# Spectrum of Ornstein-Uhlenbeck operators in $L^p$ spaces with respect to invariant measures

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**Abstract**

Let $A = \sum_{i,j=1}^N q_{ij}D_{ij} + \sum_{i,j=1}^N b_{ij}x_jD_i$ be a possibly degenerate Ornstein-Uhlenbeck operator in $\mathbf{R}^N$ and assume that the associated Markov semigroup has an invariant measure $\mu$. We compute the spectrum of $A$ in $L^p_\mu$ for $1 \le p < \infty$.

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## 1 Introduction

In this paper we study the spectrum of the Ornstein-Uhlenbeck operator

$$A = \sum_{i,j=1}^N q_{ij}D_{ij} + \sum_{i,j=1}^N b_{ij}x_jD_i = \mathrm{Tr}(QD^2) + \langle Bx, D\rangle, \qquad x \in \mathbf{R}^N, \tag{1.1}$$

where $Q = (q_{ij})$ is a real, symmetric and nonnegative matrix and $B = (b_{ij})$ is a non-zero real matrix. The associated Markov semigroup $(T(t))_{t\ge 0}$ has the following explicit representation, due to Kolmogorov

$$(T(t)f)(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbf{R}^N} e^{-<Q_t^{-1}y,y>/4} f(e^{tB}x - y)\, dy, \tag{1.2}$$

where

$$Q_t = \int_0^t e^{sB}Qe^{sB^*}\, ds$$

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and $B^*$ denotes the adjoint matrix of $B$, see for instance [6]. We assume that the spectrum of $B$ is contained in $\mathbf{C}^- = \{\lambda \in \mathbf{C} : \text{Re } (\lambda) < 0\}$. Moreover we require that $\det Q_t > 0$ for any $t > 0$ (that is, $Q_t$ is positive definite); this is clearly true, in particular, if $Q$ is invertible. We point out that the condition $\det Q_t > 0$, $t > 0$, is equivalent to the hypoellipticity of $A$, see [12], and it can be also expressed by saying that the kernel of $Q$ does not contain any invariant subspaces of $B^*$ (see [12], [13], [15], [18]).

Assuming that $\det(Q_t) > 0$, in [7, section 11.2.3] it is proved that $\sigma(B) \subset \mathbf{C}^-$ is equivalent to the existence of an invariant measure $\mu$ for $T_t$, i.e. a probability measure on $\mathbf{R}^N$ such that

$$\int_{\mathbf{R}^N} (T(t)f)(x)\, d\mu(x) = \int_{\mathbf{R}^N} f(x)\, d\mu(x)$$

for every $t \geq 0$ and $f \in C_b(\mathbf{R}^N)$, the space of all continuous and bounded functions on $\mathbf{R}^N$. Moreover the invariant measure $\mu$ is unique and it is given by $d\mu(x) = b(x)\, dx$ where

$$b(x) = \frac{1}{(4\pi)^{N/2}(\det Q_\infty)^{1/2}} e^{-<Q_\infty^{-1}x,x>/4} \tag{1.3}$$

and

$$Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*}\, ds.$$

For more information on invariant measures we refer to [8] and [19]. It is well known that $(T(t))_{t\geq 0}$ extends to a strongly continuous semigroup of positive contractions in $L^p_\mu = L^p(\mathbf{R}^N, d\mu)$ for every $1 \leq p < \infty$. Remark that, since $Q_t < Q_\infty$ in the sense of quadratic forms, the integral in (1.2) converges for every $f \in L^p_\mu$ and $x \in \mathbf{R}^N$, so that the extension of $(T(t))_{t\geq 0}$ to $L^p_\mu$ is still given by (1.2).

Let us denote by $(A_p, D_p)$ the generator of $(T(t))_{t\geq 0}$ in $L^p_\mu$. The main aim of this paper is the computation of the spectrum of $(A_p, D_p)$ for $1 \leq p < \infty$. If $1 < p < \infty$, it is known that the spectrum is discrete and consists of eigenvalues of finite multiplicities, since the resolvent is compact, see [3]. We first prove that all the eigenfunctions are polynomials and then we arrive at a complete characterization of the spectrum, see Section 3. Our method shows that it is possible to reduce the computation of the spectrum of $A$ to that of its drift term $\langle Bx, D\rangle$, no matter what the diffusion term $\text{Tr}(QD^2)$ is, see in particular Lemma 3.3.

As a by-product of our proof, we also show that the spectrum is independent of $p \in ]1, \infty[$ (the $p$-independence of the spectrum is however a consequence of the compactness of the resolvent, see e.g. [1]). For $p = 1$ we obtain that the spectrum is completely different, see Section 4. The spectrum in $L^1_\mu$ is the closed left half-plane and moreover every complex number with negative real part is an eigenvalue. Let us stress that we allow $Q$ to have rank strictly less than $N$; however our main result seems to be new even in the non-degenerate case, that is when $Q$ is positive definite.

Let us mention another result of the paper. Assuming that $A$ is nondegenerate in [10] it is shown that $T_t$ is analytic in $L^p_\mu$ even in the infinite dimensional setting, $1 < p < \infty$ (see also [6], [14] and [11]). Under our assumptions, in Section 2 we show that the semigroup $T_t$ is differentiable in $L^p_\mu$, for $1 < p < \infty$; obviously, it is not so in $L^1_\mu$ (see also Corollary 4.2).

We remark that in the particular case $Q = I$, $B = -I$, it is known that the spectrum in $L^2_\mu$ consists of the negative integers and that the Hermite polynomials form a complete system of eigenfunctions. Moreover, the operator $-A_2$ on $L^2_\mu$ is unitarily equivalent to a Schrödinger operator $-\Delta + V$ on $L^2(\mathbf{R}^N, dx)$, where $V$ is a quadratic potential (see [17] and [2]). Finally we refer to [16] for the spectrum of $A$ in $L^p(\mathbf{R}^N, dx)$ and in spaces of continuous functions.

**Notation.** If $C$ is a linear operator, we denote by $\sigma(C)$, $P\sigma(C)$ and $\rho(C)$, the spectrum, the point-spectrum and the resolvent set of $C$, respectively. The spectral bound $s(C)$ is defined by $s(C) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(C)\}$. $C_b(\mathbf{R}^N)$ stands for the Banach space of all real continuous and bounded functions on $\mathbf{R}^N$. $C_0(\mathbf{R}^N)$ is the closed subspace of $C_b(\mathbf{R}^N)$ of functions vanishing at infinity, $C_0^\infty(\mathbf{R}^N)$ is the space of $C^\infty$-functions with compact support and $\mathcal{S}(\mathbf{R}^N)$ is the Schwartz class. $\mathcal{P}_n$ is the space of all polynomials of degree less than or equal to $n$. For $1 \le p < \infty$ and $k \in \mathbf{N}$, $W^{k,p}(\mathbf{R}^N)$ are the usual Sobolev spaces, and we define

$$W^{k,p}_\mu = \{u \in W^{k,p}_{\mathrm{loc}}(\mathbf{R}^N) : D^\alpha u \in L^p_\mu \text{ for } |\alpha| \le k\}. \tag{1.4}$$

The norm in $L^p_\mu$ will be denoted by $\|\cdot\|_p$. Sometimes we write $A_p$ for $(A_p, D_p)$. Throughout this paper $\mathbf{N}$ indicates the set of nonnegative integers and $\mathbf{C}^-$, $\mathbf{C}^+$ the open left and right half-planes, respectively.

# 2 Properties of $(T(t))_{t\ge 0}$

In this section we collect some properties of $(T(t))_{t\ge 0}$ and of its generator $(A_p, D_p)$ needed in the sequel.

We observe that $C_0^\infty(\mathbf{R}^N)$ is dense in $W^{k,p}_\mu$, $1 \le p < \infty$ . Indeed, a simple truncation argument shows that the set of $W^{k,p}_\mu$-functions with compact support is dense and, given $u \in W^{k,p}_\mu$ with compact support, the usual approximating functions $\phi_\varepsilon * u$ converge to $u$, as $\varepsilon \to 0$, in $W^{k,p}(\mathbf{R}^N)$ and hence in $W^{k,p}_\mu$.

As regards the domains $D_p$, we remark that $D_p \subset D_q$ if $p \ge q$ and $A_p u = A_q u$ for $u \in D_p$. If $Q$ is non-degenerate, the domain $D_2$ is nothing but the weighted Sobolev space $W^{2,2}_\mu$ and $A_2 u = Au$ for $u \in D_2$ (see [14]). A similar result seems not to be known in the general case when $p \ne 2$. However, $D_p = W^{2,p}_\mu$ if $(A_2, D_2)$

is self-adjoint; this fact turns out to be equivalent to the identity $BQ = QB^*$ and implies $Q$ positive definite (see [4] and [5]).

For our purposes, we only need the following simple lemma.

**Lemma 2.1** *Let $1 \le p < \infty$. If $u \in C^\infty(\mathbf{R}^N)$ is such that $D_{ij}u \in L^p_\mu$ for $i,j = 1,\dots,N$ and $|x||Du| \in L^p_\mu$, then $u \in D_p$ and $A_p u = Au$. Moreover, the Schwartz class $\mathcal{S}(\mathbf{R}^N)$ is a core for $(A_p, D_p)$.*

PROOF. Observe that $Au \in L^p_\mu$. Let $0 \le \phi \in C_0^\infty(\mathbf{R}^N)$ be such that $\phi(x) = 1$ if $|x| \le 1$ and define $u_n(x) = \phi(x/n)u(x)$. It is easily seen, using dominated convergence, that $u_n \to u$ and $Au_n \to Au$ in $L^p_\mu$. Since $u_n \in C_0^\infty(\mathbf{R}^N)$, it is elementary to check that $(T(t)u_n - u_n)/t \to Au_n$ uniformly (hence in $L^p_\mu$) as $t \to 0$. Therefore, $u_n \in D_p$ and the equality $Au_n = A_p u_n$ holds. Letting $n \to \infty$ we obtain that $u \in D_p$ and that $A_p u = Au$, since $(A_p, D_p)$ is closed. Finally, since $\mathcal{S}(\mathbf{R}^N)$ is contained in $D_p$ and is $T(t)$-invariant, it is a core for $(A_p, D_p)$. $\square$

We discuss now some smoothing properties of $(T(t))_{t\ge 0}$, depending upon the hypoellipticity condition $\det Q_t > 0$. To this purpose, it is useful to recall that the above condition is also equivalent to the well-known Kalman rank condition

$$\operatorname{rank}\left[Q^{1/2}, BQ^{1/2}, \dots, B^{N-1}Q^{1/2}\right] = N,$$

arising in control theory (see e.g. [21]). In the above formula, the $N \times N^2$ matrix in the left-hand-side is obtained by writing consecutively the columns of the matrices $B^i Q^{1/2}$. Moreover, if $0 \le m \le N-1$ is the smallest integer such that $\operatorname{rank}\left[Q^{1/2}, BQ^{1/2}, \dots, B^m Q^{1/2}\right] = N$, then

$$\|Q_t^{-1/2} e^{tB}\| \le \frac{C}{t^{1/2+m}}, \qquad t \in (0,1] \tag{2.1}$$

(see [20]). Of course $m = 0$ if and only if $Q$ is invertible.

The following lemma is a slight modification of a result proved, in the infinite-dimensional setting, in [3, Lemma 3]. We give the proof for completeness. The number $m$ which appears in the statement is that defined above.

**Lemma 2.2** *Let $1 < p < \infty$. For every $t > 0$, $T(t)$ maps $L^p_\mu$ into $C^\infty(\mathbf{R}^N) \cap W^{k,p}_\mu$ for every $k \in \mathbf{N}$. Moreover, there exists $C = C(k,p) > 0$ such that for every $f \in L^p_\mu$ the inequality*

$$\|D^\alpha T(t) f\|_p \le \frac{C}{t^{|\alpha|(1/2+m)}} \|f\|_p, \qquad t \in (0,1)$$

*holds for every multiindex $\alpha$ with $|\alpha| = k$.*

Proof. Let us fix $t > 0$ and set

$$b_t(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} e^{-<Q_t^{-1}x,x>/4}.$$

Since $Q_t < Q_\infty$, in the sense of quadratic forms, it is easily seen that there exist $K, \varepsilon > 0$ (depending upon $t$) such that $b_t(x) \le K e^{-\varepsilon |x|^2} b(x)$, where $b$ (defined in (1.3)) is the density of $\mu$. It follows that one can differentiate under the integral sign in (1.2) for every $f \in L^p_\mu$ thus obtaining

$$(DT(t)f)(x) = -\frac{1}{2}\int_{\mathbf{R}^N} e^{tB^*} Q_t^{-1} y f(e^{tB}x - y) b_t(y)\, dy$$

for every $x \in \mathbf{R}^N$ and hence $T(t)f \in C^1(\mathbf{R}^N)$. By Hölder inequality and (2.1)

$$\begin{aligned} |(D_iT(t))f(x)| &\le \frac{1}{2}\Big(\int_{\mathbf{R}^N} |\langle Q_t^{-1/2} e^{tB} e_i, Q_t^{-1/2} y\rangle|^{p'} b_t(y)\, dy\Big)^{1/p'} \Big((T(t)|f|^p)(x)\Big)^{1/p} \\ &\le \frac{1}{2}|Q_t^{-1/2} e^{tB} e_i| \Big(\int_{\mathbf{R}^N} |Q_t^{-1/2} y|^{p'} b_t(y)\, dy\Big)^{1/p'} \Big((T(t)|f|^p)(x)\Big)^{1/p} \\ &\le C_p t^{-1/2-m} \Big((T(t)|f|^p)(x)\Big)^{1/p} \end{aligned}$$

and the thesis follows for $k = 1$ raising to the power $p$ and integrating the above inequality with respect to $\mu$. The proof for $k \ge 1$ proceeds as in [14, Lemma 3.2] using the equality $DT(t)u = e^{tB^*} T(t) Du$, which holds for every $u \in W^{1,p}_\mu$. This identity is easily verified in $C_0^\infty(\mathbf{R}^N)$ and extends to $W^{1,p}_\mu$ by density. □

The compactness of $(T(t))_{t\ge 0}$ for $p = 2$ easily follows from the above lemma and the compactness of the imbedding of $W^{1,2}_\mu$ into $L^2_\mu$, see [8]. If $1 < p < \infty$, the same holds by interpolation (see [3, Lemma 2]).

If $Q$ is non degenerate, the analyticity of $(T(t))_{t\ge 0}$ in $L^2_\mu$ was proved in [10] (see also [6], [14]). From the Stein interpolation theorem it follows that $(T(t))_{t\ge 0}$ is analytic in $L^p_\mu$ for $1 < p < \infty$. On the other hand, $(T(t))_{t\ge 0}$ is not analytic in $L^2_\mu$ (hence in $L^p_\mu$) if $Q$ is degenerate, see [11]. We show that in any case $(T(t))_{t\ge 0}$ is differentiable in $L^p_\mu$, if $1 < p < \infty$. To prove this we need the following lemma which is probably known (it generalises [14, Lemma 2.1]). We include the proof for the sake of completeness.

**Lemma 2.3** *If $1 < p < \infty$, for every $h = 1, \ldots, N$ the map $u \mapsto x_h u$ is bounded from $W^{1,p}_\mu$ to $L^p_\mu$.*

Proof. It suffices to show that there is a constant $K_p$ such that for every $u \in C_0^\infty(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} |x_h u(x)|^p\, d\mu(x) \le K_p \int_{\mathbf{R}^N} (|u(x)|^p + |Du(x)|^p)\, d\mu(x). \tag{2.2}$$

By a linear change of variables we may assume that $Q_\infty$ is diagonal with eigenvalues $\mu_1, \ldots, \mu_N$ and hence that

$$b(x) = \frac{1}{(4\pi)^{N/2}(\mu_1 \cdots \mu_N)^{1/2}} \exp\Big\{-\sum_{i=1}^N x_i^2/(4\mu_i)\Big\}.$$

First case, assume $p \geq 2$. If $u \in C_0^\infty(\mathbf{R}^N)$, then one has with $C = 2\max\{\mu_1, \ldots, \mu_N\}$

$$\begin{aligned}
&\int_{\mathbf{R}^N} |x_h u(x)|^p \, d\mu(x) \leq -C \int_{\mathbf{R}^N} |u(x)|^p |x_h|^{p-2} x_h \cdot D_h b(x)\, dx \\
=\ & C \int_{\mathbf{R}^N} (p x_h u(x) |x_h u(x)|^{p-2} D_h u(x) + (p-1)|x_h|^{p-2}|u(x)|^p)\, d\mu(x) \\
\leq\ & C_1 \int_{\mathbf{R}^N} |x_h|^{p-2}|u(x)|^p \, d\mu(x) + C_2 \Big(\int_{\mathbf{R}^N} |x_h u(x)|^p \, d\mu(x)\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbf{R}^N} |D_h u(x)|^p \, d\mu(x)\Big)^{\frac{1}{p}} \\
\leq\ & \varepsilon \int_{\mathbf{R}^N} |x_h u(x)|^p \, d\mu(x) + C_\varepsilon \int_{\mathbf{R}^N} (|u(x)|^p + |D_h u(x)|^p)\, d\mu(x),
\end{aligned}$$

for every $\varepsilon > 0$, with a suitable $C_\varepsilon$ (in the last line we have used Young's inequality and the estimate $|x_h|^{p-2} \leq C_\varepsilon + \varepsilon |x_h|^p$). Choosing $\varepsilon < 1$ we deduce (2.2).

Let us deal with the case $1 < p < 2$. We proceed as before but we have to estimate in a different way the term

$$\int_{\mathbf{R}^N} |x_h|^{p-2}|u(x)|^p \, d\mu(x).$$

To simplify the notation, take $h = N$ and write $x' = (x_1, \ldots, x_{N-1})$, $b(x) = b'(x') \frac{e^{-x_N^2/4\mu_N}}{(4\pi\mu_N)^{1/2}}$, and $d\mu' = b'(x')dx'$, $d\mu'' = (4\pi\mu_N)^{-1/2}\exp\{-x_N^2/4\mu_N\}dx_N$, so that

$$\begin{aligned}
\int_{\mathbf{R}^N} |x_N|^{p-2}|u(x)|^p \, d\mu(x) &= \int_{\mathbf{R}^{N-1}} d\mu'(x') \int_{\mathbf{R}} |x_N|^{p-2}|u(x', x_N)|^p \, d\mu''(x_N) \\
&= \int_{\mathbf{R}^{N-1}} d\mu'(x') \int_{|x_N| \geq 1} |x_N|^{p-2}|u(x', x_N)|^p \, d\mu''(x_N) \\
&\quad + \int_{\mathbf{R}^{N-1}} d\mu'(x') \int_{-1}^{1} |x_N|^{p-2}|u(x', x_N)|^p \, d\mu''(x_N) \\
&:= J_1 + J_2
\end{aligned}$$

Clearly, $J_1 \leq \int_{\mathbf{R}^N} |u(x)|^p \, d\mu(x)$. Let us estimate $J_2$. For every $x' \in \mathbf{R}^{N-1}$ we have, by the Sobolev embedding $W^{1,p}(-1,1) \hookrightarrow L^\infty(-1,1)$,

$$\begin{aligned}
\int_{-1}^{1} |x_N|^{p-2}|u(x', x_N)|^p \, d\mu''(x_N) &\leq C\Big(\sup_{|x_N| \leq 1} |u(x', x_N)|\Big)^p \int_{-1}^{1} |x_N|^{p-2}\, dx_N \\
&\leq C_1 \int_{-1}^{1} (|u(x', x_N)|^p + |D_N u(x', x_N)|^p)\, dx_N \\
&\leq C_2 \int_{\mathbf{R}} (|u(x', x_N)|^p + |D_N u(x', x_N)|^p)\, d\mu''(x_N)
\end{aligned}$$

whence, integrating on $\mathbf{R}^{N-1}$,

$$J_2 \le C_2 \int_{\mathbf{R}^N} (|u(x)|^p + |Du(x)|^p)\, d\mu(x),$$

and this completes the proof. □

It follows, in particular, that the map $Lu = \langle Bx, Du\rangle$ is bounded from $W^{2,p}_\mu$ into $L^p_\mu$ for $1 < p < \infty$.

**Proposition 2.4** *For $1 < p < \infty$ the semigroup $(T(t))_{t\ge 0}$ is differentiable in $L^p_\mu$.*

Proof. If $f \in \mathcal{S}(\mathbf{R}^N)$ then $T(t)f \in \mathcal{S}(\mathbf{R}^N) \subset D_p$. From Lemmas 2.3, 2.2 it follows as in [14, Proposition 3.3] that

$$\|A_pT(t)f\|_p = \|AT(t)f\|_p \le \frac{C}{t^{2m+1}}\|f\|_p, \quad 0 < t \le 1,$$

hence $A_pT(t)$ extends to a bounded operator in $L^p_\mu$ and the thesis follows. □

We shall see that the above result is false for $p = 1$, see Section 4.

# 3 Spectrum in $L^p_\mu$ for $1 < p < \infty$

In this section we assume that $1 < p < \infty$. The following estimate is the main step to show that the eigenfunctions of $A_p$ are polynomials.

**Lemma 3.1** *Let $k \in \mathbf{N}$ and $\varepsilon > 0$ be given, with $s(B)+\varepsilon < 0$. Then there exists $C = C(k,\varepsilon)$ such that for every $u \in W^{k,p}_\mu$*

$$\sum_{|\alpha|=k} \|D^\alpha T(t)u\|_p \le Ce^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p, \quad t \ge 0. \tag{3.1}$$

Proof. Let $C_1 = C_1(\varepsilon)$ be such that $\|e^{tB^*}\| \le C_1e^{t(s(B)+\varepsilon)}$ and recall that $DT(t)u = e^{tB^*}T(t)Du$ for every $u \in W^{1,p}_\mu$. Since $(T(t))_{t\ge0}$ is contractive in $L^p_\mu$ the statement is proved for $k = 1$ with $C = C_1$. Suppose that the statement is true for $k$ with a suitable constant $C_k$ and consider $u \in W^{k+1,p}_\mu$. Then, if $|\alpha| = k$,

$$\begin{aligned}\|DD^\alpha T(t)u\|_p &= \|D^\alpha DT(t)u\|_p = \|D^\alpha e^{tB^*}T(t)Du\|_p \\ &\le C_1e^{t(s(B)+\varepsilon)}\|D^\alpha T(t)Du\|_p \\ &\le C_1C_ke^{t(k+1)(s(B)+\varepsilon)}\|DD^\alpha u\|_p.\end{aligned}$$

□

Observe that $\sigma(A_p) \subset \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \le 0\}$, since $(T(t))_{t\ge 0}$ is a semigroup of contractions in $L^p_\mu$ and that $0 \in \sigma(A)$. Moreover, every eigenfunction corresponding to the eigenvalue 0 is constant (this holds also for $p = 1$). In fact, if $u \in D_p$ and $A_p u = 0$, then $T(t)u = u$. On the other hand (see [8, Theorem 4.2.1])

$$T(t)u \to \int_{\mathbf{R}^N} u \, d\mu$$

as $t \to \infty$ and therefore $u$ is constant. We now show that all the eigenfunctions are polynomials.

**Proposition 3.2** *Suppose that $u \in D_p$ satisfies $A_p u = \lambda u$. Then $u$ is a polynomial.*

Proof. Since $T(t)u = e^{\lambda t}u$, from Lemma 2.2 we deduce that $u \in W^{k,p}_\mu \cap \mathcal{C}^\infty(\mathbf{R}^N)$, for every $k$. Clearly $D^\alpha T(t)u = e^{\lambda t} D^\alpha u$ for every multiindex $\alpha$. Given $\varepsilon \in (0, |s(B)|)$, from Lemma 3.1 it follows that

$$e^{t\,\mathrm{Re}\lambda} \sum_{|\alpha|=k} \|D^\alpha u\|_p \le C(k,\varepsilon) e^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p$$

and hence $D^\alpha u = 0$ if $|\alpha||s(B)| \ge |\mathrm{Re}\lambda|$. It follows that $u$ is a polynomial of degree less than or equal to $\frac{\mathrm{Re}(\lambda)}{s(B)}$. This concludes the proof. □

Let us denote by

$$Lu = \langle Bx, Du \rangle$$

the drift term in (1.1). We reduce the computation of the spectrum of $A_p$ to that of $L$.

**Lemma 3.3** *The following statements are equivalent.*

(i) $\lambda \in \sigma(A_p)$.

(ii) *There exists a homogeneous polynomial $u \ne 0$ such that $Lu = \lambda u$.*

Proof. First we observe that $A_p u = Au$ if $u$ is a polynomial (see Lemma 2.1) and that both $A$ and $L$ map $\mathcal{P}_n$ into itself. Moreover $A = L$ on $\mathcal{P}_1$ and hence we may consider only polynomials of degree greater than or equal to 2.

Suppose that (i) holds and let $u$ be a polynomial of degree $n \ge 2$ such that $A_p u = \lambda u$, that is $\lambda u - \sum_{i,j} q_{ij} D_{ij} u - Lu = 0$. If $\lambda - L$ is bijective on $\mathcal{P}_{n-2}$ we can find $v \in \mathcal{P}_{n-2}$ such that $\lambda v - Lv = \sum_{i,j} q_{ij} D_{ij} u$ and hence $z = u - v \in \mathcal{P}_n$, satisfies $\lambda z - Lz = 0$ and $z \ne 0$. If $\lambda - L$ is not bijective on $\mathcal{P}_{n-2}$ we consider a function $z$ in its kernel. In any case we find $0 \ne z \in \mathcal{P}_n$ such that $\lambda z - Lz = 0$. To find a (nonzero) homogeneous polynomial $u$ such that $\lambda u - Lu = 0$ it is sufficient to observe that $L$ maps homogeneous polynomials into homogeneous polynomials so that all homogeneous addends $u$ of $z$ satisfy $\lambda u - Lu = 0$.

Assume now that (ii) holds with $u$ homogeneous polynomial of degree $n \ge 2$. If $\lambda - A_p$ is not injective on $\mathcal{P}_{n-2}$ clearly (i) is true. Otherwise we find $v \in \mathcal{P}_{n-2}$ such that $\lambda v - Av = \sum_{i,j} q_{ij} D_{ij} u$ and then $0 \ne w = u + v \in \mathcal{P}_n$ satisfies $\lambda w - A_p w = 0$. □

We study now the equation $\gamma u - Lu = 0$ with $u$ polynomial, $\gamma \in \mathbf{C}$. If $B = -I$ this is the well-known Euler equation satisfied by all regular functions homogeneous of degree $(-\gamma)$. If we require that $u$ is a polynomial, we obtain $(-\gamma) \in \mathbf{N}$, hence all negative integers are eigenvalues of $L$ and, for every $n \in \mathbf{N}$, all homogeneous polynomials of degree $n$ are eigenfunctions.

The equation with a general $B$ is much more complicated and we shall not characterise all polynomial solutions but only the values of $\gamma$ for which such a solution exists. Observe that a differentiable function $u$ satisfies $\gamma u - Lu = 0$ if and only if

$$u(e^{tB}x) = e^{t\gamma}u(x) \qquad t \geq 0,\ x \in \mathbf{R}^N. \tag{3.2}$$

Let $u$ be a (nonzero) homogeneous polynomial of degree $n$ satisfying (3.2): in this case the same equality holds for every *complex* point $x \in \mathbf{C}^N$. Let now $M$ be a non-singular complex $N \times N$ matrix, such that $MBM^{-1} = C$, where $C$ is the canonical Jordan form of $B$. Introduce a new homogeneous polynomial $v(z) = u(M^{-1}z)$, $z \in \mathbf{C}^N$, so that $u(x) = v(Mx)$. Since $v(Me^{tB}M^{-1}z) = e^{t\gamma}\,v(z)$, we obtain that

$$v(e^{tC}z) = e^{t\gamma}\,v(z), \qquad z \in \mathbf{C}^N,$$

and we find the values of $\gamma$ for which a solution exists working with the Jordan matrix $C$. Before proving the main result of this section, we present in a particular case the argument we use in the proof. Let us suppose that $C$ consists of a unique Jordan block of size $N$ relative to an eigenvalue $\lambda$, that is

$$C = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix}$$

and write $C = \lambda I + R$ with $R$ nilpotent. Hence $e^{tR}$ has polynomial entries and we obtain

$$e^{t\gamma}v(z) = v(e^{tB}z) = v(e^{t\lambda}e^{tR}z) = e^{n\lambda t}v(e^{tR}z) = e^{n\lambda t}q(t,z) \tag{3.3}$$

where $q(t,z) = \sum_{|\alpha|=n} c_\alpha(t)z^\alpha$ and the $c_\alpha(t)$ are polynomials. Now fix $\hat{z} \neq 0$ in (3.3) such that $v(\hat{z}) \neq 0$ and look at the variable $t$. It follows that $\gamma = n\lambda$, i.e., the eigenvalues of $L$ are multiples of the (unique) eigenvalue of $B$. In the general case, we have the following result.

**Theorem 3.4** *Let $\lambda_1, \dots, \lambda_r$ be the (distinct) eigenvalues of $B$. Then*

$$\sigma(A_p) = \Big\{\lambda = \sum_{j=1}^{r} n_j\lambda_j : n_j \in \mathbf{N}\Big\}.$$

Proof. We keep the above notation (recall that $M$ is a non-singular complex $N \times N$ matrix, such that $MBM^{-1} = C$ and $C$ is the canonical Jordan form of $B$). Let $C_j$, for $j = 1, \dots r$, be the Jordan block of $C$ associated with $\lambda_j$ and denote by $k_j$ ($1 \le k_j \le N$, $\sum_{j=1}^r k_j = N$) the size of $C_j$. We may write $C_j = \lambda_j I + R_j$ where $R_j$ is a nilpotent matrix. Let us decompose $\mathbf{C}^N$ into the direct sum of the invariant subspaces corresponding to the Jordan blocks of $C$ and write $z \in \mathbf{C}^N$ in the form $z = (z_1, \dots, z_r)$, with $z_j \in \mathbf{C}^{k_j}$.

Assume that $\gamma \in \sigma(A_p)$. Then, according to Lemma 3.3, there exists a nonzero homogeneous polynomial $u$ such that $Lu = \gamma u$ or, in an equivalent way, $u(e^{tB}x) = e^{\gamma t} u(x)$. Introducing the homogeneous polynomial $v(z) = u(M^{-1}z)$, we know that $v(e^{tC}z) = e^{t\gamma} v(z)$ for every $z \in \mathbf{C}^N$. Let us write $v$ in the following way:

$$v(z) = \sum_{|\alpha_1| + \ldots + |\alpha_r| = n} c_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^r z_j^{\alpha_j},$$

and prove that $\gamma = \sum_j \lambda_j |\alpha_j|$, for suitable $(\alpha_j)$. We have

$$\begin{aligned} e^{t\gamma} v(z) = v(e^{tC}z) &= v(e^{tC_1} z_1, \ldots, e^{tC_r} z_r) \\ &= \sum_{|\alpha_1| + \ldots + |\alpha_r| = n} c_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^r (e^{tC_j} z_j)^{\alpha_j} \\ &= \sum_{|\alpha_1| + \ldots + |\alpha_r| = n} c_{\alpha_1, \ldots, \alpha_r} \, e^{t(\lambda_1 |\alpha_1| + \ldots + \lambda_r |\alpha_r|)} \prod_{j=1}^r (e^{tR_j} z_j)^{\alpha_j}. \end{aligned}$$

Now fix $\hat{z} \neq 0$ such that $v(\hat{z}) \neq 0$ and look at the variable $t$. Since $\prod_{j=1}^r (e^{tR_j} \hat{z}_j)^{\alpha_j}$ is a polynomial in $t$ for any $(\alpha_1, \ldots, \alpha_r)$, it follows that there exists some $(\alpha_1, \ldots, \alpha_r)$ such that $\gamma = \lambda_1 |\alpha_1| + \ldots + \lambda_r |\alpha_r|$. This means that

$$\gamma = \sum_{j=1}^r n_j \lambda_j, \quad n_j \in \mathbf{N}. \tag{3.4}$$

Conversely, let $\gamma = \sum_{j=1}^r n_j \lambda_j$, with arbitrary $n_j \in \mathbf{N}$. Let us write $z \in \mathbf{C}^N$ in the form

$$z = (z_1, \ldots, z_r) = (z_1, \ldots, z_{k_1}, z_{k_1+1}, \ldots, z_{k_1+k_2}, \ldots, z_{k_1+\ldots+k_r}).$$

Consider the polynomial

$$v(z) = z_{k_1}^{n_1} \cdot z_{k_1+k_2}^{n_2} \cdots z_{k_1+\ldots k_r}^{n_r},$$

depending only upon the $r$ complex variables $z_{k_1}, z_{k_1+k_2}, \ldots, z_{k_1+\ldots k_r}$ (the last variable in each block). It is easy to verify that $v(e^{tC}z) = e^{t\gamma} v(e^{tR_1} z_1, \ldots, e^{tR_r} z_r) = e^{t\gamma} v(z)$, $z \in \mathbf{C}^N$. The polynomial $u(z) = v(Mz)$, $z \in \mathbf{C}^N$, satisfies $u(e^{tB}x) = e^{t\gamma} u(x)$, $x \in \mathbf{R}^N$. It follows that $Lu = \gamma u$ and hence $\gamma \in \sigma_p(A)$, by Lemma 3.3. □

# 4 Spectrum in $L^1_\mu$

We show that the spectrum of $A_1$ is the left half-plane. In particular $(T(t))_{t\geq 0}$ is not norm-continuous in $L^1_\mu$, hence not analytic, nor differentiable, nor compact (see [9, Ch. II, Sec. 4]).

**Theorem 4.1** *The spectrum of* $(A_1, D_1)$ *is the left half-plane* $\{\lambda \in \mathbf{C} : \mathrm{Re}\ \lambda \leq 0\}$. *Each complex number* $\lambda$ *with* $\mathrm{Re}\ \lambda < 0$ *is an eigenvalue.*

Proof. Let $b$ be the density of $\mu$ with respect to the Lebesgue measure, given by (1.3), and set $h = 1/b$. Let $\Phi : L^1 = L^1(\mathbf{R}^N, dx) \to L^1_\mu$ be the isometry defined by

$$(\Phi u)(x) = u(x)h(x), \qquad u \in L^1, \quad x \in \mathbf{R}^N.$$

We define an operator $(G, D_G)$ on $L^1$ by $D_G = \Phi^{-1}(D_1)$ and $G = \Phi^{-1}A_1\Phi$. If $u \in C_0^\infty(\mathbf{R}^N)$, then $u \in D_G$ and

$$Gu(x) = b(x)(A(uh))(x) = Au(x) + 2b(x)\sum_{i,j=1}^N q_{ij}D_ih(x)D_ju(x) + b(x)u(x)Ah(x).$$

A direct computation shows that

$$2b(x)\sum_{i,j=1}^N q_{ij}D_ih(x)D_ju(x) = \langle QQ_\infty^{-1}x, Du(x)\rangle$$

and

$$\begin{aligned} b(x)Ah(x) &= \Big[\frac{1}{2}\mathrm{Tr}(QQ_\infty^{-1}) + \frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle B^*Q_\infty^{-1}x, x\rangle\Big] \\ &= \Big[\frac{1}{2}\mathrm{Tr}(QQ_\infty^{-1}) + \frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle BQ_\infty Q_\infty^{-1}x, Q_\infty^{-1}x\rangle\Big]. \end{aligned}$$

Using the identity $BQ_\infty + Q_\infty B^* = -Q$, which implies $2\langle BQ_\infty x, x\rangle = -\langle Qx, x\rangle$, it follows that $\frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle BQ_\infty Q_\infty^{-1}x, Q_\infty^{-1}x\rangle = 0$ and hence, setting $k = \frac{1}{2}\mathrm{Tr}(QQ_\infty^{-1})$,

$$\begin{aligned} Gu(x) &= Au(x) + \langle QQ_\infty^{-1}x, Du(x)\rangle + ku(x) \\ &= \mathrm{Tr}(QD^2u(x)) + \langle (B + QQ_\infty^{-1})x, Du(x)\rangle + ku(x) \\ &= \mathrm{Tr}(QD^2u(x)) - \langle (Q_\infty B^*Q_\infty^{-1})x, Du(x)\rangle + ku(x). \end{aligned}$$

The operator $G_0 = \mathrm{Tr}(QD^2) - \langle (Q_\infty B^*Q_\infty^{-1})x, D\rangle$, with a suitable domain $D_{G_0}$, is the generator of an Ornstein-Uhlenbeck semigroup in $L^1$. Even though an explicit description of $D_{G_0}$ is not known, we point out that $C_0^\infty(\mathbf{R}^N)$ is a core of $(G_0, D_{G_0})$ (see [16, Proposition 3.2]). The above computation shows that $G = G_0 + kI$ on $C_0^\infty(\mathbf{R}^N)$ and therefore $D_{G_0} \subset D_G$ and $G = G_0 + kI$ on $D_{G_0}$, since $(G, D_G)$ is closed. On the other hand, if $\lambda$ is sufficiently large, $\lambda - G$ is invertible on $D_G$ and also on $D_{G_0}$, because it coincides therein with $G_0 + kI$. Therefore $D_G = D_{G_0}$.

Observe now that the identity $B + Q_\infty B^* Q_\infty^{-1} = -QQ_\infty^{-1}$ yields $\mathrm{Tr}(B) + \mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = -\mathrm{Tr}(QQ_\infty^{-1})$ and hence $\mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = \mathrm{Tr}(B) = -k$. Moreover $G_0$ satisfies the hypoellipticity condition. Indeed, if $E$ is an invariant subspace of $Q_\infty^{-1} B Q_\infty$, contained in $\mathrm{Ker}(Q)$, the equation $BQ_\infty + Q_\infty B^* = -Q$ easily implies that $B^*(E) \subset E$. It follows that $E = \{0\}$, since $A$ is hypoelliptic.

Since $\sigma(-Q_\infty B^* Q_\infty^{-1}) = -\sigma(B) \subset \mathbf{C}^+$, from [16, Theorem 4.7] it follows that the spectrum of $(G_0, D_{G_0})$ is the half-plane

$$\{\lambda \in \mathbf{C} : \mathrm{Re}\ \lambda \le \mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = -k\}$$

and that every complex number $\lambda$ with $\mathrm{Re}\ \lambda < -k$ is an eigenvalue. Since $G = G_0 + kI$ and the spectra of $(A_1, D_1)$ and $(G, D_G)$ coincide, the proof is complete. $\square$

Observe that the eigenvalues associated to polynomial eigenfunctions are the same for all $p \ge 1$. In fact, assuming that the eigenfunctions are polynomials, the arguments in Section 3 can be used also for $p = 1$ in order to determine the eigenvalues. However in $L^1_\mu$ there are nonpolynomial eigenfunctions and the spectrum is much larger. Moreover we have

**Corollary 4.2** *The semigroup $(T(t))_{t\ge 0}$ does not map $L^1_\mu$ into $W^{1,1}_\mu$, for any $t > 0$.*

Proof. Assume by contradiction that $T(t_0)(L^1_\mu)$ is contained in $W^{1,1}_\mu$ for some $t_0 > 0$. This implies that $T(t)(L^1_\mu) \subset W^{1,1}_\mu$ for every $t \ge t_0$. Proceeding as in Lemma 2.2, we find that $T(t)(L^1_\mu) \subset C^k(\mathbf{R}^N) \cap W^{k,1}_\mu$ for every $k \in \mathbf{N}$, $t \ge kt_0$. Remark that Lemma 3.1 holds also if $p = 1$. Arguing as in Proposition 3.2, we infer that all the eigenfunctions of $A_1$ are polynomials. Thus, by Lemma 3.3, we deduce that the point spectrum of $A_1$ is discrete. This is the desired contradiction. $\square$

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