Surface measures and tightness of (r,p)–capacities on Poisson space

V.I. Bogachev[1], O.V. Pugachev[2], M. Röckner[3]

[1], [2]: Department of Mechanics and Mathematics,
Moscow State University, 119899 Moscow, Russia

[3]: Department of Mathematics, Bielefeld University,
D–33615 Bielefeld, Germany

Abstract. We prove tightness of (r,p)–Sobolev capacities on configuration spaces equipped with Poisson measure. By using this result we construct surface measures on configuration spaces in the spirit of the Malliavin calculus. A related Gauss–Ostrogradskii formula is obtained.

AMS 1991 Subject classification:
Primary: 46G12, 60H07
Secondary: 28C20
Keywords and phrases: configuration space, capacity, tightness, Sobolev classes, surface measure, Gauss–Ostrogradskii formula
Running head: Surface measures and tightness of capacities

1 Introduction

In recent years, there has been a growing interest in the analysis and measure theory on configuration spaces of Riemannian manifolds (see, e.g., [3], [4], [6], [12], [24], [25], [26], and the references therein). Configuration spaces provide interesting examples of infinite dimensional manifolds that possess a relatively simple description and have rich analytic, measure-theoretic, and geometric structures. In particular, such spaces have natural Sobolev classes. In this work, we study capacities and surface measures on level sets of functions in Sobolev classes over configuration spaces equipped with Poisson measure. Our first main result is the tightness of the corresponding $C_{r,p}$-capacities. Apart from being interesting in its own right, this result is an efficient tool in the study of surface measures on level sets of Sobolev functions that are nondegenerate in the Malliavin sense. The idea to employ capacities for the study of surface measures is due to P. Malliavin who has successfully applied it in the Gaussian case. As an application, we obtain a version of the Gauss–Ostrogradskii formula extending a result from [12],[26]; formulas of this kind will be useful in the study of boundary value problems on configuration spaces.

Finally, we mention that the same questions in the case of Wiener space have been studied extensively by several authors. We refer, in particular, to [1], [13], [16], and P. Malliavin's monograph [18]. So, this paper provides the analogue for the Poisson space.

2 Configuration space and its tangent tensor bundles

Let M be a connected smooth complete Riemannian manifold of dimension d endowed with a σ-finite, but not finite measure σ which has a density ρ, such that $\sqrt{\rho} \in W^{1,2}_{loc}(M)$, with respect to the Riemannian volume on M. In particular, σ is locally finite. We denote by $C^k(M,TM)$, $k = 0, \dots, \infty$, the class of all C^k-vector fields on M and by $C_0^k(M,TM)$ its subclass formed by all compactly supported ones. By $(\nabla^M)^k$ we will denote the iterated gradient of k-th order on M defined as usual in terms of the Levi–Civita connection (see e.g. [5]).

Definition 2.1. *The configuration space $\Gamma = \Gamma_M$ over M is the space of $\mathbb{Z}_+ \cup \{+\infty\}$-valued measures γ on M, such that for any compact set $K \subset M$ one has $\gamma(K) < \infty$. In other words, γ has the form*

$$\gamma = \sum_{i=1}^{N} k_i \delta_{x_i}, \quad N \in \mathbb{N} \cup \{+\infty\},$$

where $x_i \in M$, δ_{x_i} is the Dirac measure concentrated in the point x_i, $k_i \in \mathbb{N}$ is the multiplicity of the point x_i, and the sequence $\{x_i\}$ has no cluster points.

Let suppγ be the support of the measure γ.

We equip the configuration space with the vague topology, i.e., the topology generated by all functions

$$\langle \varphi, \gamma \rangle \equiv \int \varphi(x)\gamma(dx), \qquad \varphi \in C_0^\infty(M),$$

which is known to be metrizable.

Definition 2.2. *A probability measure $\pi = \pi_\sigma$ on the configuration space $\Gamma = \Gamma_M$ is called Poisson measure with intensity σ if, for any $A, B \subset M$ with finite σ-measure, one has*

(i) $\pi\{\gamma : \gamma(A) = k\} = \frac{\sigma(A)^k}{k!} e^{-\sigma(A)}$,

(ii) *if A and B are disjoint, then the random variables $\gamma(A)$ and $\gamma(B)$ are independent.*

Note that the measure π has full support if and only if σ has full support. It is also well-known that π has full measure on the set of configurations without multiple points.

In the sequel we can consider Γ as the space of all configurations or as the space of configurations without multiple points. The results we are proving are valid for both cases.

Definition 2.3. *We say that a function $f\colon \Gamma \mapsto \mathbb{R}$ is smooth cylindrical and write $f \in \mathcal{F}\mathcal{C}_b^\infty$ if it has the form*

$$f(\gamma) = u(\langle \varphi_1, \gamma\rangle, \dots \langle \varphi_n, \gamma\rangle), \quad \varphi_j \in C_0^\infty(M),\ u \in C_b^\infty(\mathbb{R}^n),\ n \in \mathbb{N}.$$

The class $\mathcal{F}\mathcal{C}_b^\infty$ is dense in $L^p(\pi)$ for every $p \geq 1$ (see [3]).

Definition 2.4. *The tangent tensor space of order n at the point $\gamma \in \Gamma$ is the space*

$$T_\gamma^n\Gamma := L^2\big(M^n \mapsto T^n M^n, \gamma^n\big),$$

where $T^n M^n$ denotes the vector bundle over M^n with fiber at $(x_1, \dots, x_n) \in M^n$ given by

$$(T^n M^n)_{(x_1, \dots x_n)} := T_{x_1} M \otimes \dots \otimes T_{x_n} M.$$

So, $T_\gamma^n\Gamma$ is the space of all measurable sections Y of $T^n M^n$ such that $Y(x_1, \dots, x_n) \in (T^n M^n)_{(x_1, \dots x_n)}$ and the following norm of Y is finite:

$$\|Y\|_{T_\gamma^n\Gamma} = \left(\int \dots \int [Y(x_1, \dots x_n), Y(x_1, \dots x_n)]_n\, \gamma(dx_1) \dots \gamma(dx_n)\right)^{1/2},$$

where $[\cdot,\cdot]_n$ denotes the Riemannian scalar product of tensors of order n.

It is natural to set $T_\gamma^0\Gamma = \mathbb{R}$. In the case $n = 1$, the $T_\gamma^1\Gamma$-norm coincides on all vectors from $T_\gamma\Gamma$ with the norm defined in [3], where this tangent space was defined for the first time. It is easy to check that $T_\gamma^n\Gamma = (T_\gamma\Gamma)^{\otimes n}$.

The space $L^p(\pi, T^n\Gamma)$ of π-measurable tensor-valued sections F, $F(\gamma) \in T_\gamma^n\Gamma$, is equipped with the norm

$$\|F\|_p = \left(\int_\Gamma \|F(\gamma)\|_{T_\gamma^n\Gamma}^p \pi(d\gamma)\right)^{1/p}.$$

A vector field $V \in C_0^\infty(M, TM)$ generates a flow of diffeomorphisms ψ_t of M. This flow can be lifted to Γ by the following formula:

$$\text{if} \quad \gamma = \sum_i k_i \delta_{x_i}, \qquad \text{then} \quad \psi_t(\gamma) = \sum_i k_i \delta_{\psi_t(x_i)}.$$

Definition 2.5. *The gradient of a section $F: \Gamma \mapsto T^n\Gamma$ is the section*

$$\nabla^\Gamma F: \quad \Gamma \longmapsto T^{n+1}\Gamma,$$

defined as follows: let $x \in \operatorname{supp}\gamma$, let V_x be a smooth vector field on M such that in a neighbourhood of x the flow ψ generated by V_x moves points along geodesical lines with constant velocity, and $V_x = 0$ in some neighbourhoods of all points of $\operatorname{supp}\gamma \setminus \{x\}$*. Let $v_x := V_x(x)$. Then*

$$\left[\nabla^\Gamma F(\gamma)(x; x_1, \dots, x_n), (v_x \otimes v_1 \otimes \dots \otimes v_n)\right]_{n+1} =$$

$$= \frac{\partial}{\partial t}\Big|_{t=0} \left[F\big(\psi_t(\gamma)\big)\big(\psi_t(x_1), \dots, \psi_t(x_n)\big), \big(\Psi_t(v_1) \otimes \dots \otimes \Psi_t(v_n)\big)\right]_n,$$

for any $v_i \in T_{x_i}M$, where $\Psi_t(v) \in T_{\psi_t(x)}M$ is the parallel shift of the vector $v \in T_xM$ along ψ_t, provided such limits exist.

In the case $n = 0$, this gives the same definition of the gradient of a real-valued function as in [3]. By means of Definition 2.5 we can define gradients $(\nabla^\Gamma)^k$ of higher orders of real-, vector-, and tensor-valued sections on Γ. Then, clearly, for a section $F: \Gamma \mapsto T^n\Gamma$ we obtain a section $(\nabla^\Gamma)^k F: \Gamma \mapsto T^{n+k}\Gamma$.

Every smooth cylindrical function has gradients of all orders. Here we write explicitly the first three gradients of the function $f(\gamma) = u(\langle\varphi_1, \gamma\rangle, \dots \langle\varphi_n, \gamma\rangle)$ (we will omit the arguments of u): for x, y, $z \in \operatorname{supp}\gamma$

$$\nabla^\Gamma f(\gamma)(x) = \sum_{i=1}^n \partial_i u(\dots)\nabla^M\varphi_i(x);$$

$$(\nabla^\Gamma)^2 f(\gamma)(x,y) = \sum_{i,j=1}^n \partial_i\partial_j u(\dots)\nabla^M\varphi_i(x)\otimes\nabla^M\varphi_j(y) + \sum_{i=1}^n \partial_i u(\dots)(\nabla^M)^2\varphi_i(x)\mathbf{1}_{\{x\}}(y);$$

$$(\nabla^\Gamma)^3 f(\gamma)(x,y,z) = \sum_{i,j,k=1}^n \partial_i\partial_j\partial_k u(\dots)\nabla^M\varphi_i(x)\otimes\nabla^M\varphi_j(y)\otimes\nabla^M\varphi_k(z)$$

$$+ \sum_{i,j=1}^n \partial_i\partial_j u(\dots)\Big((\nabla^M)^2\varphi_i(x)\otimes\nabla^M\varphi_j(z)\mathbf{1}_{\{x\}}(y) + (\nabla^M)^2\varphi_i(x)\otimes\nabla^M\varphi_j(y)\mathbf{1}_{\{x\}}(z)$$

$$+\nabla^M\varphi_i(x)\otimes(\nabla^M)^2\varphi_j(y)\mathbf{1}_{\{y\}}(z)\Big) + \sum_{i=1}^n \partial_i u(\dots)(\nabla^M)^3\varphi_i(x)\mathbf{1}_{\{x\}}(y)\mathbf{1}_{\{x\}}(z),$$

where $\partial_i u$ denotes the partial derivative in the i-th argument of u.

Let us consider an example explaining the appearance of extra fields on the surfaces $\{x_1 = \dots = x_k\}$ even in the simplest cases. Let $M = \mathbb{R}^d$ and let

$$f(\gamma) = \int g(x)\,\gamma(dx),$$

where $g \in C_0^\infty(\mathbb{R}^d)$. Then

$$(\nabla^\Gamma)^2 f(\gamma)(x,y) = 0 \quad \text{if } x \neq y,$$

and

$$(\nabla^\Gamma)^2 f(\gamma)(x,x) = (\nabla^M)^2 g(x).$$

If we consider $(\nabla^\Gamma)^2 f(\gamma)(x,y)$ as a 2-tensor on M^2, then it is discontinuous due to the extra term $(\nabla^M)^2 g(x)$ on the surface $\{x=y\}$. Looking at $(\nabla^\Gamma)^3 f$, we obtain extra terms on $\{x=y\}$, $\{x=z\}$, $\{y=z\}$, and $\{x=y=z\}$.

Definition 2.6. *A section $F: \Gamma \mapsto T^n\Gamma$ is said to belong to the class $\mathcal{F}C_b^\infty(T^n\Gamma)$ of smooth cylindrical tensor fields of order n if it has the following form, where we assume without loss of generality that the points $x_1, \dots, x_n \in \operatorname{supp}\gamma$ are numbered so that the coinciding ones are neighbours:*

$$F(\gamma)(x_1,\dots,x_n) = \sum_{i=1}^N \varphi_i(\gamma) v_i(x_1,\dots,x_n) \tag{2.1}$$

$$+ \sum_{k_1+\dots+k_m=n, k_i\in\mathbb{N}} F_{k_1,\dots,k_m}(x_1,\dots,x_n) \mathbf{1}_{\{x_1=\dots=x_{k_1}\}} \cdots \mathbf{1}_{\{x_{k_1+\dots+k_{m-1}+1}=\dots=x_n\}},$$

where $v_i \in C_0^\infty(M^n, T^nM^n)$, $N \in \mathbb{N}$, and

$$F_{k_1,\dots,k_m}(x_1,\dots,x_n) = \sum_{j=1}^N \chi_j^{k_1,\dots,k_m}(\gamma) w_j^{k_1,\dots k_m}(x_1,\dots,x_n),$$

where $w_j^{k_1,\dots,k_m} \in C_0^\infty(M^n, T^nM^n)$, and the functions φ_i and $\chi_j^{k_1,\dots,k_m}$ are smooth cylindrical.

It should be stressed that the multiplicities k_i mentioned in (2.1) do not depend on the multiplicities of points in γ. They naturally arise in our situation since, e.g., the derivative of a cylindrical vector field cannot be written as a linear combination of smooth 2-tensor fields on M^2 multiplied by cylindrical functions, but involves additional terms as we have seen above. So one has to define the classes $\mathcal{F}C_b^\infty(T^n\Gamma)$ as in Definition 2.6 to make them invariant under ∇^Γ.

In the case $n=1$ the smooth cylindrical vector fields from Definition 2.6 are the same as those defined in [3].

Definition 2.7. *A tensor $A(\gamma) \in T_\gamma^n\Gamma$ is said to be symmetric if for any permutation $(x_{s_1},\dots,x_{s_n})$ of the points $x_1,\dots,x_n$ one has*

$$[A(\gamma)(x_{s_1},\dots,x_{s_n}), (v_{s_1}\otimes\cdots\otimes v_{s_n})]_n = [A(\gamma)(x_1,\dots,x_n), (v_1\otimes\cdots\otimes v_n)]_n,$$

for any $v_i \in T_{x_i}M$.

It is easy to see that a gradient of any order n of a smooth cylindrical function is symmetric in each γ.

Definition 2.8. *Let $A \in T^m_\gamma\Gamma$, $B \in T^n_\gamma\Gamma$. The symmetrized tensor product of A and B is defined by the following formula:*

$$\left(A\widetilde{\bigotimes}B\right)(x_1, \ldots, x_{m+n}) :=$$
$$= \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} A(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \otimes B(x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n)}).$$

Concerning higher order gradients on configuration spaces, see also [2].

Let V be a vector field on Γ, i.e., $V : \gamma \to \{V(\gamma)(x) \mid x \in \mathrm{supp}\gamma\}$, $V(\gamma)(x) \in T_xM$. The derivative of a function f along V is defined by

$$\partial_V f(\gamma) = \big(V(\gamma), \nabla^\Gamma f(\gamma)\big)_{T_\gamma\Gamma} \equiv \big\langle \big(V(\gamma)(\cdot), \nabla^\Gamma f(\gamma)(\cdot)\big)_{TM}, \gamma\big\rangle.$$

3 Sobolev spaces

The Sobolev norms $\|\cdot\|_{r,p}$, $r \in \mathbb{N}$, $p \geq 1$, for smooth cylindrical sections of $T^n\Gamma$, $n = 0, 1, \ldots$, are defined as follows:

$$\|F\|_{r,p} := \left(\int_\Gamma \left(\sum_{k=0}^r \|(\nabla^\Gamma)^k F(\gamma)\|^p_{T^{n+k}_\gamma\Gamma}\right) \pi(d\gamma)\right)^{1/p}. \tag{3.1}$$

Smooth cylindrical sections have finite Sobolev norms of any order. It is sufficient to prove that such sections have finite L^p-norms, because the k-th gradient of a section $g \in \mathcal{F}\mathcal{C}^\infty_b(T^n\Gamma)$ is a section from $\mathcal{F}\mathcal{C}^\infty_b(T^{n+k}\Gamma)$. Indeed, let F be of the form (2.1). Denote by Ω the union of supports of all functions on M occuring in representation (2.1). There are finitely many of them, hence Ω is compact. Since all these functions on M have bounded derivatives of all orders, the norm $\|F(\gamma)\|_{T^n_\gamma\Gamma}$ is dominated by $const \cdot \gamma(\Omega)^{n/2}$ which belongs to $L^p(\pi)$ for any $p \geq 1$, since $\int(\gamma(\Omega))^q\pi(d\gamma) < \infty$ for any $q > 0$.

Definition 3.1. *A section $F \in L^p(\pi, T^n\Gamma)$ is said to belong to the Sobolev class $W^{r,p}(T^n\Gamma)$ if there exists a sequence of smooth cylindrical sections F_m of $T^n\Gamma$ converging to F in $L^p(\pi, T^n\Gamma)$ such that $\{F_m\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{r,p}$. The k-th gradient of F is the $L^p(\pi, T^{n+k})$-limit of the corresponding gradients of F_m.*

That the k-th gradient of F in this definition is well-defined, i.e. independent of the sequence $\{F_m\}$, follows from Lemma 3.3 below.

Definition 3.2. *Let V be a vector field, i.e., a section of $T\Gamma$. We say that V possesses a divergence $\delta V \in L^1(\Gamma, \pi)$ if for any $f \in \mathcal{FC}_b^\infty$ one has*

$$\int_\Gamma \partial_V f(\gamma)\pi(d\gamma) = -\int_\Gamma f(\gamma)\delta V(\gamma)\pi(d\gamma). \tag{3.2}$$

It has been proved in [3] that for $V \in \mathcal{FC}_b^\infty(T\Gamma)$ δV exists and $\delta V \in L^2(\Gamma, \pi)$.

Note that if $V \in L^p(\pi, T\Gamma)$ and $\delta V \in L^p(\pi)$, $p > 1$, then formula (3.2) remains valid for any function $f \in W^{1,q}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 3.3. *The gradient of order r is closable, hence the above Sobolev classes are well-defined.*

Proof. In order to show that this definition is consistent, we shall prove that the values of the gradients do not depend on the choice of an approximating sequence F_m. Let F_m and G_m be two approximating sequences for some $F \in W^{r,p}(T^n\Gamma)$. Set $H_m := F_m - G_m$. Then $H_m \to 0$ in $L^p(\pi, T^n\Gamma)$ and $\|H_l - H_m\|_{r,p} \to 0$ as $m, l \to \infty$. We have to prove that $(\nabla^\Gamma)^k H_m \to 0$ in $L^p(\pi, T^{n+k}\Gamma)$.

Let $k = 1$. Let us fix a "test section" $\xi \in \mathcal{FC}_b^\infty(T^n\Gamma)$ and set

$$\big(H_m, \xi\big)(\gamma) = \int \cdots \int \left[H_m(\gamma)(x_1, \ldots x_n), \xi(\gamma)(x_1, \ldots x_n)\right]_n \gamma(dx_1) \ldots \gamma(dx_n). \tag{3.3}$$

The absolute value of the right-hand side is estimated by $\|H_m(\gamma)\|_{T^n_\gamma} \cdot \|\xi(\gamma)\|_{T^n_\gamma}$. Let $V \in \mathcal{FC}_b^\infty(T\Gamma)$. Then we have

$$\partial_V\big(H_m(\gamma), \xi(\gamma)\big) = \big(\partial_V H_m(\gamma), \xi(\gamma)\big) + \big(H_m(\gamma), \partial_V \xi(\gamma)\big).$$

Applying the integration by parts formula (see [3]) to the left-hand side, we obtain

$$-\int \big(H_m(\gamma), \xi(\gamma)\big)\delta V(\gamma)\pi(d\gamma) - \int \big(H_m(\gamma), \partial_V \xi(\gamma)\big)\pi(d\gamma) = \int \big(\partial_V H_m(\gamma), \xi(\gamma)\big)\pi(d\gamma) =: l(\xi). \tag{3.4}$$

Since ξ, $\partial_V \xi \in L^q(\pi, T^n\Gamma)$ for any $q \geq 1$, $\delta V \in L^2(\pi)$, the left-hand side of (3.4) tends to 0 as $m \to \infty$. But $\partial_V H_m \to (V, Q)_{T\Gamma}$ in $L^p(\pi, T^n\Gamma)$, since $\nabla^\Gamma H_m$ converges in $L^p(\pi, T^{n+1}\Gamma)$ to some Q. We have to show that if $\int ((V,Q)_{T\Gamma}, \xi)\, d\pi = 0$ for any $\xi \in \mathcal{FC}_b^\infty(T^n\Gamma)$, then $(V,Q)_{T\Gamma} = 0$ π-a.e. Since $(V,Q)_{T\Gamma}$ belongs to the closure of $\mathcal{FC}_b^\infty(T^n\Gamma)$ in L^p, our claim is true in the case $p = 2$, hence also in the case $p > 2$. However, if $1 \leq p < 2$, we construct an $L^\infty(\pi)$-function

$$\kappa(\gamma) = \big(\|(V(\gamma), Q(\gamma))_{T_\gamma\Gamma}\|_{T^n_\gamma} + 1\big)^{-1}$$

and replace $(V,Q)_{T\Gamma}$ by $(V,Q)_{T\Gamma}\cdot\kappa\in L^\infty(\pi,T^n\Gamma)$. Using the density of smooth cylindrical functions in $L^p(\pi)$, we obtain that $(V,Q)_{T\Gamma}\cdot\kappa$ belongs to the closure of $\mathcal{FC}_b^\infty(T^n\Gamma)$ in L^2, and we prove that $(V,Q)_{T\Gamma}\cdot\kappa=0$ π-a.e., therefore $(V,Q)_{T\Gamma}=0$ π-a.e.

Since $\mathcal{FC}_b^\infty(T\Gamma)$ is separable, we obtain that for π-a.e. γ $(V,Q)_{T\Gamma}=0$ for any $V\in\mathcal{FC}_b^\infty(T\Gamma)$. Therefore, $Q=0$ π-a.e.

Proceeding by the induction in $k=1,2,\dots r$, we obtain the desired. □

We observe that the above Sobolev classes are exact analogues of the Sobolev classes considered in [5] in the case of a finite dimensional Riemannian manifold; in that case, an alternative construction is developped in [27].

Lemma 3.4. *Let* $1/p+1/q=1/s\le 1$; $r\in\mathbb{N}$. *Then*

(i) *if* $f\in W^{r,p}$, $g\in W^{r,q}$, *then* $fg\in W^{r,s}$, *and* $\|fg\|_{r,s}\le const(r)\cdot\|f\|_{r,p}\|g\|_{r,q}$;

(ii) *if* $f\in W^{2,p}$, $V\in W^{1,q}(T\Gamma)$, *then* $\partial_V f\in W^{1,s}$, *and*

$$\|\partial_V f\|_{1,s}\le const\cdot\|f\|_{2,p}\|V\|_{1,q}.$$

Proof. (i) Assume first that $f,g\in\mathcal{FC}_b^\infty$. For gradients of the product of two functions the following formula is true:

$$(\nabla^\Gamma)^k(fg)(\gamma)=\sum_{i=0}^k C_k^i(\nabla^\Gamma)^i f(\gamma)\widetilde{\bigotimes}(\nabla^\Gamma)^{k-i}g(\gamma).$$

Hence

$$\|(\nabla^\Gamma)^k(fg)\|_{T_\gamma^k}\le const(r)\cdot\sum_{i=0}^k\|(\nabla^\Gamma)^i f\|_{T_\gamma^i}\cdot\sum_{j=0}^k\|(\nabla^\Gamma)^j g\|_{T_\gamma^j},$$

whence we obtain by Hölder's inequality that $\|fg\|_{r,s}\le const(k)\cdot\|f\|_{r,p}\|g\|_{r,q}$. This inequality remains valid when we pass to limits $f_n\to f$, $g_n\to g$ in the corresponding Sobolev norms.

(ii) The vector field $Y=\nabla^\Gamma F$ belongs to $W^{1,p}(T\Gamma)$; $\partial_V F(\gamma)=(V(\gamma),Y(\gamma))_{T_\gamma\Gamma}$. Consider first the case of smooth cylindrical V and Y. We have

$$\left|(V(\gamma),Y(\gamma))_{T_\gamma\Gamma}\right|\le\|V(\gamma)\|_{T_\gamma}\cdot\|Y(\gamma)\|_{T_\gamma}.$$

Let $y\in\operatorname{supp}(\gamma)$, let $\eta\in C_0^\infty(M,TM)$ be such that $\eta(y)=v\ne 0$, $\eta=0$ in a neighbourhood of each point from $\operatorname{supp}\gamma\setminus\{y\}$. Denote the flow of diffeomorphisms generated by η by ψ_t. Then

$$\left[\nabla^\Gamma\left(V(\gamma),Y(\gamma)\right)(y),v\right]_1=\frac{\partial}{\partial t}\,|_{t=0}\,\left(V(\psi_t(\gamma)),Y(\psi_t(\gamma))\right)_{T_{\psi_t(\gamma)}\Gamma}$$

$$= \left[\int \left(\nabla^\Gamma V(\gamma)(y,x) Y(\gamma)(x) + V(\gamma)(x) \nabla^\Gamma Y(\gamma)(y,x) \right) \gamma(dx),\ v \right]_{T_y M} =: Q_{y,v}.$$

Then, taking at each point $y \in \operatorname{supp}\gamma$ orthonormal vectors $e_1^y, \dots e_d^y \in T_y M$, we evaluate

$$\| \nabla^\Gamma \left(V(\gamma), Y(\gamma) \right)_{T_\gamma \Gamma} \|_{T_\gamma}^2 = \int \sum_{i=1}^d Q_{y,e_i^y}^2 \gamma(dy)$$

$$\leq 2 \left(\|V(\gamma)\|_{T_\gamma}^2 + \|\nabla^\Gamma V(\gamma)\|_{T_\gamma^2}^2 \right) \left(\|Y(\gamma)\|_{T_\gamma}^2 + \|\nabla^\Gamma Y(\gamma)\|_{T_\gamma^2}^2 \right).$$

Applying the Hölder inequality, we get

$$\|(V,Y)\|_{1,s} \leq const \cdot \|V\|_{1,p} \|Y\|_{1,q},$$

and this inequality extends to all $V \in W^{1,p}(T\Gamma)$ and $Y \in W^{1,q}(T\Gamma)$ by approximation. □

We shall also need the following elementary estimates concerning Poissonian random variables.

Lemma 3.5. *Let ξ be a random variable with the Poisson distribution with mean $\lambda \geq 1$. Then*

(i) *for $M \geq 3\lambda$ one has $\mathbf{P}\{\xi > M\} < 2^{2\lambda - M}$;*

(ii) *$\mathbf{E}(\xi^k) \leq c_k \lambda^k$, for any $k \geq 1$.*

Proof. (i) One has

$$\mathbf{P}\{\xi > M\} = \sum_{n=[M]+1}^\infty \frac{e^{-\lambda}\lambda^n}{n!}$$

$$\leq \frac{e^{-\lambda}\lambda^{[2\lambda]}}{[2\lambda]!} \cdot \frac{\lambda^{[M]-[2\lambda]}}{(2\lambda)^{[M]-[2\lambda]}} \cdot \sum_{m=1}^\infty \frac{\lambda^m}{(3\lambda)^m} < \left(\frac{1}{2}\right)^{M-2\lambda-1} \cdot \sum_{m=1}^\infty \left(\frac{1}{3}\right)^m = 2^{2\lambda - M}.$$

(ii) It is sufficient to prove the claim for integer k. We can choose coefficients $\theta_1, \dots, \theta_n$ such that

$$\mathbf{E}(\xi^k) = \sum_{n=0}^\infty \frac{e^{-\lambda}\lambda^n}{n!} n^k = \sum_{j=1}^k \sum_{n=0}^\infty \frac{e^{-\lambda}\lambda^n \cdot \theta_j \cdot n(n-1)\dots(n-j+1)}{n!}$$

$$= \sum_{j=1}^k \theta_j \lambda^j \sum_{n=j}^\infty \frac{e^{-\lambda}\lambda^{n-j}}{(n-j)!} = \sum_{j=1}^k \theta_j \lambda^j \leq \sum_{j=1}^k |\theta_j| \cdot \lambda^k.$$

□

4 Sobolev functions with compact supports and tightness of Sobolev capacities

In this section, we construct Sobolev functions with compact supports and use them for the proof of tightness of the $C_{r,p}$-capacities. Related ideas in different situations have been used in many papers (see, e.g., [9], [13], [17], [19], [23]; in particular, the Gaussian case is discussed in detail in [10], [11], and [18], where one can find an extensive bibliography).

Note that if a set $\mathbb{K} \subset \Gamma$ is compact in the vague topology, then for every function $\varphi \in C_0^\infty(M)$, the function $\gamma \to \langle \varphi, \gamma \rangle$ is bounded on $\mathbb{K}$. Conversely, we have:

Lemma 4.1. *Let $U_1 \subset U_2 \subset \dots$ be relatively compact regions in M such that*

$$dist_M(\overline{U_n}, M \backslash U_{n+1}) > 1 \qquad \text{and} \qquad \bigcup_n U_n = M. \tag{4.1}$$

Let $\varphi_n \in C_0^\infty(M)$ be such that $\varphi_n \mid_{U_n} = 1$, $\varphi_n \geq 0$, $\operatorname{supp} \varphi_n \subset U_{n+1}$. Then the set

$$\mathbb{K} = \{\gamma : \ \langle \varphi_n, \gamma \rangle \leq M_n, \ \forall \, n \in \mathbb{N}\}, \quad M_n > 0,$$

is compact in the vague topology.

Proof. Let $\{\gamma_j\}_{j \in \mathbb{N}}$ be a sequence of configurations in $\mathbb{K}$. For every j one has the estimate $\gamma_j(\overline{U_1}) \leq \langle \varphi_1, \gamma_j \rangle \leq M_1$, hence $\{\gamma_j(\overline{U_1})\}_{j \in \mathbb{N}}$ takes only finitely many values, therefore, one of these values, say k_1, occurs infinitely many times. Among those j with value k_1, we choose a subsequence $\{\gamma_j^1\}$ such that the local configurations $\gamma_j^1 \mid_{\overline{U_1}}$ converge to some configuration γ^1 on $\overline{U_1}$ having k_1 points (we always count points with their multiplicities).

Repeating this procedure for the compact sets $\overline{U_2}$, $\overline{U_3}, \dots$, we construct the subsequences $\{\gamma_j^2\} \supset \{\gamma_j^3\} \supset \dots$ converging locally to the configurations γ^2 on $\overline{U_2}$, γ^3 on $\overline{U_3}$, $\dots$, respectively, and it is obvious that $\gamma^n \leq \gamma^{n+1}$. Finally, we choose the diagonal subsequence $\{\gamma_1^1, \gamma_2^2, \dots\}$. This sequence converges to the configuration

$$\gamma = \lim_{n \to \infty} \gamma^n \in \mathbb{K}$$

in the vague topology, since each compact subset of M is contained in some of the regions U_n. $\square$

Let $U_1 \subset U_2 \subset \dots$ be bounded regions with smooth boundaries in M such that (4.1) holds. Set $s_n := \sigma(U_n)$; we can choose U_n such that $s_n \geq 2 s_{n-1}$; $s_0 := 1$.

We shall assume that the following condition **(C)** holds:

There exist functions $\varphi_n \in C_0^r(M)$ taking values in $[0;1]$ such that $\varphi_n \mid_{\overline{U_n}} = 1$, $\operatorname{supp} \varphi_n \subset U_{n+1}$, and constants $\alpha_k > 1$ such that

$$\sup_n \sup_{x\in M} [(\nabla^M)^k \varphi_n(x), (\nabla^M)^k \varphi_n(x)]_k \le \alpha_k^2 \quad \forall\, k = 1, 2, \dots, r. \tag{4.2}$$

In the case $r = 1$ condition (4.2) follows from condition (4.1) due to Gaffney's lemma (see [15] or [27]). In the general case, it is a restriction on M.

Example 4.2. *Fix some $r \in \mathbb{N}$, $r > 1$.*

(i) Let $M = \mathbb{R}^d$; $R_n > R_{n-1} + 1$, $n \in \mathbb{N}$, $R_0 := 0$. Take functions $f_n \in C_b^\infty(\mathbb{R})$, such that $f_n(t) = 1$ if $t \le R_n$, $f_n(t) = 0$ if $t \ge R_n + 1$,

$$f_n(R_n + s) = (1 + \exp(-\mathrm{cotan}(\pi s)))^{-1} \quad \text{if} \quad 0 < s < 1,$$

and the regions $U_n := \{|x| < R_n\}$. Then the functions $\varphi_n(x) := f_n(|x|)$ satisfy condition **(C)**.

(ii) Let M be the d-dimensional hyperbolic space. Consider polar coordinates $\rho, \phi_1, \dots, \phi_{d-1}$, where $\rho \ge 0$, $-\frac{\pi}{2} \le \phi_i \le \frac{\pi}{2}$ if $i < d-1$, and $0 \le \phi_{d-1} < 2\pi$. Its Riemannian metric is

$$ds^2 = d\rho^2 + \sinh^2\rho \Big(d\phi_1^2 + cos^2\phi_1 (d\phi_2^2 + \dots \\ + cos^2\phi_{d-3}(d\phi_{d-2}^2 + cos^2\phi_{d-2} \cdot d\phi_{d-1}^2) \dots) \Big).$$

For this manifold we can take bounded regions

$$U_n := \{\rho < R_n\}, \text{ where } R_n > R_{n-1} + 1, \quad n \in \mathbb{N}, \qquad R_0 = 0,$$

and functions $\varphi_n := f_n(\rho)$, where f_n are the same as in (i).
Then $\varphi_n \in C_0^\infty$ satisfy condition **(C)** because the curvature of M and the curvatures of spheres of radii $R_n > 1$ are bounded.

Definition 4.3. *The Sobolev capacity $C_{r,p}$, $r \in \mathbb{N}$, $p \ge 1$, is defined as follows:*

$$C_{r,p}(U) := \inf\big\{ \|f\|_{r,p} : \quad f \in W^{r,p},\ f \ge 0;\ f \ge 1 \textit{ on } U\ \pi\textit{-almost everywhere} \big\}$$

if $U \subset \Gamma$ is open;

$$C_{r,p}(B) := \inf\big\{ C_{r,p}(U) : \ B \subset U,\ U \textit{ is open} \big\}$$

for arbitrary $B \subset \Gamma$.

Definition 4.4. *A capacity C in a topological space X is said to be tight if for any $\varepsilon > 0$ one can find a compact set K_ε such that $C(X \setminus K_\varepsilon) < \varepsilon$.*

Theorem 4.5. *Let $r \in \mathbb{N}$ be such that condition* **(C)** *holds for r. Then the capacity $C_{r,p}$ generated by the Sobolev class $W^{r,p}$ on Γ is tight for every $p \geq 1$.*

Proof. Consider the functions φ_n specified above. We have

$$\langle \varphi_n, \gamma \rangle \leq \gamma(U_{n+1}),$$

and $\gamma(U_{n+1})$ is a Poissonian random value with mean $s_{n+1} \geq 2^{n+1}$. Consider the compact sets $\mathbb{K}_N = \{\gamma : \ \langle \varphi_n, \gamma \rangle \leq N s_{n+2}, \quad \forall n \in \mathbb{N}\}$, $N \geq 3$. Then by Lemma 3.5(i)

$$\pi(\Gamma \setminus \mathbb{K}_N) \leq \sum_{n=1}^{\infty} \pi\{\gamma : \ \langle \varphi_n, \gamma \rangle > N s_{n+2}\} \leq \sum_{n=1}^{\infty} 2^{-(N-2)s_{n+2}} \to 0 \quad \text{as } N \to \infty.$$

Let $\zeta_m \in C_b^\infty(\mathbb{R})$ be such that $\zeta_m \mid_{[0;m]} = 1$, $\zeta_m \mid_{[2m;+\infty)} = 0$, $0 \geq \zeta_m' \geq -2/m$, and $|\zeta_m^{(k)}| \leq \beta_k m^{-k}$, where β_k are increasing positive constants. Set

$$\Xi_n^N(\gamma) := \zeta_{N s_{n+2}}(\langle \varphi_n, \gamma \rangle).$$

Then $\Xi_n^N \in \mathcal{F}\mathcal{C}_b^\infty$, $0 \leq \Xi_n^N \leq 1$, and

$$\pi\{\Xi_n^N < 1\} \leq \pi\{\langle \varphi_n, \cdot \rangle > N s_{n+2}\} \leq 2^{-(N-2)s_{n+2}}.$$

Recall that by the chain rule

$$\frac{d}{dt} f(g(t)) = f'(g(t))g'(t); \quad \frac{d^2}{dt^2} f(g(t)) = f''(g(t))\left(g'(t)\right)^2 + f'(g(t))g''(t); \ \ldots$$

It can be checked inductively that $\frac{d^k}{dt^k} f(g(t))$ consists of not more than $k!$ summands of the form

$$f^{(j)}(g(t)) \cdot g^{(i_1)}(t) \cdot \cdots \cdot g^{(i_m)}(t),$$

where $1 \leq j \leq k$, $i_1 + \ldots i_m = k$.

In order to simplify the calculations, we may increase the constants α_k in condition **(C)** so that $\alpha_{m+k} \geq \alpha_m \cdot \alpha_k$ $\forall m, k \in \mathbb{N}$. For any $\gamma \in \Gamma$, $x_1, \ldots x_k \in \operatorname{supp}\gamma$, and orthonormal $e_1^j, \ldots e_d^j \in T_{x_j} M$, we have

$$\left| \left[(\nabla^\Gamma)^k \Xi_n^N(\gamma)(x_1, \ldots x_k), (e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k) \right]_k \right|$$

$$\leq k! \cdot \max_{1 \leq j \leq k} |\zeta_{N s_{n+2}}^{(j)}| \cdot \max_{\sum_i l_i = k} \sup_{x \in M} \sqrt{\prod_i \left[(\nabla^M)^{l_i} \varphi_n(x), (\nabla^M)^{l_i} \varphi_n(x) \right]_{l_i}} \leq k! \cdot \frac{\beta_k \cdot \alpha_k}{N s_{n+2}}$$

Note that all gradients of the function $\gamma \mapsto \langle \varphi_n, \gamma \rangle$ vanish at all those points of any configuration γ that lie outside the relatively compact set $U_{n+1} \setminus U_n$. Hence we have

$$\|(\nabla^\Gamma)^k \Xi_n^N(\gamma)\|_{T_\gamma^k}^2 \leq d^k \left(k! \frac{\beta_k \alpha_k}{N s_{n+2}} \right)^2 \cdot \gamma(U_{n+1} \setminus U_n)^k, \quad 1 \leq k \leq r,$$

for π-almost every γ (namely, for any γ without multiple points). Since by Lemma 3.5(ii)

$$\int \gamma(U_{n+1}\backslash U_n)^{kp}\pi(d\gamma) < \int \gamma(U_{n+1})^{kp}\pi(d\gamma) \le c_{kp}s_{n+1}^{kp},$$

we have the estimate

$$\|(\nabla^\Gamma)^k \Xi_n^N\|^p_{L^p(\pi,\mathcal{H}_k)} \le const(k,p)\cdot s_{n+1}^{-p}\cdot s_{n+1}^{kp/2}\cdot \pi\{\Xi_n^N < 1\} \le \frac{const(k,p)\cdot s_{n+1}^{(k/2-1)p}}{2^{(N-2)s_{n+2}}}.$$

Now set $\Xi_{(n)}^N := \Xi_1^N \cdots \Xi_n^N$. These functions also belong to $\mathcal{FC}_b^\infty$, and the sequence $\{\Xi_{(n)}^N\}_{n\in\mathbb{N}}$ is Cauchy with respect to the norm $\|\cdot\|_{r,p}$ for any p. Indeed, let $\gamma\in\Gamma$, $x_1,\dots x_k \in \operatorname{supp}\gamma$. The point x_j may belong only to one of the sets $U_{n+1}\setminus\overline{U_n}$, say to the one with $n=n_j$. So, we have

$$(\nabla^\Gamma)^k\left(\Xi_{(n)}^N(\gamma)-\Xi_{(n+1)}^N(\gamma)\right)(x_1,\dots x_k) =$$

$$= \sum_{j=1}^k \Big(\mathbf{1}_{\{n_j\le n\}}\cdot(\nabla^\Gamma)^k\Xi_{n_j}^N(\gamma)(x_1,\dots x_k)\cdot \prod_{1\le m\le n,\ m\ne n_j}\Xi_m^N(\gamma)\cdot\left(1-\Xi_{n+1}^N(\gamma)\right) -$$

$$-\mathbf{1}_{\{n_j=n+1\}}\cdot(\nabla^\Gamma)^k\Xi_{n+1}^N(\gamma)(x_1,\dots x_k)\cdot\Xi_{(n)}^N(\gamma)\Big).$$

Since $0\le \Xi_n^N\le 1$ for all N, n, one has the following estimate:

$$\left\|(\nabla^\Gamma)^k\left(\Xi_{(n)}^N(\gamma)-\Xi_{(n+1)}^N(\gamma)\right)\right\|_{T_\gamma^k} \le \mathbf{1}_{\{\Xi_{n+1}^N\ne 1\}}(\gamma)\cdot k\cdot\left(\max_{m=1}^n\|(\nabla^\Gamma)^k\Xi_m^N\|_{T_\gamma^k}+\|(\nabla^\Gamma)^k\Xi_{n+1}^N\|_{T_\gamma^k}\right)\le$$

$$\le \mathbf{1}_{\{\Xi_{n+1}^N\ne 1\}}(\gamma)\cdot k\cdot k!\cdot d^{k/2}\cdot\beta_k\alpha_k\left(\max_{m=1}^n\frac{\sqrt{\gamma(U_{m+1}\setminus U_m)}}{N\cdot s_{m+2}}+\frac{\sqrt{\gamma(U_{n+2}\setminus U_{n+1})}}{N\cdot s_{n+3}}\right)$$

$$\le \mathbf{1}_{\{\Xi_{n+1}^N\ne 1\}}(\gamma)\cdot\frac{k\cdot k!\cdot d^{k/2}\cdot\beta_k\alpha_k}{N}\cdot\sqrt{\gamma(U_{n+2})}.$$

Hence by Lemma 3.5(ii)

$$\sum_{n=1}^\infty\left\|(\nabla^\Gamma)^k\left(\Xi_{(n)}^N-\Xi_{(n+1)}^N\right)\right\|_{L^p(\pi,T_\gamma^k)} \le \sum_{n=1}^\infty\left(\pi\left\{\Xi_{n+1}^N\ne 1\right\}\cdot const(k,p)\cdot\frac{s_{n+2}^{p/2}}{N}\right)^{1/p} \le$$

$$\le const(k,p)\cdot\sum_{n=1}^\infty\left(\frac{s_{n+2}^{p/2}}{N2^{(N-2)s_{n+3}}}\right)^{1/p} \underset{N\to\infty}{\longrightarrow} 0.$$

This implies that

$$\|\Xi_{(n)}^N-\Xi_{(m)}^N\|_{r,p}\to 0 \qquad \text{as}\ \ n>m\to\infty.$$

Therefore,

$$\Xi^N := \lim_{n\to\infty} \Xi^N_{(n)} = \prod_{n=1}^{\infty} \Xi^N_n \in \bigcap_{p\geq 1} W^{r,p}.$$

Moreover, since $\Xi^N_{(0)} \equiv 1$,

$$\|1 - \Xi^N\|_{r,p} \leq const(k,p) \cdot \sum_{n=0}^{\infty} \left(\frac{s_{n+2}^{p/2}}{N 2^{(N-2)s_{n+3}}} \right)^{1/p} \to 0$$

as $N \to \infty$. It is obvious that $\Xi^N = 1$ on $\mathbb{K}_N$ and that $\operatorname{supp} \Xi^N \subset \mathbb{K}_{2N}$.

Now consider the functions $f_N := 1 - \Xi^N$, $N \in \mathbb{N}$. f_N satisfies the following conditions: $0 \leq f_N \leq 1$; $f_N = 1$ on $\Gamma \setminus \mathbb{K}_{2N}$ π-almost everywhere. Hence for the open set $\mathbb{U}_N = \Gamma \setminus \mathbb{K}_{2N}$, we have

$$C_{r,p}(\mathbb{U}_N) \leq \|f_N\|_{r,p} \to 0 \qquad \text{as} \quad N \to \infty.$$

So, $C_{r,p}$ is tight. □

Note that this result is a generalization of the result in [17], where tightness of $C_{r,p}$ was proved in the case $r = 1$, $p = 2$.

Remark 4.6. There is another way of defining Sobolev classes, which involves Markovian semigroups. We will denote the corresponding spaces (Bessel spaces) by $H^{r,p}$ (cf. [14], [18]). In the Gaussian case, the spaces $H^{r,p}$ are constructed by using the Ornstein–Uhlenbeck semigroup. In that case, $H^{r,p} = W^{r,p}$ and the Meyer equivalence holds for the corresponding norms:

$$a_{r,p} \cdot \|f\|_{W^{r,p}} \leq \|f\|_{H^{r,p}} \leq A_{r,p} \cdot \|f\|_{W^{r,p}}, \tag{4.3}$$

where $A_{r,p} > a_{r,p} > 0$ are constants, $f \in W^{r,p} = H^{r,p}$ (see, e.g., [10] and the references therein). Hence such inequalities hold also for the corresponding capacities $C_{W^{r,p}}$ and $C_{H^{r,p}}$. On configuration spaces over manifolds M, as studied in this paper, one can also define Bessel spaces by means of the corresponding heat semigroups. If (4.3) holds for the corresponding norms (e.g., if $M = \mathbb{R}^d$ with Lebesgue measure, see [29]), then we have also equivalence of the two types of capacities; if the capacity $C_{W^{r,p}}$ is tight, then the capacity $C_{H^{r,p}}$ is tight as well.

5 Surface measures

In this section, we construct surface measures related to a Poisson measure on configuration space by employing some ideas of the Malliavin calculus and the theory of differentiable measures. The principal features and main ideas of the method are found in [1] and [18]. In the case of configuration spaces, the methods of the Malliavin calculus have been employed in [6], [12], [25], [26]. Similar techniques have been used in [8], [9], [19], [20], [21] in the case of measures on locally convex spaces. Under more restrictive assumptions, the Gauss–Ostrogradskii formula on a configuration space has been proved in [26], [12].

In this paper, as well as in [20] and [21], the Sobolev smoothness conditions imposed on the functions and vector fields are weaker than those in [1]. Namely, we consider only the first and second order gradients of the function F determining the surface (and consider an alternative definition which involves only the first gradient of F), and use only Sobolev capacities of the first order, hence we do not need the restriction on the manifold, formulated in condition **(C)**.

Lemma 5.1. (i) *Let $F \in W^{2,p}$ be such that there exists a vector field $V \in W^{1,s}(T\Gamma)$ with divergence $\delta V \in L^s(\pi)$ and $(\partial_V F)^{-1} \in L^q(\pi)$. If $g \in W^{1,r}$ and the constants p, q, r, and s satisfy the inequality*

$$\frac{1}{p} + \frac{2}{q} + \frac{1}{r} + \frac{2}{s} \leq 1, \tag{5.1}$$

then the measure $(g\pi) \circ F^{-1}$ admits an absolutely continuous density k_g with respect to Lebesgue measure such that

$$\operatorname{Var} k_g = \int_{-\infty}^{+\infty} |k_g'(a)|\, da \leq const(F,V) \cdot \|g\|_{1,r}. \tag{5.2}$$

In particular, $|k_g(a)| \leq const(F,V) \cdot \|g\|_{1,r}$ for every $a \in \mathbb{R}$.

(ii) *Let $F \in W^{1,p}$ be such that there exists a vector field $V \in L^s(\pi, T\Gamma)$ with divergence $\delta V \in L^s(\pi)$ and $1/p + 1/s \leq 1$. Then, for any $g \in W^{1,r}$ with $1/r + 1/s \leq 1$, the measure $\big(\partial_V F \pi\big) \circ F^{-1}$ admits an absolutely continuous density $k_{g,V}$ such that*

$$\operatorname{Var} k_{g,V} = \int_{-\infty}^{+\infty} |k_{g,V}'(a)|\, da \leq \big(\|V\|_{L^s} + \|\delta V\|_{L^s}\big)\|g\|_{1,r}. \tag{5.3}$$

In particular, $|k_{g,V}(a)| \leq \big(\|V\|_{L^s} + \|\delta V\|_{L^s}\big)\|g\|_{1,r}$ for every $a \in \mathbb{R}$.

Proof. (i) Let us prove that the generalized derivative of the measure $(g\pi) \circ F^{-1}$ on $\mathbb{R}$ is a bounded measure. Let $\phi \in C_b^1(\mathbb{R})$ and

$$\Psi := \frac{g\delta V + \partial_V g}{\partial_V F} - \frac{g\partial_V^2 F}{(\partial_V F)^2}.$$

By Lemma 3.4 and Hölder's inequality, we have $\Psi \in L^1(\pi)$ and

$$\|\Psi\|_{L^1(\pi)} \leq const(F,V) \cdot \|g\|_{1,r},$$

where the constant $const(F,V)$ is expressed through $\|F\|_{2,p}$, $\|(\partial_V F)^{-1}\|_{L^q}$, $\|V\|_{1,s}$, and $\|\delta V\|_{L^s}$. Then we obtain

$$\int_{\mathbb{R}} \phi'(t)\,\big((g\pi)\circ F^{-1}\big)(dt) = \int_{\Gamma} \phi'\big(F(\gamma)\big) g(\gamma)\,\pi(d\gamma) =$$

$$= \int_{\Gamma} \partial_V(\phi\circ F)(\gamma)\big(\partial_V F(\gamma)\big)^{-1} g(\gamma)\,\pi(d\gamma) = \tag{5.4}$$

$$= -\int_{\Gamma} \phi\circ F(\gamma)\frac{g(\gamma)}{\partial_V F(\gamma)}\delta V(\gamma)\,\pi(d\gamma) - \int_{\Gamma} \phi\circ F(\gamma)\partial_V \frac{g(\gamma)}{\partial_V F(\gamma)}\,\pi(d\gamma) =$$

$$= -\int_{\Gamma} \phi\circ F(\gamma)\Psi(\gamma)\,\pi(d\gamma) = -\int_{\mathbb{R}} \phi(t)\,\big((\Psi\pi)\circ F^{-1}\big)\,(dt).$$

Hence the measure $(\Psi\pi)\circ F^{-1}$ is the generalized derivative of the measure $(g\pi)\circ F^{-1}$. We have

$$\|(\Psi\pi)\circ F^{-1}\| \le const(F,V)\cdot \|g\|_{1,r}.$$

Therefore, the measure $(g\pi)\circ F^{-1}$ is absolutely continuous. In particular, so is $\pi\circ F^{-1}$. In addition, $(\Psi\pi)\circ F^{-1} \ll \pi\circ F^{-1}$. This implies that the Radon–Nikodym density of $(g\pi)\circ F^{-1}$ with respect to Lebesgue measure has an absolutely continuous modification k_g whose variation is majorized by $\|(\Psi\pi)\circ F^{-1}\|$.

(ii) When we replace π by $\partial_V F\cdot\pi$, which is possible, because $\partial_V F \in L^1(\pi)$ due to the assumption $1/p+1/s \le 1$, the above reasoning simplifies and does not involve the second derivative, since we have

$$\int_{\Gamma} \phi'\circ F\, g\,\partial_V F\,d\pi = \int_{\Gamma} \partial_V(\phi\circ F) g\,d\pi = -\int_{\Gamma} \phi\circ F[g\delta V + \partial_V g]\,d\pi.$$

□

Remark 5.2. *If we impose stronger conditions of smoothness: F, g, δV, $(\partial_V F)^{-1} \in W^{\infty,\infty}$, $V \in W^{\infty,\infty}(T\Gamma)$, then the density k_g will belong to $C_b^\infty(\mathbb{R})$. This can be proved by applying Malliavin's method like in (5.4) infinitely many times.*

Theorem 5.3. (i) *Let $F \in W^{2,p}$ and let $V \in W^{1,s}\big(T\Gamma\big)$ be such that $\delta V \in L^s(\pi)$ and $(\partial_V F)^{-1} \in L^q(\pi)$. Let $2/p + 2/q + 2/s \le 1$. Then there exists a family of measures $\{\nu_a\}_{a\in\mathbb{R}}$ such that*

$$\int f(\gamma)\,\nu_a(d\gamma) = k_f(a), \qquad \forall a\in\mathbb{R},\ \forall f \in \mathcal{F}C_b^\infty. \tag{5.5}$$

Moreover, one has the following estimate:

$$\nu_a(A) \le const(F,V) \cdot C_{1,r}(A), \qquad \forall A \in \mathcal{B}(\Gamma), \tag{5.6}$$

where $const(F,V)$ *is a positive constant depending only on* F *and* V, *and* r *satisfies condition* (5.1).

(ii) *In the situation of assertion* (ii) *of Lemma* 5.1, *there exists a family of measures* $\{\nu_{a,V}\}_{a\in\mathbb{R}}$ *such that*

$$\int f(\gamma)\, \nu_{a,V}(d\gamma) = k_{f,V}(a), \qquad \forall a \in \mathbb{R},\ \forall f \in \mathcal{FC}_b^\infty, \tag{5.7}$$

and

$$\nu_a(A) \le \big(\|V\|_{L^s} + \|\delta V\|_{L^s}\big) C_{1,r}(A), \qquad \forall A \in \mathcal{B}(\Gamma). \tag{5.8}$$

Proof. We first fix $N \in \mathbb{N}$ and consider the measures

$$\nu^N_{a,\varepsilon}(d\gamma) = \frac{1}{2\varepsilon}\Xi^N(\gamma) I_{\{|F(\gamma)-a|<\varepsilon\}}\pi(d\gamma), \quad \varepsilon > 0.$$

These measures are positive, their variations do not exceed $\sup k_1$, and they are uniformly tight, being concentrated on $\mathbb{K}_{N+1}$. By virtue of Prohorov's theorem (see [7, Ch. 1, Theorem 6.1]) the family of measures $\{\nu^N_{a,\frac{1}{k}}\}_{k\in\mathbb{N}}$ is weakly relatively compact. Therefore, one can choose a subsequence $\{\nu^N_{a,\varepsilon_m}\}_{m\in\mathbb{N}}$, $\varepsilon_m \to 0$, convergent weakly to some positive measure ν^N_a concentrated on $\mathbb{K}_{N+1}$. From Lemma 5.1 and the weak convergence we get for any function $f \in \mathcal{FC}_b^\infty \subset C_b(\Gamma)$ that

$$\int f(\gamma)\nu^N_a(d\gamma) = \lim_{m\to\infty} \int f(\gamma)\nu^N_{a,\varepsilon_m}(d\gamma) = \lim_{m\to\infty} \frac{1}{2\varepsilon_m} \int\limits_{a-\varepsilon_m}^{a+\varepsilon_m} k_{f\Xi^N}(t)\, dt = k_{f\Xi^N}(a).$$

Since finite measures on Γ are uniquely determined by their integrals on smooth cylindrical functions, the measure ν^N_a does not depend on the choice of a weakly converging subsequence.

Next consider the sequence of measures $\{\nu^N_a\}_{N\in\mathbb{N}}$. Their variations still do not exceed $\sup k_1$. Let us prove their uniform tightness. Since $r = p$ satisfies (5.1), one has

$$\begin{aligned}
\nu^N_a(\Gamma \setminus \mathbb{K}_{M+1}) &\le \liminf_{m\to\infty} \nu^N_{a,\varepsilon_m}(\Gamma \setminus \mathbb{K}_{M+1}) \\
&\le \liminf_{m\to\infty} \int \big(1 - \Xi^M(\gamma)\big)\, \nu^N_{a,\varepsilon_m}(d\gamma) = \liminf_{m\to\infty} \frac{1}{2\varepsilon_m} \int\limits_{a-\varepsilon_m}^{a+\varepsilon_m} k_{\Xi^N(1-\Xi^M)}(t)\, dt \\
&\le \sup_{t\in\mathbb{R}} k_{(1-\Xi^M)}(t) \le const(F,V) \cdot \|1 - \Xi^M\|_{1,p}
\end{aligned}$$

and this value tends to zero as $M \to \infty$ independently of N. Therefore, we can use Prohorov's theorem once again and choose $N_m \to \infty$ such that the sequence of measures $\{\nu_a^{N_m}\}_{m \in \mathbb{N}}$ converges weakly to some positive measure ν_a. Equality (5.5) holds true for this measure, since $\sup_t k_{(1-\Xi^N)}(t) \to 0$ as $N \to \infty$, hence $k_{f\Xi^N}(t) \to k_f(t)$ uniformly in t for any $f \in \mathcal{FC}_b^\infty$. By the same arguments as before we prove uniqueness of the measure ν_a.

In order to prove estimate (5.6), it is sufficient to consider the case of open A. Let $\alpha > 1$. There exists a function $\chi \in W^{1,r}$ such that $\chi = 1$ on A π-almost everywhere, $\chi \geq 0$, and $\|\chi\|_{1,r} \leq \alpha C_{1,r}(A)$. Then for any $N \in \mathbb{N}$, $a \in \mathbb{R}$, and $\varepsilon > 0$, we have

$$\nu_{a,\varepsilon}^N(A) \leq \int \chi(\gamma)\, \nu_{a,\varepsilon}^N(d\gamma) = \frac{1}{2\varepsilon} \int\limits_{a-\varepsilon}^{a+\varepsilon} k_{\chi\Xi^N}(t) dt \leq \sup |k_\chi| \leq const(F,V) \cdot \alpha C_{1,r}(A).$$

The same estimate remains true for the measures ν_a^N and ν_a, since, for any sequence of positive measures P_n converging weakly to a measure P, for any open set U, one has $P(U) \leq \liminf_{n\to\infty} P_n(U)$. It remains to note that α can be taken arbitrarily close to 1. In the case of $k_{a,V}$ a similar reasoning applies. □

Definition 5.4. *A function f is said to be $C_{r,p}$-quasicontinuous if there exist increasing closed sets $\mathbb{F}_n$ such that $f|_{\mathbb{F}_n}$ is continuous for any n and $C_{r,p}(\Gamma \setminus \mathbb{F}_n) \to 0$ as $n \to \infty$.*

It is known that any function $f \in W^{r,p}$ has a $C_{r,p}$-quasicontinuous π-version (the proof is analogous to the proof for the Gaussian case, see [10, Ch. 5]).

Corollary 5.5. *Suppose that the hypotheses of Theorem* 5.3 (i) *are fulfilled and that for the function F we have chosen a $C_{1,p}$-quasicontinuous π-version. Then the measures*

$$\pi(d\gamma \mid F(\gamma) = a) := \begin{cases} k_1(a)^{-1}\nu_a(d\gamma) & \text{if } k_1(a) > 0 \\ 0 & \text{if } k_1(a) = 0 \end{cases}$$

are conditional measures of π with respect to the σ-field generated by the function F. Furthermore, in the situation of assertion (ii) *of Theorem* 5.3, *the measures $k_{1,V}(a)^{-1}\nu_{a,V}$ coincide with the conditional measures $[\partial_V F \cdot \pi](\,\cdot\,|F = a)$ provided that a $C_{1,p}$-quasicontinuous version of F is chosen.*

Proof. For every $a \in \mathbb{R}$, one has $\|\nu_a\| = k_1(a)$. Hence the measures under consideration are probability measures for $(\pi \circ F^{-1})$-almost every a. Let $\mathbb{F}_n$ be closed sets of continuity of F, $C_{1,p}(\Gamma \setminus \mathbb{F}_n) < 2^{-n}$. On each of the sets $\mathbb{F}_n$, the measure ν_a is concentrated on the closed set $F^{-1}(a) \cap \mathbb{F}_n$ by the continuity of F. The set $\Gamma \setminus \bigcup_{n=1}^{\infty} \mathbb{F}_n$

has zero $C_{1,p}$ capacity, hence ν_a vanishes on it. For any $f \in \mathcal{FC}_b^\infty$, we have

$$\int\limits_\Gamma f(\gamma)\,\pi(d\gamma) = \int\limits_{-\infty}^{+\infty} k_f(t)dt = \int\limits_{-\infty}^{+\infty}\int\limits_\Gamma f(\gamma)\,\nu_t(d\gamma)dt$$
$$= \int\limits_{-\infty}^{+\infty} k_1(t)\int\limits_\Gamma f(\gamma)\,\pi(d\gamma \mid F(\gamma)=t)\,dt.$$

Therefore, the measures π and $\int_{-\infty}^{+\infty} k_1(t)\,\pi(\cdot \mid F(\cdot)=t)\,dt$ coincide. The second claim is proved analogously. □

Note that the formula of repeated integration obtained in Corollary 5.5 is the analogue of Malliavin's coarea formula ([1], [18]).

Corollary 5.6. *Suppose that the hypotheses of Theorem* 5.3 (i) *are fulfilled and that*

$$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}+\frac{2}{s}\le 1.$$

Then, for every $C_{1,r}$-quasicontinuous version of a function $g \in W^{1,r}$, one has

$$\int g(\gamma)\,\nu_a(d\gamma) = k_g(a), \qquad \forall\, a \in \mathbb{R}. \tag{5.9}$$

In addition, under the assumptions of assertion (ii) *of Theorem* 5.3*, one has*

$$\int g(\gamma)\,\nu_{a,V}(d\gamma) = k_{g,V}(a), \qquad \forall\, a \in \mathbb{R}. \tag{5.10}$$

Proof. Without loss of generality we may assume that g is bounded. One can find a uniformly bounded sequence of functions $g_k \in \mathcal{FC}_b^\infty$ such that $\|g-g_k\|_{W^{1,r}} \to 0$. Hence for any $\varepsilon > 0$

$$C_{1,r}\left\{\gamma :\ |g_k(\gamma)-g(\gamma)| \ge \varepsilon\right\} \underset{k\to\infty}{\longrightarrow} 0.$$

Then $k_{g_k}(a) \to k_g(a)$ for every a, since

$$(g_k\pi)\circ F^{-1} - (g\pi)\circ F^{-1} = \big((g_k-g)\pi\big)\circ F^{-1},$$

which by Lemma 5.1 implies the uniform convergence of densities. On the other hand, it follows by (5.6) that $g_k \to g$ in measure ν_a, which yields the convergence of the integrals $\int g_k\,d\nu_a = k_{g_k}(a)$ to $\int g\,d\nu_a$. □

It is natural to call ν_a and $\nu_{a,V}$ the surface measures on the sets $\{F = a\}$. The advantage of ν_a is that it does not depend on V (just the existence of V is required). The advantage of $\nu_{a,V}$ is that it involves only the first derivative of F. In order to obtain more geometric surface measures one can consider (see [10, Ch. 6] for the Gaussian case) the normalized surface measures $\widetilde{\nu}_a$ defined by

$$\widetilde{\nu}_a := |\nabla^\Gamma F|\, \nu_a,$$

where a $C_{1,p}$-quasicontinuous version of $|\nabla^\Gamma F|$ is chosen. For example, $\widetilde{\nu}_0$ does not change if we replace F by $2F$, whereas the previously defined measures do not have this invariance property. In a similar manner, the normalized surface measures

$$\widetilde{\nu}_{a,V} := |\nabla^\Gamma F|\, \nu_{a,V}$$

can be considered if $|\nabla^\Gamma F|$ has a $C_{1,p}$-quasicontinuous version.

6 The Gauss–Ostrogradskii formula

From now on we assume that the hypotheses in Theorem 5.3 and Corollaries 5.5, 5.6 are fulfilled. Let us consider the case $r = p/2$, i.e., the constants p, q, s are such that $6/p + 2/q + 2/s \leq 1$.

Theorem 6.1. *Let $F \in W^{2,p}$ be $C_{1,p}$-quasicontinuous and let the vector field $V \in W^{1,s}(T\Gamma)$ be such that $\delta V \in L^s(\pi)$ and $(\partial_V F)^{-1} \in L^q(\pi)$. Let Y be any vector field from the class $W^{1,p}(T\Gamma)$ such that δY exists. Then the function*

$$\gamma \mapsto \big(\nabla^\Gamma F(\gamma), Y(\gamma)\big)_{T_\gamma\Gamma} \in W^{1,p/2}$$

is ν_0-integrable, and if $(\nabla^\Gamma F, Y)$ is a $C_{1,p/2}$-quasicontinuous version of this function, then the following Gauss–Ostrogradskii formula holds:

$$\int_{F^{-1}((-\infty;0))} \delta Y(\gamma)\, \pi(d\gamma) = \int_{F^{-1}(0)} (\nabla^\Gamma F, Y)(\gamma)\, \nu_0(d\gamma). \tag{6.1}$$

In particular, the right-hand side of (6.1) *does not depend on the choice of a quasi-continuous version of* $(\nabla^\Gamma F, Y)$.

Proof. Let

$$J_\varepsilon(t) := \left(\int_{-\varepsilon}^{\varepsilon} \exp\left(\frac{s^2}{s^2-\varepsilon^2}\right) ds \right)^{-1} \cdot I_{(-\varepsilon;\varepsilon)}(t) \cdot \exp\left(\frac{t^2}{t^2-\varepsilon^2}\right), \quad \varepsilon > 0;$$

$$\chi_N(\gamma) := \int\limits_{F(\gamma)}^{+\infty} J_{\frac{1}{N}}(t)\, dt.$$

It is easy to see that $\chi_N \in W^{2,p}$, $\chi_N = 1$ on the set $F^{-1}((-\infty; -1/N])$, and $\chi_N = 0$ on $F^{-1}([1/N; +\infty))$. Let $\chi_{N,k} \in \mathcal{FC}_b^\infty$, $k \in \mathbb{N}$, be such that $\sup|\chi_{N,k}| \le 2$ and $\|\chi_{N,k} - \chi_N\|_{2,p} \to 0$ as $k \to \infty$. Passing to a subsequence, we may also assume that $\chi_{N,k} \to \chi_N$ π-almost everywhere. By the integration by parts formula (see Definition 3.2), we have

$$\int \chi_{N,k}(\gamma)\delta Y(\gamma)\, \pi(d\gamma) = -\int \partial_Y \chi_{N,k}(\gamma)\, \pi(d\gamma).$$

Both sides of this equality converge to the analogous expressions for χ_N, so we have

$$\int \chi_N(\gamma)\delta Y(\gamma)\, \pi(d\gamma) = -\int \partial_Y \chi_N(\gamma)\, \pi(d\gamma). \tag{6.2}$$

Since $\pi\{F^{-1}(0)\} = 0$ and for every $\gamma \notin F^{-1}(0)$ one has $\chi_N(\gamma) \to I_{F^{-1}(-\infty;0)}(\gamma)$, the left-hand side of (6.2) tends to the left-hand side of (6.1). According to the chain rule, the right-hand side of (6.2) equals

$$\int \partial_Y F(\gamma) J_{\frac{1}{N}}(F(\gamma))\, \pi(d\gamma) = \int k_{\partial_Y F}(t) J_{\frac{1}{N}}(t)\, dt,$$

which converges to

$$k_{\partial_Y F}(0) = \int (\nabla^\Gamma F, Y)(\gamma)\, \nu_0(d\gamma)$$

as $N \to \infty$ by Corollary 5.6. □

Remark 6.2. Let $J_{\frac{1}{N}}$ be the functions considered in the above proof. Then, for every bounded $C_{1,r}$-quasicontinuous function χ, one has

$$\lim_{N\to\infty} \int \chi(\gamma) J_{\frac{1}{N}}(F(\gamma))\, \pi(d\gamma) = \int \chi(\gamma)\, \nu_0(d\gamma). \tag{6.3}$$

In particular, our surface measure ν_0 is the weak limit of the measures $J_{\frac{1}{N}} \circ F \cdot \pi$ concentrated on the sets $\{|F| \le 1/N\}$.

Indeed, this is true if $\chi \in W^{1,r}$, in particular, if $\chi \in \mathcal{FC}_b^\infty$. We may assume that $|\chi| \le 1$. Given $\varepsilon > 0$, one can find a compact set $K \subset \Gamma$ such that $C_{1,r}(\Gamma \backslash K) < \varepsilon$. Note that

$$\int\limits_{\Gamma\backslash K} J_{\frac{1}{N}}(F(\gamma))\, \pi(d\gamma) \le const(F, V)\varepsilon, \tag{6.4}$$

since for any $g \in W^{1,r}$ such that $g \geq 1$ π-a.e. on $\Gamma \backslash K$, we have

$$\int g(\gamma) J_{\frac{1}{N}}(F(\gamma))\, \pi(d\gamma) = \int k_g(t) J_{\frac{1}{N}}(t)\, dt \leq \sup k_g \leq const(F,V) \|g\|_{1,r}$$

and $\|g\|_{1,r}$ can be made arbitrarily close to $C_{1,r}(\Gamma \backslash K)$. Therefore, the sequence of measures $(J_{\frac{1}{N}} \circ F) \cdot \pi$ is uniformly tight, hence it converges weakly to the measure ν_0. Together with (6.4) this yields also the convergence of integrals of the function χ, since we can choose K as above such that χ is continuous on K and then find a continuous function χ_K such that $\chi_K = \chi$ on K and $|\chi_K| \leq 1$. It remains to note that the integrals of $|\chi - \chi_K|$ over the complement of K with respect to $(J_{\frac{1}{N}} \circ F) \cdot \pi$ and ν_0 are majorized by $2const(F,V)\varepsilon$ and that we have convergence on χ_K.

In a similar manner, in the situation of assertion (ii) of Theorem 5.3, one has

$$\lim_{N \to \infty} \int \chi(\gamma) J_{\frac{1}{N}}(F(\gamma))\, \partial_V F(\gamma)\, \pi(d\gamma) = \int \chi(\gamma)\, \nu_{0,V}(d\gamma), \tag{6.5}$$

which is proved along the same lines.

Remark 6.3. According to Lemma 3.4, the function

$$\gamma \mapsto \|\nabla^\Gamma F(\gamma)\|^2_{T_\gamma} = \left(\nabla^\Gamma F(\gamma), \nabla^\Gamma F(\gamma)\right)_{T_\gamma \Gamma} = \partial_{\nabla^\Gamma F} F(\gamma)$$

belongs to the class $W^{1,p/2}$. If we choose a $C_{1,p/2}$-quasicontinuous version $|\nabla^\Gamma F|^2$ of this function and use the measure $\widetilde{\nu}_0 = |\nabla^\Gamma F| \nu_0$, which has already been considered above, then (6.1) can be rewritten as

$$\int\limits_{F^{-1}((-\infty;0))} \delta Y(\gamma)\, \pi(d\gamma) = \int\limits_{F^{-1}(0)} (n_F, Y)(\gamma)\, \widetilde{\nu}_0(d\gamma), \tag{6.6}$$

where $n_F = |\nabla^\Gamma F|^{-1} \nabla^\Gamma F$ is the normal unit vector for the surface $F^{-1}(0)$. If we have another $C_{1,p}$-quasicontinuous function $\widehat{F}$ satisfying all the conditions in Theorem 6.1 with $\{\widehat{F} < 0\} = \{F < 0\}$ and $\{\widehat{F} = 0\} = \{F = 0\}$, then it follows from (6.6) that the measure $\widetilde{\nu}_0$ will not change if we replace F by $\widehat{F}$.

We have defined surface measures on the level sets of scalar Sobolev class functions (i.e., on certain hypersurfaces), but in much the same manner one can define surface measures on the level sets of the Sobolev class mappings $F = (F_1, \dots, F_d)$ with values in $\mathbb{R}^d$, i.e., on surfaces of codimension d. Furthermore, one can consider more general surfaces by taking subsets of countable unions of the above considered level sets.

Now let us compare our definition of surface measures with that of [12, §15], where elementary surfaces have been defined as level sets of suitably defined C^1-functions satisfying certain strong nondegeneracy conditions (the space M in [12,

§15] is $\mathbb{R}^k$, but the corresponding construction makes sense also in the manifold case). The definition in [12] is given explicitly in terms of the surface Lebesgue measures on finite dimensional surfaces (which is possible due to the assumptions made on the elementary surfaces in [12]). The fact that our definition of the normalized surface measures $\widetilde{\nu}_a$ agrees with that of [12], in the case when F satisfies the hypotheses of that definition, is seen from the Gauss–Ostrogradskii formula (6.6) and formula (15.11) in [12]. In fact, when F is of class C^1, in order to compare our approach with that of [12], one has to deal with the surface measures $|\nabla^\Gamma F|(\partial_V F)^{-1}\nu_{a,V}$ for a suitably chosen vector field V of class C^1 (the Gauss–Ostrogradskii formula in this case is proved by the reasoning from Remark 6.2). However, if F is a more general Sobolev class function which is not C^1, then the associated surface measures cannot be defined by means of finite dimensional surface measures. It is clear from what has been said that (6.6) and (6.1) give a generalization of the Gauss–Ostrogradskii formula obtained in [12, §15]. Finally, it should be noted that the construction of surface measures given in [28] for certain surfaces in linear spaces is too restrictive in our situation, since it requires the continuity of the gradients of determining functions.

Acknowledgements. The first two named authors are grateful to the Department of Mathematics of Bielefeld University and the SFB 343 for the hospitality during their very pleasant stay. This work has been supported in part by the Russian Foundation of Fundamental Research projects 00-15-99267, 01-01-00858, the INTAS-RFBR Grant 95-0099, and the DFG Grant 436 RUS 113/343/0(R).

References

[1] Airault H., Malliavin P., *Intégration géométrique sur l'espace de Wiener*, Bull. Sci. Math., 2e série, 112 (1988), 3–52.

[2] Albeverio S., Daletskii A., Lytvynov E., *Laplace operators on differential forms over configuration spaces*, to appear in J. Geom. Phys.

[3] Albeverio S., Kondratiev Yu.G., Röckner M., *Analysis and geometry on configuration spaces*, J. Funct. Anal. 154 (1998), no. 2, 444–500.

[4] Albeverio S., Kondratiev Yu.G., Röckner M., *Analysis and geometry on configuration spaces: the Gibbsian case*, J. Funct. Anal. 157 (1998), no. 1, 242–291.

[5] Aubin T., *Some nonlinear problems in Riemannian geometry*, Springer, Berlin, 1998.

[6] Bichteler K., Gravereaux J.B., Jacod J., *Malliavin calculus for processes with jumps*, Gordon and Breach science Publishers, 1987.

[7] Billingsley P., *Convergence of probability measures*, Wiley, New – York, 1968.

[8] Bogachev V.I., *Smooth measures, the Malliavin calculus and approximation in infinite dimensional spaces*, Acta Univ. Carol., Math. et Phys., 31 (1990), no. 2, 9–23.

[9] Bogachev V.I., *Differentiable measures and the Malliavin calculus*, J. Math. Sci. 87 (1997), no. 5, 3577–3731.

[10] Bogachev V.I., *Gaussian measures*, Amer. Math. Soc., Providence, Rhode Island, 1998.

[11] Bogachev V.I., Röckner M., *Mehler formula and capacities for infinite dimensional Ornstein–Uhlenbeck processes with general linear drift*, Osaka J. Math., **32** (1995), 237–274.

[12] Davydov Yu.A., Lifshits M.A., Smorodina N.V., *Local properties of distributions of stochastic functionals*, Fizmatlit, Moscow, 1995 (in Russian); English transl.: Amer. Math. Soc., Providence, Rhode Island, 1998.

[13] Feyel D., de La Pradelle A., *Capacités gaussiens*, Ann. Inst. Fourier 41 (1991), no. 1, 49–76.

[14] Fukushima M., Kaneko K., *(r,p)-capacities for general Markovian semigroups*, Res. Notes in Math. 124, *Infinite-dimensional analysis and stochastic processes (Bielefeld, 1983)*, 41–47.

[15] Gaffney M.P., *A special Stokes's theorem for complete Riemannian manifolds*, Ann. of Math. (2) 60 (1954), 140–145.

[16] Lescot P., *Une théoréme de désintégration en analyse quasi-sure*, Sém. Probab. XXVII, Springer, Lecture Notes Math. 1557 (1993), 256–275.

[17] Ma Zhi-Ming, Röckner M., *Construction of diffusions on configuration spaces*, Osaka J. Math., 37 (2000), 273–314.

[18] Malliavin P., *Stochastic analysis*, Springer, Berlin — New York, 1997.

[19] Pugachev O.V., *Tightness of Sobolev capacities in infinite dimensional spaces*, Infinite Dimensional Analysis, Quantum Probability and Related Topics 2 (1999), no. 3, 427–440.

[20] Pugachev O.V., *Surface measures in infinite-dimensional spaces*, Mat. Zametki 63 (1998), no. 1, 106–114 (in Russian); English transl.: Math. Notes 63 (1998), no. 1-2, 94–101.

[21] Pugachev O.V., *The Gauss–Ostrogradskii formula in infinite dimensional spaces*, Sbornik Math. 189 (1998), no. 5, 757–770.

[22] Röckner M., *Stochastic analysis on configuration spaces: basic ideas and recent results*, In: *New directions in Dirichlet forms*, Studies in advanced mathematics, vol. 8, Amer. Math. Soc., 1998.

[23] Röckner M., Schmuland B., *Tightness of general $C_{1,p}$ capacities on Banach space*, J. Funct. Anal. 108 (1992), no. 1, 1–12.

[24] Röckner M., Schmuland B., *A support property for infinite-dimensional interacting diffusion processes*, C. R. Acad. Sci. Paris, Série I, 326 (1998), 359–364.

[25] Smorodina N.V., *Differential calculus in a configuration space, and stable measures. I*, Theor. Probab. Appl. 33 (1988), 522–534.

[26] Smorodina N.V., *The Gauss–Ostrogradsky formula for a configuration space*, Theory Prob. Appl. 35 (1990), no. 4, 725–736.

[27] Strichartz R.S., *Analysis of the Laplacian on the complete Riemannian manifold*, J. Funct. Anal. 52 (1983), 48–79.

[28] Uglanov A.V., Integration on infinite-dimensional surfaces and its applications, Kluwer Academic Publishers, Dordrecht, 2000.

[29] Wu Li-Ming, *Inégalité de Sobolev sur l'espace de Poisson*, Sém. Probab. XXI, Springer, Lecture Notes Math. 1247(1987), 114–137.