Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term *

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Abstract

Following classical work by M. I. Freidlin and subsequent works by R. Sowers and S. Peszat, we prove large deviation estimates for the small noise limit of systems of stochastic reaction-diffusion equations with globally Lipschitz but unbounded diffusion coefficients, however, assuming the reaction terms to be only locally Lipschitz with polynomial growth. This generalizes results of the above mentioned authors. Our results apply, in particular, to systems of stochastic Ginzburg-Landau equations with multiplicative noise.

1 Introduction

In his pioneering paper [10] M. I. Freidlin studied large deviations for the small noise limit of stochastic reaction-diffusion equations. Moreover, he described an entire programme to obtain the desired estimates for such stochastic equations for various levels of generality of assumptions on the coefficient functions. Subsequently, several authors have taken up the challenge of realizing this programme under less and less restrictive conditions (see [18] and [16] and, more recently, [12] and [3] or in case of particularly interesting special cases [2]). Our contribution, which is specially motivated by the study of the works of S. Peszat [16] and R. Sowers [18], goes in the direction of taking systems with reaction term having polynomial growth (and in particular not globally Lipschitz-continuous) and noise of multiplicative type, without any assumptions of boundedness. As we will see in what follows, these two things together create some difficulties, even in the proof of existence of solutions.

There are two approaches to stochastic partial differential equations such as the ones in the focus of this paper: first, the martingale approach initiated by J. Walsh [19], which is pursued in [18] and also e.g. in [3]; second, the semigroup approach presented in [7], which is taken in [16]. Because of our background in infinite dimensional stochastic

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analysis, in this paper we are taking the latter approach, which has a more infinite dimensional-analytic flavour.

The precise type of stochastic reaction-diffusion equations we are interested in are systems of the following type:

$$
\left\{
\begin{aligned}
\frac{\partial u_i}{\partial t}(t,\xi) = {} & \mathcal{A}_i\, u_i(t,\xi) + f_i(t,\xi,u_1(t,\xi),\ldots,u_r(t,\xi)) \\
& + \epsilon \sum_{j=1}^{r} g_{ij}(t,\xi,u_1(t,\xi),\ldots,u_r(t,\xi)) B_j \frac{\partial^2 w_j}{\partial t \partial \xi}(t,\xi), \qquad t \geq 0, \qquad \xi \in \overline{\mathcal{O}}, \\
u_i(0,\xi) = {} & x_i(\xi), \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}_i\, u_i(t,\xi) = 0, \qquad t \geq 0, \qquad \xi \in \partial\mathcal{O}.
\end{aligned}
\right.
\tag{1.1}
$$

Here $\mathcal{O}$ is a bounded open set of $\mathbb{R}^d$, with $d \geq 1$, having smooth boundary. For each $i = 1,\ldots,r$, we have

$$
\mathcal{A}_i(\xi, D) = \sum_{h,k=1}^{d} a^i_{hk}(\xi) \frac{\partial^2}{\partial \xi_h \partial \xi_k} + \sum_{h=1}^{d} b^i_h(\xi) \frac{\partial}{\partial \xi_h}, \qquad \xi \in \overline{\mathcal{O}}.
$$

The coefficients a^i_{hk} are taken in $C^1(\overline{\mathcal{O}})$ and the coefficients b^i_h are taken in $C(\overline{\mathcal{O}})$ and for any $\xi \in \overline{\mathcal{O}}$ the matrix $\left[a^i_{hk}(\xi)\right]$ is non-negative, symmetric and fulfills a uniform ellipticity condition. The operator $\mathcal{B}_i$ acts on the boundary of $\mathcal{O}$ and is assumed either of Dirichlet or of co-normal type. Below, for each $\epsilon > 0$ we denote by $u^x_\epsilon = (u^x_{\epsilon,1},\ldots,u^x_{\epsilon,r})$ the solution of (1.1) in the space E of continuous functions on $\overline{\mathcal{O}}$, with some boundary conditions related to the boundary operators $\mathcal{B}_i$ (see Section 2 for precise definitions).

Under Hypotheses 1 to 4 on the coefficients which are specified in Section 2 below (see Theorems 6.2 and 6.3 for the precise estimates), we shall prove the following large deviations result (cf. e.g. [10] for the Freidlin-Wentzell formulation)

1. for any $x \in E$, $z \in C([0,T];E)$ and $\delta, \gamma > 0$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$
$$
\mathbb{P}\left(|u^x_\epsilon - z|_{C([0,T];E)} \leq \delta\right) \geq \exp\left(-\frac{I_{x,T}(z)+\gamma}{\epsilon^2}\right);
$$
2. for any $x \in E$, $r \geq 0$ and $\delta, \gamma > 0$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$
$$
\mathbb{P}\left(|u^x_\epsilon - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{r-\gamma}{\epsilon^2}\right).
$$

Note that we also prove that estimate 1 is uniform with respect to $|x|_E \leq R$ and $z \in K_{x,T}(r)$, for all $r, R > 0$, and estimate 2 is uniform with respect to $|x|_E \leq R$, for all $R > 0$.

Here, the action functional is given by

$$
I_{x,T}(z) := \frac{1}{2} \inf \left\{ |\varphi|^2_{L^2(0,T;H)}\,;\; z = z^x(\varphi) \right\},
$$

where $z^x(\varphi)$ is the solution of the skeleton equation corresponding to (1.1) (cf. (4.1) and Section 4 below), with the usual convention that $\inf \emptyset := +\infty$. Furthermore,

$$K_{x,T}(r) := \{I_{x,T} \leq r\}\,.$$

Let us list some special characteristics of this result in comparison with some of the above mentioned earlier papers:

1. Unlike in [3], [16] and [18], where global Lipschitz assumptions were imposed, here the functions f_i in (1.1) are only locally Lipschitz and of polynomial growth (see Hypothesis 4 and Remark 2.1 below). The most relevant consequence of this fact is that we cannot work with mild solutions and contractions theorem, so that our work becomes much more delicate.

2. $g = [g_{ij}]$ in (1.1) is not assumed to be globally bounded (as e.g. done in [2], [16] and [18]) and just assumed to be globally Lipschitz (see Hypothesis 3, but also Hypothesis 4-3). Moreover, unlike in [18], g may be degenerate. This means that we can consider for example $g_{ij}(u) = \lambda_{ij} u_j$, with $\lambda_{ij} \in \mathbb{R}$.

3. We consider systems of r coupled stochastic reaction-diffusion equations, ruling out maximum principle and hence comparison techniques commonly used in case $r = 1$.

4. We provide large deviations for paths in the space of continuous functions $C(\overline{\mathcal{O}};\mathbb{R}^r)$ and we can allow $\mathcal{O}$ to be a bounded open subset of $\mathbb{R}^d$ for arbitrary $d \geq 1$ (which does not seem to be possible e.g. under the conditions imposed in [16]).

Though we can still follow the classical approach in [16] (which is based on fundamental ideas from R. Azencott [1], P. Priouret [17] and also R. Leandre [13]), to get the above results, various severe technical difficulties have to be overcome. In particular, we cannot work on $L^2\big(\mathcal{O};\mathbb{R}^r\big)$, but have to work on $C\big(\overline{\mathcal{O}};\mathbb{R}^d\big)$. Otherwise, the necessary trace conditions on the semigroup e^{tA}, $t \geq 0$, generated by $(\mathcal{A}_i)_{1\leq i\leq r}$ do not allow $d \geq 2$ and in contrast to our case e.g. polynomials for f_i would not be included. So, we have to leave the Hilbert space framework of [16], and have to use heat kernel estimates, i.e. precise estimates for the density function of e^{tA}, for $t > 0$.

As we have already said, the motivation for investing so much effort into achieving large deviation estimates in the above general situations is mainly to be able to include polynomial reaction terms and, in addition, having a multiplicative, but unbounded noise. In particular, stochastic Ginzburg-Landau equations are included.

Finally, we would like to point out some specifics of our proof, in particular, in comparison with [16] to which we are mostly indebted, and subsequently give an overview of the single sections of this paper:

1. As in the additive noise case, subtracting the noise part and working "ω by ω", we use the dissipativity conditions (see Hypothesis 4 below) to get the desired estimates. A careful analysis of the noise part, which is technically delicate under our general conditions and does not work *pathwise*, but only in the mean, makes it possible to return to the initial system and to get the necessary estimates.

2. The exponential estimates for the noise part are more difficult since we need them for the sup-norm rather than the L^2-norm. So, we have to work out the intermediate estimates for every $\xi \in \overline{\mathcal{O}}$.

The organization of this paper is as follows. In Section 2, we describe our framework precisely, introduce notations and discuss some preliminaries. Section 3 is devoted to exponential type-estimates for the noise part and for the solution of system (1.1). In Section 4, we analyze the solution to the corresponding skeleton equation, while in Section 5 compactness of the level sets of the action functional is proven. The final Section 6 contains the proofs for the upper and lower bounds.

2 Notations and preliminaries

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$, $d \geq 1$, having a regular boundary. In what follows we denote by H the separable Hilbert space $L^2(\mathcal{O};\mathbb{R}^r)$, with $r \geq 1$, endowed with the usual scalar product $\langle \cdot,\cdot\rangle_H$ and the corresponding norm $|\cdot|_H$. For any $p \geq 1$, $p \neq 2$, the usual norm in $L^p(\mathcal{O};\mathbb{R}^r)$ is denoted by $|\cdot|_p$. If $\epsilon > 0$, we denote by $|\cdot|_{\epsilon,p}$ the norm in $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$.

We are here concerned with the following class of stochastic reaction-diffusion systems

$$\left\{\begin{array}{rl} \dfrac{\partial u_i}{\partial t}(t,\xi) = & \mathcal{A}_i\, u_i(t,\xi) + f_i(t,\xi,u_1(t,\xi),\ldots,u_r(t,\xi)) \\[3mm] & \displaystyle +\sum_{j=1}^r g_{ij}(t,\xi,u_1(t,\xi),\ldots,u_r(t,\xi))B_j\frac{\partial^2 w_j}{\partial t\partial\xi}(t,\xi), \qquad t\geq 0, \qquad \xi\in\overline{\mathcal{O}}, \\[3mm] u_i(0,\xi) = & x_i(\xi), \quad \xi\in\overline{\mathcal{O}}, \qquad \mathcal{B}_i\, u_i(t,\xi)=0, \qquad t\geq 0, \qquad \xi\in\partial\mathcal{O}. \end{array}\right. \tag{2.1}$$

For each $i = 1,\ldots,r$, we have

$$\mathcal{A}_i(\xi,D) = \sum_{h,k=1}^d a^i_{hk}(\xi)\frac{\partial^2}{\partial\xi_h\partial\xi_k} + \sum_{h=1}^d b^i_h(\xi)\frac{\partial}{\partial\xi_h}, \qquad \xi\in\overline{\mathcal{O}}.$$

The coefficients a^i_{hk} are taken in $C^1(\overline{\mathcal{O}})$ and the coefficients b^i_h are taken in $C(\overline{\mathcal{O}})$. For any $\xi \in \overline{\mathcal{O}}$ the matrix $[a^i_{hk}(\xi)]$ is non negative and symmetric and fulfills a uniform ellipticity condition. The operator $\mathcal{B}_i$ acts on the boundary of $\mathcal{O}$ and is assumed either of Dirichlet or of co-normal type.

In what follows we shall denote by A the realization in H of the differential operator $\mathcal{A} = (\mathcal{A}_1,\ldots,\mathcal{A}_r)$ endowed with the boundary condition $\mathcal{B} = (\mathcal{B}_1,\ldots,\mathcal{B}_r)$. As well known (see [8] and [14] for all details), A generates an analytic semigroup e^{tA} in H, which can be extended in a consistent way to all spaces $L^p(\mathcal{O};\mathbb{R}^r)$, for $1 \leq p \leq \infty$, and to the space $C(\overline{\mathcal{O}};\mathbb{R}^r)$.

Notice that as shown in [4, Chapter 6] and [5, Sections 4.1 and 6.1], it will not be restrictive to assume that A generates a semigroup of contractions e^{tA} in each $L^p(\mathcal{O};\mathbb{R}^r)$, with $1 \leq p \leq \infty$, which are self-adjoint on H. Actually, if this is not the case, due to the regularity of coefficients, A can be decomposed as $C+G$, where C is a second order operator which fulfills the desired properties and G is a lower order operator which can be treated as a "good" perturbation of C.

In what follows we shall set

$$E := \overline{D(A)}^{\,C(\overline{\mathcal{O}};\mathbb{R}^r)} = \overline{D(A_1)}^{\,C(\overline{\mathcal{O}};\mathbb{R})} \times \cdots \times \overline{D(A_r)}^{\,C(\overline{\mathcal{O}};\mathbb{R})}.$$

Each set $\overline{D(A_i)}^{\,C(\overline{\mathcal{O}};\mathbb{R})}$ coincides with $C(\overline{\mathcal{O}};\mathbb{R})$ or $C_0(\overline{\mathcal{O}};\mathbb{R})$, if respectively $\mathcal{B}_i$ is a co-normal or Dirichlet boundary condition. In any case, with this definition of the space E, endowed with the sup-norm $|\cdot|_E$ and the duality $\langle \cdot,\cdot\rangle_E := {}_{E^\star}\langle \cdot,\cdot\rangle_E$, it turns out that the part of e^{tA} in E (which we will still denote by e^{tA}) is strongly continuous.

For any $t,\epsilon>0$ and $p\geq 1$ we have that e^{tA} maps $L^p(\mathcal{O};\mathbb{R}^r)$ into $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$ and

$$|e^{tA}x|_{\epsilon,p} \leq c\,(t\wedge 1)^{-\frac{\epsilon}{2}}|x|_p, \qquad x\in L^p(\mathcal{O};\mathbb{R}^r), \tag{2.2}$$

for some constant c independent of p. Due to the Sobolev embedding theorem and to the Riesz-Thorin theorem, this implies

$$|e^{tA}x|_p \leq c\,(t\wedge 1)^{-\frac{d(p-q)}{2pq}}|x|_q, \qquad x\in L^q(\mathcal{O};\mathbb{R}^r).$$

In particular,

$$|e^{tA}x|_E \leq c\,(t\wedge 1)^{-\frac{d}{4}}|x|_H. \tag{2.3}$$

Moreover, as proved e.g. in [8, Lemma 2.1.2], this also implies that e^{tA} admits an integral kernel $K:(0,+\infty)\times\mathcal{O}\times\mathcal{O}\to\mathbb{R}^r$, that is

$$e^{tA}x(\xi) = \left(\int_{\mathcal{O}} K_1(t,\xi,\eta)x_1(\eta)\,d\eta\,,\ldots,\int_{\mathcal{O}} K_r(t,\xi,\eta)x_r(\eta)\,d\eta\right) \qquad t>0,$$

for any $x\in L^1(\mathcal{O};\mathbb{R}^r)$ and $\xi\in\overline{\mathcal{O}}$. Concerning the kernels, there exist two positive constants c_1 and c_2 such that for each $i=1,\ldots,r$

$$0\leq K_i(t,\xi,\eta)\leq c_1\,t^{-\frac{d}{2}}\exp\left(-c_2\,\frac{|\xi-\eta|^2}{t}\right)$$

(see [8, Corollary 3.2.8 and Theorem 3.2.9]). Moreover, as the kernel functions are continuous on $(0,+\infty)\times\overline{\mathcal{O}}\times\overline{\mathcal{O}}$ (for a proof see e.g. [8, Theorem 2.1.4]) it follows that the mapping

$$(0,+\infty)\to\mathcal{L}(E), \qquad t\mapsto e^{tA},$$

is continuous.

Furthermore, e^{tA} is compact on $L^p(\mathcal{O};\mathbb{R}^r)$ for all $1\leq p\leq\infty$ and $t>0$. The spectrum $\{-\alpha_k\}_{k\in\mathbb{N}}$ of A is independent of p and e^{tA} is analytic on $L^p(\mathcal{O};\mathbb{R}^r)$, for all $1\leq p\leq\infty$. In the sequel we shall assume the following condition on the eigenfunctions of A.

Hypothesis 1. *The complete orthonormal system of H which diagonalizes A is equi-bounded in the sup-norm, that is*

$$\sup_{k\in\mathbb{N}}|e_k|_E<\infty. \tag{2.4}$$

Next, we define $B = (B_1, \ldots, B_r) : H \to H$. In the sequel we shall assume that B fulfills the following conditions.

Hypothesis 2. *The operator B belongs to $\mathcal{L}(H)$, is non-negative and diagonal with respect to the complete orthonormal system $\{e_k\}$ which diagonalizes A, with eigenvalues $\{\lambda_k\}$. Moreover, if $d \geq 2$*

$$\exists\, \varrho \in \left(2, \frac{2d}{d-2}\right) \quad \text{such that} \quad \|B\|_\varrho := \left(\sum_{k=1}^{\infty} \lambda_k^\varrho\right)^{1/\varrho} < \infty, \tag{2.5}$$

where $2d/(d-2) := +\infty$, if $d = 2$.

For consistency, in what follows we shall set $\varrho := \infty$ when $d = 1$ and in this case $\|B\|_\varrho = \|B\|_{\mathcal{L}(H)}$.

The uniform bound on the sup-norms of the eigenfunctions e_k is satisfied e.g. by the Laplace operator with Dirichlet boundary conditions on the cube. But there are several important cases in which it is not satisfied and it is only possible to say that

$$|e_k|_\infty \leq c\, k^\alpha,$$

for some $\alpha \geq 0$. In this more general situation what one has to do is "colouring" the noise a bit more. More precisely, one has to assume that the summability condition (2.5) imposed on the eigenvalues of B is satisfied for some constant ϱ' less than the constant ϱ introduced in Hypothesis 2.

Concerning the diffusion coefficient g, we assume the following conditions.

Hypothesis 3. *The mapping $g : [0,\infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is continuous and the mapping $g(t,\xi,\cdot) : \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is Lipschitz-continuous, uniformly with respect to $\xi \in \overline{\mathcal{O}}$ and t in bounded sets of $[0,\infty)$, i.e.*

$$\sup_{\xi \in \overline{\mathcal{O}}} \sup_{\substack{\sigma,\rho \in \mathbb{R}^r \\ \sigma \neq \rho}} \frac{|g(t,\xi,\sigma) - g(t,\xi,\rho)|}{|\sigma - \rho|} \leq \Psi(t), \qquad t \geq 0,$$

for some $\Psi \in L^\infty_{loc}[0,\infty)$.

Now, we specify the conditions on $f = (f_1, \ldots, f_r)$. To this end we define for any $t \geq 0$ the composition operator $F(t,\cdot)$ by setting for any $x : \overline{\mathcal{O}} \to \mathbb{R}^r$

$$F(t,x)(\xi) := f(t,\xi,x(\xi)), \qquad \xi \in \overline{\mathcal{O}}.$$

Hypothesis 4.

1. *The mapping $F(t) : E \to E$ is locally Lipschitz-continuous , locally uniformly for $t \geq 0$ and there exist $m \geq 1$ and $\Phi \in L^\infty_{loc}[0,\infty)$ such that*

$$|F(t,x)|_E \leq \Phi(t)\,(1 + |x|_E^m)\,, \qquad x \in E, \quad t \geq 0. \tag{2.6}$$

2. *There exists $\Lambda \in L^\infty_{loc}[0,\infty)$ such that for each $x, h \in E$ and $t \geq 0$*

$$\langle F(t,x+h) - F(t,x), \delta_h \rangle_E \leq \Lambda(t)\,(1 + |h|_E + |x|_E)\,,$$

for some $\delta_h \in \partial |h|_E = \{h^\star \in E^\star\,;\, |h^\star|_{E^\star} = 1,\ \langle h^\star, h\rangle_E = |h|_E\}$.

3. One of the following two conditions holds:

i) either

$$\sup_{\xi\in\overline{\mathcal{O}}} |g(t,\xi,\sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \beta_1(t)\left(1+|\sigma|^{\frac{1}{m}}\right), \qquad \sigma\in\mathbb{R}^r, \quad t\geq 0,$$

where $m\geq 1$ *is as in* (2.6) *and* $\beta_1\in L^\infty_{loc}[0,\infty)$,

ii) or there exist some $a>0$, $m\geq 1$ *and* $\beta_2\in L^\infty_{loc}[0,\infty)$ *such that for each* $x,h\in E$

$$\langle F(t,x+h)-F(t,x),\delta_h\rangle_E \leq -a\,|h|_E^m+\beta_2(t)\,(1+|x|_E^m)\,, \qquad t\geq 0, \quad (2.7)$$

for some $\delta_h\in\partial|h|_E$.

Remark 2.1. For each $i=1,\ldots,r$ and $(t,\xi,\sigma_1,\ldots,\sigma_r)\in[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}^r$ assume

$$f_i(t,\xi,\sigma_1,\ldots,\sigma_r)=k_i(t,\xi,\sigma_i)+h_i(t,\xi,\sigma_1,\ldots,\sigma_r),$$

with $k_i(t,\cdot,\cdot):\overline{\mathcal{O}}\times\mathbb{R}\to\mathbb{R}$ and $h_i(t,\cdot,\cdot):\overline{\mathcal{O}}\times\mathbb{R}^r\to\mathbb{R}$ continuous for almost all $t\geq 0$ and where:

1. the function $h_i(t,\xi,\cdot):\mathbb{R}^r\to\mathbb{R}$ is locally Lipschitz-continuous with linear growth, uniformly with respect to $\xi\in\overline{\mathcal{O}}$ and t in bounded sets of $[0,\infty)$;

2. there exist $m\geq 1$ and $\Phi\in L^\infty_{\text{loc}}[0,\infty)$ such that

$$\sup_{\xi\in\overline{\mathcal{O}}}|k_i(t,\xi,\sigma_i)|\leq\Phi(t)\,(1+|\sigma_i|^m), \qquad \sigma_i\in\mathbb{R}, \qquad t\geq 0;$$

3. for any $\xi\in\overline{\mathcal{O}}$, $\sigma_i,\rho_i\in\mathbb{R}$ and $t\geq 0$

$$k_i(t,\xi,\sigma_i)-k_i(t,\xi,\rho_i)=\lambda_i(t,\xi,\sigma_i,\rho_i)(\sigma_i-\rho_i),$$

   for some locally bounded measurable function $\lambda_i:[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\sup_{\substack{\xi\in\overline{\mathcal{O}}\\ \sigma_i,\rho_i\in\mathbb{R},\ t\geq 0}}\lambda_i(t,\xi,\sigma_i,\rho_i)<\infty.$$

Then, as shown e.g. in [5] (see also [4, Chapter 6]) the corresponding composition operator F satisfies conditions 1 and 2 in Hypothesis 4. If in addition we assume that for any $\xi\in\overline{\mathcal{O}}$, $\sigma_i,h\in\mathbb{R}$ and $t\geq 0$ the functions k_i verify the condition

$$(k_i(t,\xi,\sigma_i+h)-k_i(t,\xi,\sigma_i))\,h\leq -a\,|h|^{m+1}+\beta(t)\left(1+|\sigma_i|^{m+1}\right),$$

for some $a>0$ and $\beta\in L^\infty_{\text{loc}}[0,\infty)$, then condition 3-ii) is also verified.

An example of such functions k_i is given by

$$k_i(t,\xi,\sigma_i)=-c_i(t,\xi)\sigma_i^{2n+1}+\sum_{j=0}^{2n}c_{ij}(t,\xi)\sigma_i^j,$$

where $c_i,c_{ij}:[0,\infty)\times\overline{\mathcal{O}}\to\mathbb{R}$ are continuous functions and there exists $a>0$ such that $c_i(t,\xi)\geq a$, for each $t\geq 0$, $\xi\in\overline{\mathcal{O}}$ and $i=1,\ldots,r$.

Finally, we denote by $\{\partial^2 w_i/\partial t\partial\xi\}$ a sequence of r independent space-time white noises defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. This means that we can write

$$\frac{\partial w}{\partial \xi}(t,\xi) = \left(\frac{\partial w_1}{\partial \xi}, \dots, \frac{\partial w_r}{\partial \xi}\right)(t,\xi) = \sum_{k=1}^{\infty} e_k(\xi)\beta_k(t),$$

where $\{e_k\}$ is a complete orthonormal basis in H and $\{\beta_k(t)\}$ is a sequence of independent standard Brownian motions. As well known the series above converges in $L^2(\Omega)$ with values in any Hilbert space U containing H, with Hilbert-Schmidt embedding (see e.g. [7]). Defining for $x, y \in E$ and $t \geq 0$

$$G(t,x)y(\xi) := g(t,\xi,x(\xi))y(\xi), \qquad \xi \in \overline{\mathcal{O}},$$

system (2.1) can be rewritten in the following abstract form

$$du(t) = [Au(t) + F(t,u(t))]\, dt + G(t,u(t))B\, dw(t), \qquad u(0) = x. \tag{2.8}$$

In [5] it is shown that under Hypotheses 1 to 4 for any $x \in E$ and for any $p \geq 1$ and $T > 0$ such a problem admits a unique *mild* solution in $L^p(\Omega; C([0,T]; E))$, the Banach space of all processes u in $C([0,T]; E)$, such that

$$|u|^p_{L_{T,p}} := \mathbb{E} \sup_{t \in [0,T]} |u(t)|^p_E < \infty.$$

This means that there exists a unique process $u^x \in L^p(\Omega; C([0,T]; E))$ such that

$$u^x(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(s, u^x(s))\, ds + \int_0^t e^{(t-s)A} G(s, u^x(s))B\, dw(s).$$

More precisely, we have the following existence and uniqueness result (see [5, Theorems 5.3 and 5.5]).

Theorem 2.2. *Assume Hypotheses 1 to 4. Then for any initial datum $x \in E$ there exists a unique mild solution u^x in $L^p(\Omega; C([0,T]; E))$, with $p \geq 1$ and $T > 0$. Furthermore,*

$$|u^x|_{L_{T,p}} \leq c_p(T)\, \left(1 + |x|_E\right), \qquad x \in E. \tag{2.9}$$

One of the key points in the proof of the theorem above is the study of the system

$$dv(t) = Av(t)\, dt + G(t, v(t))B\, dw(t), \qquad v(0) = 0. \tag{2.10}$$

For this purpose, in [5] for any $u \in L^p(\Omega; C([0,T]; E))$ the H-valued continuous process

$$\gamma(u)(t) := \int_0^t e^{(t-s)A} G(s, u(s))B\, dw(s), \qquad t \in [0,T], \tag{2.11}$$

is defined and the following crucial fact is proved.

Theorem 2.3. *Under Hypotheses 1, 2 and 3 there exists $p_\star \geq 1$ such that γ maps $L^p(\Omega; C([0,T]; E))$ into itself for any $p \geq p_\star$ and*

$$|\gamma(u) - \gamma(v)|_{L_{T,p}} \leq c_p(T)\, |u - v|_{L_{T,p}}, \tag{2.12}$$

for some continuous increasing function $c_p : [0,\infty) \to [0,\infty)$ vanishing at $t = 0$. In particular, there exists some $T_0 > 0$ such that γ is a contraction on $L^p(\Omega; C([0,T_0]; E))$. Furthermore, there exists a unique mild solution for problem (2.10) which belongs to $L^p(\Omega; C([0,T]; E))$, for any $p \geq 1$.

Notice that, as explained in [5, Remark 4.3], if we assume that

$$\sup_{\xi \in \overline{\mathcal{O}}} |g(t,\xi,\sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \Psi(t)\ (1 + |\sigma|^\alpha)\,, \qquad \sigma \in \mathbb{R}^r, \quad t \geq 0, \tag{2.13}$$

for some $\alpha \in [0,1]$ and $\Psi \in L^\infty_{\text{loc}}[0,\infty)$, then for any $p \geq p_\star$

$$|\gamma(u)|^p_{L_{T,p}} \leq c_p(T) \left(1 + \mathbb{E} \sup_{t \in [0,T]} |u(t)|_E^{\alpha p}\right). \tag{2.14}$$

In particular, if g is bounded (that is $\alpha = 0$) the norm of $\gamma(u)$ in $L^p(\Omega; C([0,T]; E))$ is uniformly bounded with respect to u.

3 Exponential estimates for the solution

In this section we extend some exponential estimates proved in [6] and [15] for the stochastic convolution $\gamma(u)$ and for u itself. Chow-Menaldi and Peszat are concerned with the case of space dimension $d = 1$. Here, we obtain their results in case of higher space dimension. This allows us to prove an exponential estimate for the solution of problem (2.10) and hence for the solution of problem (2.1).

In [6] the following result is proved.

Theorem 3.1. *Let H and V be two Hilbert spaces and let ζ be a $\mathcal{L}(H;V)$-valued predictable process such that*

$$\int_0^T \|\zeta(s)\|^2_{HS}\, ds \leq \eta, \qquad \mathbb{P} - a.s.$$

for some constant $\eta > 0$. Then, for any $\delta > 0$

$$\mathbb{P}\left(\sup_{t \in [0,T]} \left|\int_0^t \zeta(s)\, dw(s)\right|_V \geq \delta\right) \leq 3 \exp\left(-\frac{\delta^2}{6\eta}\right). \tag{3.1}$$

As shown for example in [15, Proposition 2.1] this implies that

$$\mathbb{E}\, \exp\left(\frac{1}{9\eta} \sup_{t \in [0,T]} \left|\int_0^t \zeta(s)\, dw(s)\right|^2_V\right) \leq 7. \tag{3.2}$$

We apply this result to the proof of the following result.

Theorem 3.2. *Let h_t be a $C(\overline{\mathcal{O}};\mathcal{L}(\mathbb{R}^r))$-valued predictable process such that*

$$\sup_{(t,\xi)\in\,[0,T]\times\overline{\mathcal{O}}}|h_t(\xi)|_{\mathcal{L}(\mathbb{R}^r)}\in L^\infty(\Omega)$$

and for any $t\in[0,T]$ let $H(t)$ be the operator on E defined by

$$(H(t)x)\,(\xi)=h_t(\xi)x(\xi),\qquad \xi\in\overline{\mathcal{O}},\quad x\in E.$$

Then, under Hypotheses 1 and 2 there exist $c_1,c_2,\lambda>0$ independent of A, h_t and B such that for any $t_0\in[0,T)$ and $\delta>0$

$$\mathbb{P}\left(\sup_{t\in\,[t_0,T]}\left|\int_{t_0}^t e^{(t-s)A}H(s)B\,dw(s)\right|_E\geq\delta\right)\leq c_1\,\exp\left(-\frac{\delta^2}{c_2(T-t_0)^\lambda\,\eta}\right),\tag{3.3}$$

where

$$\eta:=\sup_{\omega\in\Omega}\left(\sup_{(t,\xi)\in\,[t_0,T]\times\overline{\mathcal{O}}}|h_t(\omega)(\xi)|^2_{\mathcal{L}(\mathbb{R}^r)}\right)\,\|B\|^2_{\varrho}.$$

Proof. Here for any $0\leq r\leq t\leq T$ we denote

$$v_r(t)=\int_r^t e^{(t-s)A}H(s)B\,dw(s).$$

If we show that there exist $c_1,c_2,\lambda>0$ such that

$$\mathbb{E}\,\exp\left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2(T-t_0)^\lambda\eta}\right)\leq c_1,\tag{3.4}$$

then, thanks to the Chebyshev inequality, we immediately obtain (3.3).

By using a factorization argument (see [7, Section 5.3] for a proof), we have

$$v_{t_0}(t)=\frac{\sin\pi\beta}{\pi}\int_{t_0}^t(t-s)^{\beta-1}e^{(t-s)A}\varrho_{t_0,\beta}(s)\,ds,$$

where the process $\varrho_{t_0,\beta}$ is defined by

$$\varrho_{t_0,\beta}(s)=\int_{t_0}^s(s-r)^{-\beta}e^{(s-r)A}H(r)B\,dw(r),$$

and $\beta\in(0,1/2)$. As shown in [5, Theorem 4.2], according to (2.2) and to the Hölder inequality for any $\beta>1/p$ and $\epsilon<2(\beta-1/p)$ we have for some $c_\beta>0$

$$|v_{t_0}(t)|_{\epsilon,p}\leq c_\beta\int_{t_0}^t(t-s)^{\beta-\frac{\epsilon}{2}-1}|\varrho_{t_0,\beta}(s)|_p\,ds\leq c_\beta\,(T-t_0)^{\lambda/2}\,|\varrho_{t_0,\beta}|_{L^p([t_0,T]\times\mathcal{O};\mathbb{R}^r)},$$

for some constant $\lambda>0$. Thus, if $\epsilon>d/p$, that is $\beta>(d+2)/2p$, due to the Sobolev embedding theorem $v_{t_0}\in C([t_0,T];E)$, $\mathbb{P}$-a.s. (cf. [5, proof of Theorem 4.2] and there exist some constants $c,\lambda>0$ such that for all p large enough

$$|v_{t_0}|^2_{C([t_0,T];E)}\leq c\,(T-t_0)^\lambda\,|\varrho_{t_0,\beta}|^2_{L^p([t_0,T]\times\mathcal{O};\mathbb{R}^r)}.$$

Now, if ϱ is the constant introduced in Hypothesis 2, we can find some $p_\star$ large enough such that for any $p \geq p_\star$

$$\frac{d+2}{p} + \frac{d(\varrho-2)}{2\varrho} < 1.$$

This implies that there exists some $\beta_\star \in (0, 1/2)$ such that for any $p \geq p_\star$

$$\beta_\star > \frac{d+2}{2p} \qquad \text{and} \qquad 2\beta_\star + \frac{d(\varrho-2)}{2\varrho} < 1.$$

In correspondence of such $\beta_\star$, for any integer $k \geq k_\star := [p_\star/2] + 1$ and for any constant $c_3 > 0$ and $c_2 := c_{\beta_\star}\, c_3$ we have

$$\left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2 (T-t_0)^\lambda \eta} \right)^k \leq \frac{|\varrho_{t_0,\beta_\star}|^{2k}_{L^{2k}([t_0,T]\times \mathcal{O};\mathbb{R}^r)}}{(c_3\eta)^k} = \int_{t_0}^T \int_{\mathcal{O}} \frac{|\varrho_{t_0,\beta_\star}(t,\xi)|^{2k}}{(c_3\eta)^k}\, d\xi\, dt.$$

Moreover, since h_t is bounded, by similar arguments which proved (2.14) with $\alpha = 0$, and we can find $c_{k_\star} > 0$ such that

$$\mathbb{E} \sum_{k=0}^{k_\star - 1} \frac{1}{k!} \left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2 (T-t_0)^\lambda \eta} \right)^k \leq c_{k_\star}.$$

Hence

$$\begin{aligned}
&\mathbb{E} \exp \left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2 (T-t_0)^\lambda \eta} \right) \\
&= \mathbb{E} \sum_{k=0}^{k_\star - 1} \frac{1}{k!} \left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2 (T-t_0)^\lambda \eta} \right)^k + \mathbb{E} \sum_{k=k_\star}^{\infty} \frac{1}{k!} \left(\frac{|v_{t_0}|^2_{C([t_0,T];E)}}{c_2 (T-t_0)^\lambda \eta} \right)^k \\
&\leq c_{k_\star} + \mathbb{E} \sum_{k=k_\star}^{\infty} \frac{1}{k!} \int_{t_0}^T \int_{\mathcal{O}} \frac{|\varrho_{t_0,\beta_\star}(t,\xi)|^{2k}}{(c_3\eta)^k}\, d\xi\, dt \leq c_{k_\star} + \int_{t_0}^T \int_{\mathcal{O}} \mathbb{E}\exp \left(\frac{|\varrho_{t_0,\beta_\star}(t,\xi)|^2}{c_3\eta} \right) d\xi\, dt.
\end{aligned}$$

Therefore, in order to obtain (3.4) we have to show that there exists some constant $c_3 > 0$ such that

$$\int_{t_0}^T \int_{\mathcal{O}} \mathbb{E}\exp \left(\frac{|\varrho_{t_0,\beta_\star}(t,\xi)|^2}{c_3\eta} \right) d\xi\, dt < \infty. \tag{3.5}$$

To this end, we are going to approximate $\varrho_{t_0,\beta_\star}$ in a certain way. So, define for any $h \in \mathbb{N}$

$$\zeta^h_{t,\xi}(r) := (t-r)^{-\beta_\star} \left(\lambda_j\, e^{(t-r)A}[H(r)e_j](\xi) \right)_{1 \leq j \leq h}.$$

By the choice of $\beta_\star$, using the same arguments used in [5, proof of Theorem 4.2], for any $q \geq 1$ we obtain that

$$\lim_{h_1, h_2 \to +\infty} \int_{t_0}^T \int_{\mathcal{O}} \mathbb{E} \sup_{s \in [t_0,t]} \left| \int_{t_0}^s \left\langle \zeta^{h_1}_{t,\xi}(r), dw^{h_1}(r) \right\rangle_{\mathbb{R}^{h_1}} - \int_{t_0}^s \left\langle \zeta^{h_2}_{t,\xi}(r), dw^{h_2}(r) \right\rangle_{\mathbb{R}^{h_2}} \right|^q d\xi\, dt = 0,$$

where $w^h(t) = (\beta_1(t), \dots, \beta_h(t))$, $t \geq 0$, and

$$\int_{t_0}^T |\zeta^h_{t,\xi}(r)|^2\, dr \leq c\, \|B\|^2_\varrho \sup_{(t,\xi)\in [t_0,T]\times\overline{\mathcal{O}}} |h_t(\xi)|^2_{\mathcal{L}(\mathbb{R}^r)} \leq c\,\eta, \qquad \mathbb{P}-\text{a.}s.$$

for some constant c independent of k and $(t,\xi) \in [t_0,T]\times\overline{\mathcal{O}}$. Thus, according to Theorem 3.1 and to (3.2) if we set $c_3 := 9\,c$ we obtain

$$\sup_{h\in\mathbb{N}} \mathbb{E} \exp\left((c_3\eta)^{-1} \sup_{s\in [t_0,T]} \left|\int_{t_0}^s \left\langle \zeta^h_{t,\xi}(r), dw^h(r)\right\rangle_{\mathbb{R}^h}\right|^2\right) \leq 7.$$

It is easy to check that

$$\lim_{h\to\infty} \int_{t_0}^T \int_{\mathcal{O}} \mathbb{E}\left|\varrho_{t_0,\beta_\star}(t,\xi) - \int_{t_0}^t \langle \zeta^h_{t,\xi}(s), dw^h(s)\rangle_{\mathbb{R}^h}\right|^q d\xi\, dt = 0,$$

thus, from Fatou's lemma we can conclude that

$$\int_{t_0}^T \int_{\mathcal{O}} \mathbb{E}\exp\left(\frac{|\varrho_{t_0,\beta_\star}(t,\xi)|^2}{c_3\eta}\right) d\xi\, dt$$

$$\leq \liminf_{h\to\infty} \int_{t_0}^T \int_{\mathcal{O}} \mathbb{E}\, \exp\left((c_3\eta)^{-1} \sup_{s\in [t_0,T]} \left|\int_{t_0}^s \left\langle \zeta^h_{t,\xi}(r), dw^k(r)\right\rangle\right|^2\right) d\xi\, dt \leq c\,T\,7,$$

which yields (3.5) and hence (3.4). □

From the proof above we immediately see that a stronger estimate holds. Actually, it is possible to show that there exists some $\theta > 0$ such that

$$\mathbb{P}\left(\sup_{t\in [t_0,T]} \left|\int_{t_0}^t e^{(t-s)A} H(s) B\, dw(s)\right|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} \geq \delta\right) \leq c_1\, \exp\left(-\frac{\delta^2}{c_2(T-t_0)^\lambda\,\eta}\right).$$

Now, we can apply the result above to get for any $\epsilon \in (0,1]$ exponential estimates for the mild solution u^x_ϵ of the problem

$$du(t) = [Au(t) + F(t,u(t))]\, dt + \epsilon\, G(t,u(t)) B\, dw(t), \qquad u(0) = x, \qquad (3.6)$$

in the particular case G is bounded.

Theorem 3.3. *Assume that the mapping $g(t,\cdot,\cdot) : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is bounded, uniformly with respect to t in bounded sets of $[0,\infty)$. Then, under Hypotheses 1 to 4, for any $\alpha, R > 0$ there exists $\delta > 0$ such that*

$$\sup_{|x|_E\leq R} \mathbb{P}\left(|u^x_\epsilon|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \qquad (3.7)$$

for any $\epsilon \in (0,1]$.

Proof. By setting $v(t) := u^x_\epsilon(t) - \gamma_\epsilon(t)$, with γ_ϵ defined by

$$\gamma_\epsilon(t) := \epsilon \int_0^t e^{(t-s)A} G(s, u^x_\epsilon(s)) B \, dw(s),$$

we have

$$\frac{dv}{dt}(t) = Av(t) + F(t, u^x_\epsilon(t)), \qquad v(0) = x.$$

Thanks to conditions 1 and 2 in Hypothesis 4, there exists some $\delta_{v(t)} \in \partial |v(t)|_E$ such that for any $t \in [0,T]$

$$\frac{d}{dt}^{-} |v(t)|_E \leq \left\langle Av(t), \delta_{v(t)} \right\rangle_E + \left\langle F(t, v(t) + \gamma_\epsilon(t)) - F(t, \gamma_\epsilon(t)), \delta_{v(t)} \right\rangle_E$$

$$+ \left\langle F(t, \gamma_\epsilon(t)), \delta_{v(t)} \right\rangle_E \leq c_T \, |v(t)|_E + c_T \left(1 + |\gamma_\epsilon|^m_{C([0,T];E)}\right).$$

Since $u^x_\epsilon(t) = v(t) + \gamma_\epsilon(t)$, by comparison we easily obtain for $c := c_T$

$$|u^x_\epsilon|_{C([0,T];E)} \leq e^{cT} \left(|x|_E + c \left(1 + |\gamma_\epsilon|^m_{C([0,T];E)}\right)\right).$$

This implies that for any $\delta > 0$

$$\mathbb{P}\left(|u^x_\epsilon|_{C([0,T];E)} \geq \delta\right) \leq \mathbb{P}\left(|\gamma^\epsilon|^m_{C([0,T];E)} \geq \frac{\delta - e^{cT}|x|_E}{c\, e^{cT}} - 1\right).$$

Now, if we fix any $\delta > e^{cT}R + c\, e^{cT}$ and if we set

$$\delta' := \left(\frac{\delta - e^{cT}R}{c\, e^{cT}} - 1\right)^{\frac{1}{m}},$$

as $\gamma_\epsilon(t) = \epsilon\, \gamma(u^x_\epsilon)(t)$, with γ defined in (2.11) we get

$$\mathbb{P}\left(|u^x_\epsilon|_{C([0,T];E)} \geq \delta\right) \leq \mathbb{P}\left(|\gamma(u^x_\epsilon)|_{C([0,T];E)} \geq \frac{\delta'}{\epsilon}\right).$$

According to Theorem 3.2 we can find c_1, c_2, λ and η_T such that for all $x \in E$, with $|x|_E \leq R$, and all $\epsilon > 0$ and δ' as above we have

$$\mathbb{P}\left(|\gamma(u^x_\epsilon)|_{C([0,T];E)} \geq \frac{\delta'}{\epsilon}\right) \leq c_1 \exp\left(-\frac{\delta'^2}{\epsilon^2\, c_2 T^\lambda \eta_T}\right) = \exp\left(-\frac{1}{\epsilon^2}\left(\frac{\delta'^2}{c_2 T^\lambda \eta_T} - \epsilon^2 \log c_1\right)\right). \tag{3.8}$$

Indeed, if we set

$$h_t(\xi) = g(t, \xi, u(t)(\xi)), \qquad (t,\xi) \in [0,T] \times \overline{\mathcal{O}},$$

for any $u \in L^p(\Omega; C([0,T];E))$ the process h_t fulfills the conditions of Theorem 3.2, so that there exist $c_1, c_2, \lambda > 0$ such that for any $t_0 \in [0,T)$ and $\delta > 0$

$$\mathbb{P}\left(\sup_{t \in [t_0,T]} \left| \int_{t_0}^t e^{(t-s)A} G(s, u(s)) B \, dw(s)\right|_E \geq \delta\right) \leq c_1 \exp\left(-\frac{\delta^2}{c_2 (T-t_0)^\lambda \, \eta_T}\right), \tag{3.9}$$

where

$$\eta_T := \sup_{\omega\in\Omega}\left(\sup_{\substack{(\xi,\sigma)\in\overline{\mathcal{O}}\times\mathbb{R}^r\\ t\in[0,T]}} |g(t,\xi,\sigma)|^2_{\mathcal{L}(\mathbb{R}^r)}\right)\|B\|^2_{\varrho}.$$

Hence, due to (3.8), if we take any $\delta > 0$ such that

$$\frac{1}{c_2 T^\lambda \eta_T}\left(\frac{\delta - e^{cT}R}{c\,e^{c_T}} - 1\right)^{\frac{2}{m}} - \log c_1 \geq \alpha,$$

recalling how δ' is defined, we can conclude that (3.7) holds. □

4 The skeleton equation

In this section we study the deterministic problem

$$\frac{dz}{dt}(t) = Az(t) + F(t,z(t)) + G(t,z(t))B\varphi(t), \qquad z(0) = x, \tag{4.1}$$

where the operators A and B and the mappings F and G are those already introduced in Section 2 and φ is any function in $L^2(0,T;H)$. Our aim is to prove an estimate for the $C([0,T];E)$-norm of the unique mild solution $z^x(\varphi)$ of problem (4.1).

For any fixed $\varphi \in L^2(0,T;H)$ and $z \in C([0,T];E)$ we define

$$\gamma_\varphi(z)(t) := \int_0^t e^{(t-s)A}G(s,z(s))B\varphi(s)\,ds, \qquad t \in [0,T].$$

By proceeding as in the proof of Theorem 2.3 (see [5, Theorem 4.2]) it is possible to prove that the mapping γ_φ is continuous from $C([0,T];E)$ into itself for any $\varphi \in L^2(0,T;H)$ and if g fulfills (2.13) then

$$|\gamma_\varphi(z)|_{C([0,T];E)} \leq c(T)\left(1 + |z|^\alpha_{C([0,T];E)}\right)|\varphi|_{L^2(0,T;H)}, \tag{4.2}$$

for some continuous increasing function $c(t)$ such that $c(0) = 0$. Moreover,

$$|\gamma_\varphi(z_1) - \gamma_\varphi(z_2)|_{C([0,T];E)} \leq c(T)\,\|B\|_\varrho\,|z_1 - z_2|_{C([0,T];E)}\,|\varphi|_{L^2(0,T;H)}, \tag{4.3}$$

again with c continuous such that $c(0) = 0$. In particular, by a contraction argument it is immediate to check that for any $\varphi \in L^2(0,T;H)$ there exists a unique fixed point for the mapping γ_φ, which is clearly the unique mild solution of the problem

$$\frac{dz}{dt}(t) = Az(t) + G(t,z(t))B\varphi(t), \qquad z(0) = 0.$$

Now, we can go to the skeleton equation (4.1).

Theorem 4.1. *Assume Hypotheses 1 to 4. Then for any $\varphi \in L^2(0,T;H)$ and $x \in E$ there exists a unique mild solution $z^x(\varphi)$ to problem (4.1) in $C([0,T];E)$ and for any $r \geq 0$ and $x \in E$*

$$\sup_{|\varphi|_{L^2(0,T;H)}\leq r} |z^x(\varphi)|_{C([0,T];E)} \leq c_{r,T}\left(1 + |x|_E\right). \tag{4.4}$$

Proof. Since F is locally Lipschitz-continuous, problem (4.1) has a unique solution $z^x(\varphi)$ defined in some maximal interval $[0,T')$. In order to prove that the solution is global, we have to prove an a-priori estimate.

We give our proof under condition 3-ii) in Hypothesis 4; the proof under condition 3-i) is simpler. If we set $v = z^x(\varphi) - \gamma_\varphi(z^x(\varphi))$, we have that v is the solution of the problem

$$\frac{dv}{dt}(t) = Av(t) + F(t, v(t) + \gamma_\varphi(z^x(\varphi))(t)), \qquad v(0) = x.$$

Then, if $\delta_{v(t)}$ is the element of $\partial|v(t)|_E$ introduced in Hypothesis 4-condition 3-ii), due to (2.6), (2.7) and (4.2), with $\alpha = 1$, for any $t \in [0,T')$ we have

$$\frac{d}{dt}^- |v(t)|_E \leq \left\langle Av(t), \delta_{v(t)} \right\rangle_E + \left\langle F(t, v(t) + \gamma_\varphi(z^x(\varphi))(t)) - F(t, \gamma_\varphi(z^x(\varphi))(t)), \delta_{v(t)} \right\rangle_E$$

$$+ \left\langle F(t, \gamma_\varphi(z^x(\varphi))(t)), \delta_{v(t)} \right\rangle_E \leq -a\, |v(t)|_E^m + (\beta_2(t) + \Phi(t))\, (1 + |\gamma_\varphi(z^x(\varphi))(t)|_E^m)$$

$$\leq -a\, |v(t)|_E^m + c_t \left(1 + c(t) \left(|z^x(\varphi)|_{C([0,t];E)} + 1\right)^m |\varphi|^m_{L^2(0,T;H)}\right),$$

for some increasing continuous function $c(t)$ such that $c(0) = 0$. By comparison, if $|\varphi|_{L^2(0,T;H)} \leq r$ this yields

$$|v(t)|_E \leq |x|_E + c(t) \left(|z^x(\varphi)|_{C([0,t];E)} + 1\right) r + c_t,$$

so that, by using again (4.2), as $z^x(\varphi) = v + \gamma_\varphi(z^x(\varphi))$, we get

$$|z^x(\varphi)|_{C([0,t];E)} \leq |\gamma_\varphi(z^x(\varphi))|_{C([0,t];E)} + |x|_E + c_t + c(t) \left(|z^x(\varphi)|_{C([0,t];E)} + 1\right) r$$

$$\leq |x|_E + c_{r,t} + 2\, c(t) r |z^x(\varphi)|_{C([0,t];E)}.$$

Now, since $c(t)$ is continuous and vanishes at $t = 0$, there exists some $t_0 > 0$ such that $c(t_0) r \leq 1/4$, so that the estimate (4.4) follows in the time interval $[0, t_0]$. As the same argument can be repeated in the time intervals $[t_0, 2t_0]$, $[2t_0, 3t_0]$ and so on, we get the estimate in the whole maximal interval $[0,T')$. This implies that the solution $z^x(\varphi)$ is defined in the whole interval $[0,T]$ and estimate (4.4) holds globally. □

For any $x \in E$, $\varphi \in L^2(0,T;H)$, $z \in C([0,T];E)$ and $s \geq 0$ we define

$$\mathcal{R}^x_s(z,\varphi)(t) := e^{(t-s)A} x + \int_s^t e^{(t-r)A} F(r, z(r))\, dr + \gamma_{s,\varphi}(z)(t), \qquad t \geq s, \tag{4.5}$$

where

$$\gamma_{s,\varphi}(z)(t) := \int_s^t e^{(t-r)A} G(r, z(r)) B\varphi(r)\, dr.$$

If $s = 0$ we will set $\mathcal{R}^x_0 = \mathcal{R}^x$. Next proposition follows from Theorem 4.1 and will be useful in what follows.

Proposition 4.2. *For any $T, R, r > 0$ there exists $t_0 > 0$ such that*

$$\mathcal{R}^x_s(\cdot,\varphi) : \left\{ z \in C([s,s+t_0];E)\,;\, |z|_{C([s,s+t_0];E)} \leq 2R \right\} =: \Lambda_{R,s,t_0} \to \Lambda_{R,s,t_0}$$

is a contraction, uniformly for $|x|_E \leq R$, $|\varphi|_{L^2(0,T;H)} \leq r$ and $s \in [0, T-t_0]$.

Proof. If we go back to the proof of [5, Theorem 4.2] it is possible to check that

$$|\gamma_{s,\varphi}(z_1) - \gamma_{s,\varphi}(z_2)|_{C([s,T];E)} \leq c(T-s)\,\|B\|_\rho\,|z_1 - z_2|_{C([s,T];E)}|\varphi|_{L^2(0,T;H)}, \tag{4.6}$$

for some continuous increasing function c such that $c(0) = 0$.

If we take $z \in \Lambda_{R,s,t_1}$, for some $t_1 > 0$, we obtain for $t \in [s, s+t_1]$

$$\left| \int_s^t e^{(t-r)A} F(r, z(r))\, dr \right|_E \leq c_T\,(1 + 2^m R^m)\,(t-s),$$

and due to (4.6)

$$|\gamma_{s,\varphi}(z)(t)|_E \leq \tilde{c}(t-s)\,(1+2R)\,r.$$

Therefore, collecting all terms, we have

$$|\mathcal{R}^x_s(z,\varphi)(t)|_E \leq R + c_T\,(1 + 2^m R^m)\,(t-s) + \tilde{c}(t-s)\,(1+2R)\,r.$$

This implies that if we take $t_1 > 0$ such that

$$c_T\,(1 + 2^m R^m)\,t_1 + \tilde{c}(t_1)\,(1+2R)\,r \leq R$$

we have that $\mathcal{R}^x_s(\cdot,\varphi)$ maps Λ_{R,s,t_1} into itself. Note that up to now t_1 depends only on R, r and T and not on s.

Next, in order to conclude, we have to show that $\mathcal{R}^x_s(\cdot,\varphi)$ is a contraction in Λ_{R,s,t_0}, for some $t_0 \leq t_1$. Since we are assuming $F(t)$ to be locally Lipschitz-continuous on E, uniformly for $t \in [0,T]$, there exists $L_R > 0$ such that

$$x, y \in B_{2R}(E) \Longrightarrow |F(t,x) - F(t,y)|_E \leq L_R|x-y|_E, \qquad t \in [0,T].$$

Therefore, if $z_1, z_2 \in \Lambda_{R,s,t_0}$, thanks to (4.6) we obtain

$$|\mathcal{R}^x_s(z_1,\varphi) - \mathcal{R}^x_s(z_2,\varphi)|_{C([s,s+t_0];E)} \leq (L_R t_0 + \tilde{c}(t_0) r)\,|z_1 - z_2|_{C([s,s+t_0];E)}.$$

Thus, we can conclude by taking $t_0 \leq t_1$ such that $L_R t_0 + \tilde{c}(t_0) r \leq 1/2$. □

5 The action functional

Fix $x \in E$ and $T > 0$. For any $z \in C([0,T];E)$ we define

$$I_{x,T}(z) := \frac{1}{2} \inf \left\{ |\varphi|^2_{L^2(0,T;H)}\,;\, z = z^x(\varphi) \right\},$$

with the usual convention that $\inf \emptyset = +\infty$.

Moreover, for each $x \in E$ and $T > 0$ we introduce the level sets of the functional $I_{x,T}$

$$K_{x,T}(r) := \{ z \in C([0,T];E)\,;\, I_{x,T}(z) \leq r \}, \qquad r \geq 0.$$

As in [16] we want to prove that the laws of the solutions of the problem (3.6) fulfill a large deviations principle with action functional $I_{x,T}$. Thus, first we have to prove the following result.

Theorem 5.1. *For each $x \in E$ the level sets $K_{x,T}(r)$ are compact, for all $r \geq 0$. In particular, the functional $I_{x,T} : C([0,T]; E) \to [0,\infty]$ is lower semi-continuous.*

Proof. Step 1. We show that for each $r \geq 0$ the level set $K_{x,T}(r)$ is closed, so that in particular $I_{x,T}$ is lower semi-continuous.

Let $\{z_n\} \subset K_{x,T}(r)$ be a sequence converging to z in $C([0,T];E)$. Since $z_n \in K_{x,T}(r)$, there exists $\varphi_n \in L^2(0,T;H)$ such that

$$z_n = z^x(\varphi_n) \quad \text{and} \quad \frac{1}{2}\,|\varphi_n|^2_{L^2(0,T;H)} \leq r + \frac{1}{n}.$$

In particular the sequence $\{\varphi_n\}$ is bounded in $L^2(0,T;H)$, so that there exists $\varphi \in L^2(0,T;H)$ and $\{\varphi_{n_k}\} \subseteq \{\varphi_n\}$ such that $\varphi_{n_k} \rightharpoonup \varphi$ weakly and $|\varphi|^2_{L^2(0,T;H)} \leq 2r$. If we show that $z = z^x(\varphi)$, we conclude that $z \in K_{x,T}(r)$ and we are done.

If $\mathcal{R}^x$ is the mapping introduced in (4.5), with $s = 0$, for any fixed $h \in H$ we easily have

$$\langle z^x(\varphi_{n_k})(t) - \mathcal{R}^x(z,\varphi)(t), h\rangle_H = \int_0^t \left\langle e^{(t-s)A}\left[F(s, z^x(\varphi_{n_k})(s)) - F(s,z(s))\right], h\right\rangle_H ds$$

$$+\left\langle \left[\gamma_{\varphi_{n_k}}(z^x(\varphi_{n_k})) - \gamma_{\varphi_{n_k}}(z)\right](t), h\right\rangle_H + \int_0^t \left\langle \varphi_{n_k}(s) - \varphi(s), BG^\star(s,z(s))e^{(t-s)A}h\right\rangle_H ds. \tag{5.1}$$

Now, if we verify that all terms on the right hand side converge to zero, as k goes to infinity, we have that $z = \mathcal{R}^x(z,\varphi)$ and then $z = z^x(\varphi)$. For each $s \in [0,t]$ we have

$$\lim_{k\to\infty} \left\langle e^{(t-s)A}\left[F(s, z^x(\varphi_{n_k})(s)) - F(s,z(s))\right], h\right\rangle_H = 0.$$

Moreover, thanks to (2.6) and (4.4)

$$\left|\left\langle e^{(t-s)A}\left[F(s, z^x(\varphi_{n_k})(s)) - F(s,z(s))\right], h\right\rangle_H\right|$$

$$\leq \Phi(t)\ \left(1 + |z^x(\varphi_{n_k})(s)|_E^m + |z(s)|_E^m\right)|h|_H \leq c_T\ \left(c^m_{r,T}\,(1+|x|_E)^m + |z|^m_{C([0,T];E)}\right)|h|_H,$$

so that, due to the dominated convergence theorem we can conclude that the first term on the right hand side of (5.1) goes to zero. Concerning the second term, due to (4.3), if we set $c_T := c(T)\,\|B\|_\varrho$ we have

$$\left|\left\langle \left[\gamma_{\varphi_{n_k}}(z^x(\varphi_{n_k})) - \gamma_{\varphi_{n_k}}(z)\right](t), h\right\rangle_H\right| \leq c_T |z^x(\varphi_{n_k}) - z|_{C([0,T];E)} |\varphi_{n_k}|_{L^2(0,T;H)} |h|_H,$$

whose right hand side goes to zero, as $z^x(\varphi_{n_k}) \to z$ in $C([0,T];E)$ and $|\varphi_{n_k}|^2_{L^2(0,T;H)} \leq 2(r + 1/n_k)$. Finally, the last term in (5.1) goes to zero as φ_{n_k} converges weakly to φ.

Step 2. We show that the level set $K_{x,T}(r)$ is relatively compact in $C([0,T];E)$. Namely, we prove that for any sequence $\{z^x(\varphi_n)\} \subset K_{x,T}(r)$ there exists z in $C([0,T];E)$ and $\{z^x(\varphi_{n_k})\} \subset \{z^x(\varphi_n)\}$ such that $\{z^x(\varphi_{n_k})\}$ converges to z in $C([0,T];E)$.

Let $x \in E$ and $\varphi \in L^2(0,T;H)$ be fixed and define inductively

$$z^x_0(\varphi)(t) = e^{tA}x, \qquad z^x_{j+1}(\varphi) = \mathcal{R}^x(z^x_j(\varphi),\varphi)(t),$$

where $\mathcal{R}^x$ is the mapping introduced in (4.5) corresponding to $s = 0$. Note that if we take $R = |z^x(\varphi)|_{C([0,T];E)} + 1$, and t_0 as in Proposition 4.2, we have

$$z^x_{j+1}(\varphi)(t) = \mathcal{R}^{z^x_{j+1}(\varphi)(it_0)}_{it_0}(z^x_j(\varphi), \varphi)(t), \qquad t \in [it_0, (i+1)t_0).$$

Due to Proposition 4.2 we have that $\{z^x_j(\varphi)\}$ converges to $z^x(\varphi)$, uniformly with respect to $\varphi \in L^2(0,T;H)$, with $|\varphi|^2_{L^2(0,T;H)} \leq 2r$. Actually, since $z^x_j(\varphi) \in \Lambda_{R,0,t_0}$, from Proposition 4.2 we have that $z^x_j(\varphi)$ converges to $z^x(\varphi)$ and in particular there exists $j_1 \in \mathbb{N}$ such that $|z^x_j(\varphi)(t_0)|_E \leq R$, for and $j \geq j_1$. Thus $z^x_j(\varphi) \in \Lambda_{R,t_0,t_0}$ for any $j \geq j_1$ and by using again Proposition 4.2 we obtain that $z^x_j(\varphi)$ converges to $z^x(\varphi)$ in $C([t_0, 2t_0];E)$. By repeating this argument in the intervals $[2t_0, 3t_0]$ and so on, we obtain that $z^x_j(\varphi)$ converges to $z^x(\varphi)$ in $C([0,T];E)$.

Therefore, if we show that there exists a subsequence $\{\varphi_{n_k}\} \subseteq \{\varphi_n\}$ such that for any j the sequence $\{z^x_j(\varphi_{n_k})\}$ converges to some z_j in $C([0,T];E)$ and moreover there exists z such that the sequence $\{z_j\}$ converges to z in $C([0,T];E)$, then we can conclude that $\{z^x(\varphi_{n_k})\}$ converges to z in $C([0,T];E)$.

For this purpose we need the following preliminary result.

Lemma 5.2. *The set* $\{\mathcal{R}^x(z,\varphi)\,;\, z \in \mathcal{K},\, |\varphi|^2_{L^2(0,T;H)} \leq r\}$ *is relatively compact in* $C([0,T];E)$*, for any set* $\mathcal{K}$ *relatively compact in* $C([0,T];E)$ *and for any* $r > 0$.

Proof. Let $\{z_n\} \subset \mathcal{K}$ and $\{\varphi_n\} \subset L^2(0,T;H)$ such that $|\varphi_n|^2_{L^2(0,T;H)} \leq r$. Since $\mathcal{K}$ is relatively compact, there exists $\{z_{n_k}\} \subseteq \{z_n\}$ converging to some z_0 in $C([0,T];E)$. Moreover, as proved for example in [7, Proposition 8.4] (see also [16, Lemma 4.1]) for any fixed $z \in C([0,T];E)$ the mapping

$$L^2(0,T;H) \to C([0,T];E), \qquad \varphi \mapsto \gamma_\varphi(z),$$

is compact. Thus there exists $\{\varphi_{n_k}\} \subseteq \{\varphi_n\}$ and $\bar{z}$ such that $\gamma_{\varphi_{n_k}}(z_0)$ converges to $\bar{z}$ in $C([0,T];E)$, as $k \to \infty$.

We have

$$\mathcal{R}^x(z_{n_k}, \varphi_{n_k})(t) - \left(e^{tA}x + \int_0^t e^{(t-s)A}F(s, z_0(s))\,ds + \bar{z}(t)\right)$$

$$= \int_0^t e^{(t-s)A}\left[F(s, z_{n_k}(s)) - F(s, z_0(s))\right]\,ds + \gamma_{\varphi_{n_k}}(z_{n_k})(t) - \bar{z}(t)$$

As $z_{n_k} \to z_0$, we have $|z_{n_k}|_{C([0,T];E)} \leq R$, for some $R > 0$ and then, since $F(t)$ is locally Lipschitz continuous, uniformly for $t \in [0,T]$, there exists $c_R > 0$ such that for any $t \in [0,T]$

$$\left|\int_0^t e^{(t-s)A}\left[F(s, z_{n_k}(s)) - F(s, z_0(s))\right]\,ds\right|_E \leq c_R(T)|z_{n_k} - z_0|_{C([0,T];E)}.$$

Concerning the other term, thanks to (4.3) for any $t \in [0,T]$ we have

$$\left|\gamma_{\varphi_{n_k}}(z_{n_k})(t) - \bar{z}(t)\right| \leq \left|\gamma_{\varphi_{n_k}}(z_{n_k})(t) - \gamma_{\varphi_{n_k}}(z_0)(t)\right| + \left|\gamma_{\varphi_{n_k}}(z_0)(t) - \bar{z}(t)\right|$$

$$\leq c(T)\sqrt{r}\,|z_{n_k} - z_0|_{C([0,T];E)} + \left|\gamma_{\varphi_{n_k}}(z_0) - \bar{z}\right|_{C([0,T];E)} \to 0,$$

as k goes to infinity. This means that the sequence $\{\mathcal{R}^x(z_{n_k}, \varphi_{n_k})\}$ has a limit in $C([0,T];E)$ and the lemma is proved. □

From the lemma above it is possible to prove by induction that for each $j \in \mathbb{N}$ the set

$$\mathcal{A}_j := \{\, z_j^x(\varphi)\,;\ |\varphi|^2_{L^2(0,T;H)} \leq 2r \,\}$$

is relatively compact in $C([0,T];E)$. Actually, this is clearly true for the set $\mathcal{A}_0$. Moreover, if we assume the set $\mathcal{A}_j$ to be relatively compact, since $\mathcal{A}_{j+1} = \{\, \mathcal{R}^x(z,\varphi)\,;\ z \in \mathcal{A}_j,\ |\varphi|^2_{L^2(0,T;H)} \leq 2r \,\}$, it follows that $\mathcal{A}_{j+1}$ is relatively compact.

By a diagonal argument, this implies that there exists a subsequence $\{\varphi_{n_k}\} \subseteq \{\varphi_n\}$ such that for any $j \in \mathbb{N}$ the sequence $\{z_j^x(\varphi_{n_k})\}$ converges to some z_j in $C([0,T];E)$.

In order to conclude, we have to show that the sequence $\{z_j\}$ is a Cauchy sequence in $C([0,T];E)$. For any $i, j \in \mathbb{N}$ we have

$$|z_j - z_i|_{C([0,T];E)} \leq |z_j - z_j^x(\varphi_{n_k})|_{C([0,T];E)} + |z_j^x(\varphi_{n_k}) - z^x(\varphi_{n_k})|_{C([0,T];E)}$$

$$+|z^x(\varphi_{n_k}) - z_i^x(\varphi_{n_k})|_{C([0,T];E)} + |z_i^x(\varphi_{n_k}) - z_i|_{C([0,T];E)}.$$

Then we can conclude, as

$$\lim_{j\to\infty} |z_j^x(\varphi_{n_k}) - z^x(\varphi_{n_k})|_{C([0,T];E)} + |z^x(\varphi_{n_k}) - z_i^x(\varphi_{n_k})|_{C([0,T];E)} = 0,$$

uniformly with respect to φ_{n_k}. □

6 The large deviations estimates

For any $\epsilon > 0$ we consider the problem

$$du(t) = [Au(t) + F(t,u(t))]\ dt + \epsilon\, G(t,u(t))B\, dw(t), \qquad u(0) = x. \tag{6.1}$$

We denote by u_ϵ^x its unique mild solution (see Theorem 2.2), which is a process in $L^p(\Omega; C([0,T];E))$, for any $p \geq 1$ and $T > 0$, fulfilling an exponential tail estimate (see Theorem 3.3). The aim of this paper is to prove that the family of probability measures

$$\mu^x_{\epsilon,T} := \mathcal{L}(u^x_\epsilon), \qquad \epsilon > 0,$$

satisfies a *large deviations principle* with rate functional given by $I_{x,T}$. In Section 5 we have shown that the functional $I_{x,T}$ is lower semi-continuous and its level sets are compact in $C([0,T];E)$. Thus we only have to prove that the Freidlin-Ventcel upper and lower exponential estimates hold true.

6.1 The lower bounds

We start from the lower estimate. For this purpose, fix $\varphi \in L^2(0,T;H)$ and for any $\epsilon > 0$ define

$$w^\epsilon(t) := w(t) - \frac{1}{\epsilon}\int_0^t \varphi(s)\,ds. \tag{6.2}$$

Due to Girsanov's theorem, w^ϵ is a cylindrical Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^\epsilon)$, where $\mathbb{P}^\epsilon$ is a probability measure absolutely continuous with respect to $\mathbb{P}$, with density

$$\frac{d\mathbb{P}^\epsilon}{d\mathbb{P}} = \exp\left(\frac{1}{\epsilon}\int_0^T \langle\varphi(s), dw(s)\rangle_H - \frac{1}{2\epsilon^2}|\varphi|^2_{L^2(0,T;H)}\right).$$

In fact $\mathbb{P}$ and $\mathbb{P}^\epsilon$ are equivalent and

$$\frac{d\mathbb{P}}{d\mathbb{P}^\epsilon} = \exp\left(-\frac{1}{\epsilon}\int_0^T \langle\varphi(s), dw^\epsilon(s)\rangle_H - \frac{1}{2\epsilon^2}|\varphi|^2_{L^2(0,T;H)}\right).$$

Note that due to the definition of w^ϵ, the solution u^x_ϵ of problem (6.1) is also a solution of the problem

$$du(t) = [Au(t) + F(t,u(t)) + G(t,u(t))B\varphi(t)]\,dt + \epsilon\, G(t,u(t))B\,dw^\epsilon(t), \qquad u(0) = x. \tag{6.3}$$

As a first preliminary result, we prove that if we replace the expectation $\mathbb{E}$ with respect to the probability $\mathbb{P}$ by the expectation $\mathbb{E}^\epsilon$ with respect to the probability $\mathbb{P}^\epsilon$, the first moment of u^x_ϵ verifies an estimate analogous to (2.9), uniformly for φ in bounded sets of $L^2(0,T;H)$.

Lemma 6.1. *For any $r > 0$ and $p \geq 1$ there exists a constant $c_{r,p} > 0$ such that*

$$\mathbb{E}^\epsilon |u^x_\epsilon|^p_{C([0,T];E)} \leq c_{r,p}\left(1 + |x|^p_E\right), \qquad \epsilon \leq 1, \tag{6.4}$$

for any $|\varphi|_{L^2(0,T;H)} \leq r$.

Proof. If we set $v := u^x_\epsilon - \gamma_\varphi(u^x_\epsilon) - \epsilon\,\gamma^\epsilon(u^x_\epsilon)$, with $\gamma^\epsilon(u^x_\epsilon)$ defined by

$$\gamma^\epsilon(u^x_\epsilon)(t) := \int_0^t e^{(t-s)A} G(s, u^x_\epsilon(s))B\,dw^\epsilon(s), \tag{6.5}$$

we have that v solves the problem

$$\frac{dv}{dt}(t) = Av(t) + F(t, u^x_\epsilon(t)), \qquad v(0) = x.$$

By assuming condition 3-ii) in Hypothesis 4 (the proof under condition 3-i) is simpler), with the same arguments used in the proof of Theorem 4.1, for any $t \in [0,T]$ we easily have

$$\frac{d}{dt}^- |v(t)|_E \leq -a|v(t)|^m_E + c_T\left(1 + |\gamma_\varphi(u^x_\epsilon)(t)|^m_E + \epsilon^m\,|\gamma^\epsilon(u^x_\epsilon)(t)|^m_E\right).$$

By comparison, for any $0 \leq t \leq T_0 \leq T$ and $p \geq 1$ we obtain

$$
\begin{aligned}
|u^x_\epsilon(t)|^p_E &\leq c_p \left(|v(t)|^p_E + |\gamma_\varphi(u^x_\epsilon)(t)|^p_E + \epsilon^p\, |\gamma^\epsilon(u^x_\epsilon)(t)|^p_E\right)\\
&\leq c_p \left(|x|^p_E + c_{p,T_0}\left(1 + |\gamma_\varphi(u^x_\epsilon)|^p_{C([0,T_0];E)} + \epsilon^p\, |\gamma^\epsilon(u^x_\epsilon)|^p_{C([0,T_0];E)}\right)\right).
\end{aligned}
$$

Now, according to (2.12) and (4.2), there exists $p_\star \geq 1$ such that for any $\epsilon \leq 1$, $p \geq p_\star$ and $|\varphi|_{L^2(0,T;H)} \leq r$ we have

$$
\mathbb{E}^\epsilon \left(|\gamma_\varphi(u^x_\epsilon)|^p_{C([0,T_0];E)} + \epsilon^p\, |\gamma^\epsilon(u^x_\epsilon)|^p_{C([0,T_0];E)}\right) \leq c_p(T_0)(1+r^p)\left(1 + \mathbb{E}^\epsilon\, |u^x_\epsilon|^p_{C([0,T_0];E)}\right),
$$

with $c_p(t)$ a continuous increasing function such that $c_p(0) = 0$. Thus, by choosing T_0 small enough such that $c_p c_{p,T_0} c_p(T_0)(1+r^p) \leq 1/2$, we obtain

$$
\mathbb{E}^\epsilon\, |u^x_\epsilon(t)|^p_{C([0,T_0];E)} \leq 2\, c_p |x|^p_E + 2 c_p\, c_{p,T_0}\left(1 + c_p(T_0)(1+r^p)\right) \leq c_{r,p}(1 + |x|^p_E).
$$

As we can repeat the same arguments in the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$ and so on, the global estimate (6.4) follows, for any $p \geq p_\star$.

The case for $1 \leq p < p_\star$ follows then by Hölder's inequality. □

As an immediate consequence of (6.4) we have

$$
\lim_{K \to \infty} \mathbb{P}^\epsilon \left(|u^x_\epsilon(t)|_{C([0,T];E)} > K\right) = 0, \tag{6.6}
$$

uniformly for $|x|_E \leq R$, $|\varphi|_{L^2(0,T;H)} \leq r$ and $\epsilon \leq 1$.

Theorem 6.2 (LDP-Lower bounds). *Assume Hypotheses 1 to 4. Then, for each $R, T > 0$, $r \geq 0$ and $\delta, \gamma > 0$ there exists $\epsilon_0 > 0$ such that for any $x \in E$ with $|x|_E \leq R$ and for any $z \in \mathcal{K}_{x,T}(r)$*

$$
\mathbb{P}\left(|u^x_\epsilon - z|_{C([0,T];E)} \leq \delta\right) \geq \exp\left(-\frac{I_{x,T}(z) + \gamma}{\epsilon^2}\right), \qquad \epsilon \leq \epsilon_0. \tag{6.7}
$$

Proof. Clearly, in order to prove (6.7) it is sufficient to prove that there exists $\epsilon_0 > 0$ such that for any $\varphi \in L^2(0,T;H)$, with $|\varphi|^2_{L^2(0,T;H)} \leq 2r$

$$
\mathbb{P}\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \leq \delta\right) \geq \exp\left(-\frac{|\varphi|^2_{L^2(0,T;H)} + \gamma}{2\epsilon^2}\right), \qquad \epsilon \leq \epsilon_0.
$$

Fixing $0 < \bar{\gamma} < \gamma$, we easily have

$$
\begin{aligned}
\mathbb{P}\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \leq \delta\right) &= \mathbb{E}^\epsilon\left(\frac{d\mathbb{P}}{d\mathbb{P}^\epsilon};\ |u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \leq \delta\right)\\
&\geq \exp\left(-\frac{|\varphi|^2_{L^2(0,T;H)} + \gamma}{2\epsilon^2}\right)\exp\left(\frac{\gamma - \bar{\gamma}}{2\epsilon^2}\right)\mathbb{P}^\epsilon\left(\mathcal{A}_\epsilon\right),
\end{aligned}
$$

where

$$\mathcal{A}_\epsilon := \left\{ \epsilon \left| \int_0^T \langle \varphi(s), dw^\epsilon(s) \rangle_H \right| \leq \frac{\bar{\gamma}}{2};\ |u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} \leq \delta \right\}.$$

If we show that

$$\liminf_{\epsilon \to 0} \mathbb{P}^\epsilon\left(\mathcal{A}_\epsilon\right) > 0,$$

uniformly with respect to $|x|_E \leq R$ and $|\varphi|^2_{L^2(0,T;H)} \leq 2r$, we are done. In fact, we prove that such lim inf is 1.

By the Chebyshev inequality we have

$$\mathbb{P}^\epsilon \left(\left| \int_0^T \langle \varphi(s), dw^\epsilon(s) \rangle_H \right| > \frac{\bar{\gamma}}{2\epsilon} \right) \leq \left(\frac{2\epsilon}{\bar{\gamma}} \right)^2 \mathbb{E}^\epsilon \left| \int_0^T \langle \varphi(s), dw^\epsilon(s) \rangle_H \right|^2$$

$$= \left(\frac{2\epsilon}{\bar{\gamma}} \right)^2 \int_0^T |\varphi(s)|_H^2 \, ds \leq \frac{8r\epsilon^2}{\bar{\gamma}^2}.$$

Then, it follows

$$\mathbb{P}^\epsilon(\mathcal{A}_\epsilon) \geq 1 - \mathbb{P}^\epsilon \left(\left| \int_0^T \langle \varphi(s), dw^\epsilon(s) \rangle_H \right| > \frac{\bar{\gamma}}{2\epsilon} \right) - \mathbb{P}^\epsilon \left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta \right)$$

$$\geq 1 - \frac{8r\epsilon^2}{\bar{\gamma}^2} - \mathbb{P}^\epsilon \left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta \right).$$

Due to (6.6), for any $\alpha > 0$ we can fix $\bar{K}_\alpha > 0$ such that

$$\mathbb{P}^\epsilon \left(|u_\epsilon^x|_{C([0,T];E)} > \bar{K}_\alpha \right) \leq \alpha,$$

uniformly with respect to $|x|_E \leq R$, $|\varphi|^2_{L^2(0,T;H)} \leq 2r$ and $\epsilon \leq 1$. This implies that

$$\mathbb{P}^\epsilon \left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta \right)$$

$$\leq \mathbb{P}^\epsilon \left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta;\ |u_\epsilon^x|_{C([0,T];E)} \leq \bar{K}_\alpha \right) + \mathbb{P}^\epsilon \left(|u_\epsilon^x|_{C([0,T];E)} > \bar{K}_\alpha \right)$$

$$\leq \mathbb{P}^\epsilon \left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta;\ |u_\epsilon^x|_{C([0,T];E)} \leq \bar{K}_\alpha \right) + \alpha.$$

For fixed $\bar{K}_\alpha$, thanks to (4.4) and to the local Lipschitz-continuity of $F(t)$, uniformly for $t \in [0,T]$, we can find a constant $L > 0$ such that if $|u_\epsilon^x|_{C([0,T];E)} \leq \bar{K}_\alpha$

$$|F(s, u_\epsilon^x(s)) - F(s, z^x(\varphi)(s))|_E \leq L\, |u_\epsilon^x(s) - z^x(\varphi)(s)|_E\,, \qquad s \in [0,T],$$

for any $|x|_E \leq R$ and $|\varphi|^2_{L^2(0,T;H)} \leq 2r$. Therefore, since

$$u_\epsilon^x(t) - z^x(\varphi)(t)$$

$$= \int_0^t e^{(t-s)A} \left[F(s, u_\epsilon^x(s)) - F(s, z^x(\varphi)(s)) \right] ds + \left[\gamma_\varphi(u_\epsilon^x) - \gamma_\varphi(z^x(\varphi)) \right](t) + \epsilon \gamma^\epsilon(u_\epsilon^x)(t),$$

by using (4.3) for any $0 < T_0 \leq T$ we obtain

$$\sup_{t\in[0,T_0]} |u_\epsilon^x(t) - z^x(\varphi)(t)|_E$$

$$\leq \left(LT_0 + c(T_0)\|B\|_\varrho \sqrt{2r}\right) \sup_{t\in[0,T_0]} |u_\epsilon^x(t) - z^x(\varphi)(t)|_E + \epsilon \sup_{t\in[0,T_0]} |\gamma^\epsilon(u_\epsilon^x)(t)|_E\,.$$

Thus, if we choose T_0 such that $LT_0 + c(T_0)\|B\|_\varrho \sqrt{2r} \leq 1/2$, we have

$$\sup_{t\in[0,T_0]} |u_\epsilon^x(t) - z^x(\varphi)(t)|_E \leq 2\,\epsilon \sup_{t\in[0,T]} |\gamma^\epsilon(u_\epsilon^x)(t)|_E\,.$$

If we repeat the same argument in the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$ and so on, we obtain

$$|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} \leq c\,\epsilon\,|\gamma^\epsilon(u_\epsilon^x)|_{C([0,T];E)}\,. \tag{6.8}$$

By using (2.12) and (6.4), this implies that

$$\mathbb{P}^\epsilon\left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} > \delta;\ |u_\epsilon^x|_{C([0,T];E)} \leq \bar{K}_\alpha\right) \leq \mathbb{P}^\epsilon\left(|\gamma^\epsilon(u_\epsilon^x)|_{C([0,T];E)} > \frac{\delta}{c\epsilon}\right)$$

$$\leq \left(\frac{c\epsilon}{\delta}\right)^{p_\star} \mathbb{E}^\epsilon\,|\gamma^\epsilon(u_\epsilon^x)|^{p_\star}_{C([0,T];E)} \leq \left(\frac{c\epsilon}{\delta}\right)^{p_\star} c_{p_\star}(T)\,\left(\mathbb{E}^\epsilon\,|u_\epsilon^x|^{p_\star}_{C([0,T];E)} + 1\right) \leq \epsilon^{p_\star}\,c_{p_\star,r,T,R},$$

for any $|x|_E \leq R$, $|\varphi|^2_{L^2(0,T;H)} \leq r$ and $\epsilon \leq 1$.

Therefore, collecting all terms we can conclude that

$$\liminf_{\epsilon\to 0} \mathbb{P}^\epsilon(\mathcal{A}_\epsilon) \geq 1 - \alpha.$$

Due to the arbitrariness of α we can conclude that such lim inf is one.

□

6.2 The upper bounds

Now we estimate the probability that the trajectories of u_ϵ^x move far away from the set of small values of the action functional $I_{x,T}$. To this purpose we introduce some notations.

For any $T > 0$, $r \geq 0$ and $n \in \mathbb{N}$ we define the set

$$C_{n,T}(r) := \left\{ u = \int_0^\cdot P_n\varphi(s)\,ds;\ \ \frac{1}{2}\,|P_n\varphi|^2_{L^2(0,T;H)} \leq r \right\},$$

where P_n is the projection of H onto the space generated by $\{e_1, \ldots, e_n\}$.

Clearly $C_{n,T}(r)$ is the r-level set corresponding to the functional

$$J_{n,T}(u) := \frac{1}{2}\,\inf\left\{ |P_n\varphi|^2_{L^2(0,T;H)}\,;\ u = \int_0^\cdot P_n\varphi(s)\,ds \right\}$$

Note that as $P_n(H)$ is a finite dimensional space, $C_{n,T}(r)$ is compact in $C([0,T];H)$.

As an immediate consequence of the large deviations estimates for the standard Brownian motion on $\mathbb{R}^n$ (for a proof see e.g. [11, Theorem 3.2.2]), the family of probability measures $\{\mathcal{L}(\epsilon P_n w)\}_{\epsilon>0}$, where w is the cylindrical Wiener process introduced above, fulfills a large deviations principle with action functional $J_{n,T}$. In particular, for any $r \geq 0$ and $\delta, \gamma > 0$ there exists $\epsilon_0 > 0$ such that

$$\mathbb{P}\left(|\epsilon P_n w - C_{n,T}(r)|_{C([0,T];H)} \geq \delta\right) \leq \exp\left(-\frac{r-\gamma}{2\epsilon^2}\right), \qquad \epsilon \leq \epsilon_0. \tag{6.9}$$

In this section our aim is to prove an estimate analogous to (6.9) for the laws of the processes u_ϵ^x.

Theorem 6.3 (LDP-Upper bounds). *Assume Hypotheses 1 to 4. For any $R, T > 0$, $r \geq 0$ and $\delta, \gamma > 0$, there exists $\epsilon_0 > 0$ such that for any $x \in E$ with $|x|_E \leq R$,*

$$\mathbb{P}\left(|u_\epsilon^x - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{r-\gamma}{\epsilon^2}\right), \qquad \epsilon \leq \epsilon_0.$$

Proof. We assume here that that for any $\alpha, \delta, R, r > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for any $\varphi \in L^2(0,T;H)$, with $|\varphi|^2_{L^2(0,T;H)} \leq 2r$, and $|x|_E \leq R$ there exist $\beta_\varphi, \epsilon_\varphi > 0$ such that

$$\mathbb{P}\left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} \geq \delta\ ;\ \left|\epsilon P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi(s)\,ds\right|_{C([0,T];H)} < \beta_\varphi\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \tag{6.10}$$

for any $\epsilon \leq \epsilon_\varphi$. This crucial estimate is formulated and proved as Theorem 6.4 below.

As $C_{\bar{n},T}(r)$ is compact in $C([0,T];H)$, there exist $\varphi_1, \ldots, \varphi_k \in L^2(0,T;H)$, with $|P_{\bar{n}}\varphi_i|^2_{L^2(0,T;H)} \leq 2r$, such that

$$C_{\bar{n},T}(r) \subset \bigcup_{i=1}^k \left\{ u \in C([0,T];H);\ \left|u - \int_0^\cdot P_{\bar{n}}\varphi_i(s)\,ds\right|_{C([0,T];H)} < \beta_{\varphi_i} \right\} =: \mathcal{B}.$$

Moreover, there exists $\delta' > 0$ such that

$$\left\{u \in C([0,T];H)\ ;\ |u - C_{\bar{n},T}(r)|_{C([0,T];H)} < \delta'\right\} \subset \mathcal{B}.$$

According to (6.9), this means that for any $0 < \bar{\gamma} < \gamma$ there exists $\epsilon_1 > 0$ such that

$$\mathbb{P}\left(\epsilon P_{\bar{n}} w \notin \mathcal{B}\right) \leq \mathbb{P}\left(|\epsilon P_{\bar{n}} w - C_{\bar{n},T}(r)|_{C([0,T];H)} \geq \delta'\right) \leq \exp\left(-\frac{r-\bar{\gamma}}{\epsilon^2}\right), \qquad \epsilon \leq \epsilon_1.$$

Now, we have

$$\begin{aligned}
&\mathbb{P}\left(|u_\epsilon^x - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\right)\\
&\leq \mathbb{P}\left(\epsilon P_{\bar{n}} w \notin \mathcal{B}\right) + \mathbb{P}\left(|u_\epsilon^x - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\ ;\ \epsilon P_{\bar{n}} w \in \mathcal{B}\right),
\end{aligned}$$

so that for any $\epsilon \leq \epsilon_1$ we obtain

$$\mathbb{P}\left(|u_\epsilon^x - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{r-\bar{\gamma}}{\epsilon^2}\right)$$

$$+\sum_{i=1}^k \mathbb{P}\left(|u_\epsilon^x - z^x(\varphi_i)|_{C([0,T];E)} \geq \delta\,;\ \left|\epsilon\, P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi_i(s)\,ds\right|_{C([0,T];H)} < \beta_{\varphi_i}\right).$$

Thanks to (6.10), this implies that for any fixed $\alpha > 0$ we can find $\epsilon_0 \leq \epsilon_1$ such that

$$\mathbb{P}\left(|u_\epsilon^x - K_{x,T}(r)|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{r-\bar{\gamma}}{2\epsilon^2}\right) + k\exp\left(-\frac{\alpha}{\epsilon^2}\right),$$

for any $\epsilon \leq \epsilon_0$. This allows to conclude, by choosing α large enough and $\bar{\gamma}$ small enough. □

Therefore, in order to complete the proof of Theorem 6.3 we have to prove the following result.

Theorem 6.4. *Let $T > 0$ be fixed. Then, under the Hypotheses 1 to 4, for any $\alpha, \delta > 0$ and $R, r > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for any $\varphi \in L^2(0,T;H)$, with $|\varphi|^2_{L^2(0,T;H)} \leq 2r$, there exist $\bar{\epsilon}, \beta > 0$ such that*

$$\mathbb{P}\left(|u_\epsilon^x - z^x(\varphi)|_{C([0,T];E)} \geq \delta\ ;\ \left|\epsilon\, P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi(s)\,ds\right|_{C([0,T];H)} < \beta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \tag{6.11}$$

for any $\epsilon \leq \bar{\epsilon}$ and $|x|_E \leq R$.

Proof. We can assume that the mapping $g(t,\cdot,\cdot) : \overline{\mathcal{O}}\times\mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is bounded, uniformly with respect to $t \in [0,T]$, so that we can use Theorems 3.2 and 3.3. Actually, if this is not the case, for any $\delta > 0$ and $x \in E$ we introduce the stopping time

$$\tau_\delta^x := \inf\{\, t \geq 0\,;\ |u^x(t) - z^x(\varphi)(t)|_E \geq \delta\,\}$$

and we note that if $t \leq \tau_\delta^x$, due to (4.4) we have

$$|u^x(t)|_E \leq |u^x(t) - z^x(\varphi)(t)|_E + |z^x(\varphi)(t)|_E \leq \delta + c_{r,T}\,(1+R) := K.$$

Now, defining the mapping

$$g_K(t,\xi,\sigma) := \begin{cases} g(t,\xi,\sigma) & \text{if } |\sigma| \leq K \\ \\ g\left(t,\xi,K\sigma/|\sigma|\right) & \text{if } |\sigma| > K, \end{cases}$$

we have that $g_K(t,\xi,\cdot)$ is Lipschitz-continuous and bounded, uniformly with respect to $\xi \in \overline{\mathcal{O}}$ and $t \in [0,T]$. Moreover, if we set $G_K(t,x)(\xi) = g_K(t,\xi,x(\xi))$ and denote by

$u^x_{\epsilon,K}$ the solution of system (3.6) corresponding to the diffusion term G_K, we have that $u^x_{\epsilon,K}(t) = u^x_\epsilon(t)$, for any $t \leq \tau^x_\delta$. This implies that for any $\delta, \beta > 0$

$$\mathbb{P}\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \geq \delta\ ;\ \left|\epsilon\, P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi(s)\, ds\right|_{C([0,T];H)} < \beta\right)$$

$$\leq \mathbb{P}\left(\sup_{t\leq\tau^x_\delta} |u^x_\epsilon(t) - z^x(\varphi)(t)| \geq \delta\ ;\ \left|\epsilon\, P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi(s)\, ds\right|_{C([0,T];H)} < \beta\right)$$

$$\leq \mathbb{P}\left(\left|u^x_{\epsilon,K} - z^x(\varphi)\right|_{C([0,T];E)} \geq \delta\ ;\ \left|\epsilon\, P_{\bar{n}} w - \int_0^\cdot P_{\bar{n}}\varphi(s)\, ds\right|_{C([0,T];H)} < \beta\right).$$

Thus, it is sufficient to prove the theorem for $u^x_{\epsilon,K}$, that is under the assumption that g is bounded.

For this purpose, we will need several lemmas.

Lemma 6.5. *For any $\alpha, \delta > 0$ there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we can find $\epsilon_k > 0$ such that for any $x \in E$ and $\epsilon \leq \epsilon_k$*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \epsilon \left|\int_{\sigma_k(t)}^t e^{(t-s)A} G(s, u^x_\epsilon(s)) B\, dw(s)\right|_E \geq \delta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \tag{6.12}$$

where $\sigma_k(t) = iT/k$, if $t \in I_{i,k} := [iT/k, (i+1)T/k)$, with $i = 0, \ldots, k-1$.

Proof. For any $0 \leq r < t \leq T$ and $u \in L^p(\Omega; C([0,T];E))$, we set

$$\gamma_r(u)(t) := \int_r^t e^{(t-s)A} G(s, u(s)) B\, dw(s).$$

For any $\delta > 0$ and $k \in \mathbb{N}$ we have

$$\mathbb{P}\left(\epsilon|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)|_{C([0,T];E)} \geq \delta\right) \leq \sum_{i=0}^{k-1} \mathbb{P}\left(\epsilon\left|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)\right|_{C(I_{i,k};E)} \geq \delta\right)$$

$$\leq k \sup_{i=0,\ldots,k-1} \mathbb{P}\left(\epsilon\left|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)\right|_{C(I_{i,k};E)} \geq \delta\right).$$

Note that if $t \in I_{i,k}$, we have $\sigma_k(t) = i\,T/k$. Then, according to (3.9), since $|I_{i,k}| = T/k$ we get

$$\mathbb{P}\left(\epsilon\left|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)\right|_{C(I_{i,k};E)} \geq \delta\right) \leq c_1 \exp\left(-\frac{\delta^2 k^\lambda}{\epsilon^2 c_2\, T^\lambda \eta_T}\right).$$

This implies that

$$\mathbb{P}\left(\epsilon|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)|_{C([0,T];E)} \geq \delta\right)$$

$$\leq k\, c_1 \exp\left(-\frac{\delta^2 k^\lambda}{\epsilon^2 c_2\, T^\lambda \eta_T}\right) = \exp\left(-\frac{\delta^2 k^\lambda}{\epsilon^2 c_2\, T^\lambda \eta_T} + \log k + \log c_1\right).$$

Now, if we fix $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$

$$\frac{\delta^2 k^\lambda}{c_2\, T^\lambda \eta_T} \geq 2\alpha,$$

and in correspondence to each $k \geq k_0$ we set

$$\epsilon_k := \left(\frac{\alpha}{\log k + \log c_1}\right)^{1/2},$$

we obtain (6.12). □

Lemma 6.6. *For any $\alpha, \delta, R > 0$ there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we can find $\epsilon_k > 0$ such that for any $|x|_E \leq R$ and $\epsilon \leq \epsilon_k$*

$$\mathbb{P}\left(\left|u^x_\epsilon - u^x_{\epsilon,k}\right|_{C([0,T];E)} \geq \delta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \tag{6.13}$$

where

$$u^x_{\epsilon,k}(t) := e^{(t-\sigma_k(t))A} u^x_\epsilon(\sigma_k(t)).$$

Proof. Clearly we have

$$u^x_\epsilon(t) = u^x_{\epsilon,k}(t) + \int_{\sigma_k(t)}^t e^{(t-s)A} F(s, u^x_\epsilon(s))\, ds + \epsilon \int_{\sigma_k(t)}^t e^{(t-s)A} G(s, u^x_\epsilon(s)) B\, dw(s).$$

Thus, according to the growth condition on F, as $|t - \sigma_k(t)| \leq T/k$ for any $t \in [0,T]$, by (2.6) we obtain

$$\left|u^x_\epsilon(t) - u^x_{\epsilon,k}(t)\right|_E \leq c_T \int_{\sigma_k(t)}^t \left(1 + |u^x_\epsilon(s)|^m_E\right)\, ds + \epsilon \left|\gamma_{\sigma_k(t)}(u^x_\epsilon)(t)\right|_E$$

$$\leq \frac{c_T\, T}{k}\left(1 + |u^x_\epsilon|^m_{C([0,T];E)}\right) + \epsilon \left|\gamma_{\sigma_k(\cdot)}(u^x_\epsilon)\right|_{C([0,T];E)} := J_{k,1}(\epsilon) + J_{k,2}(\epsilon).$$

This means that

$$\mathbb{P}\left(\left|u^x_\epsilon - u^x_{\epsilon,k}\right|_{C([0,T];E)} \geq \delta\right) \leq \mathbb{P}\left(J_{k,1}(\epsilon) \geq \delta/2\right) + \mathbb{P}\left(J_{k,2}(\epsilon) \geq \delta/2\right).$$

Concerning the first term we have

$$\mathbb{P}\left(J_{k,1}(\epsilon) \geq \delta/2\right) = \mathbb{P}\left(|u^x_\epsilon|^m_{C([0,T];E)} \geq \frac{k\delta}{2\, c_T\, T} - 1\right)$$

and then, if k is large enough we obtain

$$\mathbb{P}\left(J_{k,1}(\epsilon) \geq \delta/2\right) = \mathbb{P}\left(|u^x_\epsilon|_{C([0,T];E)} \geq \left(\frac{k\delta}{2\, c_T\, T} - 1\right)^{\frac{1}{m}}\right).$$

Due to (3.7) this means that once we fix $\alpha' > \alpha$ we can find $k_1 \in \mathbb{N}$ such that

$$\mathbb{P}\left(J_{k,1}(\epsilon) \geq \delta/2\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right),$$

for any $k \geq k_1$, $\epsilon \leq 1$ and $|x|_E \leq R$.

Concerning the second term, due to (6.12) we can find $k_2 \in \mathbb{N}$ such that for any $k \geq k_2$ there exists $\epsilon_k > 0$ such that

$$\mathbb{P}\left(J_{k,2}(\epsilon) \geq \delta/2\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right),$$

for any $k \geq k_2$, $\epsilon \leq \epsilon_k$ and $|x|_E \leq R$. Thus, if we take $k_0 := k_1 \vee k_2$ and $\alpha' := \alpha + \log 2$ we obtain (6.13).

□

Lemma 6.7. *For any $\alpha, \delta, R > 0$ there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we can find some $\epsilon_k > 0$ such that for any $\epsilon \leq \epsilon_k$, $|x|_E \leq R$ and $0 \leq t_1 < t_2 \leq T$*

$$\mathbb{P}\left(\epsilon\,\left|\gamma_{t_1}(u^x_\epsilon)(t_2) - \gamma_{t_1}(u^x_{\epsilon,k})(t_2)\right|_E \geq \delta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right). \tag{6.14}$$

Proof. If we define

$$\tau^x_{\epsilon,k}(\vartheta) := \inf\left\{t \geq 0\ :\ \left|u^x_\epsilon(t) - u^x_{\epsilon,k}(t)\right|_E \geq \vartheta\right\},$$

we have

$$\mathbb{P}\left(\epsilon\,\left|\gamma_{t_1}(u^x_\epsilon)(t_2) - \gamma_{t_1}(u^x_{\epsilon,k})(t_2)\right|_E \geq \delta\right)$$

$$\leq \mathbb{P}\left(\tau^x_{\epsilon,k}(\vartheta) \leq T\right) + \mathbb{P}\left(\epsilon\,\left|\gamma_{t_1}(u^x_\epsilon)(t_2) - \gamma_{t_1}(u^x_{\epsilon,k})(t_2)\right|_E \geq \delta\ ,\ \tau^x_{\epsilon,k}(\vartheta) > T\right).$$

If $\tau^x_{\epsilon,k}(\vartheta)(\omega) > T$, as g is Lipschitz-continuous in the third variable, uniformly for $(s,\xi) \in [0,T] \times \overline{\mathcal{O}}$, there exists $L > 0$ such that

$$\sup_{(s,\xi)\in[0,T]\times\overline{\mathcal{O}}} |g(s,\xi,u^x_\epsilon(s,\xi)(\omega)) - g(s,\xi,u^x_{\epsilon,k}(s,\xi)(\omega))|_E \leq L\,\vartheta.$$

Then, according to Theorem 3.2 applied to

$$h_s := g(s,\cdot,u^x_\epsilon(s)) - g(s,\cdot,u^x_{\epsilon,k}(s)) \wedge L\,\vartheta \vee (-L\,\vartheta),$$

if we fix $\alpha' > \alpha$ and if we take $\bar{\vartheta}$ small enough for any $k \in \mathbb{N}$ we have

$$\mathbb{P}\left(\epsilon\,\left|\gamma_{t_1}(u^x_\epsilon)(t_2) - \gamma_{t_1}(u^x_{\epsilon,k})(t_2)\right|_E \geq \delta\ ,\ \tau^x_{\epsilon,k}(\bar{\vartheta}) > T\right)$$

$$\leq c_1 \exp\left(-\frac{\delta^2}{\epsilon^2\, c_2(t_2-t_1)^\lambda\, L^2\,\bar{\vartheta}^2 \|B\|_\varrho^2}\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right).$$

Concerning the other term, thanks to (6.13) in correspondence to such $\bar{\vartheta}$ there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ there exists $\epsilon_k > 0$ such that

$$\mathbb{P}\left(\tau^x_{\epsilon,k}(\bar{\vartheta}) \leq T\right) \leq \mathbb{P}\left(\left|u^x_\epsilon - u^x_{\epsilon,k}\right|_{C([0,T];E)} \geq \bar{\vartheta}\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right),$$

for any $\epsilon \leq \epsilon_k$ and $|x|_E \leq R$. By taking $\alpha' = \alpha + \log 2$ we get (6.14). □

Lemma 6.8. *For any $\alpha, \delta, R > 0$ and $0 \leq t_1 < t_2 \leq T$ there exist $\bar{n} \in \mathbb{N}$ and $\beta, \bar{\epsilon} > 0$ such that*

$$\mathbb{P}\left(\epsilon\left|\gamma_{t_1}\left(e^{(\cdot-t_1)A}u_\epsilon^x(t_1)\right)(t_2)\right|_E \geq \delta,\ \epsilon\,|P_{\bar{n}}w|_{C([t_1,t_2];H)} \leq \beta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right), \tag{6.15}$$

for any $\epsilon \leq \bar{\epsilon}$ and $|x|_E \leq R$.

Proof. We fix $\alpha' > \alpha$. By using Theorem 3.2 we can find $t_1 < t_1' < t_2' < t_2$ such that for any $x \in E$ and $\epsilon \leq 1$

$$\begin{aligned}\mathbb{P}\Bigg(\epsilon\Bigg|\int_{t_1}^{t_2} & e^{(t_2-s)A}G(s, e^{(s-t_1)A}u_\epsilon^x(t_1))B\,dw(s)\\ &-\int_{t_1'}^{t_2'} e^{(t_2-s)A}G(s, e^{(s-t_1)A}u_\epsilon^x(t_1))B\,dw(s)\Bigg|_E \geq \frac{\delta}{4}\Bigg) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right).\end{aligned} \tag{6.16}$$

Since by (2.2) and the Sobolev embedding theorem there exists $\theta > 0$ such that for any $s \in [t_1', t_2']$

$$\left|e^{(s-t_1)A}u_\epsilon^x(t_1)\right|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} \leq c(s-t_1)^{-\frac{\theta}{2}}|u_\epsilon^x(t_1)|_E \leq c(t_1'-t_1)^{-\frac{\theta}{2}}|u_\epsilon^x|_{C([0,T];E)},$$

due to (3.7) there exists $K > 0$ such that by setting

$$\mathcal{K} := \left\{\, u \in C([t_1', t_2']; C^\theta(\overline{\mathcal{O}};\mathbb{R}^r))\,;\ |u|_{C([t_1',t_2'];C^\theta(\overline{\mathcal{O}};\mathbb{R}^r))} \leq K \,\right\},$$

for any $|x|_E \leq R$ and $\epsilon \leq 1$

$$\begin{aligned}\mathbb{P}\left(e^{(\cdot-t_1)A}u_\epsilon^x(t_1) \notin \mathcal{K}\right) &= \mathbb{P}\left(\left|e^{(\cdot-t_1)A}u_\epsilon^x(t_1)\right|_{C([t_1',t_2'];C^\theta(\overline{\mathcal{O}};\mathbb{R}^r))} > K\right)\\ &\leq \mathbb{P}\left(|u_\epsilon^x|_{C([0,T];E)} > \frac{K}{c}(t_1'-t_1)^{\frac{\theta}{2}}\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right)\end{aligned}$$

Now, since the set $\mathcal{K}$ is compact in $C([t_1', t_2']; E)$, for any fixed $\rho > 0$ there exist $z_1, \ldots, z_{i_0} \in C([t_1', t_2']; E)$ such that $\mathcal{K}$ can be covered by the union of balls in $C([t_1', t_2']; E)$ of radius ρ and center z_i. This implies that for any $n \in \mathbb{N}$

$$\begin{aligned}&\mathbb{P}\left(\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-s)A}G(s, e^{(s-t_1)A}u_\epsilon^x(t_1))(I-P_n)B\,dw(s)\right|_E \geq \frac{\delta}{4}\right)\\ &\leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right) + \sum_{i=1}^{i_0}\mathbb{P}\Bigg(\left|e^{(\cdot-t_1)A}u_\epsilon^x(t_1) - z_i\right|_{C([t_1',t_2'];E)} \leq \rho,\\ &\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-s)A}G(s, e^{(s-t_1)A}u_\epsilon^x(t_1))(I-P_n)B\,dw(s)\right|_E \geq \frac{\delta}{4}\Bigg) =: \exp\left(-\frac{\alpha'}{\epsilon^2}\right) + \sum_{i=1}^{i_0} I_{i,n}\end{aligned}$$

For any $i=1,\dots,i_0$ we have for $B_n := P_n B$

$$
\begin{aligned}
I_{i,n} \leq \mathbb{P}\Bigg(& \left|e^{(\cdot - t_1)A} u_\epsilon^x(t_1) - z_i\right|_{C([t_1',t_2'];E)} \leq \rho, \\
& \epsilon \left| \int_{t_1'}^{t_2'} e^{(t_2-s)A}\left[G(s, e^{(s-t_1)A}u_\epsilon^x(t_1)) - G(s, z_i(s))\right] B\, dw(s)\right|_E \geq \frac{\delta}{12}\Bigg) \\
+\mathbb{P}\Bigg(& \left|e^{(\cdot - t_1)A} u_\epsilon^x(t_1) - z_i\right|_{C([t_1',t_2'];E)} \leq \rho, \\
& \epsilon \left| \int_{t_1'}^{t_2'} e^{(t_2-s)A}\left[G(s, e^{(s-t_1)A}u_\epsilon^x(t_1)) - G(s, z_i(s))\right] B_n\, dw(s)\right|_E \geq \frac{\delta}{12}\Bigg) \\
+\mathbb{P}\Bigg(& \epsilon \left| \int_{t_1'}^{t_2'} e^{(t_2-s)A} G(s, z_i(s))(I-P_n) B\, dw(s)\right|_E \geq \frac{\delta}{12}\Bigg) =: J_i^1 + J_{i,n}^2 + J_{i,n}^3
\end{aligned}
$$

If $|e^{(s-t_1)A}u_\epsilon^x(t_1) - z_i(s)|_E \leq \rho$, for any $s \in [t_1', t_2']$, as $g(s,\xi,\cdot)$ is Lipschitz-continuous, uniformly with respect to $s \in [0,T]$ and $\xi \in \overline{\mathcal{O}}$, we can apply Theorem 3.2 to the function $h_s(\xi) := g(s,\xi,e^{(s-t_1)A}u_\epsilon^x(t_1,\xi)) - g(s,\xi,z_i(s,\xi))$ and we can find some constants $c_1, c_2 > 0$ such that

$$
J_i^1 + J_{i,n}^2 \leq 2\, c_1 \, \exp\left(-\frac{\delta^2}{\epsilon^2\, c_2 \rho (t_2' - t_1')^\lambda}\right),
$$

for any $n \in \mathbb{N}$. Then, if ρ is sufficiently small we have

$$
J_i^1 + J_{i,n}^2 \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right),
$$

for any $\epsilon \leq 1$. Concerning $J_{i,n}^3$, with the notations introduced in the proof of Theorem 3.2 we have

$$
\int_{t_1'}^{t_2'} e^{(t_2-s)A} G(s, z_i(s))(I-P_n)B\, dw(s) = c_{\beta_\star} \int_{t_1'}^{t_2'} (t_2'-s)^{\beta_\star - 1} e^{(t_2'-s)A} \rho_{t_1', \beta_\star}(s)\, ds,
$$

where

$$
\rho_{t_1,\beta_\star}(s,\xi) = \sum_{k=n+1}^\infty \lambda_k \int_{t_1'}^s (s-\sigma)^{-\beta_\star} e^{(s-\sigma)A}\left[e^{(t_2-t_2')A} G(\sigma, z_i(\sigma)) e_k\right](\xi) d\beta_k(\sigma).
$$

As in the proof of [5, Theorem 4.2] it follows that for all $i=1,\dots,i_0$

$$
\sup_{(s,\xi)\in[t_1',t_2']\times\overline{\mathcal{O}}} \sum_{k=n+1}^\infty \lambda_k^2 \int_{t_1'}^s (s-\sigma)^{-2\beta_\star} \left| e^{(s-\sigma)A}\left[e^{(t_2-t_2')A} G(\sigma, z_i(\sigma)) e_k\right](\xi)\right|^2 ds \leq \eta_n,
$$

for some η_n which go to zero, as n goes to infinity. Hence, by using the same arguments as in the proof of [5, Theorem 4.2] and that of Theorem 3.2 we can find some constants $c_1, c_2 > 0$ such that

$$J^3_{i,n} \leq c_1 \exp\left(-\frac{\delta^2}{\epsilon^2\, c_2 \eta_n (t_2 - t_1)}\right).$$

This means that for some $\bar{n}$ sufficiently large

$$J^3_{i,\bar{n}} \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right).$$

Therefore, collecting all terms, in correspondence to $\bar{n}$ we have

$$\mathbb{P}\left(\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-s)A} G(s, e^{(s-t_1)A} u^x_\epsilon(t_1))(I - P_{\bar{n}}) B\, dw(s)\right|_E \geq \frac{\delta}{4}\right) \leq (3i_0 + 1)\exp\left(-\frac{\alpha'}{\epsilon^2}\right). \tag{6.17}$$

According to (3.7) there exists $\delta' > 0$ such that for any $|x|_E \leq R$ and $\epsilon \leq 1$

$$\mathbb{P}\left(|u^x_\epsilon(t_1)|_E \geq \delta'\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right). \tag{6.18}$$

Next, we fix any partition $\pi = \{\sigma_0 = t_1' < \sigma_1 < \ldots < \sigma_k = t_2'\}$ of the interval $[t_1', t_2']$ and we set $\pi(s) = \sigma_i$, if $s \in [\sigma_i, \sigma_{i+1})$. We have

$$\left\{\left|\int_{t_1'}^{t_2'} \left[e^{(t_2-s)A} G(s, e^{(s-t_1)A} u^x_\epsilon(t_1)) - e^{(t_2-\pi(s))A} G(\pi(s), e^{(\pi(s)-t_1)A} u^x_\epsilon(t_1))\right] B_{\bar{n}}\, dw(s)\right|_E \right.$$

$$\left.\geq \frac{\delta}{4\epsilon}\right\} \subseteq \left\{\sup_{t\in[t_1',t_2']}\left|\int_{t_1'}^{t} e^{(t-s)A} H(s) B_{\bar{n}}\, dw(s)\right|_E \geq \frac{\delta}{4\epsilon}\right\},$$

where

$$H(s) = e^{(t_2-t_2')A} G(s, e^{(s-t_1)A} u^x_\epsilon(t_1)) - e^{(t_2-t_2')A} e^{(s-\pi(s))A} G(\pi(s), e^{(\pi(s)-t_1)A} u^x_\epsilon(t_1)).$$

We remark that, since the mapping $(0,T] \to \mathcal{L}(E)$, $s \mapsto e^{sA}$, is continuous and the mapping g fulfills Hypothesis 3, if $|u^x_\epsilon(t_1)|_E \leq \delta'$ we have

$$\sup_{s\in[t_1',t_2']}\left|e^{(t_2-t_2')A} G(s, e^{(s-t_1)A} u^x_\epsilon(t_1)) - e^{(t_2-t_2')A} e^{(s-\pi(s))A} G(\pi(s), e^{(\pi(s)-t_1)A} u^x_\epsilon(t_1))\right|_E$$

$$\leq \sup_{s\in[t_1',t_2']}\left|G(s, e^{(s-t_1)A} u^x_\epsilon(t_1)) - e^{(s-\pi(s))A} G(\pi(s), e^{(\pi(s)-t_1)A} u^x_\epsilon(t_1))\right|_E \leq c_{\pi,\delta'},$$

for some constant $c_{\pi,\delta'}$ going to zero, as $|\pi| = \max\{\sigma_{i+1} - \sigma_i\}$ goes to zero. By using once again Theorem 3.2 this implies that there exists some partition π_0 sufficiently small such that for any $x \in E$ and $\epsilon \leq 1$

$$\mathbb{P}\left(|u^x_\epsilon(t_1)|_E \leq \delta',\ \epsilon\left|\int_{t_1'}^{t_2'} \left[e^{(t_2-s)A} G(s, e^{(s-t_1)A} u^x_\epsilon(t_1))\right.\right.\right.$$

$$\left.\left.\left. - e^{(t_2-\pi_0(s))A} G(\pi_0(s), e^{(\pi_0(s)-t_1)A} u^x_\epsilon(t_1))\right] B_{\bar{n}}\, dw(s)\right|_E \geq \frac{\delta}{4}\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right). \tag{6.19}$$

Therefore, combining all together the estimates (6.16), (6.17), (6.18) and (6.19), for any $\beta > 0$ we obtain

$$\mathbb{P}\left(\epsilon\left|\gamma_{t_1}\left(e^{(\cdot-t_1)A}u_\epsilon^x(t_1)\right)(t_2)\right|_E \geq \delta,\ \epsilon\,|P_{\bar{n}}w|_{C([0,T];H)} \leq \beta\right) \leq (4+3i_0)\exp\left(-\frac{\alpha'}{\epsilon^2}\right)$$

$$+\mathbb{P}\left(\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-\pi_0(s))A}G(\pi_0(s), e^{(\pi_0(s)-t_1)A}u_\epsilon^x(t_1))B_{\bar{n}}dw(s)\right|_E \geq \frac{\delta}{4},\ \epsilon\,|P_{\bar{n}}w|_{C([0,T];H)} \leq \beta\right).$$

If we determine $\beta > 0$ such that the last term above is zero, by taking ϵ small enough, we have

$$(4+3i_0)\exp\left(-\frac{\alpha'}{\epsilon^2}\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right),$$

so that (6.15) follows.

If $|P_{\bar{n}}w|_{C([0,T];H)} \leq \beta/\epsilon$ and if $\pi_0 = \{t_1' = \sigma_0 < \ldots < \sigma_{k_0} = t_2'\}$ is the partition which realizes (6.19) we have

$$\left|\int_{t_1'}^{t_2'} e^{(t_2-\pi_0(s))A}G(\pi_0(s), e^{(\pi_0(s)-t_1)A}u_\epsilon^x(t_1))B_{\bar{n}}\,dw(s)\right|_E$$

$$\leq \sum_{i=0}^{k_0-1}\left|e^{(t_2-\sigma_i)A}G(\sigma_i, e^{(\sigma_i-t_1)A}u_\epsilon^x(t_1))B\left(P_{\bar{n}}w(\sigma_{i+1}) - P_{\bar{n}}w(\sigma_i)\right)\right|_E$$

$$\leq 2\,|P_{\bar{n}}w|_{C([t_1',t_2'];H)}\sum_{i=0}^{k_0-1}\left|e^{(t_2-\sigma_i)A}G(\sigma_i, e^{(\sigma_i-t_1)A}u_\epsilon^x(t_1))B\right|_{\mathcal{L}(H,E)}.$$

Now, due to (2.3), for any $x \in H$ we have

$$|e^{tA}x|_E \leq c\,(t\wedge 1)^{-\frac{d}{4}}|x|_H, \qquad t > 0$$

and then, as $t_2 - \sigma_i \geq t_2 - t_1' > 0$ and $\sigma_i - t_1 \geq t_1' - t_1$, for any $\sigma_i \in \pi_0$, we have

$$\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-\pi_0(s))A}G(\pi_0(s), e^{(\pi_0(s)-t_1)A}u_\epsilon^x(t_1))B_{\bar{n}}\,dw(s)\right|_E \leq 2\epsilon\,|P_{\bar{n}}w|_{C([t_1,t_2];H)}c_{\pi_0} \leq \tilde{c}_{\pi_0}\,\beta.$$

Hence, we can find β small enough such that

$$\mathbb{P}\left(\epsilon\left|\int_{t_1'}^{t_2'} e^{(t_2-\pi_0(s))A}G(\pi_0(s), e^{(\pi_0(s)-t_1)A}u_\epsilon^x(t_1))Bdw(s)\right|_E \geq \frac{\delta}{4},\ \epsilon\,|P_{\bar{n}}w|_{C([0,T];H)} \leq \beta\right) = 0.$$

□

Lemma 6.9. *For any $\alpha, \delta, R > 0$ there exist $\bar{n} \in \mathbb{N}$ and $\beta, \bar{\epsilon} > 0$ such that*

$$\mathbb{P}\left(\epsilon\,|\gamma(u_\epsilon^x)|_{C([0,T];E)} \geq \delta,\ \epsilon\,|P_{\bar{n}}w|_{C([0,T];H)} \leq \beta\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right),$$

for any $\epsilon \leq \bar{\epsilon}$ and $|x|_E \leq R$.

Proof. Let $\alpha' > \alpha$. Due to (6.12) there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$

$$\mathbb{P}\left(\epsilon\, |\gamma(u_\epsilon^x)|_{C([0,T];E)} \geq \delta,\ \epsilon\, |P_{\bar{n}} w|_{C([0,T];H)} \leq \beta\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right)$$

$$+\mathbb{P}\left(\epsilon \sup_{t\in [0,T]} \left|\int_0^{\sigma_k(t)} e^{(t-s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E \geq \frac{\delta}{2},\ \epsilon\, |P_{\bar{n}} w|_{C([0,T];H)} \leq \beta\right).$$

Now, if we set $t_i^k = iT/k$, for any $t \in [0,T]$ we have

$$\left|\int_0^{\sigma_k(t)} e^{(t-s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E = \left|\int_0^{\sigma_k(t)} e^{(t-\sigma_k(t))A} e^{(\sigma_k(t)-s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E$$

$$\leq \sup_{i=0,\ldots,k-1} \left|\int_0^{t_i^k} e^{(t_i^k - s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E$$

$$\leq \sum_{i=0}^{k-1} \left|\int_{t_i^k}^{t_{i+1}^k} e^{(t_{i+1}^k - s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E.$$

This implies that

$$\mathbb{P}\left(\epsilon \sup_{t\in [0,T]} \left|\int_0^{\sigma_k(t)} e^{(t-s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E \geq \frac{\delta}{2},\ \epsilon\, |P_{\bar{n}} w|_{C([0,T];H)} \leq \beta\right)$$

$$\leq \sum_{i=0}^{k-1} \mathbb{P}\left(\epsilon \left|\int_{t_i^k}^{t_{i+1}^k} e^{(t_{i+1}^k - s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E \geq \frac{\delta}{2k},\ \epsilon\, |P_{\bar{n}} w|_{C([t_i^k, t_{i+1}^k];H)} \leq \beta\right).$$

According to Lemma 6.7, for large enough k there exists $\epsilon_k > 0$ such that for any $\epsilon \leq \epsilon_k$

$$\mathbb{P}\left(\epsilon \left|\int_{t_i^k}^{t_{i+1}^k} e^{(t_{i+1}^k - s)A} G(s, u_\epsilon^x(s)) B\, dw(s)\right|_E \geq \frac{\delta}{2k},\ \epsilon\, |P_{\bar{n}} w|_{C([t_i^k, t_{i+1}^k];H)} \leq \beta\right)$$

$$\leq \mathbb{P}\left(\epsilon \left|\int_{t_i^k}^{t_{i+1}^k} e^{(t_{i+1}^k - s)A} G(s, u_{\epsilon,k}^x(s)) B\, dw(s)\right|_E \geq \frac{\delta}{4k},\ \epsilon\, |P_{\bar{n}} w|_{C([t_i^k, t_{i+1}^k];H)} \leq \beta\right)$$

$$+ \exp\left(-\frac{\alpha'}{\epsilon^2}\right).$$

Therefore, since

$$\int_{t_i^k}^{t_{i+1}^k} e^{(t_{i+1}^k - s)A} G(s, u_{\epsilon,k}^x(s)) B\, dw(s) = \gamma_{t_i^k}\left(e^{(\cdot - t_i^k)A} u_\epsilon^x(t_i^k)\right)(t_{i+1}^k),$$

we conclude by applying Lemma 6.8. □

Conclusion of the proof of Theorem 6.4 Now we can prove (6.11). If w^ϵ is the Wiener process on the space $(\Omega, \mathcal{F}, \mathbb{P}^\epsilon)$ introduced in (6.2), we have to show that if we define

$$\mathcal{A}_\epsilon := \left\{ |u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \geq \delta \;;\; \epsilon\, |P_{\bar{n}} w^\epsilon|_{C([0,T];H)} < \beta \right\},$$

for some positive constants β and $\bar{\epsilon}$ we have that

$$\mathbb{P}(\mathcal{A}_\epsilon) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right),$$

for any $\epsilon \leq \bar{\epsilon}$. If we set

$$\xi_\epsilon := \exp\left(-\frac{1}{\epsilon}\int_0^T \langle \varphi(s), dw(s)\rangle_H\right),$$

for any $\lambda > 0$ we have

$$\mathbb{P}(\mathcal{A}_\epsilon) \leq \mathbb{P}\left(\mathcal{A}_\epsilon \cap \left\{\xi_\epsilon \leq \exp\left(\frac{\lambda}{\epsilon^2}\right)\right\}\right) + \mathbb{P}\left(\xi_\epsilon > \exp\left(\frac{\lambda}{\epsilon^2}\right)\right).$$

Due to (3.1) we have

$$\mathbb{P}\left(\xi_\epsilon > \exp\left(\frac{\lambda}{\epsilon^2}\right)\right) \leq \mathbb{P}\left(\left|\int_0^T \langle \varphi(s), dw(s)\rangle_H\right| > \frac{\lambda}{\epsilon}\right)$$

$$\leq 3\exp\left(-\frac{\lambda^2}{6\,\epsilon^2|\varphi|^2_{L^2(0,T;H)}}\right) \leq 3\exp\left(-\frac{\lambda^2}{12\,\epsilon^2\, r}\right).$$

Hence, we can find $\bar{\lambda}$ large enough such that for any $\epsilon \leq 1$

$$\mathbb{P}\left(\xi_\epsilon > \exp\left(\frac{\bar{\lambda}}{\epsilon^2}\right)\right) \leq \frac{1}{2}\exp\left(-\frac{\alpha}{\epsilon^2}\right). \tag{6.20}$$

In correspondence to such $\bar{\lambda}$ we have

$$\mathbb{P}\left(\mathcal{A}_\epsilon \cap \left\{\xi_\epsilon \leq \exp\left(\frac{\bar{\lambda}}{\epsilon^2}\right)\right\}\right) = \mathbb{E}^\epsilon\left(\frac{d\mathbb{P}}{d\mathbb{P}^\epsilon}\;;\; \mathcal{A}_\epsilon \cap \left\{\xi_\epsilon \leq \exp\left(\frac{\bar{\lambda}}{\epsilon^2}\right)\right\}\right)$$

$$\leq \exp\left(\frac{\bar{\lambda}}{\epsilon^2} - \frac{|\varphi|^2_{L^2(0,T;H)}}{2\epsilon^2}\right)\mathbb{P}^\epsilon(\mathcal{A}_\epsilon) \leq \exp\left(\frac{\bar{\lambda}}{\epsilon^2}\right)\mathbb{P}^\epsilon(\mathcal{A}_\epsilon).$$

Thus, in order to conclude, we have to show that there exists $\bar{\epsilon} > 0$ such that

$$\mathbb{P}^\epsilon(\mathcal{A}_\epsilon) \leq \frac{1}{2}\exp\left(-\frac{\bar{\lambda}}{\epsilon^2}\right)\exp\left(-\frac{\alpha}{\epsilon^2}\right), \qquad \epsilon \leq \bar{\epsilon}. \tag{6.21}$$

For any $K > 0$ we have

$$\mathbb{P}^\epsilon(\mathcal{A}_\epsilon) \leq \mathbb{P}^\epsilon\left(|u^x_\epsilon|_{C([0,T];E)} > K\right)$$

$$+\mathbb{P}^\epsilon\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \geq \delta;\; |u^x_\epsilon|_{C([0,T];E)} \leq K;\; \epsilon\,|P_{\bar{n}} w^\epsilon|_{C([0,T];H)} < \beta\right).$$

In the proof of the lower estimates we have seen that u^x_ϵ solves problem (6.3). Then, if we set $v := u^x_\epsilon - \gamma_\varphi(u^x_\epsilon) - \epsilon\gamma^\epsilon(u^x_\epsilon)$, with γ^ϵ defined in (6.5), we have that v solves the problem

$$\frac{dv}{dt}(t) = Av(t) + F(t, u^x_\epsilon(t)), \qquad v(0) = x.$$

By using the same arguments as in the proof of Theorem 3.3, we have

$$|u^x_\epsilon|_{C([0,T];E)} \leq e^{cT}\left(|x|_E + c_T\left(1 + |\gamma_\varphi(u^x_\epsilon)|^m_{C([0,T];E)} + \epsilon^m|\gamma^\epsilon(u^x_\epsilon)|^m_{C([0,T];E)}\right)\right).$$

Note that as we are assuming g to be bounded, due to (4.2) we get

$$|\gamma_\varphi(u^x_\epsilon)|_{C([0,T];E)} \leq c(T)|\varphi|_{L^2(0,T;H)} \leq c_T\sqrt{2r}$$

and hence

$$|u^x_\epsilon|_{C([0,T];E)} \leq e^{cT}\left(|x|_E + c(T)\left(1 + (2r)^{\frac{m}{2}} + \epsilon^m|\gamma^\epsilon(u^x_\epsilon)|^m_{C([0,T];E)}\right)\right).$$

This allows us to repeat the arguments used in the proof of Theorem 3.3 and to conclude that for any $\alpha > 0$ and $R > 0$ there exists $K > 0$ such that for any $\epsilon \leq 1$ and $|x|_E \leq R$

$$\mathbb{P}^\epsilon\left(|u^x_\epsilon|_{C([0,T];E)} > K\right) \leq \exp\left(-\frac{\alpha}{\epsilon^2}\right).$$

Therefore, if we fix $\alpha' = \bar{\lambda} + \alpha + \log 4$, we can find $\bar{K}$ such that

$$\mathbb{P}^\epsilon\left(|u^x_\epsilon|_{C([0,T];E)} > \bar{K}\right) \leq \exp\left(-\frac{\alpha'}{\epsilon^2}\right) = \frac{1}{4}\exp\left(-\frac{\bar{\lambda}}{\epsilon^2}\right)\exp\left(-\frac{\alpha}{\epsilon^2}\right). \tag{6.22}$$

If we fix $\bar{K}$, due to (6.8) we can find some constant $c > 0$ such that

$$\mathbb{P}^\epsilon\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \geq \delta\,;\ |u^x_\epsilon|_{C([0,T];E)} \leq \bar{K};\ \epsilon\,|P_{\bar{n}}w^\epsilon|_{C([0,T];H)} < \beta\right)$$

$$\leq \mathbb{P}^\epsilon\left(\epsilon\,|\gamma^\epsilon(u^x_\epsilon)|_{C([0,T];E)} \geq \frac{\delta}{c};\ \epsilon\,|P_{\bar{n}}w^\epsilon|_{C([0,T];H)} < \beta\right),$$

and then, due to Lemma 6.9 applied to w^ϵ and $\mathbb{P}^\epsilon$ instead of w and $\mathbb{P}$, we can conclude that there exist $\bar{\epsilon}, \beta > 0$ such that

$$\begin{aligned}&\mathbb{P}^\epsilon\left(|u^x_\epsilon - z^x(\varphi)|_{C([0,T];E)} \geq \delta\,;\ |u^x_\epsilon|_{C([0,T];E)} \leq \bar{K};\ \epsilon\,|P_{\bar{n}}w^\epsilon|_{C([0,T];H)} < \beta\right)\\ &\leq \frac{1}{4}\exp\left(-\frac{\bar{\lambda}}{\epsilon^2}\right)\exp\left(-\frac{\alpha}{\epsilon^2}\right).\end{aligned} \tag{6.23}$$

By (6.20), (6.22) and (6.23) we obtain (6.21). ☐

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