On closability of classical Dirichlet forms

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Abstract. We construct a measure μ on $\mathbb{R}^2$ such that the classical Dirichlet form $\mathcal{E}(f,g) = \int(\nabla f, \nabla g)d\mu$ is closable, but the partial Dirichlet form $\mathcal{E}_x(f,g) = \int \partial_x f \partial_x g d\mu$ is not. This proves the well-known conjecture of M. Röckner.

1 Introduction

The problem of closability of Dirichlet forms arises in the theory of differential operators, the theory of Sobolev spaces, and stochastic analysis (see [1], [4], [5], [7], [8], [9]).

Let $(X, \mathcal{B}, \mu)$ be a measurable space with positive measure μ; $L^2(\mu) = L^2(X, \mathcal{B}, \mu)$. Recall the definition of closability of a Dirichlet form $\mathcal{E}$ defined on the domain $\mathcal{D} \subset L^2(\mu)$ ([7, p. 28]).

Definition 1.1. *The form $\mathcal{E}$ is said to be closable on $L^2(\mu)$ if for any sequence of functions $f_n \in \mathcal{D}$ such that $\|f_n\|_{L^2(\mu)} \underset{n\to\infty}{\longrightarrow} 0$ and $\mathcal{E}(f_n - f_m, f_n - f_m) \underset{m,n\to\infty}{\longrightarrow} 0$, it follows that $\mathcal{E}(f_n, f_n) \underset{n\to\infty}{\longrightarrow} 0$.*

In applications, the closability of gradient Dirichlet forms on finite- and infinite-dimensional linear spaces and manifolds is often verified by the aid of the following fact: if for some measure μ on $\mathbb{R}^n$ the partial Dirichlet forms

$$\mathcal{E}_i(f,g) = \int \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\,\mu(dx) \tag{1.1}$$

are closable on $L^2(\mu)$ for every $i = 1, \dots n$, then the gradient Dirichlet form

$$\mathcal{E}(f,g) = \sum_{i=1}^{n} \mathcal{E}_i(f,g) \tag{1.2}$$

is closable on $L^2(\mu)$ as well; the same is true also in infinite dimensions (cf. [2, Theorem 3.2]). A necessary and sufficient condition of closability of (1.1) can be found in [2, Theorem 5.3]. In this relation, the following non-trivial question arose: is it true that if the form (1.2) is closable on $L^2(\mu)$ then the partial Dirichlet forms (1.1) are also closable on $L^2(\mu)$? About 10 years ago M. Röckner conjectured that it is not always true, but no counter-example was known.

The main aim of this paper is to show that even in the case of $\mathbb{R}^2$ the answer is negative (cf. Example 3.2 below). Moreover, the measure μ constructed as a counter-example is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$ and can even have full support (cf. Example 3.3). A positive result of this paper (employed in the justification of our counter-example) is a new sufficient condition for the closability of gradient Dirichlet forms for a special class of measures on $\mathbb{R}^n$ (see Theorem 2.3 below).

2 Extensions of Sobolev functions

Let U be an open set in $\mathbb{R}^n$ and let $p \geq 1$. The Sobolev class $W^{1,p}(U)$ is defined as the completion of the space $\left\{ f\mid_U \colon\ f \in C_b^\infty(\mathbb{R}^n)\colon\ \|f\|_{p,1} < \infty \right\}$ with respect to the Sobolev norm

$$\|f\|_{p,1} = \left(\int\limits_U (|f| + |\nabla f|)^p \, d\lambda^n \right)^{1/p},$$

where λ^n is Lebesgue measure on $\mathbb{R}^n$.

Let us recall the following theorem about extension of Sobolev functions (see [10, Ch. VI, § 3, Theorem 5] and [6]).

Theorem 2.1. *Let Q be an open set in $\mathbb{R}^n$ such that there exist numbers $\varepsilon, M, N > 0$, and a finite or countable family of open sets $\{V_i\}$ with the following properties:*

1) the ε-neighborhood of any point $x \in Q$ lies in some V_i;

2) $\sum\limits_i \mathbf{1}_{V_i} \leq N$;

3) for any i there exists a set Q_i that is isometric to some open set of the form $\{x_n < \varphi_i(x_1, \ldots x_{n-1})\}$, where φ_i is a Lipschitzian function with the Lipschitzian constant M, such that $V_i \cap Q = V_i \cap Q_i$.

Then, for any $p \geq 1$, there exists a linear operator E_1 of extension from $W^{1,p}(Q)$ to $W^{1,p}(\mathbb{R}^n)$ such that

$$E_1 f\mid_Q = f, \quad \|E_1 f\|_{L^p(\mathbb{R}^n)} \leq c_p \cdot \|f\|_{L^p(Q)},$$
$$\|\nabla(E_1 f)\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(Q, \mathbb{R}^n)}.$$

Lemma 2.2. *Let G be a connected open set in $\mathbb{R}^n$ and let G_0 be an open set such that $\overline{G_0} \subset G$ and the open set $Q = G \setminus \overline{G_0}$ satisfies the hypotheses of Theorem 2.1. Denote by T the transformation*

$$T\colon\ \mathbb{R}^n \to \mathbb{R}^n, \quad T(x) = y + r \cdot J(x), \tag{2.3}$$

where $y \in \mathbb{R}^n$ is a fixed vector, $r > 0$, and J is an orthogonal linear operator on $\mathbb{R}^n$. Then, for any $p \geq 1$, there exists a linear operator E_T of extension from $W^{1,p}(T(G \setminus \overline{G_0}))$ to $W^{1,p}(T(G))$ such that

$$E_T f\mid_{T(G \setminus \overline{G_0})} = f;$$

$$\|E_T f\|_{L^p(T(G))} \leq c_p \cdot \|f\|_{L^p(T(G\setminus\overline{G_0}))}, \tag{2.4}$$

$$\|\nabla(E_T f)\|_{L^p(T(G),\mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(T(G\setminus\overline{G_0}),\mathbb{R}^n)}, \tag{2.5}$$

where the constant c_p depends only on G and G_0 and does not depend on y, r, and J.

Proof. By Theorem 2.1 there exists an operator E_1 of extension from $W^{1,p}(G \setminus \overline{G_0})$ to $W^{1,p}(\mathbb{R}^n)$ with

$$\|E_1 f\|_{L^p(\mathbb{R}^n)} \leq c_p \cdot \|f\|_{L^p(G\setminus\overline{G_0})}, \tag{2.6}$$

$$\|\nabla(E_1 f)\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(G\setminus\overline{G_0},\mathbb{R}^n)}. \tag{2.7}$$

Set $Ef := E_1 f\,|_G$. Then estimates (2.6) and (2.7) hold also for the corresponding norms of Ef on G. Let us consider the operator

$$H : W^{1,p}(G) \to W^{1,p}(T(G)), \qquad Hf(Tx) = f(x).$$

It is easy to see that $|\nabla(Hf)(Tx)| = \frac{1}{r}|\nabla f(x)|$, and since the Jacobian of the transformation T equals r^n, we have

$$\|Hf\|_{L^p(T(G))} = r^{n/p} \cdot \|f\|_{L^p(G)}, \quad \|\nabla(Hf)\|_{L^p(T(G),\mathbb{R}^n)} = r^{n/p-1} \cdot \|\nabla f\|_{L^p(G,\mathbb{R}^n)}. \tag{2.8}$$

The same equalities are true for the L^p-norms on $T(G\setminus\overline{G_0})$ and $G\setminus\overline{G_0}$, respectively. Now we define the operator of extension from $W^{1,p}(T(G\setminus\overline{G_0}))$ to $W^{1,p}(T(G))$ by

$$E_T := H \circ E \circ H^{-1}.$$

Finally, from equalities (2.8) and estimates (2.6) and (2.7) we obtain the desired estimates (2.4) and (2.5). □

From now on we shall assume that, if a Dirichlet form is defined on some subclass $\mathcal{D}$ of $L^2(\mu)$, μ being a measure on $\mathbb{R}^d$, then

$$\mathcal{D} = \left\{\varphi\,|_{\mathrm{supp}\mu} : \ \varphi \in C_b^\infty(\mathbb{R}^d)\right\}.$$

Theorem 2.3. *Let Q be an open set in $\mathbb{R}^n$ satisfying the hypotheses of Theorem 2.1. Let the connected open sets G and $G_0 \subset G$ satisfy the hypotheses of Lemma 2.2. Consider a countable family of transformations $\{T_k\}$ of the form (2.3) such that $T_k(G) \subset Q$ and $T_k(G) \cap T_j(G) = \emptyset$ if $k \neq j$. Set*

$$S := Q \setminus \bigcup_{k=1}^{\infty} T_k(\overline{G_0}).$$

Then

(i) the Dirichlet form

$$\mathcal{E}'(f,g) = \int\limits_S (\nabla f, \nabla g)\, d\lambda^n \tag{2.9}$$

is well-defined and closable on $L^2(\lambda^n|_S)$,

(ii) the Sobolev classes $W^{1,p}(S)$, $p \geq 1$, *are well-defined.*

Proof. Fix $p \in [1; +\infty)$. Note that the Sobolev gradient $\nabla f \in L^p(S, \mathbb{R}^n)$ is well-defined for smooth functions, since if $f \in C_b^\infty(\mathbb{R}^n)$ and $f = 0$ on S, then $\nabla f = 0$ on S a.e.

Denote by E_{T_k} the operator of extension of functions from $W^{1,p}(T_k(G \setminus \overline{G_0}))$ to $W^{1,p}(T_k(G))$ constructed in Lemma 2.2. Let $f \in C_b^\infty(\mathbb{R}^n)$. Put

$$E_0 f = \begin{cases} f(x), & x \in S, \\ \left(E_{T_k}(f|_{T_k(G\setminus\overline{G_0})})\right)(x), & x \in T_k(\overline{G_0}),\ k \in \mathbb{N}; \end{cases}$$

$$\psi_j = \begin{cases} f(x), & x \in Q \setminus \bigcup\limits_{k=1}^{j} T_k(\overline{G_0}), \\ \left(E_{T_k}(f|_{T_k(G\setminus\overline{G_0})})\right)(x), & x \in T_k(\overline{G_0}),\ k = 1, \dots, j. \end{cases}$$

The function ψ_j belongs to the Sobolev class $W^{1,p}(Q)$. Indeed, it belongs to $W^{1,p}(T_k(G_0))$ for $k = 1, \dots j$; let $f_m^1, \dots, f_m^j$, $m \in \mathbb{N}$, be C_b^∞-functions approximating ψ_j in the $\|\cdot\|_{1,p}$-norm on $T_k(G)$, $k = 1, \dots j$, and $\chi_o, \dots, \chi_j \in C_b^\infty$ be such that $0 \leq \chi_k \leq 1$, $\sum_{k=0}^j \chi_k = 1$, and $\chi_k = 1$ on $T_k(G_0)$, $k = 1, \dots j$. It is easy to check that the functions $f_m^{(j)} = \sum\limits_{k=0}^{j} \chi_k f_m^k$ coincide with ψ_j on $Q \setminus \bigcup_{k=1}^j T_k(G_0)$ and approximate ψ_j in $\|\cdot\|_{1,p}$ on Q as $m \to \infty$.

We have

$$\begin{aligned} \|E_0 f\|_{L^p(Q)} &\leq \|f\|_{L^p(S)} + \sum_k \|E_{T_k} f\|_{L^p(T_k(G))} \\ &\leq \|f\|_{L^p(S)} + c_p \cdot \sum_k \|f\|_{L^p(T_k(G\setminus\overline{G_0}))} \leq (1 + c_p) \cdot \|f\|_{L^p(S)}, \end{aligned}$$

since the sets $T_k(G)$ are disjoin; by analogy,

$$\begin{aligned} \|\nabla\psi_j\|_{L^p(Q,\mathbb{R}^n)} &\leq \|\nabla f\|_{L^p(S,\mathbb{R}^n)} + \sum_k \|\nabla(E_{T_k} f)\|_{L^p(T_k(G),\mathbb{R}^n)} \\ &\leq \|\nabla f\|_{L^p(S,\mathbb{R}^n)} + c_p \cdot \sum_k \|\nabla f\|_{L^p(T_k(G\setminus\overline{G_0}),\mathbb{R}^n)} \leq (1 + c_p) \cdot \|\nabla f\|_{L^p(S,\mathbb{R}^n)}. \end{aligned}$$

We also have

$$|\psi_j(x)|^p \leq |E_0 f(x)|^p + |f(x)|^p \quad \text{and} \quad \psi_j(x) \underset{j\to\infty}{\longrightarrow} E_0 f(x) \quad \text{for a.e.} \quad x \in Q.$$

In addition, $\nabla\psi_j(x)$ converge a.e. on Q as $j \to \infty$. Therefore, $E_0 f \in W^{1,p}(Q)$.

Next we apply the operator E_1 of extension from $W^{1,p}(Q)$ to $W^{1,p}(\mathbb{R}^n)$ (which increases $W^{1,p}$-norms not more than in C_p times). We obtain

$$\|E_1 \circ E_0 f\|_{L^p(\mathbb{R}^n)} \le C_p \cdot (1 + c_p)\|f\|_{L^p(S)};$$

$$\|\nabla(E_1 \circ E_0 f)\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} \le C_p \cdot (1 + c_p)\|\nabla f\|_{L^p(S,\mathbb{R}^n)}.$$

Suppose we have a mapping $V \in L^p(S, \mathbb{R}^n)$ and a sequence of smooth functions $f_k \in C_b^\infty(\mathbb{R}^n)$ with

$$\|f_k\|_{L^p(S)} \to 0, \quad \|\nabla f_k - V\|_{L^p(S,\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$

In order to prove that the Sobolev class $W^{1,p}(S)$ is well-defined, we have to show that $V = 0$ a.e. on S. Let $g_k := E_1 \circ E_0 f_k \in W^{1,p}(\mathbb{R}^n)$; let $h_k \in C_b^\infty(\mathbb{R}^n)$, $\|h_k - g_k\|_{W^{1,p}(\mathbb{R}^n)} < \frac{1}{k}$. Then we have

$$\|h_k\|_{L^p(\mathbb{R}^n)} < C_p(1 + c_p)\|f_k\|_{L^p(S)} + \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0;$$

$$\|\nabla(h_k - h_m)\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} < C_p(1 + c_p)\|\nabla(f_k - f_m)\|_{L^p(S,\mathbb{R}^n)} + \frac{1}{m} + \frac{1}{k} \underset{m,k\to\infty}{\longrightarrow} 0.$$

By the closability of Sobolev gradients in $\mathbb{R}^n$ this implies that $\|\nabla h_k\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} \to 0$. Since $f_k = g_k$ on S, we have

$$\|\nabla f_k\|_{L^p(S,\mathbb{R}^n)} \le \|\nabla g_k\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} < \|\nabla h_k\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} + \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0,$$

which implies $V = 0$ a.e. on S. Therefore, the first order Sobolev gradient is well-defined for functions from $W^{1,p}(S)$, $p \ge 1$. In particular, if we take $p = 2$, this implies the closability of the Dirichlet form (2.9). □

3 Construction of a counter-example

In this section we apply Theorem 2.3 to the plane $\mathbb{R}^2$ and the open squares

$$G = (-3; 3) \times (-3; 3), \quad G_0 = (-2; 2) \times (-2; 2).$$

Lemma 3.1. *Given a rectangle $\Pi = [a; b] \times (0; 1)$, $0 < b - a \le 1$, there exists a set of squares*

$$\mathcal{M}_{a,b} = \left\{ (x_k - 2\varepsilon; x_k + 2\varepsilon) \times (y_k - 2\varepsilon; y_k + 2\varepsilon) \right\}_{k=0}^{m}$$

belonging to Π such that
1) $(x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon) \subset \Pi$;
2) $(x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)$, $k = 0, 1, \dots m$, are disjoint;
3) for any $h \in [1/4; 3/4]$, the line $\{y = h\}$ intersects some square from $\mathcal{M}_{a,b}$;
4) let $S := \left([a; b] \times [\frac{1}{4}; \frac{3}{4}]\right) \setminus \bigcup_{k=0}^{m} ((x_k - 2\varepsilon; x_k + 2\varepsilon) \times (y_k - 2\varepsilon; y_k + 2\varepsilon))$. Then there exists a function $g_{a,b} \in C_b^\infty(\mathbb{R}^2)$ with values in $[0; 1]$ such that

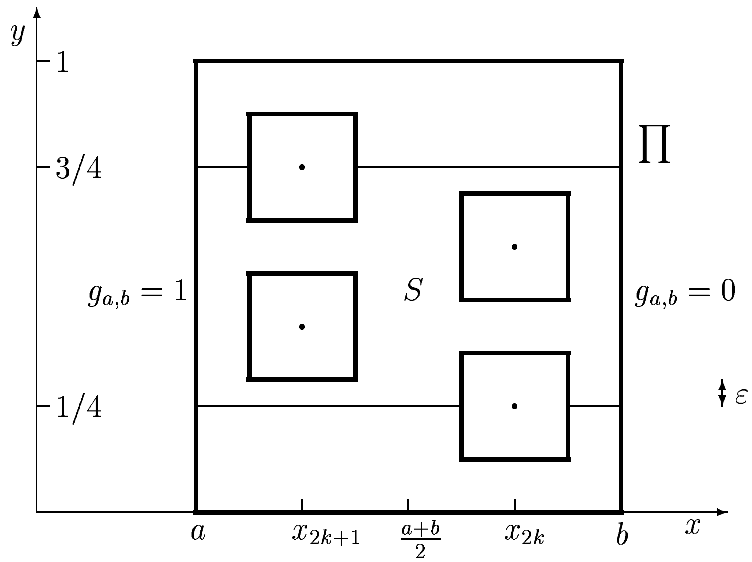

Figure 1: to Lemma 3.1.

a) $g_{a,b}(a,y) = 1$, $g_{a,b}(b,y) = 0$, *for any* y*;*
b) $\frac{\partial}{\partial x} g_{a,b} = 0$ *on* S*;*
c) $\left|\frac{\partial}{\partial x} g_{a,b}\right| \le \frac{9}{b-a}$.

Proof. Choose a natural number $m \in \left[\frac{2}{b-a}; \frac{3}{b-a}\right]$ and set

$$\varepsilon := \frac{1}{6m}; \quad x_k := \frac{a+b}{2} + (-1)^k \cdot \frac{b-a}{4}, \quad y_k := \frac{1}{4} + \frac{k}{2m} = \frac{1}{4} + k \cdot 3\varepsilon; \quad k = 0, 1, \dots, m.$$

Fig. 1 shows the location of our squares in the case $m = 3$. Condition 1) is fulfilled because for any point $(x,y) \in (x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)$ one has

$$\left|x - \frac{a+b}{2}\right| < \frac{b-a}{4} + 3\varepsilon = \frac{b-a}{4} + \frac{1}{2m} \le \frac{b-a}{2},$$

$$0 \le \frac{1}{4} - \frac{1}{2m} = \frac{1}{4} - 3\varepsilon < y < \frac{1}{4} + \frac{m}{2m} + 3\varepsilon = \frac{3}{4} + \frac{1}{2m} \le 1.$$

In order to check 2), we note that the squares $(x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)$ with even k lie in the region $\{x > (a+b)/2\}$ and those with odd k lie in the region $\{x < (a+b)/2\}$. If we take j and k such that $|k - j| \ge 2$, then $|y_k - y_j| \ge 1/m = 6\varepsilon$; due to this estimate the two squares are disjoint. Condition 3) follows from the containment

$$\left[\frac{1}{4}; \frac{3}{4}\right] \subset \bigcup_{k=0}^{m} (y_k - 2\varepsilon; y_k + 2\varepsilon). \tag{3.10}$$

4) It follows from (3.10) that there exists a set of nonnegative functions

$$\chi_0, \chi_1, \dots, \chi_m \in C_b^\infty(\mathbb{R})$$

with $\mathrm{supp}\chi_k = (-\infty; y_0 + 2\varepsilon]$ if $k = 0$, $[y_k - 2\varepsilon; y_k + 2\varepsilon]$ if $0 < k < m$, and $[y_m - 2\varepsilon; +\infty)$ if $k = m$, such that

$$\sum_{k=0}^{m} \chi_k(y) = 1 \quad \forall y \in \mathbb{R}.$$

Let ζ_α^β denote a function from the class $C_0^\infty(\mathbb{R})$ such that

$$\mathrm{supp}\zeta_\alpha^\beta = [\alpha; \beta], \quad 0 \le \zeta_\alpha^\beta \le \frac{2}{\beta - \alpha}, \quad \int \zeta_\alpha^\beta(x)dx = 1.$$

Put

$$\varphi(x, y) := \sum_{k=0}^{m} \zeta_{x_k - 2\varepsilon}^{x_k + 2\varepsilon}(x) \cdot \chi_k(y).$$

Then $\varphi \in C_b^\infty(\mathbb{R}^2)$, $\varphi = 0$ on the set S, and

$$\int_a^b \varphi(x, y)dx = \sum_{k=0}^{m} \chi_k(y) \int_a^b \zeta_{x_k - 2\varepsilon}^{x_k + 2\varepsilon}(x) = 1, \quad \forall y \in \mathbb{R}.$$

Note that

$$\sup_{x,y} |\varphi(x, y)| \le \sup_x \left|\zeta_{x_k - 2\varepsilon}^{x_k + 2\varepsilon}(x)\right| \le \frac{1}{2\varepsilon}.$$

Finally, let

$$g_{a,b}(x, y) := \int_x^b \varphi(t, y)dt.$$

This function belongs to $C_b^\infty(\mathbb{R}^2)$ and satisfies a), b), and c), since $\frac{1}{2\varepsilon} = 3m \le \frac{9}{b-a}$. □

Now let us recall the construction of a *"thick Cantor set"*.

Step 1. Begin with $I := [0; 1]$. In the center of I take the interval $(a; b)$ of the length 4^{-1}, i.e., $a = 3/8$, $b = 5/8$. Then $I \setminus (a; b)$ splits into two closed intervals $I_0 = [0; 3/8]$ and $I_1 = [5/8; 1]$.

Step 2. In the center of I_0 take the interval $(a_0; b_0)$ of the length 4^{-2}, then $I_0 \setminus (a_0; b_0)$ splits into I_{00} and I_{01}. In the center of I_1 take the interval $(a_1; b_1)$ of the length 4^{-2}, then $I_1 \setminus (a_1; b_1)$ splits into I_{10} and I_{11}.

Step 3. In the centers of I_{00}, I_{01}, I_{10}, I_{11}, take the intervals $(a_{00}; b_{00})$, $(a_{01}; b_{01})$, $(a_{10}; b_{10})$, $(a_{11}; b_{11})$, respectively, each one having the length 4^{-3}; thus we obtain 8 closed intervals

$$I_{000},\ I_{001},\ I_{010},\ I_{011},\ I_{100},\ I_{101},\ I_{110},\ I_{111}.$$

Then we proceed inductively. The closed intervals of the n-th generation are shorter than 2^{-n}, but the limiting compact set

$$\mathbf{K} = I \setminus (a; b) \setminus (a_0; b_0) \setminus (a_1; b_1) \setminus (a_{00}; b_{00}) \setminus (a_{01}; b_{01}) \setminus (a_{10}; b_{10}) \setminus (a_{11}; b_{11}) \setminus \dots$$

has Lebesgue measure 1/2.

Now we shall give an example of a measure μ such that the Dirichlet form $\mathcal{E}(f,g) = \int(\nabla f, \nabla g)d\mu$ is closable on $L^2(\mu)$, but the partial Dirichlet form $\mathcal{E}_x(f,g) = \int \partial_x f \partial_x g d\mu$ is not. This measure is the restriction of Lebesgue measure to a certain set F.

Example 3.2. Let $Q = (0;1) \times (0;1)$. By using Lemma 3.1 we construct the countable set of squares

$$\mathcal{M} = \mathcal{M}_{a,b} \cup \bigcup_{n=1}^{\infty} \left(\bigcup_{i_1=0}^{1} \cdots \bigcup_{i_n=0}^{1} \mathcal{M}_{a_{i_1 \dots i_n}, b_{i_1 \dots i_n}} \right).$$

Set

$$F := Q \setminus \bigcup_{K \in \mathcal{M}} \overline{K}.$$

By construction of the intervals (a_*, b_*) and the squares from $\mathcal{M}_{a_*,b_*}$, it is obvious that the squares from $\mathcal{M}$ satisfy the hypotheses of Theorem 2.3. It follows from the theorem that the Dirichlet form

$$\mathcal{E}(f,g) = \int_F (\nabla f, \nabla g)\, d\lambda^2$$

is closable on $L^2(\lambda^2 \mid F)$. Next we shall prove that the Dirichlet form

$$\mathcal{E}_x(f,g) = \int_F \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} d\lambda^2$$

fails to be closable on $L^2(\lambda^2 \mid_F)$. To this end, we construct a sequence of functions $f_n \in C_b^\infty(\mathbb{R}^2)$ such that $\{f_n\}$ converges to zero in $L^2(F) := L^2(\lambda^2 \mid_F)$, but $\partial f_n / \partial x$ converge in $L^2(F)$ to a function that differs from zero on a set of positive Lebesgue measure.

Denote by $\xi^\varepsilon_{(s;t)}$ a function from $C_b^\infty(\mathbb{R})$ with the following properties:

$\xi^\varepsilon_{(s;t)}(x) = 0$ if $x \le s$;

$\left(\xi^\varepsilon_{(s;t)}\right)'(x) = 0$ if $x \ge t$;

$\left(\xi^\varepsilon_{(s;t)}\right)'(x) = 1$ if $s + \varepsilon(t-s) \le x \le t - \varepsilon(t-s)$;

$0 \le \left(\xi^\varepsilon_{(s;t)}\right)'(x) \le 1$ for any x.

Fix a function $\theta \in C_0^\infty(\mathbb{R})$ with the support $[1/4;3/4]$ such that $1 \ge \theta(y) > 0$ if $1/4 < y < 3/4$. After these preparations, we begin to construct the functions f_n. We set

$$f_1(x,y) := \begin{cases} \theta(y)\xi^{4^{-1}}_{I_0}(x) & \text{if} \quad x \in I_0, \\ \theta(y)\xi^{4^{-1}}_{I_0}(a) g_{a,b}(x,y) & \text{if} \quad x \in (a;b), \\ \theta(y)\xi^{4^{-1}}_{I_1}(x) & \text{if} \quad x \in I_1; \end{cases}$$

$$f_2(x,y) := \begin{cases} \theta(y)\xi_{I_{00}}^{4^{-2}}(x) & \text{if} \quad x \in I_{00}, \\ \theta(y)\xi_{I_{00}}^{4^{-2}}(a_0)g_{a_0,b_0}(x,y) & \text{if} \quad x \in (a_0;b_0), \\ \theta(y)\xi_{I_{01}}^{4^{-2}}(x) & \text{if} \quad x \in I_{01}, \\ \theta(y)\xi_{I_{01}}^{4^{-2}}(a)g_{a,b}(x,y) & \text{if} \quad x \in (a;b), \\ \theta(y)\xi_{I_{10}}^{4^{-2}}(x) & \text{if} \quad x \in I_{10}, \\ \theta(y)\xi_{I_{10}}^{4^{-2}}(a_1)g_{a_1,b_1}(x,y) & \text{if} \quad x \in (a_1;b_1), \\ \theta(y)\xi_{I_{11}}^{4^{-2}}(x) & \text{if} \quad x \in I_{11}. \end{cases}$$

Then we proceed inductively. In the process of construction of f_n we use the functions $\xi_{I_*}^{4^{-n}}$ corresponding to the closed intervals of the n-th generation and the functions g_{a_*,b_*} corresponding to the intervals that we have taken in the first n steps (see the construction of the *"thick Cantor set"*). It is easy to check the following properties of the functions f_n:

$$0 \leq f_n(x,y) < 2^{-n}, \tag{3.11}$$

$$0 \leq \frac{\partial f_n(x,y)}{\partial x} \leq 1 \quad \text{on} \quad F, \tag{3.12}$$

on the squares from $\mathcal{M}_{a_{i_1\ldots i_k},b_{i_1\ldots i_k}}$ one has

$$\left|\frac{\partial f_n(x,y)}{\partial x}\right| < \frac{9}{4^{-(k+1)}}, \qquad k = 0,1,2,\ldots, \tag{3.13}$$

f_n with all its derivatives vanishes on the lines $\{x=0\}$ and $\{y=0\}$, and, finally,

$$\text{for a.e. } (x,y) \in Q \qquad \lim_{n\to\infty} \frac{\partial f_n(x,y)}{\partial x} = V(x,y) = \begin{cases} \theta(y) & \text{if} \quad x \in \mathbf{K}, \\ 0 & \text{if} \quad x \notin \mathbf{K}. \end{cases} \tag{3.14}$$

Therefore,

$$f_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{\partial f_n}{\partial x} \underset{n\to\infty}{\longrightarrow} V \quad \text{in } L^2(F).$$

But $V(x,y) > 0$ on the set $\mathbf{K}\times(1/4;3/4)$ that has measure $1/4$. This completes our example.

The measure constructed in Example 3.2 is supported by a set with "holes". However, it is possible to refine this measure so that its support will coincide with the square Q.

Example 3.3. Set $\mu := \rho \cdot \lambda^2$, where

$$\rho(x,y) = \begin{cases} 1 & \text{if } (x,y) \in F; \\ 16^{-1} & \text{if } (x,y) \in K \in \mathcal{M}_{a,b}; \\ 16^{-(m+1)} & \text{if } (x,y) \in K \in \mathcal{M}_{a_{i_1\ldots i_m},b_{i_1\ldots i_m}}, \ m \in \mathbb{N}. \end{cases}$$

The Dirichlet form

$$\mathcal{E}'(f,g) = \int (\nabla f, \nabla g)\, d\mu$$

is closable on $L^2(\mu)$ because it is the sum of the form $\mathcal{E}$ from Example 3.2 and the classical Dirichlet forms on the squares $K \in \mathcal{M}$. In order to prove non-closability of the form

$$\mathcal{E}'_x(f,g) = \int \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x}\, d\mu$$

on $L^2(\mu)$, consider the same sequence of functions $f_n \in C_b^\infty(\mathbb{R}^2)$ as in Example 3.2. It follows from (3.11) that $\|f_n\|_{L^2(\mu)} \underset{n\to\infty}{\longrightarrow} 0$. From the estimates (3.12) and (3.13) we see that the functions $|\partial f_n/\partial x|$ are majorized by the function

$$M(x,y) = \begin{cases} 1 & \text{if } (x,y) \in F, \\ 9/4^{-(m+1)} & \text{if } (x,y) \in K \in \mathcal{M}_{a_{i_1\ldots i_m}, b_{i_1\ldots i_m}},\ m = 0,1,2,\ldots, \end{cases}$$

but

$$\|M\|^2_{L^2(\mu)} \le 1 + \sum_{m=0}^{\infty} 16^{-(m+1)} \sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \left(\frac{9}{4^{-(m+1)}}\right)^2 \sum_{K \in \mathcal{M}_{a_{i_1\ldots i_m}, b_{i_1\ldots i_m}}} \lambda^2(K)$$

$$\le 1 + \sum_{m=0}^{\infty} 16^{-(m+1)} \cdot 2^m \cdot \left(\frac{9}{4^{-(m+1)}}\right)^2 \cdot 4^{-(m+1)} = 1 + \frac{81}{4} \sum_{m=0}^{\infty} 2^{-m} < 42.$$

Since

$$\frac{\partial f_n(x,y)}{\partial x} \underset{n\to\infty}{\longrightarrow} V(x,y) \quad \text{for } \lambda^2\text{-a.e.} \quad (x,y) \in Q,$$

by the Lebesgue dominated convergence theorem we have

$$V \in L^2(\mu), \qquad \left\| \frac{\partial f_n}{\partial x} - V \right\|_{L^2(\mu)} \underset{n\to\infty}{\longrightarrow} 0,$$

therefore, since $\mu\{(x,y):\ V(x,y) > 0\} > 0$, the form $\mathcal{E}'_x$ is not closable on $L^2(\mu)$.

Now we shall extend the result of this paper to the d-dimensional case.

Corollary 3.4. *Let $d \in \mathbb{N}$. There exists a measure ν on $\mathbb{R}^d$ such that the Dirichlet form*

$$\mathcal{E}^\nu_{h_1,\ldots,h_d}(f,g) = \sum_{k=1}^{d} h_k \int \frac{\partial f(x)}{\partial x_k} \cdot \frac{\partial g(x)}{\partial x_k} \nu(dx), \qquad h_1,\ldots,h_d \ge 0, \tag{3.15}$$

is closable on $L^2(\nu)$ if and only if $h_1 \cdot \ldots \cdot h_d > 0$.

Proof. Let F be the set constructed in Example 3.2. Set

$$F_{ij} := \left\{x = (x_1, ...x_d) \in (0;1)^d : \ (x_i, x_j) \in F\right\}, \qquad i, j = 1, \ldots, d, \quad i \neq j.$$

Denote by ν_{ij} the restriction of the d-dimensional Lebesgue measure to the set F_{ij}. Then the Dirichlet form $\mathcal{E}^{\nu_{ij}}_{h_1,\ldots,h_d}$ is closable on $L^2(\nu_{ij})$ if $h_1 \cdot \ldots \cdot h_d > 0$. Indeed, it is sufficient to consider the gradient Dirichlet form

$$\mathcal{E}^{\nu_{ij}}_{1,\ldots,1}(f,g) = \int (\nabla f(x), \nabla g(x)) \nu_{ij}(dx) \tag{3.16}$$

because one has the following estimate with some $A > a > 0$:

$$a \cdot \mathcal{E}^{\nu_{ij}}_{1,\ldots,1}(f,f) \leq \mathcal{E}^{\nu_{ij}}_{h_1,\ldots,h_d}(f,f) \leq A \cdot \mathcal{E}^{\nu_{ij}}_{1,\ldots,1}(f,f).$$

But $\nu_{ij} = \mu_{ij} \times m_{ij}$, where

$$m_{ij} = \bigotimes_{\substack{k=1 \\ k \neq i,j}}^{d} dx_k \quad \text{and} \quad \mu_{ij} = \mu(dx_i, dx_j),$$

where μ is the restriction of the two-dimensional Lebesgue measure to F. For both measures m_{ij} (on $\mathbb{R}^{n-2}$) and μ_{ij} (on $\mathbb{R}^2$) the gradient Dirichlet forms are closable, therefore, so is (3.16).

Take the measure

$$\nu := \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \widehat{\nu}_{ij},$$

where $\widehat{\nu}_{ij}$ is the image of ν_{ij} under the parallel shift along the vector $2e_i + 4e_j$. For any $h_1, \ldots, h_d$, $h_1 \cdot ... \cdot h_d > 0$, the Dirichlet form (3.15) is closable on $L^2(\nu)$ as the sum of forms closable over measures whose sum is ν. If $h_j = 0$ and $h_i > 0$, we can take the sequence of functions

$$\widetilde{f}_n(x) := \begin{cases} f_n(x_i - 2, x_j - 4) & \text{if} \quad x_i > 2, \ x_j > 4, \\ 0 & \text{if} \quad x_i \leq 2 \text{ or } x_j \leq 4, \end{cases}$$

where f_n is the function from Example 3.2. Then $\widetilde{f}_n \in C_b^\infty(\mathbb{R}^d)$; $\|\widetilde{f}_n\|_{L^2(\nu)} \underset{n\to\infty}{\longrightarrow} 0$; for $k \neq i$, $k \neq j$, we have $\frac{\partial \widetilde{f}_n}{\partial x_k} \equiv 0$; but

$$\frac{\partial \widetilde{f}_n(x)}{\partial x_i} \overset{L^2(\nu)}{\underset{n\to\infty}{\longrightarrow}} u(x) = \begin{cases} V(x_i - 2, x_j - 4) & \text{if } x_i > 2, \ x_j > 4, \\ 0 & \text{if } x_i \leq 2 \text{ or } x_j \leq 4, \end{cases}$$

where V is the function defined in (3.14) in Example 3.2. We have

$$\mathcal{E}^{\nu}_{h_1,\ldots,h_d}(\widetilde{f}_n - \widetilde{f}_m, \widetilde{f}_n - \widetilde{f}_m) = h_i \int \left| \frac{\partial(\widetilde{f}_n(x) - \widetilde{f}_m(x))}{\partial x_i} \right|^2 \nu(dx)$$

$$= h_i \int\limits_F \left| \frac{\partial (f_n(x_i, x_j) - f_m(x_i, x_j))}{\partial x_i} \right|^2 dx_i dx_j \underset{m,n\to\infty}{\longrightarrow} 0.$$

But $\nu\{u > 0\} = \mu\{V > 0\} = \frac{1}{4}$, hence

$$\mathcal{E}^{\nu}_{h_1,\ldots,h_d}(\widetilde{f}_n, \widetilde{f}_n) \underset{n\to\infty}{\longrightarrow} h_i \int\limits_F V^2(x_i, x_j) dx_i dx_j > 0.$$

Therefore, the Dirichlet form (3.15) is not closable on $L^2(\nu)$. □

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