SOME SDEs WITH DISTRIBUTIONAL DRIFT

PART II: Lyons-Zheng structure, Itô's formula and semimartingale characterization

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Summary: In dimension 1 we study a martingale problem related to a parabolic PDE operator L with continuous (non-degenerate) diffusion term and with drift being the derivative of a continuous function. We state a necessary and sufficient condition on f and the solution X such that $f(X)$ is a semimartingale. When X is a semimartingale, we also establish an Itô formula for $f(X)$ under minimal assumptions. Particular attention is devoted to the case when L is close to divergence form.

Key words: Martingale problems, Lyons-Zheng processes, time reversal, distributional drift.

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Introduction

This paper continues [18] which tried to give meaning and a basic stochastic analysis framework to stochastic differential equations of the following type:

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b'(X_s)\, ds, \tag{0.1}$$

where b, σ are continuous functions such that $\sigma > 0$. In such a case the formal operator associated with X is given by $Lf = \frac{\sigma^2}{2} f'' + b'f'$.

Equation (0.1) was considered as a martingale problem and sometimes in the sense of probability laws.

Diffusions in a generalized sense were studied by several authors. First, we mention a classical book by N.I. Portenko ([37]) which, however, remains in the framework of semimartingales.

More recently, H.J. Engelbert and J. Wolf ([16]) considered special cases of processes solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes. Those solutions are no longer semimartingales but Dirichlet processes. A special case of equation (0.1) with $\sigma = 1$ and continuous b was treated by P. Seignourel ([45]) without defining the stochastic analysis framework in relation with long time behaviour. This is the case of irregular medium; the case of b being a Brownian path appears also in the literature with the denomination "random medium".

The counterpart of the paper [18] at the level of "strong" solutions is a recent preprint of R.F. Bass and Z-Q. Chen ([4]) which examines, one-dimensional stochastic differential equations with Hölder continuous diffusion and with a drift being the derivative of a Hölder function. For that equation they establish strong existence and pathwise uniqueness.

The literature on Dirichlet processes in the framework of Dirichlet forms is huge and it is impossible to list it completely. We only want to mention some very useful monographies such as [22, 24]. The subject has shown a large development in infinite dimension starting from [2]. A later monography is [28]. Recently, the case of time-dependent Dirichlet forms has attracted a lot of interest, see [34, 46].

Our point of view of Dirichlet processes is pathwise, following [19, 5]. A (continuous) Dirichlet process is the sum of a local martingale M and a zero quadratic variation process A. In [18], we have already examined how the natural framework of the solutions to (0.1) is the class of Dirichlet processes and its $W^{1,2}_{\mathrm{loc}}$ transformations. In this paper, we first discuss the time reversal structure of those processes. This leads to show that those processes are not only Dirichlet, but also Lyons-Zheng (LZ) processes. Those processes were defined in [44] after inspiration from [27, 26] and they extend the class of reversible semimartingales. A (strong) LZ process essentially is a Dirichlet process $M + A$ whose zero quadratic variation part can be expressed as

$$A = \frac{1}{2}(M^2 - M^1) + V, \tag{0.2}$$

where $M^1 = M$, M^2 is a backward local martingale and V is a bounded variation process.

Examples of such processes are first of all C^1-functions of reversible semimartingales; furthermore, Bessel processes of arbitrary dimension belong to this class, see [44]. In this paper, we will also see that, generally, solutions to stochastic differential equations with distributional drift are LZ processes.

For a LZ process X, it is quite natural to define a stochastic integral (of symmetric type) and an Itô formula for $f(X)$, $f \in C^1$, see [27, 44]. An Itô formula under weak assumptions is also the object of this paper. We recall that two papers were simultaneously written by Föllmer, Protter, Shiryaev [21] (resp. Russo, Vallois [42]) for $f(B)$ (resp. $f(S)$), involving the covariation $[f(B), B]$ (resp. $[f(S), S]$); in the first case $f \in W^{1,2}_{loc}$ and B is a classical Brownian motion; in the second case $f \in C^1$ and S is a reversible (multidimensional) semimartingale. Subsequently, generalizations of the first case were treated in [6], [32], where S is first an elliptic diffusion with smooth coefficients and a non-degenerate martingale in the sense of Malliavin calculus. Errami, Russo and Vallois ([17]) generalized the paper [42] to the case of processes with jumps. The multidimensional situation for $f \in W^{1,2}_{loc}$ was treated in [20] for Brownian motion B and in [32] (resp. [33]) when X

is one-dimensional (resp. multidimensional) Brownian martingale which is non-degenerate regarding Malliavin calculus.

The paper is organized as follows. In the whole paper, the basic assumption is the existence, via a regularization procedure, of some function Σ which expresses formally $\Sigma(x) := \int_0^x \frac{2b'}{\sigma^2}$. In Section 2, we recall the notion of the basic operator L and consider the concept of a C^1-generalized solution to $Lf = \dot{l}$, where $\dot{l} \in C^0$, $f \in C^1$. We recall that $Lf = \dot{l}$ admits a solution for any $\dot{l} \in C^0$. $\mathcal{D}_L$ is the subset of C^1-functions f such that $Lf = \dot{l}$ for some $\dot{l} \in C^0$. Significant examples arise when $b = \alpha\sigma^2/2 + \beta$, where $\alpha \in [0,1]$ and β is a function of bounded variation. A particular situation appears when L is close to divergence type which means that

$$b = \frac{\sigma^2}{2} + \beta. \tag{0.3}$$

In Section 3, we recall the concept of martingale problem related to L. This problem has a unique solution provided that a condition of non-explosion is fulfilled. This condition is also necessary. If L is close to divergence type then it is possible to show that the martingale problem is equivalent to a stochastic differential equation in the weak sense (0.1); more precisely, the solution X to the martingale problem associated with L will solve

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A(b), \tag{0.4}$$

where $A : C^0(\mathbb{R}) \to \mathcal{C}$ is the unique extension of the map

$$l \mapsto \int_0^{\cdot} l'(X_s)\, ds$$

defined on $C^1(\mathbb{R})$; $\mathcal{C}$ denotes the metric space of continuous processes endowed with the ucp topology. The existence of such an extension is explained by the fact that the map $\mathcal{L} : \mathcal{D}_L \to C^0$, defined by

$$\mathcal{L}f(x) = \int_0^x Lf(y)\, dy$$

can be extended continuously to $C^1(\mathbb{R})$.

In Section 3, we also recall the fact that solutions of martingale problems have a law density at each time $t > 0$ fulfilling local Aronson estimates.

Moreover, we prove sharper properties on the integrability of the density when L is of divergence form.

In Section 4, we reveal that the solution to a martingale problem is in fact a LZ process, at least under some weak technical assumption (TA) on the coefficients. For this, we show that it is the C^1-transformation of a time reversible semimartingale.

Time reversal tools are essential for this task. Similar calculations have been performed by several authors in other situations, see for instance [19, 31, 8, 9, 29].

The second significant result of Section 4 is an Itô formula under weak assumptions. It applies to $f \in W^{1,2}_{loc}$ and solutions X to the equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \gamma(X_s)\, ds, \tag{0.5}$$

where $\sigma > 0$ is continuous and $\gamma \in L^{\infty}_{loc}$.

Section 5 characterizes the class of $f \in W^{1,2}_{loc}$ such that $f(X)$ is a semimartingale. In particular, we obtain a condition on b and σ ensuring that X is a semimartingale.

Semimartingale characterizations for functions of the Brownian motion were first considered by [11] with extensions to the Markov case. There is a recent result of Fukushima [23] characterizing functionals of semimartingales, in the case of a symmetric Markov process. Our methods are direct and do not make use of the Markov property. Extensions for time inhomogeneous functions of the Brownian motion have been obtained by [10].

We conclude the paper in Section 6 by examining the case when L is of divergence type

$$Lf = \frac{1}{2}(\sigma^2 f')'. \tag{0.6}$$

A first interpretation of X was given in [46] and in [38] related to a LZ type decomposition. In [38], however, no stochastic differential equation appears. If $\sigma \in C^1$ then it is immediate to show

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+W_s, \tag{0.7}$$

where d^+W denotes the backward integral.

If X solves a martingale problem associated with L then it follows from Section 5 that X is a semimartingale if and only if σ has bounded variation. In this case we show that X solves also (0.7). In the general case (σ continuous) this is no longer true.

1 Notations and recalls

If I is a real open interval then $C(I)$ will be the F-type space (according to the notations of [13, Chapter 2]) of continuous functions on I endowed with the topology of uniform convergence on compacts. For $k \geq 0$, $C^k(I)$ will be a similar space equipped with the topology of uniform convergence of the first k derivatives. If $I = \mathbb{R}$ we will simply write C, C^k instead of $C(\mathbb{R})$, $C^k(\mathbb{R})$.

Furthermore, we will work with the following F-type spaces. L^2_{loc} denotes the space of all Borel functions which are square integrable when restricted to compact subsets. $W^{1,2}_{loc}$ is the space of all absolutely continuous functions f admitting a density $f' \in L^2_{loc}$. It is equipped with the distance which sums $|f(0)|$ and the distance of f' in L^2_{loc}. Similarly, we can consider L^p_{loc} for $p \geq 1$. We denote the set of C^k real functions with compact support by C^k_c, $k \geq 0$. *const* will denote a generic positive constant.

T will be a fixed real number. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. All processes will be considered with index in $\mathbb{R}$. The F-type space of continuous processes equipped with the ucp topology is denoted by $\mathcal{C}$. We recall that a sequence of processes (H_n) in $\mathcal{C}$ converges ucp to H if, for every $T > 0$, $\sup_{t \in [0,T]} |(H_n - H)(t)|$ converges to zero in probability. Note that H belongs automatically to $\mathcal{C}$.

For convenience, we follow the framework of stochastic calculus introduced in [40] and continued in [41, 42, 43], [49, 50, 51] and [44]. Let $X = (X_t, t \in [0,T])$ be a continuous process and $Y = (Y_t, t \in [0,T])$ be a process with paths in L^1_{loc}. We recall in the sequel the most useful rules of calculus.

The forward, backward and symmetric integrals and the covariation process are defined by the following limits in the ucp (uniform convergence in probability) sense whenever they exist

$$\int_0^t Y_s\, d^- X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds \tag{1.1}$$

$$\int_0^t Y_s\, d^+ X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon}\, ds \tag{1.2}$$

$$\int_0^t Y_s\, d^0 X_s \quad := \quad \lim_{\varepsilon \to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)\vee 0}}{2\varepsilon}\, ds \tag{1.3}$$

$$[X,Y]_t \quad := \quad \lim_{\varepsilon \to 0+} C_\varepsilon(X,Y)_t, \tag{1.4}$$

where

$$C_\varepsilon(X,Y)_t := \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds.$$

For $[X,X]$ we shortly write $[X]$.

All stochastic integrals and covariation processes will be of course elements of $\mathcal{C}$.

For a given process $Z = (Z_t, t \in [0,T])$ we set $\hat{Z}_t := Z(T-t)$, $t \in [0,T]$.

Remark 1.1 *a)*

$$\int_0^t Y_s\, d^0 X_s = \frac{1}{2} \int_0^t Y_s\, d^- X_s + \frac{1}{2} \int_0^t Y_s\, d^+ X_s,$$

b) $[X,Y]_t = \int_0^t Y_s\, d^+ X_s - \int_0^t Y_s\, d^- X_s$, *provided that two of the three terms in a) and b) exist.*

c) $X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, d^- X_s + \int_0^t X_s\, d^- Y_s + [X,Y]_t$ *with similar conventions as in a) and b).*

d) $[X,Y]_t = [\hat{X}, \hat{Y}]_T - [\hat{X}, \hat{Y}]_{T-t}$.

e) If one of the two following members exists then

$$\int_0^t Y_s\, d^+ X_s = - \int_{T-t}^T \hat{Y}_s\, d^- \hat{X}_s$$

holds, where the integrals from a to b $(a, b \in \mathbb{R})$ *are analogously defined as in (1.1), ..., (1.4).*

Remark 1.2 *a) If $[X,X]$ exists then it is always an increasing process and X is called a finite quadratic variation process. If $[X,X] \equiv 0$ then X is said to be a zero quadratic variation process (or a zero energy process).*

b) If X, Y are continuous processes such that $[X,Y]$, $[X,X]$, $[Y,Y]$ exist then $[X,Y]$ has bounded (total) variation. If $f,g \in C^1$ then

$$[f(X), g(Y)]_t = \int_0^t f'(X)g'(Y)\, d[X,Y].$$

c) If A is a zero quadratic variation process and X is a finite quadratic variation process then $[X,A] \equiv 0$.

d) A bounded variation process is a zero quadratic variation process.

e) (Classical Itô's formula) If $f \in C^2$ then $\int_0^{\cdot} f'(X)\, d^- X$ exists and is equal to

$$f(X) - f(X_0) - \frac{1}{2}\int_0^{\cdot} f''(X)\, d[X].$$

f) If $f \in C^1$ and $g \in C^2$ then the forward integral $\int_0^{\cdot} f(X) d^- g(X)$ is well defined.

In this paper all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale, Y is $\mathbb{F}$-adapted with the suitable square integrability conditions, then $\int_0^{\cdot} Y\, d^- X$ is the usual Itô's integral. If Y is an $\mathbb{F}$-semimartingale then $\int_0^{\cdot} Y\, d^0 X$ is the classical Fisk-Stratonovich integral and $[X,Y]$ the usual covariation process $\langle X, Y\rangle$.

A semimartingale X such that $\hat{X}$ is again a semimartingale (with respect to some filtrations) is said to be a *time reversible semimartingale.*

A $\mathbb{F}$-*Dirichlet process* is the sum of an $\mathbb{F}$-local continuous martingale M and an $\mathbb{F}$-adapted zero quadratic variation process A, see [19], [5].

Remark 1.3 *([44]) If $X = M + A$ is a Dirichlet process and $f \in C^1$ then $f(X) = M^f + A^f$ is a Dirichlet process, where*

$$M^f = \int_0^{\cdot} f'(X_s)\, dM_s$$

and $A^f := f(X) - M^f$ has zero quadratic variation. □

A sequence (τ^N) of $\mathbb{F}$- stopping times will be said to be "suitable" if

$$\bigcup_N \{\tau^N \leq T\}$$

has probability one. We will use the notation of stopped process as usually X^τ.

Remark 1.4 *Let X a $\mathbb{F}$- adapted continuous process. X is a semimartingale (resp. Dirichlet processes) if and only if the stopped processes X^{τ^N} are also semimartingales (resp. Dirichlet processes).* □

At this stage, we recall the concept of a LZ process, see [44].

Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ be two filtrations. A process $Y = (Y_t, t \in [0,T])$ is said to be $(\mathbb{F},\mathbb{H})$-*adapted* if Y is $\mathbb{F}$-adapted and $\hat{Y}$ is $\mathbb{H}$-adapted.

A continuous $(\mathbb{F},\mathbb{H})$-adapted process $(X_t)_{t\in[0,T]}$ is called a (strong) $(\mathbb{F},\mathbb{H})$-*Lyons-Zheng process* (or simply *LZ process*) if there are $M^i = (M^i_t,\ t \in [0,T])$, $i = 1, 2$, $V = (V_t,\ t \in [0,T])$, such that

$$X = \frac{1}{2}M^1 + \frac{1}{2}M^2 + V \tag{1.5}$$

and the following conditions are satisfied:

a) M^1 is a local $\mathbb{F}$-martingale with $M^1_0 = 0$.

b) $\hat{M}^2$ is a local $\mathbb{H}$-martingale with $M^2_T = 0$.

c) V is a bounded variation process.

d) $M^1 - M^2$ is a zero quadratic variation process.

Remark 1.5 *Let $X = (X_t, t \in [0,T])$ be a $(\mathbb{F},\mathbb{H})$-LZ process.*

a) $[X,X] = \frac{1}{2}([M^1,M^1]+[M^2,M^2])$. In particular, X is a finite quadratic variation process.

b) If X is a $(\mathbb{F},\mathbb{H})$-LZ process then $\hat{X}$ is a $(\mathbb{H},\mathbb{F})$-LZ process.

c) The decomposition (1.5) is unique.

d) A time reversible semimartingale is a LZ process with respect to the natural filtrations.

e) If X is a LZ process with $X = \frac{1}{2}(M^1 + M^2) + V$ then $Y = f(X)$, where $f \in C^1$, is again a LZ process with decomposition

$$M_f^1 = \int_0^{\cdot} f'(X)\, dM^1, \quad M_f^2 = -\int_{\cdot}^{T} f'(X)\, d^+ M^2.$$

f) A LZ process which is also a semimartingale is a time reversible semimartingale.

h) A LZ process is a Dirichlet process.

i) Let X be a $(\mathbb{F}, \mathbb{H})$-adapted process admitting a decomposition of type (1.5) satisfying the conditions a), b), c). If X is a Dirichlet process with M^1 as martingale part, then it is truely a LZ process; that means that also condition d) is realized.

j) If X is a stationary symmetric Markov process associated with a Dirichlet form (see f. ex. [22]) and u belongs to the domain of the form then $u(X)$ is a LZ process (see [26]). In this case we have $V = 0$.

We recall briefly the notion of LZ stochastic integration in a specific framework. Let Y be a $(\mathbb{F}, \mathbb{H})$-adapted process and X a $(\mathbb{F}, \mathbb{H})$-Lyons Zheng process with decomposition (1.5). The LZ-symmetric integral is then defined by

$$\int_0^t Y \circ dX = \frac{1}{2} \int_0^t Y\, d^- M^1 - \frac{1}{2} \int_{T-t}^{T} \hat{Y}\, d^- \hat{M}^2 + \int_0^t Y\, dV. \tag{1.6}$$

We recall that

$$\begin{aligned} \int_0^t Y\, d^- M^1 &= \int_0^t (Y_s - Y_0)\, dM_s^1 + Y_0 M_t^1 \\ \int_0^t Y\, d^+ M^2 &= -\int_{T-t}^{T} (\hat{Y}_s - Y_T)\, d\hat{M}_s^2 + Y_T M_t^2. \end{aligned}$$

Remark 1.6 *If $[Y, M^i]$, $i = 1, 2$, exist then*

$$\int_0^t Y\, d^0 X = \int_0^t Y \circ dX + \frac{1}{4}[Y, M^1 - M^2]_t.$$

In particular, if Y is a zero quadratic variation process then $[Y, M^1 - M^2] = 0$ and so

$$\int_0^{\cdot} Y \, d^0 X = \int_0^{\cdot} Y \circ dX.$$

Remark 1.7 *Let X be a $(\mathbb{F}, \mathbb{H})$-LZ process, H, R be $(\mathbb{F}, \mathbb{H})$-adapted processes. We define $Y_t := \int_0^t R \circ dX$, $0 \le t \le T$. Then we have*

$$\int_0^t H \circ dY = \int_0^t HR \circ dX.$$

Remark 1.8 *Let $f \in C^1(\mathbb{R}^n)$, $\underline{X} = (X_1, \ldots, X_n)$ be a vector of $(\mathbb{F}, \mathbb{H})$-LZ processes. The following Itô's formula holds:*

$$f(\underline{X}_t) = f(\underline{X}_0) + \int_0^t \Big(\sum_{i=1}^n \partial_i f(\underline{X}_s) \Big) \circ dX_s^i$$

(see [44, 4.4]).

2 The basic operator L

Let σ, $b \in C^0(\mathbb{R})$ such that $\sigma > 0$. We consider formally a PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2} g'' + b' g'. \tag{2.1}$$

By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x) \, dx = 1$. We denote

$$\Phi_n(x) := n\Phi(nx), \quad \sigma_n^2 := \sigma^2 * \Phi_n, \quad b_n := b * \Phi_n.$$

We then consider

$$L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'. \tag{2.2}$$

A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

Definition A function $f \in C^1(\mathbb{R})$ is said to be a solution to

$$Lf = \dot{l}, \tag{2.3}$$

where $\dot{l} \in C^0$, (in the C^1-generalized sense) if, for any mollifier Φ, there are sequences (f_n) in C^2, $(\dot{l}_n)$ in C^0 such that

$$L_n f_n = \dot{l}_n, \quad f_n \to f \text{ in } C^1, \quad \dot{l}_n \to \dot{l} \text{ in } C^0. \tag{2.4}$$

In the whole paper, we will suppose the existence of the following function:

$$\Sigma(x) := \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)dy \tag{2.5}$$

in C^0 independently from the mollifier.

Example 2.1 *a) If $b = \alpha\frac{\sigma^2}{2}$ for some $\alpha \in]0,1]$ then*

$$\Sigma(x) = \log(\sigma^{+2\alpha}(x))$$

and

$$h'(x) = \sigma^{-2\alpha}(x).$$

b) Suppose that b is of bounded variation. Then we get

$$\int_0^x \frac{b_n'}{\sigma_n^2}(y)\,dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2},$$

since $db_n \to db$ weakly-$*$ and $\frac{1}{\sigma^2}$ is continuous.

c) If σ has bounded variation then we have

$$\Sigma(x) = -2\int_0^x b\,d(\frac{1}{\sigma^2}) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the cas where $\sigma = 1$ for any b.

Remark 2.2 *a) The existence of Σ is equivalent to the existence of a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$. In particular $h : \mathbb{R} \to \mathbb{R}$ with $h' = e^{-\Sigma}$ is a solution to $Lh = 0$. Transformation h has been introduced by Zvonkin, see [52]*

b) If $b' \in C^0, f \in C^2$ is a classical solution to $Lf = \dot{l}$, then f is immediately seen to be a C^1-generalized solution.

c) If $Lf = \dot{l}_1$ and $Lf = \dot{l}_2$ in the C^1-generalized sense, then $\dot{l}_1 = \dot{l}_2$.

d) If $\dot{l} \in C^0, x_0, x_1 \in \mathbb{R}$ then there is a unique solution to

$$Lf = \dot{l}, \quad f(0) = x_0, \quad f'(0) = x_0 \tag{2.6}$$

The solution is given by

$$\begin{cases} f(0) & = & x_0 \\ f'(x) & = & e^{-\Sigma(x)}\left(2\int_0^x \frac{\dot{l}}{\sigma^2} e^{\Sigma}(y)dy + x_1\right) \end{cases}$$

e) The definition above can be easily adapted to the case that the coefficients are in an open interval $I =]a,b[, \quad -\infty \leq a < b \leq \infty$.

We will denote by $\mathcal{D}_L$ (resp. $\mathcal{D}_L(I)$) the set of all $f \in C^1(\mathbb{R})$ (resp. $C^1(I)$) such that there exists some $\dot{l} \in C^0$ with $Lf = \dot{l}$ in the C^1-generalized sense.

In general, $f(x) = x$ does not belong to $\mathcal{D}_L$. In fact in that case X would be a semimartingale; Corollary 5.11 will show that this rarely occurs.

Remark 2.3 *$\mathcal{D}_L$ is dense in C^1 and in $W^{1,2}_{loc}$.*

Two transformations play a significant role in this paper: $h \in C^1(\mathbb{R})$ and $h' = e^{-\Sigma}$ and $k \in C^1(\mathbb{R})$ such that $k(0) = 0, \quad k' = \frac{e^{\Sigma}}{\sigma^2}$.

h eliminates the drift and k transforms the operator L into a divergence form. We set

$$\begin{aligned} I &= \text{Im } h =]a,b[\\ K &= \text{Im } k =]c,d[\end{aligned}$$

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\sigma_0^2}{2}\phi'', \tag{2.7}$$

where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : \ y \in I \\ 0 & : \ y \notin I. \end{cases}$$

L^0 is a classical PDE map; however we can also consider it in the C^1-generalized sense and introduce $\mathcal{D}_{L^0}$.

On the other hand, we define the following operator in divergence form

$$L^1 g = (\frac{\sigma_1^2}{2} g')' := \frac{\sigma_1^2}{2} g'' + (\frac{\sigma_1^2}{2})' g'$$

where

$$\sigma_1(z) = \begin{cases} (\sigma k')(k^{-1}(z)) & : \quad z \in K \\ 0 & : \quad z \notin K \end{cases}$$

Lemma 2.4 *a)* $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2 \sigma^2$,

b) $\mathcal{D}_{L^0}(I) = C^2(I)$,

c) $\phi \in \mathcal{D}_{L^0}(I)$ *holds if and only if* $\phi \circ h \in \mathcal{D}_L$. *Moreover, we have*

$$L(\phi \circ h) = (L^0 \phi) \circ h \tag{2.8}$$

for every $\phi \in C^2(I)$.

d) $g \in \mathcal{D}_{L^1(K)}$ *if and only if* $g \circ k \in \mathcal{D}_L$,

e) for every $g \in \mathcal{D}_{L^1(K)}$ *we have* $L^1 g = L(g \circ k) \circ k^{-1}$.

Definitions We say that the *non-explosion condition* (NE) is fulfilled if the solution u of

$$Lu = 1, \quad u(0) = u'(0) = 0$$

in the C^1-generalized sense is such that

$$u(-\infty) = u(+\infty) = +\infty.$$

We denote by $\mathcal{L} : \mathcal{D}_L \longrightarrow C^0$ the map

$$f \longrightarrow \int_0^{\cdot} Lf(y)dy.$$

We say that L is *close to divergence type* if

$$Lf = (\frac{\sigma^2}{2} f')' + \beta' f'$$

where β is a continuous function of bounded variation.

Remark 2.5 *If L is close to divergence type $\mathcal{L}$ admits a continuous extension from $\mathcal{D}_L$ to $W^{1,2}_{loc}$, denoted by $\hat{\mathcal{L}}$ with values in $L^2_{\rm loc}$.*

3 The martingale problem

Definition A process X is said to solve *the martingale problem* related to L with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\, ds$$

is a local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X^{s,x}_t, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if

(i) $X^{s,x}_s = x$,

(ii) for every $f \in \mathcal{D}_L$,

$$f(X^{s,x}_t) - f(x) - \int_s^t Lf(X^{s,x}_r)\, dr, \quad t \geq s$$

is a local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X^{s,x}_{t-s}$ solves the martingale problem at time 0.

Remark 3.1 *Let $\mathbb{F} = (\mathcal{F}_t)$ be the natural forward filtration of X. X is an $\mathbb{F}$-Dirichlet process with local martingale part*

$$M^X_t = \int_0^t \sigma(X_s) dW_s.$$

In particular, X is a finite quadratic variation porcess with

$$[X, X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s) ds.$$

We recall the operators

$$\mathcal{A} : \mathcal{D}_L \longrightarrow \mathcal{C}, \quad \text{given} \quad \text{by} \quad \mathcal{A}(f) = \int_0^{\cdot} Lf(X_s) ds$$

$$A : C^1 \longrightarrow \mathcal{C}, \quad \text{given} \quad \text{by} \quad A(l) = \int_0^{\cdot} l'(X_s)ds.$$

Taking in account Corollary 5.3 of Part I and its proof, we have the following.

Remark 3.2 *(i) $\mathcal{A}$ can be continuously extended to $W^{1,2}_{loc}$ with values in $\mathcal{C}$. Moreover $\mathcal{A}(f)$ is a zero quadratic variation process for any $f \in W^{1,2}_{loc}$.*

(ii) For every $f \in W^{1,2}_{loc}$,

$$f(X_t) = f(x_0) + \int_0^t (f'\sigma)(X_s)dW_s + \mathcal{A}(f) \tag{3.1}$$

(iii) If L is close to divergence type, i.e.

$$Lg = \frac{\sigma^2}{2} g'' + b'g'$$

with b' being a Radon measure then, A can be extended to L^2_{loc}. In this case, we have

$$\mathcal{A}(f) = A(\hat{\mathcal{L}}f).$$

Proposition 3.3 *a) X solves the martingale problem related to L if and only if $Y = h(X)$ is a local martingale with values in $I = h(\mathbb{R})$, which solves the stochastic differential equation*

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)dW_s$$

in the sense of probability distributions.

b) There is a unique solution to the martingale problem related to L with prescribed initial condition $x_0 \in \mathbb{R}$ if and only if the (NE) condition is verified. Moreover, X is an $\mathbb{F}$-Dirichlet process with martingale part

$$M_t = \int_0^t \sigma(X_s)dW_s.$$

d) X solves the martingale problem related to L if and only if $Z = k(X)$ solves the martingale problem related to L^1.

Remark 3.4 *Let $T > 0$ and $(Z_t)_{t\geq 0}$ be a process. We denote by $\mathbb{F} = \mathbb{F}_Z$ the natural forward filtration of Z, given by $\mathcal{F}_t = \sigma(Z_s : s \leq t)$, and by $\mathbb{H} = \mathbb{H}_Z$ the backward filtration, given by $\mathcal{H}_t = \sigma(\hat{Z}_s : s \leq t)$. Clearly, we have $\mathcal{F}_Y = \mathcal{F}_X = \mathcal{F}_Z$ and $\mathcal{H}_Y = \mathcal{H}_X = \mathcal{H}_Z$.*

Remark 3.5 *a) When L is close to divergence type, Part I shows that the martingale problem related to L is equivalent to the true stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + A(b).$$

b) We recall from Part I that, in this case, X fulfills what we called the Bouleau-Yor property, inspired to us by [7]. This means that:

- *A admits a continuous extension to C^0, always denoted by the same letter,*
- *$\int_0^\cdot g(X)d^- A(l)$ exists for every $g \in C^2, l \in C^0$.*

c) In Lemma 3.10 of Part I, we stated the following for a process X having the Bouleau-Yor property:

$$\int_0^\cdot g(X)d^- A(l) = A(\phi(g,l)), \tag{3.2}$$

where

$$\phi(g,l)(x) = (gl)(x) = (gl)(0) - \int_0^x lg'(y)dy. \tag{3.3}$$

On the other hand, a solution to the martingale problem related to L has a density at each time which fulfills locally the Aronson estimates, at least if we formulate a technical assumption, see (TA) below.

A family $(p_t(x,\cdot), t > 0, x \in \mathbb{R})$ of probability densities is said to fulfill the *local Aronson estimates* if, for every continuous function χ with compact support, there is some $M > 0$ such that

$$\begin{aligned} & \frac{1}{M\sqrt{t}} \exp\left(-\frac{|x-y|^2 M}{t}\right)\chi(x-y) \\ \leq\ & p_t(x,y)\chi(x-y) \\ \leq\ & \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right)\chi(x-y). \end{aligned} \tag{3.4}$$

Let X be the solution to the martingale problem related to L with initial condition x_0.

At this level, we need to recall a technical assumption (TA). It will suppose there are positive constants c, C, such that

$$(TA) \qquad c \le \frac{e^{\Sigma}}{\sigma} \le C.$$

Remark 3.6 *If (TA) is verified then is easy to see that the coefficients σ_0 and σ_1 related to L^0 and L^1 are also bounded above and below by a positive constant.*

From Section 5 of Part I (Theorem 5.7 and Proposition 5.10), we recall the following result.

Proposition 3.7 *Suppose that (TA) is verified.*

(i) For every $t > 0$, the law of X_t has a density $p = p_t(x_0, \cdot)$.

(ii) For every $t > 0$, p satisfies the local Aronson estimates and $(t, x, y) \mapsto p_t(x, y)$ is continuous from $]0, \infty[\times \mathbb{R}^2$ to $\mathbb{R}$.

(iii) The law of the vector (X_s, X_T), for $0 < s < T$, has the following density

$$(x_1, x_2) \mapsto p_s(x_0, x_1) p_{T-s}(x_1, x_2)$$

Suppose for a moment that L is of divergence form, i.e.

$$Lg = (\frac{\sigma^2}{2} g')',$$

and there are positive constants c, C such that $0 < c \le \sigma^2 \le C$.

According to [47] and Part I, see [18], there is a family of probability measures $\nu_t(dx, y), t \ge 0)$ enjoying

$$\frac{\partial \nu_t}{\partial t}(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y \tag{3.5}$$

and $\nu_t(dx, y) = p_t(x, y), \quad \forall t > 0$.

We will refer to it as the *fundamental solution* of $\partial_t u = Lu$. It fulfills the global Aronson estimates.

Remark 3.8 *a) (3.5) is to be understood in the following distributional way:*

$$\int \nu_t(dx,y) f(x) = f(y) - \int_0^t ds \int dx \frac{\sigma^2}{2}(x) \frac{\partial}{\partial x}(p_s(x,y)) f'(x). \quad (3.6)$$

b) The maps $(t,y) \mapsto p_t(x,y)$ are in $L^2([0,T]\times\mathbb{R}^2)$ again because of the Aronson estimates.

Lemma 3.9 *If L is of divergence form then $\frac{\partial p_t}{\partial x}(\cdot,y)$ exists in the distributional sense and satisfies:*

- $\sup_y \int_0^T \left(\int_{\mathbb{R}} |\frac{\partial}{\partial x} p_t(x,y)|^2 dx \right)^{\frac{1}{2}} dt < \infty.$
- $\sup_y \int_{[0,T]\times K} |\frac{\partial}{\partial x} p_t(x,y)| dx dt < \infty$

for every compact interval K.

Proof. Consider the following (classical) variational framework. All the definitions and properties recalled below without specific comments can be found in [25], [36], [48]. Let H and V be the Hilbert spaces

$$H = L^2(\mathbb{R}), \quad V = W^{1,2}(\mathbb{R}).$$

Let H' and V' be their dual spaces. Identifying H with H', we have the continuous dense injections

$$V \subset H \subset V'$$

and the dual pairing $\langle u,v\rangle_{V',V}$ between V' and V coincides with the scalar product in H, $\langle u,v\rangle_H$, when $u\in H$, $v \in V$.

Consider the bilinear form $a(.,.) : V\times V \to \mathbb{R}$ defined by

$$a(u,v) = \int_{\mathbb{R}} \frac{\sigma^2(x)}{2} u'(x) v'(x) dx, \quad u,v\in V.$$

It is symmetric, continuous and coercive. By Lax-Milgram Theorem, there exists an isomorphism $L : V \to V'$ such that $a(u,v) = -\langle Lu, v\rangle_{V',V}$ for all $u,v\in V$. It is given by

$$Lu = \left(\frac{\sigma^2}{2} u'\right)', \quad u \in V,$$

where we have $\frac{\sigma^2}{2}u' \in H$ (by the assumptions on σ) and the subsequent derivative is taken in the distributional sense.

Let

$$D(L) = \{u \in V;\, Lu \in H\} = L^{-1}(H).$$

The linear (unbounded) operator $L : D(L) \subset H \to H$, restriction of the previously defined operator $L : V \to V'$, is selfadjoint (it is a consequence of the symmetry of $a(.,.)$) and generates an analytic semigroup e^{tL}, $t \geq 0$, in H (all selfadjoint negative definite operators in Hilbert spaces are infinitesimal generators of analytic semigroups). Among the regularity properties of e^{tL}, let us recall that $e^{tL}(H) \subset D(L)$ for all $t > 0$, hence in particular $e^{tL}(H) \subset V$ for all $t > 0$.

The operator L^{-1} is an isomorphism between H and the Hilbert space $D(L)$ endowed with the graph norm. The space $D(L)$ is dense in V and H.

It is possible to define the fractional powers $(-L)^\alpha$, $\alpha > 0$, as linear selfadjoint strictly positive operators in H with domains $D((-L)^\alpha) \subset H$. They are isomorphisms when the domains are endowed with the graph norm. We have $V = D\left((-L)^{\frac{1}{2}}\right)$. Hence $D\left((-L)^{\frac{1}{2}}\right) = W^{1,2}(\mathbb{R})$. By interpolation, $D((-L)^\alpha) = W^{2\alpha,2}(\mathbb{R})$ for all $\alpha \in \left(0, \frac{1}{2}\right)$. We remark that the fractional powers of $-L$ commute between themselves and with e^{tL} (with proper domains of the compositions).

By duality we have the continuous dense inclusions

$$V \subset D((-L)^\alpha) \subset H \subset D((-L)^\alpha)' \subset V'$$

for all $\alpha \in \left(0, \frac{1}{2}\right)$. Since $(-L)^\alpha$ is selfadjoint, it can be extended to an isomorphism $(-L)^\alpha : H \to D((-L)^\alpha)'$. We shall use its inverse $(-L)^{-\alpha} : D((-L)^\alpha)' \to H$. Notice that $D((-L)^\alpha)' = W^{-2\alpha,2}(\mathbb{R})$, for all $\alpha \in \left(0, \frac{1}{2}\right)$.

The semigroup e^{tL}, $t \geq 0$, can be restricted to an analytic semigroup in every Hilbert space $D((-L)^\alpha)$, in particular in V. By duality, and taking into account that it is selfadjoint, it can be extended to an analytic semigroup in V'. We continue to denote these restrictions and extensions by e^{tL}, $t \geq 0$. Dualizing the regularity property $e^{tL}(H) \subset V$ for all $t > 0$, we get $e^{tL}(V') \subset H$ for all $t > 0$, and therefore, by composition (since $e^{tL} = e^{\frac{t}{2}L}e^{\frac{t}{2}L}$), we have $e^{tL}(V') \subset V$ for all $t > 0$.

Since $V \subset C^0(\mathbb{R})$ with continuous dense embedding, by Sobolev embedding Theorem, for every $y \in \mathbb{R}$ the Dirac distribution δ_y at y belongs to V'. Therefore we can compute $e^{tL}\delta_y$, $t \geq 0$, that a priori is a continuous function of t with values in V'. By the previous regularity fact, we have $e^{tL}\delta_y \in V$ for all $t > 0$. By the properties of e^{tL}, the function $(t,x) \mapsto \left(e^{tL}\delta_y\right)(x)$ is the unique solution of the parabolic PDE $v' = Lv$ with initial condition $v(0) = \delta_y$, hence it coincides with $p_t(x,y)$.

More precisely, we have $W^{2\alpha,2}(\mathbb{R}) \subset C^0(\mathbb{R})$ with continuous dense embedding, for $\alpha > \frac{1}{4}$. Therefore $\delta_y \in W^{-2\alpha,2}(\mathbb{R})$. It follows that, for any given $\alpha > \frac{1}{4}$ and $y \in \mathbb{R}$, $(-L)^{-\alpha}\delta_y \in H$. Moreover, using Fourier transform and the definition of δ_y, one can check that $\sup_y \|\delta_y\|_{W^{-2\alpha,2}(\mathbb{R})} \leq C$, and therefore

$$\sup_y \left\|(-L)^{-\alpha}\delta_y\right\|_H \leq \tilde{C}.$$

Finally, recall the basic inequality for analytic semigroups

$$\left\|(-L)^{\beta} e^{tL} h\right\|_H \leq \frac{C_{\beta,T}}{t^{\beta}} \|h\|_H$$

for every $t \in (0,T]$, $\beta > 0$, $h \in H$.

Take some $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$. We have

$$\begin{aligned}
\int_0^T \left\|e^{tL}\delta_y\right\|_V dt &= \int_0^T \left\|(-L)^{\alpha}(-L)^{-\alpha} e^{tL}\delta_y\right\|_V dt \\
&= \int_0^T \left\|(-L)^{\alpha} e^{tL}(-L)^{-\alpha}\delta_y\right\|_V dt \\
&\leq const \int_0^T \left\|(-L)^{\frac{1}{2}}(-L)^{\alpha} e^{tL}(-L)^{-\alpha}\delta_y\right\|_H dt \\
&= const \int_0^T \left\|(-L)^{\frac{1}{2}+\alpha} e^{tL}(-L)^{-\alpha}\delta_y\right\|_H dt \\
&\leq const \int_0^T \frac{1}{t^{\frac{1}{2}+\alpha}} \left\|(-L)^{-\alpha}\delta_y\right\|_H dt \\
&\leq const \left\|(-L)^{-\alpha}\delta_y\right\|_H \leq C',
\end{aligned}$$

for all $y \in \mathbb{R}$. Using this bound, we finally have

$$\sup_{y\in\mathbb{R}} \int_{[0,T]\times K} \left|\frac{\partial}{\partial x} p_t(x,y)\right| dxdt \leq C_{T,K} \sup_{y\in\mathbb{R}} \int_0^T \left(\int_{\mathbb{R}} \left|\frac{\partial}{\partial x} p_t(x,y)\right|^2 dx\right)^{\frac{1}{2}} dt$$

$$\leq C'_{T,K} \sup_{y\in\mathbb{R}} \int_0^T \left\|e^{tL}\delta_y\right\|_V dt \leq C''_{T,K}.$$

Both claims are finally proved. □

4 Lyons-Zheng processes and Itô's formula under weak assumptions

In this section we will use the same notations and make the same conventions as in Section 2.

Let X be the solution to the martingale problem related to L with initial condition x_0. From Proposition 3.3 b), we already know that X is an $\mathbb{F}$-Dirichlet process. On the other hand, if $f \in W^{1,2}_{loc}$ we know from Remark 3.2 that $f(X)$ is an $\mathbb{F}$-Dirichlet process with local martingale part.

$$M^f_t = \int_0^t f'(X)\sigma(X)dW \tag{4.1}$$

¿From Part I, we also know that Remark 1.2 b) can be extended to the case $f, g \in W^{1,2}_{loc}$. In particular

$$[f(X), g(X)]_t = \int_0^t f'(X)g'(X)d[X]. \tag{4.2}$$

In this section, we are interested in the time reversal structure of X. This knowledge will provide two results.

- The fact that X is an LZ-process.
- An Itô formula under weak assumptions when X is a semimartingale diffusion.

We set again $Y = h(X), Z = k(X), y_0 = h(x_0), z_0 = k(x_0)$. We will also write $Z = j(Y)$, where $j = k \circ h^{-1}$.

We recall that Y (resp. Z) solves the martingale problem related to L^0 (resp. L^1) with initial condition y_0 (resp. z_0).

¿From Section 3, we remind that for $t > 0$, the law of X_t has a density denoted by $p_t(x_0, \cdot)$. We denote by $r_t(z_0, \cdot)$ the law density of $Z_t, t > 0$. Section 4 of Part I gives the useful relation

$$p_t(x_0, x) = r_t(k(x_0), k(x))k'(x) \tag{4.3}$$

Similarly, writing $q_t(y_0, \cdot)$ the law density of Y_t, we have

$$q_t(y_0, y) = r_t(j(y_0), j(y))j'(y) \tag{4.4}$$

We easily obtain that $j'(y) = \frac{1}{\sigma_0^2}$.

Applying Lemma 3.9 and Proposition 3.7 to L^1 and Z we obtain the following estimates

$$\sup_y \int_0^T \left(\int_{\mathbb{R}} |\frac{\partial}{\partial z} r_t(z, y)|^2 dz \right)^{\frac{1}{2}} dt \quad < \quad \infty \tag{4.5}$$

$$\sup_y \int_{[0,T]\times K} |\frac{\partial}{\partial z} r_t(z, y)| dz dt \quad < \quad \infty \tag{4.6}$$

for every compact real interval K.

The first theorem of this section concerns the Lyons-Zheng characterization of X.

Theorem 4.1 *Suppose that (TA) is verified for L. Then X is an LZ-process.*

Proof. In view of applying Remark 1.5 d) and e), we prove that Y is a time reversible semimartingale. (4.4) gives

$$q_t(y_0, y) = \frac{r_t(j(y_0), j(y))}{\sigma_0^2(y)}. \tag{4.7}$$

Usual calculations on time reversal given for instance in [35, 31, 44] say that the time reversed process $(\hat{Y}_t, t \in [0, T])$ solves the stochastic differential equation

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \sigma_0(\hat{Y}_s)\, dB_s + \int_0^t \tilde{b}(T - s, \hat{Y}_s)\, ds,$$

where B is a classical $\mathcal{F}_{\hat{Y}}$- Brownian motion and

$$\tilde{b}(s, y) = -\left(\frac{\partial}{\partial y}(\sigma_0^2(y) q_s(y_0, y))\right) / q_s(y_0, y) \tag{4.8}$$

provided that (4.8) makes sense and

$$\int_0^T |\tilde{b}(s, Y_s)|\, ds < \infty \quad \text{a.s.} \tag{4.9}$$

For this it is enough to show that

$$\mathbb{E}\left(\int_0^T ds\, |\tilde{b}(s, Y_s)| \mathbf{1}_{\{\sup_{t\in[0,T]} |Y_t| \le M\}} \right) < \infty$$

holds for some $M > 0$. Previous expression is bounded by

$$\int_0^T ds \int_{-M}^{M} dy \left| \frac{\partial}{\partial y} (\sigma_0^2(y) q_s(y_0, y)) \right|. \tag{4.10}$$

(4.7) implies that

$$\begin{aligned} \frac{\partial}{\partial y}\Big(\sigma_0^2(y) q_s(y_0, y)\Big) &= \frac{\partial}{\partial y}(r_s(j(y_0), j(y))) \\ &= \left(\frac{\partial r_s}{\partial y}\right) (j(y_0), j(y))\, j'(y). \end{aligned} \tag{4.11}$$

(4.10) gives

$$\begin{aligned} &\int_0^T ds \int_{-M}^{M} dy \left| (\frac{\partial}{\partial y} r_t)(j(y_0), j(y)) \right| j'(y) \\ &= \int_0^T ds \int_{[-j(M), j(M)]} dz \left| \left(\frac{\partial r_s}{\partial z}\right)(j(y_0), z) \right| \end{aligned}$$

which is finite because of (4.6).

In conclusion, Y is a time reversible semimartingale and so $X = h^{-1}(Y)$ a LZ process. □

At this level we would like to relax the technical assumption, but it is not completely possible. It will however be possible for the study of Itô's formula.

For $M > 0$ and a real function f, we set

$$f^M(x) = \begin{cases} f(x) & \text{if } |x| \le M \\ f(M) & \text{if } x \ge M \\ f(-M) & \text{if } x \le -M \end{cases}$$

We can show that

$$\lim_{n\to\infty} \int_0^{\cdot} \frac{(b_n^M)'}{(\sigma_n^M)^2}(y)\, dy$$

is well-defined in C^0 (independently of the mollifier) and it equals Σ^M. It is obvious that for the PDE map $L(M)$, defined formally by

$$L(M)g = \frac{(\sigma^M)^2}{2} g'' + (b^M)' g',$$

and it fulfills assumption (TA).

We consider the event

$$\Omega_M = \{\omega : X_t(\omega) \in [-M, M], \forall t \in [0, T]\}$$

and the stopping time

$$\tau^M = \inf\{t \in [0,T] | X_t \notin [-M, M]\} \wedge (T+1)$$

(τ^M) is a "suitable" sequence of stopping times.

Remark 4.2 *Let $M > 0$ such that $x_0 \in \,]-M, M[$. On Ω_M, the process X coincides with the stopped processes X^{τ^M}. On the same event, this one coincides with the stopped process $X(M)^{\tau^M}$ for the solution $X(M)$ to a martingale problem related to $L(M)$.*

Indeed, for this, Proposition 3.3 a) allows us to consider the stochastic differential equation

$$Y_t = Y_0 + \int_0^t \sigma_0(Y_s)\, dW_s,$$

which is solved by $Y := h(X)$. The time changed process

$$B_t := Y_{T_t},$$

where $T_t = A_t^{-1}$ is the inverse of $A_t := \int_0^t \sigma_0^2(Y_s)\, ds$, is easily checked to be a Brownian motion. Furthermore, by [14, Proposition 5.2], we know

$$T_t = \int_0^t \frac{1}{\sigma_0^2}(B_s)\, ds.$$

Now we define

$$\sigma_0^{(M)}(y) = \begin{cases} \sigma_0(y) & \text{if} \quad |y| \le h(M) \\ \sigma_0(M) & \text{if} \quad y \ge h(M) \\ \sigma_0(-M) & \text{if} \quad y \le h(-M) \end{cases}$$

and consider

$$T_t^{(M)} := \int_0^t \frac{1}{(\sigma_0^{(M)})^2}(B_s)\, ds$$

and $A_t^{(M)} := T_t^{(M)^{-1}}$. By [14, Proposition 5.2], the process $Y(M)_t := B_{A_t^{(M)}}$ then solves the stochastic differential equation

$$Y(M)_t = Y_0 + \int_0^t \sigma_0^{(M)}(Y(M)_s)\, d\tilde{W}_s$$

for some Brownian motion $\tilde{W}$. From $B_t = Y_{T_t}$ we deduce $T_t^{(M)} = T_t$ on $\{t < A_{\tau_m}\}$, hence

$$A_t = A_t^{(M)} \quad \text{on} \quad \{t < \tau_m\}.$$

Thus, we conclude $Y_{t\wedge\tau_m} = Y(M)_{t\wedge\tau_m}$. For a more detailed discussion on construction of solutions to SDEs without drift we refer to [14].

Remark 4.3 *If $f \in W^{1,2}_{loc}$ then $f(X)$ is a $\mathbb{F}$-Dirichlet process with martingale part M^f defined at (4.1). If moreover, (TA) is verified then $f(X)$ obviously admits a LZ decomposition given at Remark 1.5 e) with $M^1_f = M^f$. Remark 1.5 i) entails that it is a true LZ process.*

As we said, the second result of this section is Itô's formula under weak assumptions. We recall that similar formulas were first considered independently first by [21] and [42]; further extensions have been performed in [20] and [6], [32, 33].

Here, the innovation is that we deal with non-degenerate diffusion processes with non-smooth coefficients. For that purpose, the technical assumption (TA) is not required.

Theorem 4.4 *Let $\sigma \in C^0(\mathbb{R})$ with $\sigma > 0$, γ be a locally bounded function and X a diffusion process of the type*

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \gamma(X_s)\, ds. \tag{4.12}$$

Then the following Itô formula holds for every $f \in W^{1,2}_{loc}$:

$$f(X_t) = f(x_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}[f(X), X]_t. \tag{4.13}$$

Proof. According to the notations of Sections 2 and 3, X solves the martingale problem related to L, where

$$Lf = \frac{\sigma^2}{2} f'' + b' f'$$

and $b(x) = \int_0^x \gamma(y)\, dy$. In this case, we have

$$\Sigma(x) = 2 \int_0^x \frac{\gamma}{\sigma^2}(y)\, dy.$$

In particular, Σ belongs to $W^{1,\infty}_{loc}$.

For a first step we assume again (TA). By Theorem 4.1, X is a LZ process. Since X is a semimartingale, the reversed process $\hat{X}$ is also a semimartingale by Remark 1.5 f).

Remark 4.5 *Using [42], under (TA), we have already shown (4.13) for every $f \in C^1$.*

In order to prove (4.13) for $f \in W^{1,2}_{loc}$, we need to work out explicitly the equation solved by $\hat{X}$. We set $Z := k(Y)$, where $k \in C^1(\mathbb{R})$ is defined before Lemma 2.4, which yields together with (4.3)

$$p_t(x_0, x) = r_t(k(x_0), k(x))\sigma^{-2}(x) \exp(\Sigma(x)). \tag{4.14}$$

As for (4.8), we have

$$\hat{X}_t = \hat{X}_0 + \int_0^t \sigma(\hat{X}_s)\, dB_s + \int_0^t \tilde{\gamma}(T - s, \hat{X}_s)\, ds, \tag{4.15}$$

where

$$\gamma(x) + \tilde{\gamma}(t, x) = \frac{-\frac{\partial}{\partial x}(\sigma^2(x) p_t(x_0, x))}{p_t(x_0, x)}$$

provided that

$$\int_0^T |\tilde{\gamma}(s, X_s)|\, ds < \infty \tag{4.16}$$

holds a.s. Since $\int_0^t \gamma(X_s)ds$ exists, (4.16) holds, if for every $M > 0$, we have

$$\mathbb{E}\left(\int_0^T |\gamma + \tilde{\gamma}|(X_s) 1_{\{|X_s| \leq M\}} ds\right) < \infty.$$

This happens if

$$\int_0^T ds \int_{-M}^M dx \left| \frac{\partial}{\partial x}(\sigma^2(x) p_s(x_0, x)) \right| < \infty. \tag{4.17}$$

(4.17) will be proved later. For the moment, we observe

$$\begin{aligned} \frac{\partial}{\partial x}(\sigma^2(x) p_t(x_0, x)) &= \frac{\partial}{\partial x}\Big(\exp(\Sigma(x)) r_t(k(x_0), k(x))\Big) \\ &= \exp(\Sigma(x))\Big(\Sigma'(x) r_t(k(x_0), k(x)) \\ &\quad + \frac{\partial}{\partial z} r_t(k(x_0), k(x)) k'(x)\Big). \end{aligned} \tag{4.18}$$

In order to conclude by the Banach-Steinhaus argument of [42] we have to check that

$$\int_0^t g(X)\, d^{\pm} X \tag{4.19}$$

exists for every $g \in L^2_{loc}$. The forward integral is known to coincide with the Itô's integral

$$\int_0^t g(X_s)\, dX_s = \int_0^t g(X_s)\sigma(X_s)\, dW_s + \int_0^t g(X_s)\gamma(X_s)\, ds. \tag{4.20}$$

By Remark 1.1 e), the backward integral equals

$$-\int_{T-t}^T g(\hat{X}_s)\, d\hat{X}_s = -\int_{T-t}^T (g\sigma)(\hat{X}_s)\, ds - \int_{T-t}^T g(\hat{X}_s)\tilde{\gamma}(T-s, \hat{X}_s)\, ds \tag{4.21}$$

provided that the right members of (4.20) and (4.21) exist. For this, we have to verify

$$\int_0^T g^2(X_s)\sigma^2(X_s)\, ds < \infty, \tag{4.22}$$

$$\int_0^T |g(X_s)\gamma(X_s)|\, ds < \infty, \tag{4.23}$$

$$\int_0^T |g(X_s)\tilde{\gamma}(s, X_s)|\, ds < \infty \tag{4.24}$$

a.s. Since σ and γ are locally bounded, by Cauchy-Schwarz, (4.22) and (4.23) hold if

$$\int_0^T |g|^2(X_s)\, ds < \infty \quad \text{a.s.} \tag{4.25}$$

(4.24) will be verified if, for every $M > 0$,

$$\int_0^T ds \int_{-M}^M dx\, |g(x)| \left| \frac{\partial}{\partial x}(\sigma^2(x) p_s(x_0, x)) \right| < \infty \tag{4.26}$$

holds. Therefore proving (4.26) will justify (4.17) and (4.25), simultaneously.

In view of (4.18), expression (4.26) is bounded by

$$const\Big(\int_0^T ds \int_{-M}^M dx\, |g(x)| r_s(k(x_0), k(x)) \\ + \int_0^T ds \int_{[-k(M),k(M)]} dz \left| g(k^{-1}(z)) \left(\frac{\partial r_s}{\partial z}\right)(z_0, z) \right| \Big).$$

The first integral above is finite because $g \in L^2_{loc}$ and $(s, x) \mapsto r_s(k(x_0), k(x))$ is square integrable by Remark 3.8 b).

The second integral is bounded again through Cauchy-Schwarz. It gives

$$\left(\int_{[-k(M),k(M)]} dz g^2(k^{-1}(z)) \right)^{\frac{1}{2}} \int_0^T ds \left(\int_{[-k(M),k(M)]} dz \left(\frac{\partial r_s}{\partial z}\right)^2 (z_0, z) \right)^{\frac{1}{2}} \tag{4.27}$$

This quantity is bounded because of $g \in L^2_{loc}$ and Lemma 3.9.

This shows the result when (TA) is fulfilled.

Suppose, however, that (TA) is not necessarily fulfilled. Then, we know that on the event Ω_M defined just before Remark 4.2, $X = X(M)$.

Taking in account the definition of forward and backward integrals, (4.19) will exist if

$$\int_0^t g(X(M)) d^{\pm} X(M)$$

exist. This is obvious because $X(M)$ solves the martingale problem related to $L(M)$ and this fulfills assumption (TA). □

5 The semimartingale characterization

Let L be a PDE operator, Σ and h as in Section 2, we recall that $h' = e^{-\Sigma}$.

The basic question of this section is the following. Which are the functions $f \in W^{2,1}_{loc}$ such that $f(X)$ is a semimartingale? In particular, this includes a necessary and sufficient condition for X to be a semimartingale. We believe that with respect to other semimartingale characterizations existing in the literature, see for instance [23], this uses direct stochastic analysis tools and the result is easily readable.

First of all we make some recalls about measure theory.

We denote by BV the space of continuous functions which have locally bounded variation. We equip BV with the metrizable topology that is associated with the following convergence. A sequence v_n in BV converges to v if and only if $v_n(0) \to v(0)$ and $dv_n \to dv$ holds with respect to the weak $*$-topology.

Remark 5.1 *a) The sequence (dv_n) converges to dv if and only if, for every $\alpha \in C^0(\mathbb{R})$,*

$$\int_0^t \alpha \, dv_n \to_{n\to\infty} \int_0^t \alpha \, dv$$

holds at every point of continuity t of v.

b) If (v_n) is a sequence converging in BV then the Banach-Steinhaus Theorem implies that the total variations are uniformly bounded on every compact set K, i.e.,

$$\sup_n \int_K d|v_n| < \infty. \tag{5.1}$$

c) We have $v_n \to v$ in BV if and only if $v_n(x) \to v(x)$ at every point of continuity x of v and (5.1) holds.

d) Let (v_n) be a sequence in BV such that (5.1) holds. Let v_n^+, v_n^- be increasing functions such that

$$v_n = v_n^+ - v_n^- \quad \textit{and} \quad |v_n| = v_n^+ + v_n^- .$$

Then there is a subsequence (n_k) such that $(v^+_{n_k})$ and $(v^-_{n_k})$ converge in BV. In particular, the subsequence $(|v_{n_k}|)$ of the total variations converges in BV.

In fact, (5.1) implies

$$\sup_n \int_K dv_n^\pm < \infty \tag{5.2}$$

for each compact interval K. By the Helly extraction argument, there is a subsequence (n_k) such that $(v^+_{n_k})$ and $(v^-_{n_k})$ converge respectively to some v^1, v^2 at each continuity point.

e) *C^1 is dense in BV.*

f) *We have $BV \subset L^2_{loc}$, because a locally bounded variation function is locally bounded.*

Moreover, using point d), it is not difficult to prove that the BV convergence implies the one in L^2_{loc}.

In fact, let v_n be a sequence of BV functions. We then have $v_n = v_n^+ - v_n^-$, where v_n^+, v_n^-, v^+, v^- are increasing functions vanishing at zero. We suppose that v_n converge to $v = v^+ - v^-$ in BV. Since L^2_{loc} is a metric space, it is enough to show that some subsequence (v_{n_k}) converges to v in L^2_{loc}. ¿From point d), we learn the existence of a subsequence (n_k) such that $v^\pm_{n_k}(x) \to v^\pm(x)$, for each continuity point x therefore a.e. with respect to Lebesgue measure. (5.2) implies that $(v^\pm_{n_k})$ are uniformly bounded on each compact interval K. Now, the theorem of dominated convergence yields $v^\pm_{n_k} \to v^\pm$ in L^2_{loc}.

We denote by BV^1 the set of all absolutely continuous functions whose derivative f' satisfies $f'/h' = f'e^\Sigma \in BV$. In particular, $BV^1 \subset W^{1,2}_{loc}$ holds. BV^1 becomes a Polish space of F-type when equipped with the following metrizable topology. A sequence (f_n) is defined to converge in BV^1 if $f_n \to f$ in C^0 and $(f_n'/h') \to (f'/h')$ in BV.

We recall the map $\mathcal{L} : \mathcal{D}_L \longrightarrow C^0$ defined in Section 2 by

$$\mathcal{L}f(x) = \int_0^x Lf(y)dy$$

On $\mathcal{D}_L$, the operator $\mathcal{L}$ takes values in BV. When we will consider this operator as BV valued map, we will use the notation $\mathcal{L}^{BV}$.

Lemma 5.2 *(i) The convergence in BV^1 implies the one in L^2_{loc}.*

(ii) $\mathcal{D}_L \subset BV^1$

(iii) $\mathcal{D}_L$ *is dense in* BV^1.

(iv) The mapping $f \mapsto \mathcal{L}^{BV} f$ admits a continuous extension from $\mathcal{D}_L$ to BV^1. It will be denoted by $\hat{\mathcal{L}}^{BV}$.

Proof.

(i) It is a consequence of the embedding $BV \subset L^2_{loc}$ given by Remark 5.1 f).

(ii) If $f \in \mathcal{D}_L$, we set $\dot{l} = Lf$. Remark 2.2 d) implies that

$$\frac{f'}{h'}(x) = 2\int_0^x \frac{\dot{l}}{\sigma^2 h'}(y)dy + f'(0).$$

This shows $f \in BV^1$.

(iii) Let $f \in BV^1$ and (ϕ_n) a sequence in C^2 such that $\phi_n' \to \frac{f'}{h'}$ in BV when n goes to ∞. Clearly we can define $f_n \in C^1$ such that $f_n' = \phi_n' h'$. Obviously $f_n \in \mathcal{D}_L$ and $Lf_n = \frac{\phi_n'' \sigma^2 h'}{2}$.

(iv) Let (f_n) be a sequence in $\mathcal{D}_L$ converging to zero in BV^1. We have to show that $l_n := \mathcal{L} f_n$ converges to zero in BV.

By assumption, we have $(f_n'/h') \to 0$ in BV. Again Remark 2.2 d) says that

$$f_n'(x) = h'(x)\left(2\int_0^x \frac{\dot{l}_n}{\sigma^2 h'^2}(y)\,dy + f_n'(0)\right),$$

where $h'(x) = \exp(-\Sigma(x))$ and $\dot{l}_n = Lf_n$. This implies that

$$\frac{\dot{l}_n}{\sigma^2 h'^2}(y)\,dy$$

converges in the weak-$*$-topology to zero. Since $\sigma^2 h'^2$ is a continuous function, l_n converges to zero in BV. □

Proposition 5.3 *(i) The operator $\mathcal{L}^{BV}$ is closable in $W^{1,2}_{loc}$ with values in BV.*

(ii) The domain of the smallest closure is BV^1.

Proof. The Proposition will be a consequence of the following Lemma.

Lemma 5.4 *Let (f_n) be a sequence in $\mathcal{D}_L$ such that $l_n = \mathcal{L} f_n$ converge to some l in BV and $f_n \to f$ holds in $W^{1,2}_{loc}$. Then we have $f \in BV^1$ and $\hat{\mathcal{L}}^{BV} f = l$.*

Proof. Of course, we have $l_n \in C^1$, $Lf_n = \dot{l}_n$, $\dot{l}_n$ being the derivative of l_n. Independently of the convergence of (f_n), we have

$$\frac{2\dot{l}_n}{\sigma^2 h'^2}\, dy \to \frac{2dl}{\sigma^2 h'^2} \tag{5.3}$$

in the weak-$*$-topology because $1/(\sigma^2 h'^2)$ is a continuous function. The convergence of (f_n') in L^2_{loc} and (5.3) force the convergence of the real sequence $(f_n/h')(0)$. Consequently, (f_n'/h') converges in BV and so, (f_n) converges in BV^1. Since the convergences in BV^1 and $W^{1,2}_{loc}$ must agree, we have $f = \lim_{n\to\infty} f_n$ in BV^1 so that $\hat{\mathcal{L}}^{BV} f = l$ holds by Lemma 5.2. □

Before stating the characterization Theorem, we provide a preliminary result which specifies a class of integrands of $\mathcal{A}(f), f \in W^{1,2}_{loc}$.

Lemma 5.5 *Let X be a solution to a martingale problem with respect to L satisfying (TA). Then*

$$\int_0^{\cdot} g(X) d^- \mathcal{A}(f)$$

exists for every $g, f \in W^{1,2}_{loc}$. In particular, the mapping $f \to \int_0^{\cdot} g(X) d^- \mathcal{A}(f)$ is continuous from $W^{1,2}_{loc}$ to $\mathcal{C}$.

Proof. We know

$$\mathcal{A}(f) = f(X_t) - f(X_0) - \int_0^{\cdot} (f'\sigma)(X_s)\, dW_s.$$

Because $\int_0^{\cdot}(f'\sigma)(X_s)\,dW_s$ is a local martingale, the forward integral above exists if $\int_0^{\cdot} g(X)\,d^- f(X)$ exists for every $f, g \in W^{1,2}_{loc}$. Since (4.2) ensures the existence of $[f(X), g(X)]$, the latter forward integral exists if and only if

$$\int_0^{\cdot} g(X)\,d^0 f(X)$$

exists, see Remark 1.1 a), b). But since $g(X)$ is a finite quadratic variation process, Remark 1.6 tells us that the symmetric integral above equals the LZ type integral

$$\int_0^{\cdot} g(X) \circ df(X),$$

Now, the LZ integral is well-defined because $f(X)$ is a $(\mathbb{F}, \mathbb{H})$-LZ process see Remark 1.5 e) and $g(X)$ is $(\mathbb{F}, \mathbb{H})$- adapted and square integrable. □.

Theorem 5.6 *$f(X)$ is a semimartingale if and only if $f \in BV^1$.*

Proof. We introduce again the notations introduced before Lemma 2.4.

k is a C^1 real function such that

$$k'(x) = \frac{e^{\Sigma(x)}}{\sigma^2(x)}, \quad L^1 g = \left(\frac{\sigma_1^2 g}{2}\right)', \quad \sigma_1 = (\sigma\, k') \circ k^{-1}.$$

We also set

$$q(x) = \int_0^T ds\, p_s(x_0, x), \tag{5.4}$$

where $(p_t(x_0, \cdot))$ is the density of the law of X_t, $t > 0$.

We recall that $Z_t = k(X_t)$ and the density of (Z_t) is a fundamental solution of $\partial_t u = L^1 u$, see Definition before Remark 3.8. Let $r_t(x_1, \cdot)$, $x_1 = k(x_0)$, be such a density. We recall that by (4.14), we have

$$p_t(x_0, x) = r_t(x_1, k(x))\, k'(x). \tag{5.5}$$

We set

$$r(z) = \int_0^T ds\, r_s(x_1, z).$$

We have of course

$$q(x) = r(k(x))\, k'(x). \tag{5.6}$$

Proposition 3.7 ii) says that $(q_t(x,y))$ and $(r_t(x,y))$ are continuous on $]0,\infty[\times\mathbb{R}^2$; therefore q and r are continuous, hence locally bounded. Moreover they are strictly positive because of the Aronson estimates and (4.6) yields $r \in W^{1,1}_{\text{loc}}$.

Let us consider the "suitable" sequence of stopping times (τ^M) and processes $X(M)$ solving a martingale problem related to $L(M)$ as before Remark 4.2.

Now, $f(X)$ is a semimartingale if and only if $f(X(M))$ is a semimartingale for every M. Therefore, we can suppose that X fulfills (TA).

We proceed now with the proof of necessity.

i) Let us suppose that $f(X)$ is a semimartingale for some $f \in W^{1,2}_{\text{loc}}$. Then by (3.1), $\mathcal{A}(f)$ must be a semimartingale. Since it is a zero quadratic variation process, it is forced to be of bounded variation.

ii) We set $\bar{f} = f \circ k$. Clearly $\bar{f} \in W^{1,2}_{\text{loc}}$; so $L^1\bar{f}$ is a well-defined distribution. We first prove that $L^1\bar{f}$ is a Radon measure.

By Remark 2.3, there is a sequence (f_n) in $\mathcal{D}_L$ so that $f_n \to f$ in $W^{1,2}_{\text{loc}}$. We define $\bar{f}_n = f_n \circ k$ which belong to $\mathcal{D}_{L^1}$.

Let $\phi \in C^0(\mathbb{R})$ with compact support such that

$$\tilde{\phi} = \frac{\phi \circ k}{r \circ k} \in C^1. \tag{5.7}$$

We observe that $\phi \in W^{1,2}_{loc}$. Since $\mathcal{L}^1$ is continuous from $W^{1,2}_{loc}$ to L^2_{loc}

$$\begin{aligned}
\langle L^1\bar{f}, \phi\rangle &= -\int \frac{\sigma^2}{2}\bar{f}'(z)\phi'(z)dz \\
&= -\int dz\, \mathcal{L}^1\bar{f}(z)\phi'(z) \\
&= \lim_{n\to\infty} -\int dz\, \mathcal{L}^1\bar{f}_n(z)\phi'(z) \\
&= \lim_{n\to\infty}\int dz\ \phi(z)\, L^1\bar{f}_n(z) \\
&= \lim_{n\to\infty}\int dy\, \phi\circ k(y)\, L\, f_n(y)\, k'(y) \\
&= \lim_{n\to\infty}\int dy\, \frac{\phi\circ k}{r\circ k}(y)\, L\, f_n(y)\, k'(y)\, r\circ k(y)
\end{aligned}$$

Using (5.6), we obtain

$$\begin{aligned}
& \lim_{n\to\infty} \int dy\, \tilde{\phi}(y)\, L\, f_n(y)\, q(y) \\
= & \lim_{n\to\infty} \int ds \int dy\, p_s(x_0, y)\, \tilde{\phi}(y) L f_n(y) \\
= & \lim_{n\to\infty} \mathbb{E}\left\{ \int_0^T ds\, L\, f_n(X_s)\, \tilde{\phi}(X_s) \right\}
\end{aligned}$$

This equals

$$\lim_{n\to\infty} \mathbb{E}\left\{ \int_0^T d^-\, \mathcal{A}(f_n)\, \tilde{\phi}(X) \right\} \tag{5.8}$$

Lemma 5.5 and the fact that X is a LZ process imply that

$$\int_0^\cdot \tilde{\phi}(X) d^- \mathcal{A}(f_n) \longrightarrow \int_0^\cdot \tilde{\phi}(X) d^- \mathcal{A}(f) = \int_0^\cdot \tilde{\phi}(X) d\mathcal{A}(f)$$

holds in ucp. The equality above is explained by the fact that $\mathcal{A}(f)$ is a bounded variation process. Therefore (5.8) converges to

$$\mathbb{E}\left(\int_0^T d\mathcal{A}(f) \tilde{\phi}(X) \right)$$

provided that the sequence

$$\int_0^T L f_n(X_s) \tilde{\phi}(X_s) = \int_0^T L^1 \bar{f}_n(Z_s) \phi_1(Z_s),$$

where $\phi_1 = \frac{\phi}{r} = \phi \circ k^{-1}$ is uniformly integrable. This can be established by verifying that the sequence of the expectations of squares is bounded.

In fact, using Proposition 3.7 iii)

$$\begin{aligned}
& \mathbb{E}\left(\int_0^T ds L^1 \bar{f}_n(Z_s) \phi_1(Z_s) \right)^2 \\
= \; & 2 \int_0^T ds_1 \int_{s_1}^T ds_2 \int dy_1 \int dy_2 L^1 \bar{f}_n(y_1) \phi_1(y_1) L^1 \bar{f}_n(y_2) \phi_1(y_2) \\
& r_{s_1}(x_0, y_1) r_{s_2 - s_1}(y_1, y_2) \\
= \; & 2 \int_0^T ds_1 \int_{s_1}^T ds_2 \int dy_1 \frac{\sigma_1^2}{2} \bar{f}_n'(y_1) \frac{\partial}{\partial y_1} \left(\phi_1(y_1) r_{s_1}(x_0, y_1) \right) \\
& \int dy_2 \frac{\sigma_1^2}{2}(y_2) \bar{f}_n'(y_2) \frac{\partial}{\partial y_2} \left(\phi_1(y_2) r_{s_2 - s_1}(y_1, y_2) \right)
\end{aligned}$$

Using Cauchy-Schwarz, this quantity is bounded by

$$const(\tilde{\phi}) \int_0^T ds_1 \left\{ \int_{-M}^{M} dy_1 \bar{f}_n'^2(y_1) \int_{-M}^{M} \left[(\frac{\partial r_{s_1}}{\partial y_1}(x_0, y_1))^2 + (r_{s_1}(x_0, y_1))^2 \right] dy_1 \right\}^{\frac{1}{2}}$$
$$\int_0^{T-s_1} ds_2 \left\{ \int_{-M}^{M} dy_2 \bar{f}_n'^2(y_2) \int_{-M}^{M} \left[(\frac{\partial r_{s_2}}{\partial y_2}(y_1, y_2))^2 + (r_{s_2}(y_1, y_2))^2 \right] dy_2 \right\}^{\frac{1}{2}}$$

for some $M > 0$ such that $[-M, M]$ contains the support of ϕ_1.

The latter quantity is bounded by

$$const(\tilde{\phi}) \int_{-M}^{M} dy \bar{f}_n'^2(y) \int_0^T ds_1 \left\{ \int_{-M}^{M} \left[\left(\frac{\partial r_{s_1}}{\partial y_1}(x_0, y_1) \right)^2 + (r_{s_1}(x_0, y_1))^2 \right] dy_1 \right\}^{\frac{1}{2}}$$
$$\sup_{y_1 \in \mathbb{R}} \int_0^T ds_2 \left\{ \int_{-M}^{M} \left[\left(\frac{\partial r_{s_2}}{\partial y_2}(y_1, y_2) \right)^2 + (r_{s_2}(y_1, y_2))^2 \right] dy_2 \right\}^{\frac{1}{2}}$$

The fact that $\bar{f}_n' \to \bar{f}'$ in L^2_{loc}, Aronson estimates and Lemma 3.9 imply that the quantity above is finite.

We have now established the identity

$$\langle L^1 \bar{f}, \phi \rangle = \mathbb{E} \left(\int_0^T d\mathcal{A}(f) \tilde{\phi}(X) \right). \tag{5.9}$$

Now, the right member can be extended by continuity to $\phi \in C^0$; in fact functions ϕ fulfilling (5.7) are dense in C^0. This shows that $L^1 \bar{f}$ is a Radon measure.

iii) $L^1 \bar{f}$ being a Radon measure, $\mathcal{L}^1 \bar{f}$ is of bounded variation; this means that $\frac{\sigma_1^2 \bar{f}'}{2}$ has bounded variation. This equals

$$(\sigma k')^2 \circ k^{-1} (f \circ k^{-1})' = (\sigma^2 k' f') \circ k^{-1},$$

which implies that

$$\sigma^2 k' f' = e^{\Sigma} f'$$

must be of bounded variation. This shows the necessity.

We now proceed to the converse implication. Let $f \in BV^1$. By Lemma 5.2, there is a sequence (f_n) in $\mathcal{D}_L$ such that $f_n \to f$ in BV^1 and so $L f_n dx \to d\mu$ weakly- $*$ for some Radon measure μ.

Remark 5.1 d) guarantees the existence of a subsequence (n_k) such that

$$Lf_{n_k}^{+}dx \to \nu^{+} \text{ and } Lf_{n_k}^{-} \to \nu^{-}$$

weakly, where $\nu^{\pm}$ are two Radon measures on $\mathbb{R}$.

We want to prove that $\mathcal{A}(f)$ has bounded variation. By (3.1), $f \mapsto \mathcal{A}(f)$ is continuous from C^1 to $\mathcal{C}$. Again by Lemma 5.2, the sequence (f_n) converges to f in $W_{loc}^{1,2}$ so that $\mathcal{A}(f_n) \to \mathcal{A}(f)$ holds in ucp. Now, we have

$$\mathcal{A}(f_n) = \int_0^{\cdot} (Lf_n)(X_s)\,ds = \int_0^{\cdot} (Lf_n)^{+}(X_s)\,ds - \int_0^{\cdot} (Lf_n)^{-}(X_s)\,ds.$$

We define $\tilde{f}_n^{+}$, $\tilde{f}_n^{-}$ in C^1 such that $\tilde{f}_n^{\pm}(0) = 0$ and

$$(\tilde{f}_n^{\pm})' = h'(x)\left(2\int_0^x \frac{(Lf_n)^{\pm}}{(\sigma h')^2}(y)\,dy\right)$$

hold. Since $(Lf_{n_k})^{\pm}(y)\,dy$ converges weakly-$*$ to $\nu^{\pm}$, Remark 5.1 a) says that

$$\frac{(\tilde{f}_n^{\pm})'(x)}{h'} \to 2\int_0^x \frac{d\nu^{\pm}}{(\sigma h')^2(y)}$$

holds in BV. We consider $\tilde{f}^{\pm} \in BV^1$ which are given by $\tilde{f}^{\pm}(0) = 0$ and

$$(\tilde{f}^{\pm})'(x) = 2h'(x)\int_0^x \frac{d\nu^{\pm}}{(\sigma h')^2(y)}.$$

Again according to (3.1) and the definition of $\mathcal{A}$, we have

$$\int_0^{\cdot} Lf_n^{\pm}(X_s)\,ds = f_n^{\pm}(X_t) - f_n^{\pm}(X_0) - \int_0^t ((f_n^{\pm})'\sigma)(X_s)\,dW_s. \tag{5.10}$$

Since $f_n^{\pm} \to f$ in $W_{loc}^{1,2}$ and $\mathcal{A}$ is continuous, the sequence of increasing processes $\int_0^{\cdot} Lf_{n_k}^{\pm}(X_s)\,ds$ in (5.10) converges to the increasing process

$$\tilde{f}^{\pm}(X_t) - \tilde{f}^{\pm}(X_0) - \int_0^t ((\tilde{f}^{\pm})'\sigma)(X_s)\,dW_s = \mathcal{A}(\tilde{f}^{\pm}).$$

So $\mathcal{A}(f)$ is the difference of the increasing processes $\mathcal{A}(\tilde{f}^{+})$ and $\mathcal{A}(\tilde{f}^{-})$. □

Remark 5.7 *If $f \in BV^1$ then, in particular, we have $\hat{\mathcal{L}}^{BV} f \in BV$.*

Remark 5.8 *For a Markov process X, [11] states necessary and sufficient conditions on f such that $f(X)$ is a semimartingale. Here, we specify in particular this result in a very simple way without using directly the Markov property. We observe that, for a standard Brownian motion X, we have $h \equiv id$ and $BV^1 = \{f \in W^{1,2}_{loc} : f' \in BV\}$. BV^1 is in this case constituted by functions which are a difference of convex functions.*

Corollary 5.9 *Let us suppose that L is close to divergence type. Then $f(X)$ is a semimartingale if and only if $\hat{\mathcal{L}}f$ has bounded variation.*

Remark 5.10 *Remark 2.5 allows to extend continuously $\hat{\mathcal{L}}$ to $W^{1,2}_{loc}$ so that the statement makes sense.*

Proof of the Corollary.

Let $f \in BV^1$. By Remark 2.3 and Proposition 5.3, there is a sequence (f_n) in $\mathcal{D}_L$ converging to f in $W^{1,2}_{loc}$ such that $(\mathcal{L}f_n)$ converges in BV to some $g \in BV$. By continuity of $\hat{\mathcal{L}}$ in $W^{1,2}_{loc}$, we have

$$\hat{\mathcal{L}}f = \lim_{n\to\infty} \mathcal{L}f_n \text{ in } L^2_{loc}.$$

Since the convergence in L^2_{loc} and BV must agree, we have $\hat{\mathcal{L}}f = g \in BV$.

Conversely, if $\hat{\mathcal{L}}f = g$ is BV, by a usual regularization procedure, we find $g_n \in C^1$ satisfying $g_n \to g$ in BV. Let $f_n \in \mathcal{D}_L$ such that

$$f_n(0) = 0, \; f_n'(x) = h'(x) \int_0^x \frac{2}{\sigma^2(y)} \, dg_n(y).$$

Remark 2.2 d) says that $\mathcal{L}f_n = g_n$. Taking the limit in the expression above and using the continuity of the extension of $\mathcal{L}$ (Remark 2.5) we obtain that $\frac{f_n'}{h'}$ converge to $\frac{f'}{h'}$ in BV. Thus, we have $f \in BV^1$. □

Corollary 5.11 *Let X be a solution to the martingale problem related to L. Then X is a semimartingale if and only if Σ has bounded variation.*

Proof. By Theorem 5.6, X is a semimartingale if and only if $id \in BV^1$. This means $(h')^{-1} = \exp(\Sigma) \in BV$ or, equivalently $\Sigma \in BV$. □

Remark 5.12 *(i) If $\sigma = 1$ then $\Sigma \equiv b$ and we discover the result of [45].*

(ii) Let L be close to divergence type, see before Remark 2.5, and

$$\Sigma(x) = \ln \frac{1}{\sigma^2} + 2 \int_0^x \frac{d\beta}{\sigma^2}.$$

Then Σ is of bounded variation if and only if so is σ.

6 The backward and symmetric equations

In this section we suppose that L is of divergence form, i.e.

$$Lg = \left(\frac{\sigma^2}{2} g'\right)' \tag{6.1}$$

Let X be the solution to the martingale problem related to L with initial condition x_0.

We know by Remark 3.5 that X solves the generalized stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A\left(\frac{\sigma^2}{2}\right). \tag{6.2}$$

If $\sigma \in C^1$ then X is a semimartingale and

$$A\left(\frac{\sigma^2}{2}\right) = \int_0^t \left(\frac{\sigma^2}{2}\right)'(X_s)\, ds = \int_0^t (\sigma\sigma')(X_s)\, ds = [\sigma(X), X]_t.$$

Therefore, X solves the backward stochastic differential equation

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+ W_s \tag{6.3}$$

because of Remark 1.1. Immediately, the following question arises. If σ is not smooth is X still a solution to (6.3)? The answer will be yes if X is a semimartingale. In the general case, it does not seem to be true but we do not give a rigorous argument for this.

Let us suppose again our technical assumption (TA), which corresponds here to $0 < c \leq \sigma^2 \leq C < \infty$.

First of all, we would like to understand some features of the time reversal of the $\mathcal{F}^X$-Brownian motion W.

Let us recall that the time reversed process $\hat{Y}$ of Y with respect to some horizon $T > 0$ is a solution to the stochastic differential equation given before (4.8)

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \sigma_0(\hat{Y}_s)\, dB_s + \int_0^t \tilde{b}(T-s, \hat{Y}_s)\, ds, \tag{6.4}$$

where B is a $\mathcal{F}^Y$-Brownian motion and $\tilde{b}$ is given by (4.8), (4.7), and (4.12):

$$\tilde{b}(s,y) = -\Big(\frac{\partial}{\partial x}\log p_s\Big)(x_0, h^{-1}(y));$$

$(p_t(x,y))$ is the fundamental solution associated with L. We remark that in this case p coincides with r since L is already in divergence form. As in the proof of Proposition 3.3 of Part I, if $f \in \mathcal{D}_L$ then we can apply Itô's formula to $(f \circ h^{-1})(\hat{Y})$ which equals in fact $f(\hat{X})$. We get

$$\begin{aligned} f(\hat{X}_t) &= f(\hat{X}_0) + \int_0^t (f'\sigma)(\hat{X}_s)\, dB_s + \int_0^t (Lf)(\hat{X}_s)\, ds \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s) f'(\hat{X}_s)\, ds. \end{aligned} \tag{6.5}$$

Remark 6.1 *In a sense to be precised $\hat{X}$ solves the martingale problem related to*

$$\tilde{L}f = Lf - \frac{\partial}{\partial x}(\log p_{T-s})(x_0, x) f'.$$

Lemma 3.9 and Aronson estimates tell us that the additional term due to time reversal belongs to $L^1_{loc}([0,T] \times \mathbb{R})$ in (t,x).

Even in this time reversed concept it is possible to define $\tilde{A}$ as the unique extension of the map $f \mapsto \int_0^{\cdot} Lf(\hat{X})\, ds$ to C^1. The map $\tilde{A} : C^0 \to \mathcal{C}$ will be the unique extension of $l \mapsto \int_0^{\cdot} l'(\hat{X}_s)\, ds$. It is clear that (6.5) can be extended to C^1 (and even to $W^{1,2}_{loc}$) by

$$\begin{aligned} f(\hat{X}_t) &= f(\hat{X}_0) + \int_0^t (f'\sigma)(\hat{X}_s)\, dB_s + \tilde{A}(f)_t \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s) f'(\hat{X}_s)\, ds. \end{aligned} \tag{6.6}$$

Remark 6.2 *For $f \in C^1$, $l \in C^0$, we have*

(i) $\tilde{\mathcal{A}}(f)_t = \mathcal{A}(f)_T - \mathcal{A}(f)_{T-t}$,

(ii) $\tilde{A}(l)_t = A(l)_T - A(l)_{T-t}$.

(iii) For $f = id$, (6.6) yields

$$\begin{aligned}\hat{X}_t &= \hat{X}_0 + \int_0^t \sigma(\hat{X}_s)\, dB_s - \hat{A}\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad + A\Big(\frac{\sigma^2}{2}\Big)_T - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)\, ds.\end{aligned}$$

By time reversal, we get

$$\begin{aligned}X_t &= X_T - \int_t^T \sigma(X_s)\, d^+\hat{B}_s - A\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad + A\Big(\frac{\sigma^2}{2}\Big)_T - \int_t^T \frac{\partial}{\partial x}(\log p_{T-s})(x_0, X_s)\, ds.\end{aligned}$$

Thus, we have

$$\begin{aligned}X_t &= x_0 - \int_0^t \sigma(X_s)\, d^+B_s + A\Big(\frac{\sigma^2}{2}\Big)_t \\ &\quad - \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, X_s)\, ds.\end{aligned} \tag{6.7}$$

We consider now $f_0 \in C^1$ such that $f_0(0) = f_0'(0) = 0$ and $f_0' = 1/\sigma$. By regularization, it is not difficult to see that $\hat{\mathcal{L}} f_0 = \frac{\sigma}{2}$. We now apply Remark 3.2 ii) and iii) and obtain

$$W_t = f_0(X_t) - f_0(x_0) - \frac{1}{2} A(\sigma)_t. \tag{6.8}$$

Using (6.6) with $f = f_0$, we get

$$\begin{aligned}B_t &= f_0(\hat{X}_t) - f_0(\hat{X}_0) - \frac{1}{2}\tilde{A}(\sigma)_t \\ &\quad + \int_0^t \frac{\frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)}{\sigma(\hat{X}_s)}\, ds.\end{aligned} \tag{6.9}$$

Proposition 6.3 *$\hat{W}_t - W_T$ is a $\mathbb{H}$ semimartingale if and only if σ is of bounded variation.*

Remark 6.4 *Proposition 6.3 means that $\hat{W}_t - W_T$ is a $\mathbb{H}$ semimartingale if and only if X is a semimartingale.*

Proof of Proposition 6.3. ¿From (6.8) and Remark 6.2 we get

$$\hat{W}_t = f_0(\hat{X}_t) - f_0(x_0) - \frac{1}{2}A(\sigma)_T + \frac{1}{2}\tilde{A}(\sigma)_t.$$

Subtracting (6.9) from (6.8), we obtain

$$\begin{aligned} B_t - \hat{W}_t &= f_0(x_0) - f_0(X_T) + \hat{A}(\sigma)_t - \frac{1}{2}A(\sigma)_T \\ &\quad + \int_0^t \frac{\partial}{\partial x}(\log p_{T-s})(x_0, \hat{X}_s)\frac{1}{\sigma(\hat{X}_s)}\, ds. \end{aligned} \tag{6.10}$$

Since $f_0(X_T) = f_0(x_0) + W_T + \frac{1}{2}A(\sigma)_T$, we get

$$B_t - \hat{W}_t + W_T = \hat{A}(\sigma)_t + \int_0^t \frac{\partial}{\partial x}\log p_{T-s}(x_0, \hat{X}_s)\frac{1}{\sigma(\hat{X}_s)}\, ds. \tag{6.11}$$

We recall that $(\hat{W}_t - W_T)$ and B are both $\mathbb{H}$-adapted. Since B is a $\mathbb{H}$-Brownian motion and $A(\sigma)$, hence $\hat{A}(\sigma)$, a zero quadratic variation process, (6.11) shows that $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-Dirichlet process.

Now, $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-semimartingale if and only if $A(\sigma)$ has bounded variation. By Section 5, this happens if and only if σ is of bounded variation. □

We go on with the study of the backward equation. Let X be a solution to the martingale problem related to L. Let us suppose that

$$\int_0^t g(X_s)\, d^+W_s \text{ exists for all } g \in C^0. \tag{6.12}$$

Then, using (6.11), we see

$$\int_0^t g(X_s)\, d^+W_s = -\int_{T-t}^T g(\hat{X}_s)\, d^-\hat{W}_s$$

$$= -\int_{T-t}^{T} g(\hat{X}_s)\, dB_s - \int_{T-t}^{T} g(\hat{X}_s)\, d^- \hat{A}(\sigma)_s \quad (6.13)$$
$$- \int_{T-t}^{T} \frac{\partial}{\partial x} \log p_{T-s}(x_0, \hat{X}_s) \frac{g(\hat{X}_s)}{\sigma(\hat{X}_s)}\, ds.$$

In particular, $\int_{T-t}^{T} g(\hat{X}_s)\, d^- \hat{A}(\sigma)_s$ exists. Therefore we realize the following

Remark 6.5 *$\int_0^t g(X_s)\, d^+ W_s$ exists if and only if $\int_0^t g(X_s)\, d^+ A(\sigma)_s$ exists. We recall that, by Lemma 5.5, this is always true if $g \in W^{1,2}_{loc}$ because X is a LZ process.*

Proposition 6.6 *Under the assumption (6.12) X is a solution to*

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^+ W_s. \quad (6.14)$$

Corollary 6.7 *If X is a semimartingale then (6.12) is always verified.*

Proof of Corollary 6.7. If X is a semimartingale then $\hat{X}$ is a $\mathbb{H}$-semimartingale because X is also a LZ process. In this case $(\hat{W}_t - W_T)$ is a $\mathbb{H}$-semimartingale by Remark 6.4. At this point, $\int_0^t g(X_s)\, d^+ W_s = -\int_{T-t}^{T} g(\hat{X}_s)\, d\hat{W}_s$ is a classical Itô's integral. □

Proof of Proposition 6.6. Relation (6.11) implies

$$\hat{B}_t - W_t + W_T = A(\sigma)_t + \int_t^T \frac{\partial}{\partial x} \log p_{T-s}(x_0, X_s)\, ds$$

so that

$$d^+ \hat{B}_t = d^+ W_t + d^+ A(\sigma)_t + \frac{\frac{\partial}{\partial x} \log p_{T-t}(x_0, X_t)}{\sigma(X_t)}\, dt \quad (6.15)$$

holds. By (6.15), we evaluate

$$\int_0^t \sigma(X_s)\, d^+ W_s = \int_0^t \sigma(X_s)\, d^+ \hat{B}_s - \int_0^t \sigma(X_s)\, d^+ A(\sigma)_s$$
$$- \int_0^t \frac{\partial}{\partial x} \log(p_{T-s})(x_0, X_s)\, ds. \quad (6.16)$$

Comparing (6.7) and (6.16) yields

$$X_t - \int_0^t \sigma(X_s)\, d^+W_s = -A(\sigma^2/2)_t + \int_0^t \sigma(X_s)\, d^+A(\sigma)_s. \qquad (6.17)$$

□

Remark 6.8 *X fulfills (6.17) without assumption (6.12), provided that* $\int_0^t \sigma(X_s)\, d^+W_s$ *exists.*

If (6.12) is realized then

$$A(\sigma^2/2)_t = \int_0^t \sigma(X_s)\, d^+A(\sigma)_s \qquad (6.18)$$

holds. Assumption (6.12), Remark 6.5 and Banach-Steinhaus Theorem imply that

$$g \mapsto \int_0^{\cdot} g(X_s)\, d^+A(\sigma)_s$$

is continuous from C^0 to $\mathcal{C}$. Therefore

$$\int_0^t \sigma(X_s)\, d^+A(\sigma)_s = \lim_{n\to\infty} \int_0^t \sigma_n(X_s)\, d^+A(\sigma)_s$$

holds for the usual regularizations of σ^2.

Since $\sigma_n(X)$ is a finite quadratic variation process and $A(\sigma)$ a zero quadratic variation process, we get

$$\int_0^t \sigma_n(X_s)\, d^+A(\sigma)_s = \int_0^t \sigma_n(X_s)\, d^-A(\sigma)_s.$$

Using the fact that $\sigma_n \in C^2$, Remark 3.5 c) says that previous integral equals $A(\Phi(\sigma_n, \sigma))_t$, where $\Phi(g,l)(x) = (gl)(x) - gl(0) - \int_0^x g'l(y)\, dy$. So, we have

$$\int_0^t \sigma_n(X_s)\, d^+A(\sigma)_s = A(\Phi(\sigma_n, \sigma))_t.$$

The problem here is that $(\Phi(\sigma_n, \sigma))$ does not necessarily tend to σ^2. Using the additivity of A, we get

$$\begin{aligned} A(\Phi(\sigma_n, \sigma)) &= A(\sigma_n \sigma) - A\left(\int_0^{\cdot} (\frac{\sigma_n^2}{2})(y)\, dy\right) \\ &= A(\frac{\sigma_n^2}{2}) - \int_0^{\cdot} \sigma_n'(\sigma - \sigma_n)(X_s)\, ds. \end{aligned}$$

Clearly $A(\sigma_n^2) \to A(\sigma^2)$ ucp so that $A(\Phi(\sigma_n, \sigma))$ converge to $A(\frac{\sigma^2}{2})$ if and only if

$$\int_0^t \sigma_n'(\sigma - \sigma_n)(X_s)ds \to 0 \tag{6.19}$$

holds in probability for any t.

Proposition 6.9 *If X is a semimartingale then (6.19) holds.*

Remark 6.10 *In the general case there is no reason for (6.19) to be fulfilled.*

Proof of the Proposition. Using localization techniques, we may assume σ, σ_n to have compact support. By Proposition 3.7 iii), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \sigma_n'(\sigma - \sigma_n)(X_s)\,ds\right)^2 \\
&= \int_0^t ds_1 \int_0^t ds_2\, \mathbb{E}\left(\sigma_n'(\sigma - \sigma_n)(X_{s_1})\sigma_n'(\sigma - \sigma_n)(X_{s_2})\right) \\
&= 2\int_0^t ds_1 \int_0^{s_1} ds_2 \int dy_1\, dy_2\, \sigma_n'(y_1)\sigma_n'(y_2)(\sigma - \sigma_n)(y_1)(\sigma - \sigma_n)(y_2) \\
&\quad p_{s_2}(x_0, y_2)p_{s_1 - s_2}(y_2, y_1).
\end{aligned}$$

This equals

$$\begin{aligned}
&2\int_0^t ds_1 \int_0^{s_1} ds_2 \int d\sigma_n(y_1)d\sigma_n(y_2)(\sigma - \sigma_n)(y_1)(\sigma - \sigma_n)(y_2) \\
&\quad p_{s_2}(x_0, y_2)p_{s_1 - s_2}(y_2, y_1) \\
&= 2\int d\sigma_n(y_1) \int d\sigma_n(y_2)(\sigma - \sigma_n)(y_1)(\sigma - \sigma_n)(y_2) \\
&\quad \int_0^t ds_1 \int_0^{s_1} ds_2 p_{s_2}(x_0, y_2)p_{s_1 - s_2}(y_2, y_1).
\end{aligned}$$

By Corollary 5.11 and Remark 5.12 ii), σ is of bounded variation, so $\sigma_n \to \sigma$ in BV since $\sigma_n^2 \to \sigma^2$ in BV. The fact that $d\sigma_n \to$ weak-$*$ and that $\sigma_n \to \sigma$ in C^0, implies that the expression above converges to zero. $\square$

We finish the paper with some remarks on the symmetric case; this corresponds to the case $\alpha = \frac{1}{2}$ in Example 2.1 a).

We consider

$$Lf = \frac{\sigma^2}{2}f'' + \frac{\sigma^{2\prime}}{4}f'.$$

Let X be a solution to the martingale problem related to L.

Proposition 6.11 *X is the unique solution to the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(X_s) \circ dW_s. \tag{6.20}$$

Proof. We first prove that (6.20) has a unique solution given by $F(W_t, X_0)$, where F is the deterministic flow given by

$$\frac{\partial F}{\partial r}(r, x_0) = \sigma(F(r, x_0)), \quad F(0, x_0) = x_0. \tag{6.21}$$

We recall that $F \in C^1(\mathbb{R}_+ \times \mathbb{R})$.

Let $H : \mathbb{R}_+ \times \mathbb{R}$ be the inverse flow. Let X be a $(\mathbb{F}, \mathbb{H})$-adapted process solving (6.20). Then the Itô formula of Remark 1.8 yields

$$H(W_t, X_t) = X_0 + \int_0^t \frac{\partial H}{\partial r}(W_s, X_s) \circ dW_s + \int_0^t \frac{\partial H}{\partial x}(W_s, X_s) \circ dX_s.$$

Remark 1.7, (6.20) and the fact that $\frac{\partial u}{\partial r}(r, x) = -\sigma(x)\frac{\partial u}{\partial x}(r, x)$ show that $H(W_t, X_t) \equiv X_0$. Therefore, the solution X must be equal to $F(W_t, X_0)$ and hence unique.

The fact that $F(W_t, X_0)$ solves (6.20) is a direct consequence of the one-dimensional LZ Itô formula of Remark 1.7.

Remark 6.12 *The proof of Proposition 6.11 yields something more. If W is a $(\mathbb{F}, \mathbb{H})$ reversible semimartingale then there exists a unique $(\mathbb{F}, \mathbb{H})$ adapted solution to (6.20).*

In order to conclude the proof of Proposition 6.11 we have to show that $F(W_t, X_0)$ solves the martingale problem.

First of all, we observe that in this case, setting $h(0) = x_0$, $h'(r) = \sigma^{-1}(r)$, we get $h^{-1}(r) = F(r, x_0)$. This means that the process Y of Section 3 is in fact the Brownian motion W. We recall that $L^0 f = f''/2$. Lemma 2.4 b) says that

$$\mathcal{D}_L = \{f \in C^1 : f \circ h^{-1} \in C^2\}.$$

Moreover, $Lf = \frac{1}{2}(f \circ h^{-1})''$ holds. For $f \in \mathcal{D}_L$, the Itô formula yields

$$\begin{aligned}
f(X_t) &= (f \circ h^{-1})(W_t) \\
&= f(x_0) + \int_0^t (f \circ h^{-1})'(W_s)\, dW_s + \frac{1}{2} \int_0^t (f \circ h^{-1})''(W_s)\, ds \\
&= f(x_0) + M_t + \frac{1}{2} \int_0^t L^0 f \circ h^{-1}(W_s)\, ds \\
&= f(x_0) + M_t + \int_0^t (Lf) \circ h^{-1}(W_s)\, ds.
\end{aligned}$$

Therefore, X solves the martingale problem related to L. □

Remark 6.13 *X solves also the symmetric equation*

$$X_t = x_0 + \int_0^t \sigma(X_s)\, d^0 W_s \tag{6.22}$$

because of Remark 1.6.

Conversely, let X be a $(\mathbb{F}, \mathbb{H})$-adapted process and W a Brownian motion (or a more general $(\mathbb{F}, \mathbb{H})$ semimartingale). If X solves (6.22) then it also solves (6.20). However, other solutions to (6.22) may exist.

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