On closability of classical
Dirichlet forms

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Abstract. We construct a measure $\mu$ on $\mathbb{R}^2$ such that the classical Dirichlet form $\mathcal{E}(f, g) = \int (\nabla f, \nabla g) d\mu$ is closable, but the partial Dirichlet form $\mathcal{E}_x(f, g) = \int \partial_x f \partial_x g d\mu$ is not. This proves the well-known conjecture of M. Röckner.

1 Introduction

The problem of closability of Dirichlet forms arises in the theory of differential operators, the theory of Sobolev spaces, and stochastic analysis (see [1], [4], [5], [7], [8], [9]).

Let $(X, \mathcal{B}, \mu)$ be a measurable space with positive measure $\mu$; $L^2(\mu) = L^2(X, \mathcal{B}, \mu)$. Recall the definition of closability of a Dirichlet form $\mathcal{E}$ defined on the domain $\mathcal{D} \subset L^2(\mu)$ ([7, p. 28]).

Definition 1.1. The form $\mathcal{E}$ is said to be closable on $L^2(\mu)$ if for any sequence of functions $f_n \in \mathcal{D}$ such that $\|f_n\|_{L^2(\mu)} \to 0$ and $\mathcal{E}(f_n - f_m, f_n - f_m) \to 0$, it follows that $\mathcal{E}(f_n, f_n) \to 0$.

In applications, the closability of gradient Dirichlet forms on finite- and infinite-dimensional linear spaces and manifolds is often verified by the aid of the following fact: if for some measure $\mu$ on $\mathbb{R}^n$ the partial Dirichlet forms

$$\mathcal{E}_i(f, g) = \int \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \mu(dx)$$

are closable on $L^2(\mu)$ for every $i = 1, \ldots n$, then the gradient Dirichlet form

$$\mathcal{E}(f, g) = \sum_{i=1}^n \mathcal{E}_i(f, g)$$

is closable on $L^2(\mu)$ as well; the same is true also in infinite dimensions (cf. [2, Theorem 3.2]). A necessary and sufficient condition of closability of (1.1) can be found in [2, Theorem 5.3]. In this relation, the following non-trivial question arose: is it true that if the form (1.2) is closable on $L^2(\mu)$ then the partial Dirichlet forms (1.1) are also closable.
on $L^2(\mu)$? About 10 years ago M. Röckner conjectured that it is not always true, but no counter-example was known.

The main aim of this paper is to show that even in the case of $\mathbb{R}^2$ the answer is negative (cf. Example 3.2 below). Moreover, the measure $\mu$ constructed as a counter-example is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$ and can even have full support (cf. Example 3.3). A positive result of this paper (employed in the justification of our counter-example) is a new sufficient condition for the closability of gradient Dirichlet forms for a special class of measures on $\mathbb{R}^n$ (see Theorem 2.3 below).

2 Extensions of Sobolev functions

Let $U$ be an open set in $\mathbb{R}^n$ and let $p \geq 1$. The Sobolev class $W^{1,p}(U)$ is defined as the completion of the space $\{f |_{U}: f \in C_{0}^{\infty}(\mathbb{R}^n); \|f\|_{p,1} < \infty\}$ with respect to the Sobolev norm

$$\|f\|_{p,1} = \left( \int_{U} \left( |f| + |\nabla f| \right)^{p} \, d\lambda^{n} \right)^{1/p},$$

where $\lambda^{n}$ is Lebesgue measure on $\mathbb{R}^n$.

Let us recall the following theorem about extension of Sobolev functions (see [10, Ch. VI, § 3, Theorem 5] and [6]).

**Theorem 2.1.** Let $Q$ be an open set in $\mathbb{R}^n$ such that there exist numbers $\varepsilon, M, N > 0$, and a finite or countable family of open sets $\{V_i\}$ with the following properties:

1) the $\varepsilon$-neighborhood of any point $x \in Q$ lies in some $V_i$;
2) $\sum_{i} 1_{V_i} \leq N$;
3) for any $i$ there exists a set $Q_i$ that is isometric to some open set of the form $\{x_n < \varphi_i(x_1, \ldots, x_{n-1})\}$, where $\varphi_i$ is a Lipschitzian function with the Lipschitzian constant $M$, such that $V_i \cap Q = V_i \cap Q_i$.

Then, for any $p \geq 1$, there exists a linear operator $E_1$ of extension from $W^{1,p}(Q)$ to $W^{1,p}({\mathbb{R}^n})$ such that

$$E_1 f\, |_{Q} = f, \quad \|E_1 f\|_{L^p(\mathbb{R}^n)} \leq c_p \cdot \|f\|_{L^p(Q)};$$

$$\|\nabla (E_1 f)\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(Q, \mathbb{R}^n)}.$$

**Lemma 2.2.** Let $G$ be a connected open set in $\mathbb{R}^n$ and let $G_0$ be an open set such that $\overline{G_0} \subset G$ and the open set $Q = G \setminus \overline{G_0}$ satisfies the hypotheses of Theorem 2.1. Denote by $T$ the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n, \quad T(x) = y + r \cdot J(x), \quad (2.3)$$

where $y \in \mathbb{R}^n$ is a fixed vector, $r > 0$, and $J$ is an orthogonal linear operator on $\mathbb{R}^n$. Then, for any $p \geq 1$, there exists a linear operator $E_T$ of extension from $W^{1,p}(T(G \setminus \overline{G_0}))$ to $W^{1,p}(T(G))$ such that

$$E_T f \, |_{T(G \setminus \overline{G_0})} = f;$$

2
\[ \|E_T f\|_{L^p(T (G))} \leq c_p \cdot \|f\|_{L^p(T (G \setminus G_0))}, \]  
(2.4)

\[ \|\nabla (E_T f)\|_{L^p(T (G), \mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(T (G \setminus G_0), \mathbb{R}^n)} , \]  
(2.5)

where the constant \( c_p \) depends only on \( G \) and \( G_0 \) and does not depend on \( y, r, \) and \( J. \)

**Proof.** By Theorem 2.1 there exists an operator \( E_1 \) of extension from \( W^{1, p} (G \setminus G_0) \) to \( W^{1, p} (\mathbb{R}^n) \) with

\[ \|E_1 f\|_{L^p(\mathbb{R}^n)} \leq c_p \cdot \|f\|_{L^p(G \setminus G_0)} \]  
(2.6)

\[ \|\nabla (E_1 f)\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \leq c_p \cdot \|\nabla f\|_{L^p(G \setminus G_0, \mathbb{R}^n)} . \]  
(2.7)

Set \( E f := E_1 f \mid_G. \) Then estimates (2.6) and (2.7) hold also for the corresponding norms of \( E f \) on \( G. \) Let us consider the operator

\[ H : W^{1, p} (G) \rightarrow W^{1, p} (T (G)), \quad H f (T x) = f(x). \]

It is easy to see that \( |\nabla (H f) (T x)| = \frac{1}{r} |\nabla f (x)|, \) and since the Jacobian of the transformation \( T \) equals \( r^n, \) we have

\[ \|H f\|_{L^p(T (G))} = r^{n/p} \cdot \|f\|_{L^p(G)}, \quad \|\nabla (H f)\|_{L^p(T (G), \mathbb{R}^n)} = r^{n/p-1} \cdot \|\nabla f\|_{L^p(\mathbb{R}^n)}. \]  
(2.8)

The same equalities are true for the \( L^p \)-norms on \( T (G \setminus G_0) \) and \( G \setminus G_0, \) respectively. Now we define the operator of extension from \( W^{1, p} (T (G \setminus G_0)) \) to \( W^{1, p} (T (G)) \) by

\[ E_T := H \circ E \circ H^{-1}. \]

Finally, from equalities (2.8) and estimates (2.6) and (2.7) we obtain the desired estimates (2.4) and (2.5). \( \square \)

From now on we shall assume that, if a Dirichlet form is defined on some subclass \( \mathcal{D} \) of \( L^2 (\mu), \) \( \mu \) being a measure on \( \mathbb{R}^d, \) then

\[ \mathcal{D} = \{ \varphi \mid_{\text{supp} \mu} : \ \varphi \in C_b^{\infty} (\mathbb{R}^d) \}. \]

**Theorem 2.3.** Let \( Q \) be an open set in \( \mathbb{R}^n \) satisfying the hypotheses of Theorem 2.1. Let the connected open sets \( G \) and \( G_0 \subset G \) satisfy the hypotheses of Lemma 2.2. Consider a countable family of transformations \( \{ T_k \} \) of the form (2.3) such that \( T_k (G) \subset Q \) and \( T_k (G) \cap T_j (G) = \emptyset \) if \( k \neq j. \) Set

\[ S := Q \setminus \bigcup_{k=1}^{\infty} T_k (G_0) . \]

Then

(i) the Dirichlet form
\[ \mathcal{E}'(f, g) = \int_S (\nabla f, \nabla g) \, d\lambda^n \]  

is well-defined and closable on \( L^2(\lambda^n | S) \),

(ii) the Sobolev classes \( W^{1,p}(S) \), \( p \geq 1 \), are well-defined.

**Proof.** Fix \( p \in [1; +\infty) \). Note that the Sobolev gradient \( \nabla f \in L^p(S, \mathbb{R}^n) \) is well-defined for smooth functions, since if \( f \in C_c^\infty(\mathbb{R}^n) \) and \( f = 0 \) on \( S \), then \( \nabla f = 0 \) on \( S \) a.e.

Denote by \( E_{T_k} \) the operator of extension of functions from \( W^{1,p}(T_k(G \setminus G_0)) \) to \( W^{1,p}(T_k(G)) \) constructed in Lemma 2.2. Let \( f \in C_c^\infty(\mathbb{R}^n) \). Put

\[
E_0f = \begin{cases} 
    f(x), & x \in S, \\
    \left( E_{T_k}(f |_{T_k(G \setminus G_0)}) \right)(x), & x \in T_k(G_0), \ k \in \mathbb{N}; 
\end{cases}
\]

\[
\psi_j = \begin{cases} 
    f(x), & x \in Q \setminus \bigcup_{k=1}^j T_k(G_0), \\
    \left( E_{T_k}(f |_{T_k(G \setminus G_0)}) \right)(x), & x \in T_k(G_0), \ k = 1, \ldots, j. 
\end{cases}
\]

The function \( \psi_j \) belongs to the Sobolev class \( W^{1,p}(Q) \). Indeed, it belongs to \( W^{1,p}(T_k(G_0)) \) for \( k = 1, \ldots, j \); let \( f_m^0, \ldots, f_m^j, m \in \mathbb{N} \), be \( C_c^\infty \)-functions approximating \( \psi_j \) in the \( \| \cdot \|_{1,p} \) norm on \( T_k(G) \), \( k = 1, \ldots, j \), and \( \chi_0, \ldots, \chi_j \in C^\infty_0 \) be such that \( 0 \leq \chi_k \leq 1 \), \( \sum_{k=0}^j \chi_k = 1 \), and \( \chi_k = 1 \) on \( T_k(G_0), k = 1, \ldots, j \). It is easy to check that the functions \( f_m^j = \sum_{k=0}^j \chi_k f_m \)

coincide with \( \psi_j \) on \( Q \setminus \bigcup_{k=1}^j T_k(G_0) \) and approximate \( \psi_j \) in \( \| \cdot \|_{1,p} \) on \( Q \) as \( m \to \infty \).

We have

\[
\| E_0f \|_{L^p(Q)} \leq \| f \|_{L^p(S)} + \sum_{k} \| E_{T_k}f \|_{L^p(T_k(G))} \\
\leq \| f \|_{L^p(S)} + c_p \cdot \sum_{k} \| f \|_{L^p(T_k(G \setminus G_0))} \leq (1 + c_p) \cdot \| f \|_{L^p(S)},
\]

since the sets \( T_k(G) \) are disjoint; by analogy,

\[
\| \nabla \psi_j \|_{L^p(Q, \mathbb{R}^n)} \leq \| \nabla f \|_{L^p(S, \mathbb{R}^n)} + \sum_{k} \| \nabla (E_{T_k}f) \|_{L^p(T_k(G), \mathbb{R}^n)} \\
\leq \| \nabla f \|_{L^p(S, \mathbb{R}^n)} + c_p \cdot \sum_{k} \| \nabla f \|_{L^p(T_k(G \setminus G_0), \mathbb{R}^n)} \leq (1 + c_p) \cdot \| \nabla f \|_{L^p(S, \mathbb{R}^n)}.
\]

We also have

\[
|\psi_j(x)|^p \leq |E_0f(x)|^p + |f(x)|^p \quad \text{and} \quad \psi_j(x) \xrightarrow{j \to \infty} E_0f(x) \quad \text{for a.e.} \quad x \in Q.
\]

In addition, \( \nabla \psi_j(x) \) converge a.e. on \( Q \) as \( j \to \infty \). Therefore, \( E_0f \in W^{1,p}(Q) \).
Next we apply the operator $E_1$ of extension from $W^{1,p}(Q)$ to $W^{1,p}(\mathbb{R}^n)$ (which increases $W^{1,p}$-norms not more than in $C_p$ times). We obtain
\[
\|E_1 \circ E_0 f\|_{L^p(\mathbb{R}^n)} \leq C_p \cdot (1 + c_p)\|f\|_{L^p(S)};
\]
\[
\|
abla (E_1 \circ E_0 f)\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \leq C_p \cdot (1 + c_p)\|
abla f\|_{L^p(S, \mathbb{R}^n)}.
\]
Suppose we have a mapping $V \in L^p(S, \mathbb{R}^n)$ and a sequence of smooth functions $f_k \in C_0^\infty(\mathbb{R}^n)$ with
\[
\|f_k\|_{L^p(S)} \to 0, \quad \|
abla f_k - V\|_{L^p(S, \mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.
\]
In order to prove that the Sobolev class $W^{1,p}(S)$ is well-defined, we have to show that $V = 0$ a.e. on $S$. Let $g_k := E_1 \circ E_0 f_k \in W^{1,p}(\mathbb{R}^n)$; let $h_k \in C_0^\infty(\mathbb{R}^n)$, $\|h_k - g_k\|_{W^{1,p}(\mathbb{R}^n)} < \frac{1}{k}$. Then we have
\[
\|h_k\|_{L^p(\mathbb{R}^n)} < C_p(1 + c_p)\|f_k\|_{L^p(S)} + \frac{1}{k} \quad \overset{k \to \infty}{\longrightarrow} \quad 0;
\]
\[
\|
abla (h_k - h_m)\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} < C_p(1 + c_p)\|
abla (f_k - f_m)\|_{L^p(S, \mathbb{R}^n)} + \frac{1}{m} + \frac{1}{k} \quad \overset{m,k \to \infty}{\longrightarrow} \quad 0.
\]
By the closability of Sobolev gradients in $\mathbb{R}^n$ this implies that $\|
abla h_k\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} \to 0$. Since $f_k = g_k$ on $S$, we have
\[
\|
abla f_k\|_{L^p(S, \mathbb{R}^n)} \leq \|
abla g_k\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} < \|
abla h_k\|_{L^p(\mathbb{R}^n, \mathbb{R}^n)} + \frac{1}{k} \quad \overset{k \to \infty}{\longrightarrow} \quad 0,
\]
which implies $V = 0$ a.e. on $S$. Therefore, the first order Sobolev gradient is well-defined for functions from $W^{1,p}(S)$, $p \geq 1$. In particular, if we take $p = 2$, this implies the closability of the Dirichlet form (2.9). \hfill \square

3 Construction of a counter-example

In this section we apply Theorem 2.3 to the plane $\mathbb{R}^2$ and the open squares
\[
G = (-3;3) \times (-3;3), \quad G_0 = (-2;2) \times (-2;2).
\]

Lemma 3.1. Given a rectangle $\Pi = [a;b] \times (0;1)$, $0 < b - a \leq 1$, there exists a set of squares
\[
\mathcal{M}_{a,b} = \left\{ (x_k - 2\varepsilon; x_k + 2\varepsilon) \times (y_k - 2\varepsilon; y_k + 2\varepsilon) \right\}_{k=0}^m
\]
belonging to $\Pi$ such that
1) $(x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon) \subset \Pi$;
2) $(x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)$, $k = 0, 1, \ldots, m$, are disjoint;
3) for any $h \in [1/4; 3/4]$, the line $\{y = h\}$ intersects some square from $\mathcal{M}_{a,b}$;
4) let $S := ([a;b] \times [\frac{1}{3}; \frac{4}{3}]) \setminus \bigcup_{k=0}^m ((x_k - 2\varepsilon; x_k + 2\varepsilon) \times (y_k - 2\varepsilon; y_k + 2\varepsilon))$. Then there exists a function $g_{a,b} \in C_0^\infty(\mathbb{R}^2)$ with values in $[0;1]$ such that
\begin{align*}
a) \quad & g_{a,b}(a, y) = 1, \quad g_{a,b}(b, y) = 0, \quad \text{for any } y; \\
b) \quad & \frac{\partial}{\partial y} g_{a,b} = 0 \quad \text{on } S; \\
c) \quad & \left| \frac{\partial}{\partial x} g_{a,b} \right| \leq \frac{9}{b-a}.
\end{align*}

Proof. Choose a natural number \( m \in \left[ \frac{2}{b-a}; \frac{3}{b-a} \right] \) and set

\[ \varepsilon := \frac{1}{6m}; \quad x_k := \frac{a + b}{2} + (-1)^k \cdot \frac{b - a}{4}, \quad y_k := \frac{1}{4} + \frac{k}{2m} = \frac{1}{4} + k \cdot 3\varepsilon; \quad k = 0, 1, \ldots, m. \]

Fig. 1 shows the location of our squares in the case \( m = 3 \). Condition 1) is fulfilled because for any point \((x, y) \in (x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)\) one has

\[ \left| x - \frac{a + b}{2} \right| < \frac{b - a}{4} + 3\varepsilon = \frac{b - a}{4} + \frac{1}{2m} \leq \frac{b - a}{2}, \]

\[ 0 \leq \frac{1}{4} - \frac{1}{2m} = \frac{1}{4} - 3\varepsilon < y < \frac{1}{4} + \frac{m}{2m} + 3\varepsilon = \frac{3}{4} + \frac{1}{2m} \leq 1. \]

In order to check 2), we note that the squares \((x_k - 3\varepsilon; x_k + 3\varepsilon) \times (y_k - 3\varepsilon; y_k + 3\varepsilon)\) with even \( k \) lie in the region \( \{x > (a + b)/2\} \) and those with odd \( k \) lie in the region \( \{x < (a + b)/2\} \). If we take \( j \) and \( k \) such that \(|k - j| \geq 2\), then \(|y_k - y_j| \geq 1/m = 6\varepsilon\); due to this estimate the two squares are disjoint. Condition 3) follows from the containment

\[ \left[ \frac{1}{4}; \frac{3}{4} \right] \subset \bigcup_{k=0}^{m} (y_k - 2\varepsilon; y_k + 2\varepsilon). \quad (3.10) \]

4) It follows from (3.10) that there exists a set of nonnegative functions

\[ \chi_0, \chi_1, \ldots, \chi_m \in C_0^\infty(\mathbb{R}) \]
with \( \text{supp} \chi_k = (-\infty; y_0 + 2 \varepsilon] \) if \( k = 0 \), \([y_k - 2 \varepsilon; y_k + 2 \varepsilon]\) if \( 0 < k < m \), and \([y_m - 2 \varepsilon; +\infty) \) if \( k = m \), such that

\[
\sum_{k=0}^{m} \chi_k(y) = 1 \quad \forall y \in \mathbb{R}.
\]

Let \( \zeta^\beta_\alpha \) denote a function from the class \( C_0^\infty(\mathbb{R}) \) such that

\[
\text{supp} \zeta^\beta_\alpha = [\alpha; \beta], \quad 0 \leq \zeta^\beta_\alpha \leq \frac{2}{\beta - \alpha}, \quad \int_\mathbb{R} \zeta^\beta_\alpha(x)dx = 1.
\]

Put

\[
\varphi(x, y) := \sum_{k=0}^{m} \zeta^{x_k+2\varepsilon}_{x_k-2\varepsilon}(x) \cdot \chi_k(y).
\]

Then \( \varphi \in C_b^\infty(\mathbb{R}^2) \), \( \varphi = 0 \) on the set \( S \), and

\[
\int_a^b \varphi(x, y)dx = \sum_{k=0}^{m} \chi_k(y) \int_a^b \zeta^{x_k+2\varepsilon}_{x_k-2\varepsilon}(x) = 1, \quad \forall y \in \mathbb{R}
\]

Note that

\[
\sup_{x, y} |\varphi(x, y)| \leq \sup_{x} \left| \zeta^{x_k+2\varepsilon}_{x_k-2\varepsilon}(x) \right| \leq \frac{1}{2\varepsilon}.
\]

Finally, let

\[
g_{a, b}(x, y) := \int_x^b \varphi(t, y)dt.
\]

This function belongs to \( C_b^\infty(\mathbb{R}^2) \) and satisfies a), b), and c), since \( \frac{1}{2\varepsilon} = 3m \leq \frac{9}{5-\alpha} \).

Now let us recall the construction of a "thick Cantor set".

**Step 1.** Begin with \( I := [0; 1] \). In the center of \( I \) take the interval \( (a; b) \) of the length \( 4^{-1} \), i.e., \( a = 3/8 \), \( b = 5/8 \). Then \( I \setminus (a; b) \) splits into two closed intervals \( I_0 = [0; 3/8] \) and \( I_1 = [5/8; 1] \).

**Step 2.** In the center of \( I_0 \) take the interval \( (a_0; b_0) \) of the length \( 4^{-2} \), then \( I_0 \setminus (a_0; b_0) \) splits into \( I_{00} \) and \( I_{01} \). In the center of \( I_1 \) take the interval \( (a_1; b_1) \) of the length \( 4^{-2} \), then \( I_1 \setminus (a_1; b_1) \) splits into \( I_{10} \) and \( I_{11} \).

**Step 3.** In the centers of \( I_{00}, I_{01}, I_{10}, I_{11} \), take the intervals \( (a_{00}; b_{00}), (a_{01}; b_{01}), (a_{10}; b_{10}), (a_{11}; b_{11}) \), respectively, each one having the length \( 4^{-3} \), thus we obtain 8 closed intervals \( I_{000}, I_{001}, I_{010}, I_{011}, I_{100}, I_{101}, I_{110}, I_{111} \).

Then we proceed inductively. The closed intervals of the \( n \)-th generation are shorter than \( 2^{-n} \), but the limiting compact set

\[
K = I \setminus (a; b) \setminus (a_0; b_0) \setminus (a_1; b_1) \setminus (a_{00}; b_{00}) \setminus (a_{01}; b_{01}) \setminus (a_{10}; b_{10}) \setminus (a_{11}; b_{11}) \setminus \ldots
\]
has Lebesgue measure $1/2$.

Now we shall give an example of a measure $\mu$ such that the Dirichlet form $\mathcal{E}(f, g) = \int (\nabla f, \nabla g) d\mu$ is closable on $L^2(\mu)$, but the partial Dirichlet form $\mathcal{E}_x(f, g) = \int \partial_x f \partial_x g d\mu$ is not. This measure is the restriction of Lebesgue measure to a certain set $F$.

**Example 3.2.** Let $Q = (0; 1) \times (0; 1)$. By using Lemma 3.1 we construct the countable set of squares

$$\mathcal{M} = \mathcal{M}_{a,b} \cup \bigcup_{n=1}^{\infty} \left( \bigcup_{i_1=0}^{1} \cdots \bigcup_{i_n=0}^{1} \mathcal{M}_{a_1 \ldots a_n, b_1 \ldots b_n} \right).$$

Set

$$F := Q \setminus \bigcup_{K \in \mathcal{M}} K.$$

By construction of the intervals $(a_s, b_s)$ and the squares from $\mathcal{M}_{a, b}$, it is obvious that the squares from $\mathcal{M}$ satisfy the hypotheses of Theorem 2.3. It follows from the theorem that the Dirichlet form

$$\mathcal{E}(f, g) = \int_F (\nabla f, \nabla g) d\lambda^2$$

is closable on $L^2(\lambda^2 | F)$. Next we shall prove that the Dirichlet form

$$\mathcal{E}_x(f, g) = \int_F \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} d\lambda^2$$

fails to be closable on $L^2(\lambda^2 | F)$. To this end, we construct a sequence of functions $f_n \in C^\infty_b(\mathbb{R}^2)$ such that $\{f_n\}$ converges to zero in $L^2(F) := L^2(\lambda^2 | F)$, but $\partial f_n / \partial x$ converge in $L^2(F)$ to a function that differs from zero on a set of positive Lebesgue measure.

Denote by $\xi_{(s,t)}^{\epsilon}$ a function from $C^\infty_b(\mathbb{R})$ with the following properties:

- $\xi_{(s,t)}^{\epsilon}(x) = 0$ if $x \leq s$;
- $(\xi_{(s,t)}^{\epsilon})'(x) = 0$ if $x \geq t$;
- $(\xi_{(s,t)}^{\epsilon})'(x) = 1$ if $s + \epsilon(t - s) \leq x \leq t - \epsilon(t - s)$;
- $0 \leq (\xi_{(s,t)}^{\epsilon})'(x) \leq 1$ for any $x$.

Fix a function $\theta \in C^\infty_b(\mathbb{R})$ with the support $[1/4; 3/4]$ such that $1 \geq \theta(y) > 0$ if $1/4 < y < 3/4$. After these preparations, we begin to construct the functions $f_n$. We set

$$f_1(x, y) := \begin{cases} 
\theta(y)\xi_{l_0}^{I_{a,b}}(x) & \text{if } x \in I_{0}, \\
\theta(y)\xi_{l_0}^{I_{a,b}}(a)g_{a,b}(x, y) & \text{if } x \in (a; b), \\
\theta(y)\xi_{l_1}^{I_{a,b}}(x) & \text{if } x \in I_{1};
\end{cases}$$
Then we proceed inductively. In the process of construction of \( f_n \) we use the functions \( \xi_f^{t^-n} \) corresponding to the closed intervals of the \( n \)-th generation and the functions \( g_{a,b} \) corresponding to the intervals that we have taken in the first \( n \) steps (see the construction of the "thick Cantor set"). It is easy to check the following properties of the functions \( f_n \):

\[
0 \leq f_n(x, y) < 2^{-n}, \tag{3.11}
\]

\[
0 \leq \frac{\partial f_n(x, y)}{\partial x} \leq 1 \quad \text{on} \quad F, \tag{3.12}
\]

on the squares from \( \mathcal{M}_{a_1\ldots i_k, b_1\ldots i_k} \) one has

\[
\left| \frac{\partial f_n(x, y)}{\partial x} \right| < \frac{9}{4^{(k+1)}}, \quad k = 0, 1, 2, \ldots, \tag{3.13}
\]

\( f_n \) with all its derivatives vanishes on the lines \( \{x = 0\} \) and \( \{y = 0\} \), and, finally,

\[
\text{for a.e. } (x, y) \in Q \quad \lim_{n \to \infty} \frac{\partial f_n(x, y)}{\partial x} = V(x, y) = \begin{cases} 
\theta(y) & \text{if } x \in K, \\
0 & \text{if } x \notin K.
\end{cases} \tag{3.14}
\]

Therefore,

\[
f_n \to 0 \quad \text{and} \quad \frac{\partial f_n}{\partial x} \to V \quad \text{in } L^2(F).
\]

But \( V(x, y) > 0 \) on the set \( K \times (1/4; 3/4) \) that has measure 1/4. This completes our example.

The measure constructed in Example 3.2 is supported by a set with "holes". However, it is possible to refine this measure so that its support will coincide with the square \( Q \).

**Example 3.3.** Set \( \mu := \rho \cdot \lambda^2 \), where

\[
\rho(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in F; \\
16^{-1} & \text{if } (x, y) \in K \in \mathcal{M}_{a,b}; \\
16^{-(m+1)} & \text{if } (x, y) \in K \in \mathcal{M}_{a_1\ldots i_m, b_1\ldots i_m}, \ m \in \mathbb{N}.
\end{cases}
\]
The Dirichlet form

\[ \mathcal{E}'(f, g) = \int (\nabla f, \nabla g) \, d\mu \]

is closable on \( L^2(\mu) \) because it is the sum of the form \( \mathcal{E} \) from Example 3.2 and the classical Dirichlet forms on the squares \( K \in \mathcal{M} \). In order to prove non-closability of the form

\[ \mathcal{E}'_x(f, g) = \int \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} \, d\mu \]

on \( L^2(\mu) \), consider the same sequence of functions \( f_n \in C^\infty_c(\mathbb{R}^2) \) as in Example 3.2. It follows from (3.11) that \( \|f_n\|_{L^2(\mu)} \xrightarrow{n \to \infty} 0 \). From the estimates (3.12) and (3.13) we see that the functions \( |\partial f_n/\partial x| \) are majorized by the function

\[ M(x, y) = \begin{cases} 1 & \text{if } (x, y) \in F, \\ 9/4^{-(m+1)} & \text{if } (x, y) \in K \in \mathcal{M}_{a_1 \ldots a_m, b_1 \ldots b_m}, \ m = 0, 1, 2, \ldots \end{cases} \]

but

\[ \|M\|_{L^2(\mu)}^2 \leq 1 + \sum_{m=0}^{\infty} 16^{-(m+1)} \sum_{i_1=0}^{1} \cdots \sum_{i_m=0}^{1} \left( \frac{9}{4^{-(m+1)}} \right)^2 \sum_{K \in \mathcal{M}_{a_1 \ldots a_m, b_1 \ldots b_m}} \lambda^2(K) \]

\[ \leq 1 + \sum_{m=0}^{\infty} 16^{-(m+1)} \cdot 2^m \cdot \left( \frac{9}{4^{-(m+1)}} \right)^2 \cdot 4^{-(m+1)} = 1 + \frac{81}{4} \sum_{m=0}^{\infty} 2^{-m} < 42. \]

Since

\[ \frac{\partial f_n(x, y)}{\partial x} \xrightarrow{n \to \infty} V(x, y) \text{ for } \lambda^2\text{-a.e. } (x, y) \in Q, \]

by the Lebesgue dominated convergence theorem we have

\[ V \in L^2(\mu), \quad \left\| \frac{\partial f_n}{\partial x} - V \right\|_{L^2(\mu)} \xrightarrow{n \to \infty} 0, \]

therefore, since \( \mu \{(x, y) : V(x, y) > 0\} > 0 \), the form \( \mathcal{E}'_x \) is not closable on \( L^2(\mu) \).

Now we shall extend the result of this paper to the \( d \)-dimensional case.

**Corollary 3.4.** Let \( d \in \mathbb{N} \). There exists a measure \( \nu \) on \( \mathbb{R}^d \) such that the Dirichlet form

\[ \mathcal{E}'_{h_1, \ldots, h_d}(f, g) = \sum_{k=1}^{d} h_k \int \frac{\partial f(x)}{\partial x_k} \cdot \frac{\partial g(x)}{\partial x_k} \nu(dx), \quad h_1, \ldots, h_d \geq 0, \quad (3.15) \]

is closable on \( L^2(\nu) \) if and only if \( h_1 \cdot \ldots \cdot h_d > 0 \).
Proof. Let $F$ be the set constructed in Example 3.2. Set

$$F_{ij} := \{ x = (x_1, \ldots, x_d) \in (0; 1)^d : (x_i, x_j) \in F \}, \quad i, j = 1, \ldots, d, \quad i \neq j.$$  

Denote by $\nu_{ij}$ the restriction of the $d$-dimensional Lebesgue measure to the set $F_{ij}$. Then the Dirichlet form $\mathcal{E}_{h_1, \ldots, h_d}^{\nu_{ij}}$ is closable on $L^2(\nu_{ij})$ if $h_1 \cdot \ldots \cdot h_d > 0$. Indeed, it is sufficient to consider the gradient Dirichlet form

$$\mathcal{E}_{1, \ldots, 1}^{\nu_{ij}}(f, g) = \int (\nabla f(x), \nabla g(x))\nu_{ij}(dx)$$  

(3.16)

because one has the following estimate with some $A > a > 0$: 

$$a \cdot \mathcal{E}_{1, \ldots, 1}^{\nu_{ij}}(f, f) \leq \mathcal{E}_{h_1, \ldots, h_d}^{\nu_{ij}}(f, f) \leq A \cdot \mathcal{E}_{1, \ldots, 1}^{\nu_{ij}}(f, f).$$

But $\nu_{ij} = \mu_{ij} \times m_{ij}$, where

$$m_{ij} = \bigotimes_{k=1 \atop k \neq i, j}^d dx_k \quad \text{and} \quad \mu_{ij} = \mu(dx_i, dx_j),$$

where $\mu$ is the restriction of the two-dimensional Lebesgue measure to $F$. For both measures $m_{ij}$ (on $\mathbb{R}^{n-2}$) and $\mu_{ij}$ (on $\mathbb{R}^2$) the gradient Dirichlet forms are closable, therefore, so is (3.16).

Take the measure

$$\nu := \sum_{i=1}^d \sum_{j=1 \atop j \neq i}^d \tilde{\nu}_{ij},$$

where $\tilde{\nu}_{ij}$ is the image of $\nu_{ij}$ under the parallel shift along the vector $2e_i + 4e_j$. For any $h_1, \ldots, h_d, h_1 \cdot \ldots \cdot h_d > 0$, the Dirichlet form (3.15) is closable on $L^2(\nu)$ as the sum of forms closable over measures whose sum is $\nu$. If $h_j = 0$ and $h_i > 0$, we can take the sequence of functions

$$\tilde{f}_n(x) := \begin{cases} f_n(x_i - 2, x_j - 4) & \text{if } x_i > 2, \ x_j > 4, \\ 0 & \text{if } x_i \leq 2 \text{ or } x_j \leq 4, \end{cases}$$

where $f_n$ is the function from Example 3.2. Then $\tilde{f}_n \in C^\infty_b(\mathbb{R}^d)$; $\|\tilde{f}_n\|_{L^2(\nu)} \to 0$; for $k \neq i, k \neq j$, we have $\frac{\partial f_n}{\partial x_k} \equiv 0$; but

$$\frac{\partial \tilde{f}_n(x)}{\partial x_i} \xrightarrow{n \to \infty} u(x) = \begin{cases} V(x_i - 2, x_j - 4) & \text{if } x_i > 2, \ x_j > 4, \\ 0 & \text{if } x_i \leq 2 \text{ or } x_j \leq 4, \end{cases}$$

where $V$ is the function defined in (3.14) in Example 3.2. We have

$$\mathcal{E}_{h_1, \ldots, h_d}(\tilde{f}_n - \tilde{f}_m, \tilde{f}_n - \tilde{f}_m) = h_i \int \left| \frac{\partial (\tilde{f}_n(x) - \tilde{f}_m(x))}{\partial x_i} \right|^2 \nu(dx)$$

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\[
= h_i \int_F \left| \frac{\partial (f_n(x_i, x_j) - f_m(x_i, x_j))}{\partial x_i} \right|^2 d{x_i} d{x_j} \xrightarrow{m,n \to \infty} 0.
\]

But \( \nu\{u > 0\} = \mu\{V > 0\} = \frac{1}{4} \), hence

\[
\mathcal{E}_{h_1, \ldots, h_d}^\nu(\tilde{f}_n, \tilde{f}_n) \xrightarrow{n \to \infty} h_i \int_F V^2(x_i, x_j) d{x_i} d{x_j} > 0.
\]

Therefore, the Dirichlet form (3.15) is not closable on \( L^2(\nu) \). \( \square \)

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References


