ at 0 as a natural condition from the point of view of Mathematical Physics. If $\alpha_1 < d-1$, *i.e.* the singularity of ϕ is not very big, then the system (6.1) has a unique strong solution and this solution is regular, see [**Sk96, Sk99**]. But the physically important case of a superstable potential ($\alpha_1 > d$) is hence not covered and needs the additional analysis provided by this paper.

To include this case in the general framework of the paper we introduce the following potential energy functional

$$E(x) := \sum_{1\leq k<j\leq N} V(x_k - x_j) \quad (\leq +\infty),\ x = (x_1, ..., x_N) \in \mathbb{R}^{dN},$$

and the corresponding density as

$$\varrho(x_1, ..., x_N) = \exp\{-E(x_1, ..., x_N)\}.$$

Then

$$\{\varrho > 0\} = \{x = (x_1, ..., x_N) \in \mathbb{R}^{dN} \mid x_k \neq x_j \text{ if } k \neq j\},$$

and obviously $\varrho \in C_b^2(\mathbb{R}^{dN})$. Note that now

$$\sqrt{\varrho(x)} = \exp\left(-\frac{1}{2}E(x)\right), \sqrt{\varrho} \in C_b^2\left(\mathbb{R}^{dN}\right)$$

and

$$\frac{\nabla\varrho}{\varrho}(x) = \left(-\sum_{j\neq k} \nabla V(x_k - x_j)\right)_{k=1}^{N} \in \mathbb{R}^{dN},\ x \in \{\varrho > 0\}.$$

It is not hard to show that

$$\frac{|\nabla \varrho|}{\varrho} \in L^p_{loc}\left(\mathbb{R}^{dN}, \mu\right), \ p \geq 1, \tag{6.4}$$

where as above $\mu(dx) = \varrho(x)\,dx$. Therefore, both assumptions (H1), (H2) above are satisfied. Moreover, the boundedness of the density ϱ implies that the conservativity assumptions (H3) is also true, see Remark 3.7(i) and e.g. [**Li99**], Theorem 4. Then all results above can be applied to the system under consideration. In particular, for any order of singularity of V at $0 \in \mathbb{R}^d$ there exists a conservative diffusion in S_N which gives the unique solution to the martingale problem associated with the system (6.1). This system is hence regular in the sense of [**Sk96, Sk99**]. The question whether this process gives a strong solution to (6.1) remains open.

6.2. Diffusions in a random media. Consider now another model in which a particle performs a random motion in the Euclidean space $\mathbb{R}^d, d \geq 2$, interacting with randomly distributed impurities. This model can be formalized as follows. The impurities form a locally finite subset (i.e. configuration) $\gamma = \{x_k | k \geq 1\} \subset \mathbb{R}^d$ and the interaction between the moving particle and particles from γ is given by the pair potential V as in Subsection 6.1. To simplify our considerations we assume instead of (6.3) the stronger condition that ϕ has finite support (i.e. we have a finite range of the interaction). The configurations γ are distributed according to a given random point process on $\mathbb{R}^d$. In mathematical physics this random point process usually corresponds to a Gibbs measure ν on the configuration space over $\mathbb{R}^d$. The stochastic dynamics of the considered particle is described by the following SDE:

$$d\xi(t) = -\sum_{k=1}^{\infty} \nabla V(\xi(t) - x_k)\,dt + \sqrt{2}dw(t), \tag{6.5}$$

$$\xi(0) = x \in \mathbb{R}^d \setminus \gamma,$$

where w is the standard Wiener process in $\mathbb{R}^d$. This equation describes a diffusion process with a random drift of a special type. For a review on the stochastic dynamics in random velocity fields see, e.g., [**Ol94**]. Essential difficulties in the study of the solution to (6.5) are related not only to the singularity of the potential V but also to an additional irregularity of the drift term in (6.5) coming from the configuration γ. To be able to control the drift in (6.5) we will restrict the class of admissible configurations. Let $B(x,r) := \{y \in \mathbb{R}^d |\ |y - x| < r\}$ denote the open ball of radius $r > 0$ with center at $x \in \mathbb{R}^d$. Define the set U of all configurations γ in $\mathbb{R}^d$ such that

$$\forall r > 0\ \exists c(\gamma, r) > 0 :$$

$$|\gamma \cap B(x,r)| \leq c(\gamma, r)\sqrt{\log(1 + |x|)}\ , \ \ x \in \mathbb{R}^d, \tag{6.6}$$

where $|A|$ denotes the cardinality of a set A. Note that for many classes of probability measures ν on the configuration spaces we have $\nu(U) = 1$, see [**KKK02**]. In particular, this is true for the well-known Ruelle measures corresponding to pair superstable potentials [**KY93**].

We introduce the potential energy of the particle in the configuration $\gamma \in U$ as a function $E_\gamma : \mathbb{R}^d \to \mathbb{R}$ of the form

$$E_\gamma(x) := \sum_{k=1}^{\infty} V(x - x_k) \, (\leq +\infty) .$$

Then for the density

$$\varrho_\gamma(x) := \exp\{-E_\gamma(x)\}, x \in \mathbb{R}^d,$$

we have $\varrho_\gamma \in C^2(\varrho_\gamma > 0)$ and

$$\frac{\nabla \varrho_\gamma}{\varrho_\gamma}(x) = -\sum_{k=1}^{\infty} \nabla V(x - x_k) .$$

The assumptions (H1) and (H2) are obviously satisfied, but the assumption (H3) needs more delicate considerations. Namely, to show the L^1- uniqueness (cf. Remark 3.7(i)) we still can apply Theorem 4 from [**Li99**], but the verification of the conditions of this theorem uses the a priori information (6.6) about the given configuration $\gamma \in U$ in an essential way. For the technical details we refer the reader to [**KKR02**]. Therefore, for all $\gamma \in U$ we can apply all results of this paper to the equation (6.5) and construct a conservative diffusion on the state space $\mathbb{R}^d \setminus \gamma$ which gives us the unique solution to the martingale problem associated with (6.5). The main problem remaining open in this model is related to the study of the asymptotic behavior of the particle for $t \to \infty$. In this direction we only have some conjectures based on physical intuition rather than on rigorous results. For an excellent discussion of the physical point of view concerning diffusions with a random velocity field we refer to [**BG90**] and the references therein. Rigorous mathematical results have so far only been obtained in the case of drifts which are much more regular than those considered above, see e.g. [**Ol94**].

Acknowledgement

The last named author would like to thank the Scuola Normale Superiore, in particular, his host Giuseppe Da Prato for a very pleasant stay in Pisa during which a part of this work was done. Financial support of the SNS di Pisa as well as of the DFG-Forschergruppe "Spectral Analysis, Asymptotic Distributions, and Stochastic Dynamics" is gratefully acknowledged.

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