An integrability property of positive harmonic functions

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Abstract: Let H be the set of all positive harmonic functions on a ball $B \subset \mathbb{R}^d$ and let H_0 be the (proper) subset of H consisting of all $h \in H$ that can be represented by

$$h(x) = \int_{\partial B} K(x,z) f(z)\, d\sigma(z) \quad (x \in B)$$

where K is the Poisson kernel, σ is the surface area measure on ∂B, and f is a positive σ-integrable function on ∂B. In this paper, it is shown that for any Borel measure μ on B and any real $q > 0$ the following holds:

$$H_0 \subset L^q(B,\mu) \;\Leftrightarrow\; H \subset L^q(B,\mu).$$

Further results of this kind are obtained for a general setting of harmonic spaces covering the elliptic case and the parabolic one as well. Applying these results we give, in the last section, an answer to an open question in [11] concerning a quasi-linear problem.

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1 Introduction and results

Let X be a connected locally compact space with countable base of open subsets and define $\mathcal{B}$ to be the set of all Borel measurable numerical functions on X. Throughout this paper $\mathcal{H}$ is defined to be a harmonic sheaf on X so that $(X, \mathcal{H})$ is a $\mathcal{P}-$harmonic Bauer space in the sense of [4]. We assume that every constant function is harmonic on X (i.e., $1 \in \mathcal{H}(X)$) and that $\mathcal{H}$ has the Doob convergence property, that is, the supremum of any increasing sequence of harmonic functions on an open set D of X is harmonic on D if it is finite on a dense subset of D.

We shall denote by $\mathcal{H}^+$ ($\mathcal{S}^+$ resp.) the set of all positive harmonic (superharmonic resp.) functions on X and by $\mathcal{P}$ the convex cone of all potentials on X.

Let us note that all notions and facts on abstract spaces which are used in this paper can be found in [2, 4]. The reader who is mainly interested in the solutions of the Laplace (or heat) equation may simply consider the case (i) (or (ii)) below:

(i) X is a Greenian domain in $\mathbb{R}^d$ in the sense of [5] (for instance X is a bounded domain), and the sheaf $\mathcal{H}$ is defined by: For any open subset D of X; $h \in \mathcal{H}(D)$ if and only if h is sufficiently smooth and $\Delta h = 0$ in D.
As usual, Δ denotes the Laplacian in $\mathbb{R}^d$.

(ii) X is a domain in $\mathbb{R}^d \times \mathbb{R}$ and $\mathcal{H}$ is defined by: For any open subset D of X; $h \in \mathcal{H}(D)$ if and only if h is sufficiently smooth and $(\Delta - \frac{\partial}{\partial t})h = 0$ in D.

A closed subset A of X is called an absorbing set if it contains the support of every harmonic measure μ_x^D for every $x \in A$ and every regular open relatively compact set D containing x. Let r denote a reference measure relative to the harmonic space $(X, \mathcal{H})$, that is, r is a probability measure on X such that X is the smallest absorbing set containing its support (see Remark 1 in Section 2 for an example of r). In virtue of [2, Satz 1.4.4], it is easy to see that r has the following property (P):

$$h_n \in \mathcal{H}^+ \text{ and } \sum_{n=1}^{\infty} \int_X h_n(x)\, dr(x) < \infty \;\Rightarrow\; \sum_{n=1}^{\infty} h_n \in \mathcal{H}^+.$$

For any subset $\mathcal{F}$ of $\mathcal{B}$; $\mathcal{F}_r$ will denote the set of all functions in $\mathcal{F}$ which are integrable on X with respect to r. From [10] (more details dealing with the heat equation and also the Laplace operator can be found in [5]) we know that there exists a Polish space Y and a family $(K(\cdot, y))_{y\in Y}$ of positive harmonic functions on X such that:

(α) For every $x \in X$; the function $y \mapsto K(x, y)$ is continuous on Y.

(β) For every $y \in Y$; $\int_X K(x, y)\, dr(x) = 1$.

(γ) For every $h \in \mathcal{H}_r^+$; there is a unique finite Borel measure μ on Y such that

$$h(x) = K\mu(x) := \int_Y K(x, y)\, d\mu(y) \quad (x \in X). \tag{1}$$

Conversely it is not hard to show that $K\mu \in \mathcal{H}_r^+$ for every finite Borel measure μ on Y, in particular

$$\mu(Y) = \int_X K\mu(x)\, dr(x).$$

We fix a finite Borel measure m on Y which charges all nonempty open subsets of Y and we define $\mathcal{H}_m^+$ to be the set of all harmonic functions h having the form

$$h = Kf := \int_Y K(\cdot, y) f(y)\, dm(y) \tag{2}$$

where f belongs to the set $L_+^1(Y, m)$ consisting of all positive $m-$integrable functions on Y.

Let N be a positively homogeneous increasing map on $\mathcal{B}^+$ with values in $[0,\infty]$, i.e., $N(f) \le N(g)$ for every $f, g \in \mathcal{B}^+$ such that $f(x) \le g(x)$ for all $x \in X$, and $N(\lambda f) = \lambda N(f)$ for all $\lambda \ge 0$ and all $f \in \mathcal{B}^+$. Assume moreover that N is *σ-smooth* in the sense that for any increasing sequence $(f_n) \in \mathcal{B}^+$

$$N(\sup_{n\ge1} f_n) = \sup_{n\ge1} N(f_n).$$

Our main goal is to prove the following two theorems which generalize results obtained in [12].

Theorem 1. *The following statements are equivalent:*

(a) $N(s) < \infty$ *for all* $s \in \mathcal{S}_r^+$.

(b) *There exists a constant* $C > 0$ *such that for all* $s \in \mathcal{S}_r^+$

$$N(s) \le C \int_X s(x)\, dr(x). \tag{3}$$

(c) $N(p) < \infty$ *for all* $p \in \mathcal{P}_r$.

Theorem 2. *The following statements are equivalent:*

(a) $N(h) < \infty$ *for all* $h \in \mathcal{H}_r^+$.

(b) *There exists a constant* $C > 0$ *such that for all* $h \in \mathcal{H}_r^+$

$$N(h) \le C \int_X h(x)\, dr(x). \tag{4}$$

(c) $N(h) < \infty$ *for all* $h \in \mathcal{H}_m^+$.

2 Consequences and remarks

Let $s \in \mathcal{S}^+$. Since s is finite on a dense subset of X (see [4, p. 37]) we may choose $(x_n) \in X$ and a sequence (a_n) of strictly positive reals so that

$$\sum_{n=1}^{\infty} a_n = 1 \text{ and } \sum_{n=1}^{\infty} a_n s(x_n) < \infty.$$

Therefore $r := \sum_{n=1}^{\infty} a_n \delta_{x_n}$ is a reference measure relative to $(X, \mathcal{H})$ and obviously s is r-integrable on X. Here and in the sequel, δ_x denotes the Dirac measure concentrated at the point x. Now, the following corollary follows immediately from Theorems 1 and 2.

Corollary 1. *The following statements are equivalent:*

(a) $N(u) < \infty$ *for all* $u \in \mathcal{S}^+$ ($\mathcal{H}^+$ *resp.*).

(b) *For every reference measure* r *relative to* $(X, \mathcal{H})$, *there exists* $C > 0$ *such that the inequality*

$$N(u) \leq C \int_X u(x)\, dr(x)$$

holds for all $u \in \mathcal{S}^+$ ($\mathcal{H}^+$ *resp.*).

(c) $N(p) < \infty$ *for all* $p \in \mathcal{P}$.

Remark 1. Consider the case (i) already mentioned in the introduction. In this elliptic setting, X and the empty set are the only absorbing sets and consequently all probability measures on X are reference measures relative to $(X, \mathcal{H})$.

Let $x_0 \in X$ and let q be a strictly positive real. For $r = \delta_{x_0}$ (remark that here $\mathcal{H}_r^+ = \mathcal{H}^+$) and for N defined on $\mathcal{B}^+$ by

$$N(f) = \left(\int_X f(x)^q\, dx \right)^{1/q},$$

Theorems 1 and 2 are proved by N. Suzuki [12]. Our method, especially the proof of $(a) \Leftrightarrow (b)$, seems to be very easy and it is completely different from the proof given in [12] which is based on a theorem of I. M. Gelfand in functional analysis (see [8])

Remark 2. To obtain (a)$\Leftrightarrow$(b) in Theorems 1 and 2 we do not need to assume that N is σ-smooth. Furthermore, in Theorem 2 the equivalence (a)$\Leftrightarrow$(b) is valid for any measure r on X which has the property (P). In other words, the fact that $\mathcal{H}$ possesses the Doob convergence property is not a necessary assumption. Notice that the existence of a measure r with property (P) is assured by [4, Theorem 11.1.2] if $\mathcal{H}$ has the nuclearity property. See [4, p. 276] for the definition (and various equivalent conditions) of the nuclearity property for harmonic sheaves.

We say that a harmonic function on X is quasi-bounded if it is the limit of some increasing sequence of positive bounded harmonic functions on X.

Corollary 2. *Suppose that* $X = B$ *is the open unit ball in* $\mathbb{R}^d$ *and let* $\mathcal{H}$ *be the classical sheaf of harmonic functions on* B. *If* $N(h) < \infty$ *for every quasi-bounded harmonic function* h *on* B *then* N *is finite on* $\mathcal{H}^+(B)$.

Proof. It is well known (see, e.g., [5]) that every quasi-bounded harmonic function on B is given by the formula (2) where K is the Poisson kernel, $Y = \partial B$, and m is the surface area measure on ∂B. Therefore, taking $r = \delta_0$ the corollary is a consequence of Theorem 2. □

The above corollary will be used in Section 4 in order to show that the converse statement in [11, Corollary 1.14] does not hold in general.

Remark 3. Analogous conclusion as in Corollary 2 can be shown for a wide class of harmonic spaces $(X, \mathcal{H})$. In fact, in virtue of Theorem 2, it would be enough that the measure m satisfying $Km \equiv 1$ charges every nonempty open subset of Y. (Of course Corollary 2 fails in general.)

3 Proofs of Theorems 1 and 2

Let $(E, \mathcal{E}, \nu)$ be a measure space and denote by $L^1(\nu)$ the set of all ν-integrable numerical functions on E. A convex cone $\mathcal{F}$ of positive measurable functions on E will be called Σ_ν-*stable* if for any sequence $(f_n) \in \mathcal{F}$

$$\sum_{n=1}^{\infty} f_n \in L^1(\nu) \;\Rightarrow\; \sum_{n=1}^{\infty} f_n \in \mathcal{F}.$$

Lemma 1. *Let $(E, \mathcal{E}, \nu)$ be a measure space and let $\mathcal{F}$ be a Σ_ν-stable convex cone of positive measurable functions on E. If $M : \mathcal{F} \to [0, \infty]$ is a positively homogeneous increasing functional then the following statements are equivalent:*
(a) *$M(f) < \infty$ for all $f \in \mathcal{F} \cap L^1(\nu)$.*
(b) *There exists $C > 0$ such that for every $f \in \mathcal{F}$*

$$M(f) \le C \int_E f(x)\, d\nu(x). \tag{5}$$

Proof. (b)⇒(a): Trivial.
(a)⇒(b): Under assumption (a) we claim that M is bounded on the set

$$B_{\mathcal{F}} := \left\{ f \in \mathcal{F} : \int_E f(x)\, d\nu(x) \le 1 \right\}.$$

Indeed, assuming the contrary we can find $(f_n) \in B_{\mathcal{F}}$ such that for every $n \ge 1$

$$M(f_n) \ge n\, 2^n.$$

Therefore, according to the monotone convergence theorem and the fact that $\mathcal{F}$ is Σ_ν-stable, the function $f_0 := \sum_{n=1}^{\infty} 2^{-n} f_n \in B_{\mathcal{F}}$. On the other hand, seeing that

$$M(f_0) \ge 2^{-n} M(f_n) \ge n$$

for every $n \geq 1$, we conclude that $M(f_0) = \infty$. This contradicts (a) and proves the claim. So, taking

$$C := \sup_{f \in B_{\mathcal{F}}} M(f),$$

it follows that (5) holds for every function $f \in \mathcal{F}$ with strictly positive ν-integral over E. Besides, if $f \in \mathcal{F}$ vanishes ν-almost everywhere on E then $nf \in B_{\mathcal{F}}$ for all $n \geq 1$ and consequently

$$M(f) = M(nf)/n \leq C/n.$$

Thus the proof of the lemma is complete. □

Lemma 2. *The convex cones $\mathcal{H}^+$, $\mathcal{H}^+_m$, $\mathcal{S}^+$, and $\mathcal{P}$ are Σ_r-stable for every reference measure r relative to $(X, \mathcal{H})$.*

Proof. The Σ_r-stability of $\mathcal{H}^+$ is given by the property (P). To see that $\mathcal{H}^+_m$ is Σ_r-stable it is sufficient to remark that $K(\sum_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} K f_n$ for any sequence (f_n) of positive Borel measurable functions on Y. On the other hand, if $\mathcal{S}^+$ is Σ_r-stable then $\mathcal{P}$ possesses the same property. This is a consequence of the fact that the sum of a countable family of potentials is also a potential whenever it is superharmonic on X [4, Proposition 2.2.2]. So it only remains to prove that $\mathcal{S}^+$ is Σ_r-stable.

Let $(s_n) \in \mathcal{S}^+$ and set $s := \sum_{n=1}^{\infty} s_n$. Since s is a limit of an increasing sequence in $\mathcal{S}^+$ we deduce that s is hyperharmonic on X (here, this simply means that s is superharmonic or $s \equiv \infty$ on X) and therefore

$$A := \overline{\{x \in X : s(x) < \infty\}}$$

is an absorbing set by [2, Satz 1.4.2]. So, if s is integrable on X with respect to r then A contains the support of r which yields that $A = X$ by definition of the reference measure r. Whence s is superharmonic on X in virtue the Doob convergence property. □

Proof of Theorem 1. (b)⇒(a)⇒(c): Trivial.

(c)⇒(b): Combining Lemmas 1 and 2 we find a constant $C > 0$ for which (3) is valid for any $s \in \mathcal{P}$. Since every superharmonic function is the supremum of an increasing sequence of potentials [4, Proposition 2.3.1], statement (b) follows by the monotone convergence theorem and the σ-smoothness of N. □

Let $\mu, \mu_1, \mu_2, \cdots$ be finite Borel measures on Y. We say that (μ_n) converges weakly to μ provided

$$\lim_{n \to \infty} \int_Y \varphi(y)\, d\mu_n(y) = \int_Y \varphi(y)\, d\mu(y)$$

for every bounded continuous function φ on Y. To prove the implication (c)⇒(b) in Theorem 2 we need the following lemma which is well known if Y is locally compact.

Lemma 3. *Every finite Borel measure μ on Y is a weak limit of a sequence $(f_n m)$ where $f_n \in L^1_+(Y, m)$ for all $n \geq 1$.*

Proof. Choose a metric d on Y which is compatible with the topology of Y. Take $z \in Y$ and define for every $n \geq 1$

$$V_n := \{y \in Y : d(y, z) < 1/n\}.$$

Clearly V_n is a nonempty open subset of Y and thereby $\alpha_n := m(V_n) > 0$ by assumption on m. Consider the function g_n defined on Y by $g_n(y) = 1/\alpha_n$ if $y \in V_n$ and $g_n(y) = 0$ elsewhere. Then

$$\left|\varphi(z) - \int_Y g_n(y)\varphi(y)\, dm(y)\right| \leq \int_{V_n} |\varphi(z) - \varphi(y)| g_n(y)\, dm(y) \leq \frac{1}{n}$$

for every bounded function φ on Y satisfying the Lipschitz condition

$$|\varphi(y_1) - \varphi(y_2)| \leq d(y_1, y_2) \quad (y_1, y_2 \in Y).$$

Therefore, according to Theorem 2.5.17 in [7], $(g_n m)$ converges weakly (narrowly, in the terminology of [7]) to the Dirac measure δ_z. This finishes the proof of the lemma because μ is a weak limit of some sequence of finite Borel measures on Y with finite support (see [3, Appendix III]). □

Proof of Theorem 2. (b)⇒(a)⇒(c): Trivial.

(c)⇒(b): Remark first that the fact that N is increasing and σ-smooth on $\mathcal{B}^+$ yields that for every sequence $(f_n) \in \mathcal{B}^+$

$$N(\liminf_{n\to\infty} f_n) \leq \liminf_{n\to\infty} N(f_n). \tag{6}$$

Suppose now that N is finite on $\mathcal{H}^+_m$. Since $\mathcal{H}^+_m$ is Σ_r-stable, Lemma 1 assures the existence of a constant $C > 0$ such that for every $f \in L^1_+(Y, m)$

$$N(Kf) \leq C \int_Y f(y)\, dm(y). \tag{7}$$

Let $h \in \mathcal{H}^+_r$, μ be the finite Borel measure on Y satisfying $h = K\mu$, and let $f_1, f_2, \cdots \in L^1(Y, m)$ as in Lemma 3. For any $i \geq 1$ let $K_i := \inf(i, K)$ and define $K_i f$ ($K_i\mu$ resp.) in a similar way to Kf ($K\mu$ resp.). From (7) it follows that

$$N(K_i f_n) \leq C \int_Y f_n(y)\, dm(y) \tag{8}$$

for all $i, n \geq 1$. On the other hand, since $K_i(x, \cdot)$ is bounded and continuous on Y for any fixed $x \in X, i \geq 1$, the sequence $(K_i f_n)_n$ is pointwise convergent to $K_i\mu$. Therefore letting $n \to \infty$ in (8) we conclude, in accordance with (6), that the inequality

$$N(K_i\mu) \leq C\mu(Y)$$

holds for every $i \geq 1$. Whence, recalling that $\mu(Y) = \int_X h(x)\,dr(x)$ and letting $i \to \infty$, we obtain the desired inequality (4). □

4 Application

In this section, $X = B$ is the open unit ball of $\mathbb{R}^d$, $d \geq 2$, and $\mathcal{H}$ is the classical sheaf of harmonic functions on B. The space Y is the unit sphere ∂B, the formula (1) is the Poisson integral, and m is the surface area measure on ∂B.

In [11] (see also further references in [11]), M. Marcus and L. Véron investigated the quasi-linear problem

$$\begin{aligned} \Delta u &= u^q \quad \text{in } B, \\ u &= \mu \quad \text{on } \partial B, \end{aligned} \tag{9}$$

where q is a real > 1 and μ is a finite Borel measure on ∂B. Following [11] we say that μ is a *q-trace* (of u on ∂B) if the problem (9) is solvable and define $\mathcal{M}_q^+(\partial B)$ to be the set of all finite Borel measures on ∂B which are q-traces.

A finite Borel measure μ on ∂B is called (again following [11]) *q-admissible* provided

$$\int_B h(x)^q (1 - |x|)\,dx < \infty \tag{10}$$

where $h = K\mu$. It is known [9] (see [6, 11] for a characterization of $\mathcal{M}_q^+(\partial B)$) that every q-admissible measure is a q-trace. However, whether the converse statement holds remains an open question in [11] (see *Remark.* page 497).

The following corollary shows that for

$$q \geq (d+1)/(d-1)$$

the set of q-admissible Borel measures on ∂B is always a proper subset of $\mathcal{M}_q^+(\partial B)$.

Corollary 3. *Let q be a real > 1 such that all finite Borel measures on ∂B which are q-traces are also q-admissible. Then $q < (d+1)/(d-1)$.*

Proof. We know that the class $\mathcal{M}_q^+(\partial B)$ contains every measure of the form fm where $f \in L^1_+(\partial B, m)$ (see [6] or [11]). Therefore, if all q-traces are q-admissible it follows that the mapping M given by

$$M(h) = \left(\int_B h(x)^q (1 - |x|)\, dx \right)^{1/q}$$

is finite for every quasi-bounded harmonic function h on B, and consequently M is finite on $\mathcal{H}^+(B)$ by Corollary 2. But by classical computations (as in [1]) it is not difficult to see that (10) holds for every function $h \in \mathcal{H}^+(B)$ if and only if $q < (d+1)/(d-1)$. □

We then conclude that for $q \geq (d+1)/(d-1)$ there exists a measure in the class $\mathcal{M}_q^+(\partial B)$ which is not q-admissible. Remark that this measure can be chosen of the form fm where $f \in L^1_+(\partial B, m)$.

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