Semilinear perturbations of harmonic spaces and Martin-Orlicz capacities: An approach to the trace of moderate $\mathcal{U}$-functions

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Abstract: Let $(X, \mathcal{H})$ be a harmonic space in the sense of H. Bauer [6] which has a Green function G_X. It is known [25] that to every reference measure r there corresponds a suitable integral representation of functions in $\mathcal{H}^+(X) \cap L^1(X, r)$. Let Y be the minimal Martin boundary, P the Martin kernel, and denote by $\mathcal{M}(Y)$ the set of all signed Borel measures on Y with bounded variation. In this paper we consider the perturbed (semilinear) structure $(X, \mathcal{U})$ obtained from $(X, \mathcal{H})$ by means of (γ, Ψ) where γ is a local Kato measure on X and Ψ belongs to a class of real-valued functions on $X \times \mathbb{R}$ containing $(x, t) \mapsto t|t|^{\alpha-1}$ for any $\alpha > 1$. We show that for every function $u \in \mathcal{U}_r(X) := \{u \in \mathcal{U}(X) : |u| \leq h$ for some $h \in \mathcal{H}^+(X) \cap L^1(X, r)\}$ there exists a unique signed measure $\nu \in \mathcal{M}(Y)$ such that

$$u + \int_X G_X(\cdot, \zeta)\Psi(\zeta, u(\zeta))\, d\gamma(\zeta) = \int_Y P(\cdot, y)\, d\nu(y).$$

Conversely, we prove that this integral equation admits a solution $u \in \mathcal{U}_r(X)$ whenever ν does not charge compact sets $K \subset Y$ of zero Martin-Orlicz capacity, that is, the integral $\int_X \int_X G_X(x, \zeta)\Psi(\zeta, \int_Y P(\zeta, y)\, d\mu(y))\, d\gamma(z)\, dr(x)$ is equal to 0 or ∞ for every $\mu \in \mathcal{M}^+(Y)$ such that $\mu(Y \backslash K) = 0$. In Section 6, we use our approach to investigate the trace of moderate solutions to some semilinear equations.

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1 Introduction

Let r be a reference measure relative to a given harmonic space $(X, \mathcal{H})$ in the sense of H. Bauer [6], and let $\mathcal{H}_r^+(X)$ be the set of all positive harmonic functions on X (i.e., which belong to $\mathcal{H}(X)$) which are r-integrable. Developing an integral representation of functions in $\mathcal{H}_r^+(X)$, K. Janssen determined in [25] a Polish space Y (minimal Martin boundary) and a function $P : X \times Y \to \mathbb{R}_+$ (Martin kernel) such that:

Theorem 1.1 ([25]). *Every harmonic function* $h \in \mathcal{H}_r(X) := \mathcal{H}_r^+(X) - \mathcal{H}_r^+(X)$ *has a unique representation*

$$h(x) = P\nu(x) := \int_Y P(x,y)\, d\nu(y) \quad (x \in X) \tag{1.1}$$

where ν *belongs to the set* $\mathcal{M}(Y)$ *of all signed Borel measures on* Y *with bounded variation. Conversely,* $P\nu \in \mathcal{H}_r^+(X)$ *for any* $\nu \in \mathcal{M}(Y)$.

In this paper we are interested in the analogous representation problem in a non-linear setting. To simplify the presentation of our approach let us suppose that the harmonic space $(X, \mathcal{H})$ possesses a Green function G_X (see [10, Sect. 4]), and assume that $1 \in \mathcal{H}(X)$. Standard examples of $(X, \mathcal{H})$ are:

1. (Elliptic case) X is a Greenian domain of $\mathbb{R}^d$ and $\mathcal{H}$ is the sheaf of classical harmonic functions (i.e., solutions to the Laplace equation).
2. (Parabolic case) X is a domain of $\mathbb{R}^d \times \mathbb{R}$ and $\mathcal{H}$ is the sheaf of parabolic functions in the terminology of [16] (i.e., solutions to the heat equation).

Any probability measure can serve as reference measure in Example 1, while this is not true in Example 2. However, a probability measure whose support is the whole space X is always a reference measure relative to $(X, \mathcal{H})$.

Let Ψ be a function in $\mathbb{Y}(X)$ having the doubling property (see Subsection 2.6, for instance $\Psi(x,t) = t|t|^{\alpha-1}$ where $\alpha > 1$), and consider a positive Radon measure γ on X in the local Kato class $\mathbb{K}_{loc}^+(X)$, i.e., such that $\int_K G_X(\cdot,\zeta)\, d\gamma(\zeta)$ is a bounded continuous potential on X for every compact set $K \subset X$. A continuous function u on X is called a $\mathcal{U}$-function if, for every open relatively compact subset D of X, the function $u + \int_D G_D(\cdot,\zeta)\Psi(\zeta, u(\zeta))\, d\gamma(\zeta)$ is harmonic on D. If moreover $|u| \le h$ for some $h \in \mathcal{H}_r^+(X)$, we say that u is moderate. We denote by $\mathcal{U}(X)$ the set of all $\mathcal{U}$-functions on X and by $\mathcal{U}_r(X)$ the set of all moderate functions in $\mathcal{U}(X)$. First, we establish the following existence result:

Proposition 1.2. *For every moderate* $\mathcal{U}$*-function* u *on* X*, there exists a unique measure* $\nu \in \mathcal{M}(Y)$*, which will be denoted by* $tr(u)$ *and called the trace of* u *on* Y*, such that*

$$u(x) + \int_X G_X(x,\zeta)\Psi(\zeta, u(\zeta))\, d\gamma(\zeta) = P\nu(x) \quad (x \in X). \tag{1.2}$$

Moreover, for all $u, v \in \mathcal{U}_r(X)$*,* $u \ge v$ *if and only if* $tr(u) \ge tr(v)$.

We then extend the first part of Theorem 1.1 to the perturbed semilinear structure $(X, \mathcal{U})$ (observe that for $\gamma = 0$, $\mathcal{U}_r(X) = \mathcal{H}_r(X)$ and $\nu = tr(u)$ means that $u = P\nu$). Furthermore, although it may happen that (1.2) is not solvable for a given $\nu \in \mathcal{M}(Y)$ (see [21]), the last part of the above proposition assures that (1.2) admits at most one solution $u \in \mathcal{U}_r(X)$. This function u is interpreted as the solution of the (boundary value) problem

$$u \in \mathcal{U}_r(X) \quad \text{and} \quad u = \nu \ \text{ on } Y. \tag{1.3}$$

In other words, (1.3) is considered to be equivalent to the integral equation (1.2).

The main purpose of this paper is to investigate the set $\mathcal{Q}_\Psi(Y)$ consisting of all $\nu \in \mathcal{M}(Y)$ for which (1.3) possesses a solution $u \in \mathcal{U}_r(X)$.

Remark 1.3. [Details are in Subsection 6] *Let $\gamma \in \mathbb{K}^+_{loc}(\mathbb{R}^d)$, $\Psi \in \mathbb{Y}(\mathbb{R}^d)$, and consider Example* 1 *where $X = B$ is the unit open ball of $\mathbb{R}^d$. Then $Y = \partial B$ and a continuous function u on B is a solution of* (1.3) *if and only if it is a solution of the boundary value problem*

$$\begin{aligned} \Delta u &= \Psi(\cdot, u)\gamma \quad \textit{in } B, \\ u &= \nu \qquad\qquad \textit{on } \partial B. \end{aligned} \tag{1.4}$$

In particular, (1.4) *is solvable for every $\nu = f\sigma$ where f is a continuous function on ∂B and σ is the surface area measure on ∂B. Furthermore, the boundary condition $u = \nu$ means, in this case, that $\lim_{x\to y} u(x) = f(y)$ for all $y \in \partial B$.*

By means of minimal thin subsets of X, we established in [20] necessary and sufficient conditions under which a given positive finite measure ν on Y is a trace of some moderate $\mathcal{U}$-function on X. In the present paper, we discuss the solvability of problem (1.3) by investigating some exceptional subsets of Y.

Definitions. A Borel set $E \subset Y$ is called removable if for every $\nu \in \mathcal{M}^+(E)$ (i.e., $\nu \in \mathcal{M}^+(Y)$ such that $\nu(Y \backslash E) = 0$) the following holds:

$$u \in \mathcal{U}(X) \text{ and } 0 \le u \le P\nu \quad \Rightarrow \quad u \equiv 0 \text{ on } X.$$

We say that E is c_Ψ-polar if for every $\nu \in \mathcal{M}^+(E)$ the following holds:

$$\int_X \int_X G_X(x,\zeta)\Psi(\zeta, P\nu(\zeta))\, d\gamma(z)\, dr(x) < \infty \quad \Rightarrow \quad \nu = 0.$$

In the situation of Example 1 and assuming that X is bounded and Lipschitz, it will be shown (see Subsection 6.4) that a Borel subset E of ∂X ($Y = \partial X$) is removable if and only if for every $u \in \mathcal{U}_r^+(X)$,

$$u = 0 \text{ on } \partial X \backslash E \quad \Rightarrow \quad u \equiv 0 \text{ on } X.$$

A tool of vital importance in our study (especially in the proof of Theorem 1.5 below) is the Martin-Orlicz capacity c_Ψ defined for every Borel subset $E \subset Y$ by

$$c_\Psi(E) = \sup\left\{\nu(E) : \nu \in \mathcal{M}^+(E) \text{ and } \|P\nu\|_\Psi \le 1\right\}$$

where $\|\cdot\|_\Psi$ is the Orlicz norm in the Orlicz type space $L_\Psi(X)$ consisting of all (classes of equivalent) Borel measurable functions f on X such that

$$\int_X \int_X G_X(x,\zeta)\Psi(\zeta,|f(\zeta)|)\, d\gamma(z)\, dr(x) < \infty$$

(for this characterization of $L_\Psi(X)$ the doubling property of Ψ is used).

Notice that c_Ψ-polar sets are subsets E of Y such that $c_\Psi(E) = 0$.

Among the important properties of $\mathcal{Q}_\Psi(Y)$, we shall prove that $\nu \in \mathcal{Q}_\Psi(Y)$ if and only if $|\nu| \in \mathcal{Q}_\Psi(Y)$. This allows us to restrict our study of the solvability of problem (1.3) to the case when ν is positive. In particular, it will be not difficult to prove:

Theorem 1.4. *If $\nu \in \mathcal{Q}_\Psi(Y)$ then all removable subsets of Y are ν-null sets.*

Imposing some additional assumptions on γ, we give sufficient conditions for (1.3) to be solvable. More precisely, we obtain the following result:

Theorem 1.5. *If all c_Ψ-polar subsets of Y are ν-null sets then $\nu \in \mathcal{Q}_\Psi(Y)$.*

Consider once again Example 1 where X is assumed to be bounded and sufficiently smooth. Then, for $r = \delta_{x_0}$ ($x_0 \in X$), Y can be identified with the Euclidean boundary ∂X of X, and P is the normalized ($P(x_0,\cdot) \equiv 1$) Martin kernel on X (here a possible choice for γ is the restriction of the d-dimensional Lebesgue measure λ to X, but γ might as well be singular with respect to λ).

Let $\gamma = \lambda|_X$ and $\Psi(x,t) = t|t|^{\alpha-1}$, $\alpha > 1$. Then, for every $\nu \in \mathcal{M}^+(\partial X)$, (1.3) is equivalent to the boundary value problem

$$\begin{array}{rcll} \Delta u &=& u^\alpha & \text{in } X, \\ u &=& \nu & \text{on } \partial X, \end{array} \tag{1.5}$$

which has been investigated by various techniques (see [21, 28, 18, 17, 30]). In this setting, $L_\Psi(X)$ is a classical Lebesgue space and c_Ψ coincides with the Martin capacity c_α introduced in [17]. It is shown (Le Gall [28] for $\alpha = 2$, Dynkin and Kuznetsov [18] for $\alpha \le 2$, Marcus and Véron [30] for $\alpha > 2$) that for every Borel subset E of ∂X, E is removable if and only if $c_\alpha(E) = 0$. Consequently, (1.5) has a solution if and only if ν does not charge c_α-polar subsets of ∂X. It will be shown that, in general, this condition does not characterize the class $\mathcal{Q}_\Psi(Y)$. In fact, we shall give an example (see Remark 6.5) for which the converse statement in Theorem 1.5 does not hold.

Plan of the paper. After recalling in Section 2 the basic notions and facts on harmonic spaces, we study in Section 3 semilinear perturbations of harmonic spaces. In Section 4, we introduce the trace of a moderate $\mathcal{U}$-function and give its first properties. In the last part of the same section, we investigate removable sets and prove Theorem 1.4 (Proposition 4.4). Section 5 deals with the Martin-Orlicz capacity c_Ψ and the proof of Theorem 1.5 (Theorem 5.7). Finally, as application of our work, Section 6 is devoted to a study of semilinear problems of the type (1.4).

2 Preliminaries

In the following $(X, \mathcal{H})$ will always denote a harmonic space in the sense of H. Bauer [6] such that the constant functions are harmonic on X. We shall recall in this section the basic notions and facts on harmonic spaces that we need (for more details see [6, 14, 16, 4]). The reader who is not familiar with these notions and is mainly interested in boundary value problems of the kind (1.4) may simply restrict himself to Example 1 already mentioned in the introduction. Section 6 will deal explicitly with this situation.

2.1. Basic Notations. Given a set $\mathcal{F}$ of numerical functions, $\mathcal{F}_b$ ($\mathcal{F}^+$ resp.) will denote the set of all functions in $\mathcal{F}$ which are bounded (positive resp.). For every open subset Ω of X let $\mathcal{B}(\Omega)$ ($\mathcal{C}(\Omega)$ resp.) be the set of all Borel measurable numerical (continuous real resp.) functions on Ω. By $\mathcal{B}_{bc}(\Omega)$ we shall denote the set of all functions in $\mathcal{B}_b(\Omega)$ with compact support in Ω.

For $A \subset X$ we denote by A^c the complement of A in X and define 1_A to be the characteristic function of A: $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in A^c$.

Given a topological space T, $\mathcal{M}(T)$ will denote the set of all signed Borel measures μ on T such that $\|\mu\| = |\mu|(T)$ is finite. Recall that $|\mu| = \mu^+ + \mu^-$ where $\mu^+ = \sup(\mu, 0)$ and $\mu^- = \sup(-\mu, 0)$. For any Borel set $E \subset T$, we denote by μ_E the restriction of μ to E and by $\mathcal{M}(E)$ the set of all $\mu \in \mathcal{M}(T)$ which are supported by E (i.e., $\mu(T \backslash E) = 0$). Finally, by a *kernel* on T we shall mean a family $(k(\tau, \cdot))_{\tau \in T}$ of Borel measures on T such that $\int f(t) k(\cdot, dt) =: kf \in \mathcal{B}^+(T)$ for every $f \in \mathcal{B}^+(T)$.

2.2. Harmonic Kernels. Let $\mathcal{O}$ be the set of all open relatively compact subsets of X and let $\Omega \in \mathcal{O}$. A Borel measurable function f on $\partial\Omega$ is *resolutive* if and only if f is μ_x^Ω-integrable for all $x \in \Omega$ where μ_x^Ω is the *harmonic measure* of x with respect to Ω (see [6]). To each resolutive function $f \in \mathcal{B}(\partial\Omega)$ we associate the harmonic function $H_\Omega f$ on Ω given by

$$H_\Omega f(x) = \int_{\partial\Omega} f(y)\, d\mu_x^\Omega(y)$$

If $f \in \mathcal{B}(X)$ such that the restriction of f to $\partial\Omega$ is resolutive we define

$$H_\Omega f(x) = \begin{cases} H_\Omega(f|_{\partial\Omega})(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in X\backslash\Omega. \end{cases}$$

We call H_Ω the *harmonic kernel* associated to Ω. A point $z \in \partial\Omega$ is called *regular* provided $\lim_{\Omega \ni x \to z} H_\Omega f(x) = f(z)$ for every $f \in \mathcal{C}(\partial\Omega)$, and we say that Ω is *regular* if all points $z \in \partial\Omega$ are regular.

2.3. Superharmonic Functions, Potentials. For every open subset Ω of X let $\mathcal{S}(\Omega)$ be the set of all lower semicontinuous (l.s.c) functions $s > -\infty$ on Ω such that for every $D \in \mathcal{O}$ with $\overline{D} \subset \Omega$,

$$H_D s \in \mathcal{H}(D) \ \text{ and } \ H_D s \leq s.$$

Functions in $\mathcal{S}(\Omega)$ ($-\mathcal{S}(\Omega)$ resp.) are called *superharmonic* (*subharmonic* resp.) on Ω. A *potential* on Ω is a function $p \in \mathcal{S}^+(\Omega)$ such that the constant zero is the greatest harmonic minorant of p on Ω. Let $\mathcal{P}(\Omega)$ denote the set of all potentials on Ω.

We suppose that $\mathcal{P}(X)$ contains a strictly positive function on X.

2.4. Potential Kernels. Throughout this paper we fix a *potential kernel* V_X on X, that is, V_X is a kernel on X such that for every $f \in \mathcal{B}^+_{bc}(X)$

$$V_X f \in \mathcal{P}(X) \cap \mathcal{C}_b(X) \cap \mathcal{H}\left(X\backslash\overline{\{f \neq 0\}}\right). \tag{2.1}$$

If moreover $V_X(1_D) \not\equiv 0$ on X for every nonempty open subset D of X we shall say that the potential kernel V_X is *strictly positive*. For each $\Omega \in \mathcal{O}$ (open and relatively compact) we define

$$V_\Omega := V_X - H_\Omega V_X. \tag{2.2}$$

Then V_Ω is a potential kernel on Ω and $V_\Omega(\mathcal{B}^+_b(\Omega)) \subset \mathcal{P}(\Omega) \cap \mathcal{C}_b(\Omega)$. Furthermore, it is not hard to verify that the family $(V_\Omega)_{\Omega \in \mathcal{O}}$ is *compatible*, in the sense that for any $\Omega_1, \Omega_2 \in \mathcal{O}$ and any $f \in \mathcal{B}_b(\Omega_1 \cup \Omega_2)$

$$V_{\Omega_1} f - V_{\Omega_2} f \in \mathcal{H}(\Omega_1 \cap \Omega_2).$$

Remark 2.1. Suppose that for every $\Omega \in \mathcal{O}$, W_Ω is a potential kernel on Ω so that $(W_\Omega)_{\Omega \in \mathcal{O}}$ is compatible. Then, in view of [6, Satz 5.3.6] there exists a unique potential kernel W_X on X such that $W_\Omega = W_X - H_\Omega W_X$ for every $\Omega \in \mathcal{O}$. More on potential kernels (also for balayage spaces) can be found in [23, Sect.2].

Assuming that X has a (continuous) Green function G_X (see [10] for the definition of G_X), a positive Radon measure γ on X is called a *local Kato measure* on X if V_X^γ defined by

$$V_X^\gamma f := \int_X G_X(\cdot,\zeta) f(\zeta)\, d\gamma(\zeta) \tag{2.3}$$

is a potential kernel on X. Notice that V_X^γ is strictly positive if and only if γ charges every nonempty subset of X.

2.5. Admissible Pairs. A closed subset A of X is called an *absorbing set* if it contains the support of every harmonic measure μ_x^D for any $x \in A$ and any regular open relatively compact set D containing x. We say that a probability measure on X is a *reference measure* if the only absorbing set containing its support is the whole space X. A pair (V, r) of a potential kernel V on X and a reference measure r on X will be said to be *admissible* if the following conditions are fulfilled:

(AP1) V is strictly positive.

(AP2) For every compact subset $K \subset X$, there are $\Omega \in \mathcal{O}$ and $c > 0$ such that $K \subset \Omega$ and the inequality

$$\sup_{x\in K} |h(x)| \le c \int_\Omega V_\Omega |h|\, dr \tag{2.4}$$

holds for all $h \in \mathcal{H}_b(\Omega)$.

We say that (γ, r) is an admissible pair provided γ is a local Kato measure on X and conditions (AP1)-(AP2) hold for $V = V_X^\gamma$ given by (2.3). See Section 6 for some examples of admissible pairs.

2.6. Young Functions. An odd strictly increasing function $Y : \mathbb{R} \to \mathbb{R}$ will be called a *Young function* if it is convex on $\mathbb{R}_+$, $\lim_{t\to 0} Y(t)/t = 0$ and $\lim_{t\to\infty} Y(t)/t = \infty$. Let $\mathbb{Y}_0$ be the set of all Young functions and define $\mathbb{Y}(X)$ to be the class of all Borel measurable functions $\Psi : X \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) The functions $\Psi(x, \cdot)$ are in $\mathbb{Y}_0$ for all $x \in X$.

(ii) For every compact subset K of X there exist $M_K, N_K \in \mathbb{Y}_0$ such that

$$M_K(t) \le \Psi(x,t) \le N_K(t) \;\text{ for all } (x,t) \in K \times \mathbb{R}_+.$$

Clearly $\mathbb{Y}_0 \subset \mathbb{Y}(X)$ and for any $\Psi \in \mathbb{Y}(X)$ the following holds:

(A_1) For every $x \in X$, $\Psi(x, \cdot)$ is continuous, odd, and increasing on $\mathbb{R}$.

(A_2) The function Ψ is locally bounded on $X \times \mathbb{R}$.

(A_3) $\Psi(x, t+s) \geq \Psi(x,t) + \Psi(x,s)$ for all $x \in X$ and all $t, s \geq 0$.

(A_4) For every $x \in X$, $\Psi(x, \cdot)$ is convex on $\mathbb{R}_+$.

To each $\Psi \in \mathbb{Y}(X)$ we associate the function Ψ^* defined on $X \times \mathbb{R}$ by

$$\Psi^*(x,t) = \operatorname{sgn}(t) \sup_{s \geq 0} \left(s|t| - \Psi(x,s) \right). \tag{2.5}$$

It is well known (see, e.g., [26]) that $\Psi^* \in \mathbb{Y}_0$ for any $\Psi \in \mathbb{Y}_0$. Analogously, it is easy to remark that $\Psi^* \in \mathbb{Y}(X)$ and $(\Psi^*)^* = \Psi$ if $\Psi \in \mathbb{Y}(X)$.

We shall say that a real function Ψ on $X \times \mathbb{R}$ has the *doubling property* if there exists a constant $\kappa > 0$ such that

$$\Psi(x, 2t) \leq \kappa \Psi(x,t) \quad \text{for all } (x,t) \in X \times \mathbb{R}_+. \tag{2.6}$$

In the theory of Orlicz space, this property is known as Δ_2-*condition.* If $\Psi \in \mathbb{Y}(X)$, it can be shown that Ψ^* possesses the doubling property if and only if the function Ψ satisfies the ∇_2-*condition*: There exists $\ell > 1$ such that

$$\Psi(x, \ell t) \geq 2\ell \Psi(x,t) \quad \text{for all } (x,t) \in X \times \mathbb{R}_+. \tag{2.7}$$

3 Basic tools

Assumptions of this section: Ψ is a Borel measurable real function on $X \times \mathbb{R}$ which satisfies (A_1) and (A_2).

3.1. Perturbations Generated by the Pair (V, Ψ). For every $\Omega \in \mathcal{O}$ (or $\Omega = X$) we define

$$V_\Omega^\Psi f := V_\Omega \Psi(\cdot, f) \tag{3.1}$$

whenever the right side in (3.1) has a sense. Then, for any open set D such that $\overline{D} \subset \Omega$ we easily see, in view of (2.2), that

$$V_\Omega^\Psi = V_D^\Psi + H_D V_\Omega^\Psi. \tag{3.2}$$

Notice that for $\Omega = X$ we may write V instead of V_X and V^Ψ instead of V_X^Ψ.

Proposition 3.1. (Comparison principle) *Let $\Omega \in \mathcal{O} \cup \{X\}$ and let f, g be two real Borel measurable functions on Ω such that $V_\Omega^\Psi|f|$ and $V_\Omega^\Psi|g|$ are finite potentials on Ω and the function $f - g + V_\Omega^\Psi f - V_\Omega^\Psi g$ is superharmonic on Ω. Then*

$$f \geq g \text{ if and only if } f + V_\Omega^\Psi f \geq g + V_\Omega^\Psi g.$$

Proof. Since $\Psi(x,\cdot)$ is increasing for any $x \in X$ we easily see that $f + V_\Omega^\Psi f \geq g + V_\Omega^\Psi g$ whenever $f \geq g$ on Ω. To prove the converse statement let

$$\phi = \Psi(\cdot, f) - \Psi(\cdot, g)$$

and suppose that $f + V_\Omega^\Psi f \geq g + V_\Omega^\Psi g$ on Ω. Then $s := f - g + V_\Omega \phi^+$ is a positive superharmonic function on Ω and

$$s \geq V_\Omega \phi^+ \text{ on } \{\phi^+ > 0\}. \tag{3.3}$$

Therefore, by the same arguments as in the proof of Proposition 2.4 of [10], it follows from (3.3) that s dominates $V_\Omega \phi^+$ on Ω. Thus $f \geq g$ on Ω. □

Corollary 3.2. *Let $\Omega \in \mathcal{O}$, f, g as in the previous proposition and assume moreover that* $\liminf_{x\to z}[f(x) - g(x)] \geq 0$ *for all $z \in \partial\Omega$. Then $f \geq g$ on Ω.*

Proof. We only need to prove that $s = f + V_\Omega^\Psi f - g - V_\Omega^\Psi g$ is positive on Ω. Let again $\phi = \Psi(\cdot, f) - \Psi(\cdot, g)$ then $s + V_\Omega \phi^- = f - g + V_\Omega \phi^+$. Since $s + V_\Omega \phi^-$ is superharmonic on Ω and $\liminf_{x\to z} s(x) \geq 0$ for every $z \in \partial\Omega$, the minimum principle relative to the harmonic space $(X, \mathcal{H})$ implies that $s + V_\Omega \phi^- \geq 0$ on Ω. This in turn yields that $s \geq 0$ on Ω. □

The following three theorems can be found in [20]. For the sake of completeness, the proofs are given the appendix.

Theorem 3.3. *For every $\Omega \in \mathcal{O}$ and every $f \in \mathcal{B}_b(\partial\Omega)$, there exists a unique bounded continuous function u on Ω, which will be denoted by $U_\Omega f$, satisfying*

$$u + V_\Omega^\Psi u = H_\Omega f. \tag{3.4}$$

If $\Omega \in \mathcal{O}$ and f is a Borel measurable function on a set containing $\overline{\Omega}$ such that f is bounded on $\partial\Omega$ we shall denote by $U_\Omega f$ the function which equals $U_\Omega(f|_{\partial\Omega})$ on Ω and equals f elsewhere. Clearly, the mapping U_Ω is odd and increasing.

For every open subset $\Omega \subset X$ we define $\mathcal{U}^*(\Omega)$ to be the set of all l.s.c locally bounded functions u on Ω such that

$$U_D u \leq u \ \text{ for all } D \in \mathcal{O} \text{ with } \overline{D} \subset \Omega.$$

We also define

$$\mathcal{U}_*(\Omega) := -\mathcal{U}^*(\Omega), \ \ \mathcal{U}(\Omega) := \mathcal{U}^*(\Omega) \cap \mathcal{U}_*(\Omega),$$

and we call $\mathcal{U}$ ($\mathcal{U}^*$, $\mathcal{U}_*$ resp.)-*function* on Ω every element of $\mathcal{U}(\Omega)$ ($\mathcal{U}^*(\Omega)$, $\mathcal{U}_*(\Omega)$ resp.).

Remark 3.4. *Using* (3.2) *and* (3.4) *it is easy verified that for all* $D, \Omega \in \mathcal{O}$ *such that* $\overline{D} \subset \Omega$ *we have*

$$U_D \circ U_\Omega = U_\Omega. \tag{3.5}$$

Therefore, $U_\Omega f$ *is a* $\mathcal{U}$*-function on* Ω *for every* $\Omega \in \mathcal{O}$ *and every* $f \in \mathcal{B}_b(\partial\Omega)$. *If moreover* Ω *is regular and* f *is continuous on* $\partial\Omega$ *then* $U_\Omega f$ *is the unique continuous extension of* f *to* $\overline{\Omega}$ *which is a* $\mathcal{U}$*-function on* Ω.

Theorem 3.5. *If* $\Omega \in \mathcal{O}$ *and* $u \in \mathcal{B}_b(\Omega)$ *then* $u \in \mathcal{U}(\Omega)$ ($\mathcal{U}^*(\Omega)$ *resp.) if and only if* $u + V_\Omega^\Psi u \in \mathcal{H}(\Omega)$ ($\mathcal{S}(\Omega)$ *resp.). In particular, if* $u \in \mathcal{B}(\Omega)$ *is locally bounded on* Ω *where* Ω *is an arbitrary open subset of* X, *then* $u \in \mathcal{U}(\Omega)$ ($\mathcal{U}^*(\Omega)$ *resp.) if and only if* $u + V_D^\Psi u \in \mathcal{H}(D)$ ($\mathcal{S}(D)$ *resp.) for every* $D \in \mathcal{O}$ *such that* $\overline{D} \subset \Omega$.

Combining the above theorem and Corollary 3.2 we obtain:

Corollary 3.6. *Let* $\Omega \in \mathcal{O}$ *and let* $u, v \in \mathcal{B}_b(\Omega)$ *such that* $\liminf_{x\to z}[u(x) - v(x)] \geq 0$ *for all* $z \in \partial\Omega$. *If* $u \in \mathcal{U}^*(\Omega)$ *and* $v \in \mathcal{U}_*(\Omega)$ *then* $u \geq v$ *on* Ω.

We deduce from Theorem 3.5 that $\mathcal{U}(\Omega)$ is closed under uniform convergence on compact subsets of Ω. Note also that all positive $\mathcal{U}_*$-function on Ω are subharmonic on Ω.

Theorem 3.7. *Let* $\Omega \subset X$ *be an open subset and let* (u_n) *be a sequence of* $\mathcal{U}$*-functions on* Ω *which are locally uniformly bounded on* Ω. *The following holds:*

(a) *If* (u_n) *increases to* u *then* u *is a* $\mathcal{U}$*-function on* Ω.

(b) *There exists a subsequence of* (u_n) *which converges locally uniformly on* Ω. *In particular, if* (u_n) *converges pointwise to a function* u *then* $u \in \mathcal{U}(\Omega)$ *and* (u_n) *converges uniformly to* u *on every compact subset of* Ω.

3.2. Operators L and Q.

In the following, we fix an exhaustion (Ω_n) of X, that is, $\Omega_n \in \mathcal{O}$, $\overline{\Omega}_n \subset \Omega_{n+1}$ for every $n \geq 1$, and $X = \cup_{n\geq 1}\Omega_n$. Clearly, for every $f \in \mathcal{B}^+(X)$

$$Vf = \lim_{n\to\infty} V_{\Omega_n} f.$$

The following convergence lemma follows easily from the fact that V and V_{Ω_n} are kernels.

Lemma 3.8. *Let* $f, f_n \in \mathcal{B}(X)$ *and let* $g, g_n \in \mathcal{B}^+(X)$. *The following holds:*

(a) $V(\liminf_{n\to\infty} g_n) \leq \liminf_{n\to\infty} V_{\Omega_n} g_n$.

(b) *Assume that* $|f_n| \leq g_n$ *for all* $n \geq 1$, *and* $(f_n), (g_n), (V_{\Omega_n} g_n)$ *converge pointwise to* f, g, Vg *respectively. If* $Vg < \infty$ *then* $\lim_{n\to\infty} V_{\Omega_n} f_n = Vf$.

We shall use the operators L and Q which are introduced in [20] in order to study a Liouville property related to equations of the type $\Delta u = \Psi(\cdot, u)\gamma$. For every positive harmonic function h on X we consider

$$Lh := \inf_{\Omega \in \mathcal{O}} U_\Omega h \quad \text{and} \quad Qh := \sup_{\Omega \in \mathcal{O}} H_\Omega Lh.$$

Lemma 3.9. *Let $\Omega, D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$ and let s be a positive, locally bounded, superharmonic function on a neighborhood of $\overline{\Omega}$. Then $U_D s \geq U_\Omega s$.*

Proof. From the formula $U_\Omega s + V_\Omega^\Psi s = H_\Omega s$ we have $0 \leq U_\Omega s \leq Hs$ and consequently $0 \leq U_\Omega s \leq s$. So the monotonicity of U_D and (3.5) imply that $U_\Omega s \leq U_D s$. □

Theorem 3.10. *Let $h \in \mathcal{H}^+(X)$. The following holds:*

(a) *$Lh \in \mathcal{U}^+(X)$, $Qh \in \mathcal{H}^+(X)$, and we have*

$$Lh \leq Qh \leq h, \tag{3.6}$$

$$Lh + V^\Psi Lh = Qh. \tag{3.7}$$

(b) *If $V^\Psi h < \infty$ then $Qh = h$.*

(c) *L and Q are monotone increasing on $\mathcal{H}^+(X)$.*

(d) *Lh and Qh can be characterized as follows:*

$$\begin{aligned} Lh &= \max\{u \in \mathcal{U}^+(X) : u \leq h\} && (3.8)\\ &= \max\{u \in \mathcal{U}(X) : |u| \leq h\}. && (3.9)\\ Qh &= \min\{g \in \mathcal{H}^+(X) : g \geq Lh\} && (3.10)\\ &= \max\{g \in \mathcal{H}^+(X) : g \leq h;\ Qg = g\}. && (3.11) \end{aligned}$$

(e) *$L \circ Q = L$ and $Q \circ Q = Q$.*

Proof. (a) By Lemma 3.9, the sequence $(U_{\Omega_n} h)$ is a decreasing and

$$Lh = \lim_{n\to\infty} U_{\Omega_n} h. \tag{3.12}$$

Because $0 \leq U_{\Omega_n} h \leq h$ for every $n \geq 1$, Theorem 3.7.b assures that Lh is a $\mathcal{U}$-function on X. Now, since $0 \leq Lh \leq h$ and Lh is subharmonic on X we conclude that the sequence $(H_{\Omega_n} Lh)$ is increasing and

$$Qh = \lim_{n\to\infty} H_{\Omega_n} Lh. \tag{3.13}$$

Whence, the fact that $Lh \leq H_{\Omega_n} Lh \leq h$ yields that $Qh \in \mathcal{H}^+(X)$ and the inequality (3.6) holds. To get (3.7) it suffices to pass to the limit in the formula

$$Lh + V_{\Omega_n}^\Psi Lh = H_{\Omega_n} Lh.$$

(b) Since $0 \le U_{\Omega_n} h \le h$ and

$$U_{\Omega_n} h + V^{\Psi}_{\Omega_n} U_{\Omega_n} h = h, \tag{3.14}$$

by Lemma 3.8 we obtain that $Lh + V^{\Psi} Lh = h$. Therefore, $h = Qh$ in virtue of (3.7) and the comparison principle.

(c) Trivial.

(d) To justify (3.8) and (3.9) it is enough to show that $Lh \ge |u|$ for every $u \in \mathcal{U}(X)$ satisfying $|u| \le h$. So, if u is a such function then for all $n \ge 1$

$$|u| = |U_{\Omega_n} u| \le U_{\Omega_n} h,$$

and therefore $|u| \le Lh$.

The equality (3.10) is a consequence of (3.6) and the monotonicity of the harmonic kernel H_Ω for any $\Omega \in \mathcal{O}$. To obtain (3.11) it suffices to use the fact that $Q(Qh) = Qh$ which is given by the statement (e).

(e) Since $Lh \le Qh \in \mathcal{H}^+(X)$, we conclude by (3.8) that $Lh \le LQh$ and therefore

$$Qh = Lh + V^{\Psi} Lh \le LQh + V^{\Psi} LQh = Q(Qh) \le Qh.$$

Thus $Q(Qh) = h$ and, by comparison principle, $L(Qh) = Lh$. □

Lemma 3.11. *Let $\Omega \in \mathcal{O}$, and let $\alpha, \beta \ge 0$ such that*

$$\Psi(x, \alpha t + \beta s) \ge \alpha \Psi(x, t) + \beta \Psi(x, s) \quad \textit{for all } x \in X,\ t, s \ge 0. \tag{3.15}$$

Then

$$U_\Omega(\alpha f + \beta g) \le \alpha U_\Omega f + \beta U_\Omega g \quad \textit{for all } f, g \in \mathcal{B}_b^+(\partial\Omega). \tag{3.16}$$

Furthermore, the converse inequality in (3.15) *implies the converse one in* (3.16).

Proof. Let $f, g \in \mathcal{B}_b^+(\partial\Omega)$ and denote by $u = U_\Omega f$, $v = U_\Omega g$ and $w = U_\Omega(\alpha f + \beta g)$. Then

$$\phi := \Psi(\cdot, \alpha u + \beta v) - \alpha \Psi(\cdot, u) - \beta \Psi(\cdot, v) \in \mathcal{B}_b^+(\Omega)$$

which implies that

$$V^{\Psi}_\Omega(\alpha u + \beta v) - \alpha V^{\Psi}_\Omega u - \beta V^{\Psi}_\Omega v = V_\Omega \phi \in \mathcal{P}(\Omega) \cap \mathcal{C}_b(\Omega).$$

From (3.4) it follows that

$$\begin{aligned} \alpha u + \beta v + V^{\Psi}_\Omega(\alpha u + \beta v) &= H_\Omega(\alpha f + \beta g) + V_\Omega \phi, \\ w + V^{\Psi}_\Omega w &= H_\Omega(\alpha f + \beta g). \end{aligned}$$

Therefore, applying Proposition 3.1 we get that $\alpha u + \beta v \ge w$ which finishes the proof. Clearly the second statement can be proved in a similar way. □

Corollary 3.12. (a) *If* (A_3) *holds then* L *and* Q *are subadditive on* $\mathcal{H}^+(X)$.

(b) *If* (A_4) *holds then* L *and* Q *are concave (and also subadditive) on* $\mathcal{H}^+(X)$.

Proof. (a) Assumption (A_3) means that (3.15) holds true for $\alpha = \beta = 1$. Hence, by the previous lemma, U_Ω is subadditive on $\mathcal{B}_b^+(\partial\Omega)$ for every $\Omega \in \mathcal{O}$. This, (3.12) and (3.13) prove statement (a).

(b) To see that L and Q are concave it is enough to apply again Lemma 3.11 for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. It is not hard to see that under (A_1), assumption (A_4) yields (A_3). So, if (A_4) holds we conclude by statement (a) that L and Q are subadditive on $\mathcal{H}^+(X)$. □

Corollary 3.13. *Suppose that* (A_3) *is satisfied and let* (h_n) *be an increasing sequence in* $\mathcal{H}^+(X)$ *such that* $h := \sup_{n\geq 1} h_n \in \mathcal{H}^+(X)$. *Then*

$$\sup_{n\geq 1} Lh_n = Lh \quad \text{and} \quad \sup_{n\geq 1} Qh_n = Qh.$$

Proof. By (3.6) and Corollary 3.12, we obtain for every $n \geq 1$ that

$$0 \leq Lh - Lh_n \leq h - h_n \quad \text{and} \quad 0 \leq Qh - Qh_n \leq h - h_n.$$

This completes the proof. □

The proof of the following result is given in Appendix I.

Proposition 3.14. *Suppose that* (A_3) *holds and the function* Ψ *has the doubling property. Then* Q *is linear on* $\mathcal{H}^+(X)$, *i.e., for all* $g, h \in \mathcal{H}^+(X)$ *and every* $\alpha \geq 0$,

$$Q(\alpha g + h) = \alpha Qg + Qh. \tag{3.17}$$

3.3. Martin Type Representation. From now on r is a fixed reference measure on X. Define $\mathcal{H}_r^+(X)$ to be the set of all positive harmonic functions which are integrable on X with respect to r and let

$$\mathcal{H}_r(X) := \mathcal{H}_r^+(X) - \mathcal{H}_r^+(X).$$

We know [25] that there exist a Polish space Y and a family $(P(\cdot, y))_{y\in Y}$ of positive harmonic functions on X such that:

J.1: The map $y \mapsto P(\cdot, y)$ is one-to-one from Y to the set of all minimal harmonic functions h on X satisfying $\int_X h\, dr = 1$. (Recall that a function $h \in \mathcal{H}^+(X)$ is called *minimal* if $h \not\equiv 0$ and if every harmonic function g satisfying the inequality $0 \leq g \leq h$ is a constant multiple of h.)

J.2: For every $x \in X$, the function $P(x, \cdot) : y \mapsto P(x, y)$ is continuous on Y.

J.3: The formula

$$h = P\nu := \int_Y P(\cdot, y)\, d\nu(y) \tag{3.18}$$

defines a one-to-one correspondence between $h \in \mathcal{H}_r(X)$ and $\nu \in \mathcal{M}(Y)$. Furthermore for any $\nu \in \mathcal{M}(Y)$, $|\nu|(Y) = \int_X P|\nu|\, dr$; and $\nu \geq 0$ if and only if $P\nu \geq 0$.

Remark 3.15. If X is a Greenian domain of $\mathbb{R}^d$ and $\mathcal{H}$ is the classical sheaf of harmonic functions, (Y, P) can be chosen so that Y is the minimal part of the Martin boundary and $P(\cdot, y)$ is the Martin function with pole at $y \in Y$.

4 The notion of the trace

Assumptions of this section: Ψ is a Borel measurable real-valued function on $X \times \mathbb{R}$ which satisfies $(A_1), (A_2)$ and (A_3).

4.1. An Existence Lemma. We consider the subset $\mathcal{U}_r(X)$ of $\mathcal{U}(X)$ given by

$$\mathcal{U}_r(X) := \{u \in \mathcal{U}(X) : |u| \leq h \ \text{ for some } h \in \mathcal{H}_r^+(X)\}.$$

A function $u \in \mathcal{U}_r(X)$ will be called a *moderate* $\mathcal{U}$-function on X. It is clear that a function $u \in \mathcal{U}(X)$ is moderate if and only if $|u| \leq v$ for some $v \in \mathcal{U}_r^+(X)$.

Lemma 4.1. *If $u \in \mathcal{U}_r(X)$, then $V^\Psi|u| \in \mathcal{P}(X) \cap \mathcal{C}(X)$ and $u + V^\Psi u \in \mathcal{H}_r(X)$.*

Proof. Take $u \in \mathcal{U}_r(X)$ and choose $g \in \mathcal{H}_r^+(X)$ such that $|u| \leq g$. Then $|u| \leq Lg$ by (3.9). On the other hand, $V^\Psi Lg \in \mathcal{P}(X) \cap \mathcal{C}(X)$ in view of formula (3.7). Therefore $V^\Psi|u|$ is a continuous potential on X. Put $h = u + V^\Psi u$. Combining (3.2) and (3.4) we see that $H_D h = h$ for every $D \in \mathcal{O}$ which implies that h is harmonic on X. Finally, since

$$|h| \leq |u| + V^\Psi|u| \leq Lg + V^\Psi Lg \leq g$$

we conclude that $h \in \mathcal{H}_r(X)$. □

From the above lemma it follows that the formula

$$u + V^\Psi u = P\mu \tag{4.1}$$

assigns to each moderate $\mathcal{U}$-function u on X a unique signed measure $\mu \in \mathcal{M}(Y)$. Conversely, the comparison principle assures that for each $\mu \in \mathcal{M}(Y)$ there is at most one function $u \in \mathcal{U}_r(X)$ which satisfies (4.1). We call the measure μ given by (4.1) the *trace* of u on Y and we write

$$\mu = tr(u).$$

We shall denote by $\mathcal{Q}_\Psi(Y)$ the set of all $\mu \in \mathcal{M}(Y)$ such that μ is the trace of some moderate $\mathcal{U}$-function on X. In other words, $\mu \in \mathcal{Q}_\Psi(Y)$ means that the equation (4.1) is solvable in $\mathcal{U}_r(X)$.

4.2. Properties of the Trace. Let $\mu \in \mathcal{M}^+(Y)$ and $h = P\mu$. Then (3.7) yields that the measure $\nu \in \mathcal{M}^+(Y)$ satisfying $Qh = P\nu$ belongs to the class $\mathcal{Q}_\Psi^+(Y)$. Defining

$$Q\mu := \nu$$

we obtain an increasing subadditive operator Q from $\mathcal{M}^+(Y)$ into $\mathcal{Q}_\Psi^+(Y)$. Furthermore,

$$\mathcal{Q}_\Psi^+(Y) = \{\mu \in \mathcal{M}^+(Y) : Q\mu = \mu\}. \tag{4.2}$$

In the sequel, we may write $L\mu$ to mean $L(P\mu)$.

Theorem 4.2. *Let $\mu, \nu, \mu_1, \mu_2, \cdots \in \mathcal{M}(Y)$. The following holds:*

(a) *If $|\mu| \le \nu$ and $\nu \in \mathcal{Q}_\Psi^+(Y)$ then $\mu \in \mathcal{Q}_\Psi(Y)$.*

(b) *$\mu \in \mathcal{Q}_\Psi(Y)$ if and only if $|\mu| \in \mathcal{Q}_\Psi^+(Y)$.*

(c) *If $\mu_n \in \mathcal{Q}_\Psi^+(Y)$ for all $n \ge 1$ and (μ_n) increases to μ, then $\mu \in \mathcal{Q}_\Psi^+(Y)$.*

(d) *If Ψ satisfies (A_4) then $\mathcal{Q}_\Psi(Y)$ is convex.*

(e) *If Ψ has the doubling property then $\mathcal{Q}_\Psi(Y)$ is a linear subspace of $\mathcal{M}(Y)$. In this case, $f\mu \in \mathcal{Q}_\Psi(Y)$ whenever $\mu \in \mathcal{Q}_\Psi^+(Y)$ and $f \in L^1(Y, \mu)$.*

Proof. (a) Let $h = P\mu$ and $g = P\nu$. For every $n \ge 1$ we have

$$|U_{\Omega_n} h| \le U_{\Omega_n} g \le g.$$

Then, by Theorem 3.7, there exists a subsequence (u_k) of $(U_{\Omega_n} h)$ which is uniformly convergent on every compact subset of X. So $u := \lim_{k\to\infty} u_k$ is a moderate $\mathcal{U}$-function on X. Using the monotonicity and the continuity of $\Psi(x, \cdot)$, we obtain that

$$\begin{gathered} |\Psi(\cdot, u_k)| \le \Psi(\cdot, U_{\Omega_k} g), \\ \lim_{k\to\infty} \Psi(\cdot, u_k) = \Psi(\cdot, u), \\ \lim_{k\to\infty} \Psi(\cdot, U_{\Omega_k} g) = \Psi(\cdot, Lg). \end{gathered}$$

On the other hand, the fact that $\nu \in \mathcal{Q}_\Psi^+(Y)$ implies that

$$\lim_{k\to\infty} V_{\Omega_k}^\Psi U_{\Omega_k} g = V^\Psi L g < \infty.$$

Therefore, by Lemma 3.8 we conclude that $\lim_{k\to\infty} V^{\Psi}_{\Omega_k} u_k = V^{\Psi}u$ and consequently

$$u + V^{\Psi}u = h.$$

This means that $\mu \in \mathcal{Q}_{\Psi}(Y)$ and $tr(\mu) = u$.

(b) If $|\mu| \in \mathcal{Q}^+_{\Psi}(Y)$ then $\mu \in \mathcal{Q}_{\Psi}(Y)$ by statement (a). Suppose now that $\mu \in \mathcal{Q}_{\Psi}(Y)$ and let u be the moderate $\mathcal{U}$-function on X satisfying $\mu = tr(u)$. Choose $\nu \in \mathcal{M}^+(Y)$ such that $|u| \le P\nu$. Then $|u| \le L\nu$ by (3.9) and thereby $|P\mu| \le P(Q\nu)$. This yields that $|\mu| \le Q\nu$ (recall that $P|\mu|$ is the least harmonic majorant of $|P\mu|$). So $|\mu| \in \mathcal{Q}^+_{\Psi}(Y)$ by statement (a).

(c) follows trivially from Corollary 3.13.

(d) Since, by Corollary 3.12, Q is a concave operator on $\mathcal{M}^+(Y)$ we easily deduce from (4.2) that $\mathcal{Q}^+_{\Psi}(Y)$ is a convex subset of $\mathcal{M}^+(Y)$. So statement (b) proves that $\mathcal{Q}_{\Psi}(Y)$ is also convex.

(e) By Proposition 3.14 $\mathcal{Q}^+_{\Psi}(Y)$ is a cone. In fact, for every $\mu, \nu \in \mathcal{M}^+(Y)$ and every $\alpha \ge 0$ we have

$$Q(\alpha\mu + \nu) = \alpha Q\mu + Q\nu.$$

So from (b) it follows that

$$\mathcal{Q}_{\Psi}(Y) = \mathcal{Q}^+_{\Psi}(Y) - \mathcal{Q}^+_{\Psi}(Y) \tag{4.3}$$

which proves that $\mathcal{Q}_{\Psi}(Y)$ is a linear space. The second part of (e) is a consequence of statements (b) and (c). □

Studying equations $\Delta u = u|u|^{\alpha-1}$, $\alpha > 1$, on bounded domains $\Omega \subset \mathbb{R}^d$, analogous results as in the previous theorem are obtained in [30]. To see the interest of introducing the operators L and Q, the reader may compare our proof to the proof given by M. Marcus and L. Véron [30, *Proof of Proposition A*] who used a result of H. Brézis concerning the boundary value problem $\Delta u = f$ in Ω and $u = \phi \in L^1(\partial\Omega)$ on $\partial\Omega$. We also notice that, using probabilistic tools, E. B. Dynkin and S. E. Kuznetsov proved a similar result [19, Theorem 4.3] as assertion (c) of the preceding theorem.

4.4. Removable Singularities. Let E be a Borel subset of Y. We shall say that E is *removable* if the function ϑ_E which is defined at every point $x \in X$ by

$$\vartheta_E(x) := \sup_{\mu\in\mathcal{M}^+(E)} L\mu(x) \tag{4.4}$$

is identically zero. Since $\{L\mu : \mu \in \mathcal{M}^+(E)\}$ is an upward filtering family of continuous functions, we may find an increasing sequence $(\mu_n) \in \mathcal{M}^+(E)$ such that

$$\vartheta_E = \sup_{n\ge1} L\mu_n,$$

which yields, in particular, that $\vartheta_E \in \mathcal{U}^+(X)$ if it is locally bounded on X. In the following proposition, we have collected the basic properties of the map $E \to \vartheta_E$.

Proposition 4.3. *Let $E, F, E_1, E_2, \cdots \subset Y$ be Borel subsets. Then:*
(a) *If $E \subset F$ then $\vartheta_E \leq \vartheta_F$.*
(b) *If (E_n) increases to E then $\vartheta_E = \sup_{n\geq 1} \vartheta_{E_n}$.*
(c) *If $E = \cup_{n=1}^{\infty} E_n$ then $\vartheta_E \leq \sum_{n=1}^{\infty} \vartheta_{E_n}$.*

Proof. (a) Obvious.

(b) Let $u = \sup_{n\geq 1} \vartheta_{E_n}$ and let $\mu \in \mathcal{M}^+(E)$. Seeing that $\mu_{E_n} \in \mathcal{M}^+(E_n)$ for all $n \geq 1$ and (μ_{E_n}) increases to μ, we conclude that

$$L\mu = \sup_{n\geq 1} L\mu_{E_n} \leq u.$$

Whence $\vartheta_E \leq u$. Therefore $u = \vartheta_E$ since $u \leq \vartheta_E$ by (a).

(c) For every $k \geq 1$ let $F_k = \cup_{n=1}^{k} E_n$ and $\mu \in \mathcal{M}^+(F_k)$. Because L is subadditive and $\mu \leq \sum_{n=1}^{k} \mu_{E_n}$, it follows that $L\mu \leq \sum_{n=1}^{k} L\mu_n$ and consequently $L\mu \leq \sum_{n=1}^{k} \vartheta_{E_n}$. Thus $\vartheta_{F_k} \leq \sum_{n=1}^{k} \vartheta_{E_n}$ for all $k \geq 1$, which yields the desired inequality remarking that $\vartheta_E = \sup_{k\geq 1} \vartheta_{F_k}$. □

As immediate consequences of the previous proposition, we see that every Borel subset of a removable set of Y is also removable, and $\cup_{n=1}^{\infty} E_n$ is removable whenever (E_n) is a sequence of removable subsets of Y.

Proposition 4.4. *Let E be a Borel subset of Y. The following statements are equivalent:*
(a) *E is removable.*
(b) *$\nu(E) = 0$ for all $\nu \in \mathcal{Q}_\Psi^+(Y)$.*
(c) *Every compact subset $K \subset E$ is removable.*

Proof. From the fact that $Q\mu \in \mathcal{Q}_\Psi^+(Y)$ and $L\mu = L(Q\mu)$ for every $\mu \in \mathcal{M}^+(Y)$ we obtain that

$$\vartheta_E = \sup_{\nu \in \mathcal{M}^+(E) \cap \mathcal{Q}_\Psi^+(Y)} L\nu. \tag{4.5}$$

This yields the equivalence between (a) and (b). To finish the proof it suffices to recall that every $\mu \in \mathcal{M}^+(Y)$ is inner regular (see, e.g., [7]). □

5 Polar sets

Assumption of this section: $\Psi \in \mathbb{Y}(X)$.

5.1. ORLICZ TYPE SPACES. For our purpose it will be convenient to identify all Borel measurable functions f, g on X satisfying $\int_X V(|f-g|)\,dr = 0$.

We define $\mathcal{L}_\Psi(X)$ (*Orlicz class*) to be the set of all $f \in \mathcal{B}(X)$ such that

$$\varrho_\Psi(f) := \int_X V^\Psi |f|\,dr < \infty.$$

Let $L_\Psi(X)$ (*Orlicz space*) be the smallest linear space containing $\mathcal{L}_\Psi(X)$, and let $E_\Psi(X)$ be the largest linear space contained in $\mathcal{L}_\Psi(X)$. Classical analogous definitions, for $X \subset \mathbb{R}^d$ and $\Psi \in \mathbb{Y}_0$, are well known (see, e.g., [26]). An alternative approach to the theory of Orlicz spaces can be found in [15]. Notice that if Ψ has the doubling property then

$$E_\Psi(X) = \mathcal{L}_\Psi(X) = L_\Psi(X).$$

Notation. Here and in the following, Φ denotes the function Ψ^* given by (2.5) (of course $\Phi \in \mathbb{Y}(X)$ and $\Phi^* = \Psi$).

For every Borel measurable function f on X we consider

$$\|f\|_\Psi = \sup\left\{\int_X V|fg|\,dr : g \in \mathcal{B}(X), \varrho_\Phi(g) \le 1\right\}, \tag{5.1}$$

$$\|f\|_{(\Psi)} = \inf\left\{\alpha > 0 : \varrho_\Psi\left(\alpha^{-1} f\right) \le 1\right\}. \tag{5.2}$$

Obviously, $\|\cdot\|_\Psi$ and $\|\cdot\|_{(\Psi)}$ are increasing on $\mathcal{B}^+(X)$. Furthermore,

$$\|f\|_\Psi \le 1 \;\Rightarrow\; \varrho_\Psi(f) \le \|f\|_\Psi, \tag{5.3}$$

$$\|f\|_{(\Psi)} \le 1 \;\Leftrightarrow\; \varrho_\Psi(f) \le 1. \tag{5.4}$$

We also need the following kind of *Hölder inequality* which follows from (5.4):

$$\int_X V|fg|\,dr \le \|f\|_\Psi \|g\|_{(\Phi)}. \tag{5.5}$$

From (5.3) and (5.4) we deduce that $\|f\|_{(\Psi)} \le \|f\|_\Psi \le 2\|f\|_{(\Psi)}$. Therefore, $L_\Psi(X)$ is the set of all functions $f \in \mathcal{B}(X)$ such that $\|f\|_\Psi < \infty$ and $\|\cdot\|_\Psi$ and $\|\cdot\|_{(\Psi)}$ define two equivalent norms on $L_\Psi(X)$. Moreover, it is not difficult to verify that $L_\Psi(X)$ endowed with $\|\cdot\|_\Psi$ is a Banach space. We call $\|\cdot\|_\Psi$ ($\|\cdot\|_{(\Psi)}$ resp.) the Orlicz (Luxemburg resp.) norm.

Let $f \in E_\Psi(X)$ and consider the sequence (f_n) given for every $n \ge 1$ by

$$f_n = 1_{\Omega_n} \inf(\sup(f, -n), n). \tag{5.6}$$

Seeing that $f_n \in \mathcal{B}_{bc}(X)$, $|f_n| \leq |f|$, and $\lim_{n\to\infty} f_n = f$, it follows that for every $\alpha > 0$

$$\lim_{n\to\infty} \varrho_\Psi(\alpha|f - f_n|)) = 0.$$

Therefore, $E_\Psi(X)$ coincides with the closure (relative to the convergence in norm) of $\mathcal{B}_{bc}(X)$ in $L_\Psi(X)$. Define $B_{(\Phi)}$ to be the closed unit ball in $L_\Phi(X)$ with respect to the Luxemburg norm and let

$${}^E B_{(\Phi)} = E_\Phi(X) \cap B_{(\Phi)}.$$

Clearly (5.4) means that $B_{(\Phi)} = \{f \in \mathcal{B}(X) : \varrho_\Phi(f) \leq 1\}$. Using sequences defined by (5.6) it is not difficult to see that

$$\|f\|_\Psi = \sup_{g \in {}^E B^+_{(\Phi)}} \int_X V(|f|g)\, dr. \tag{5.7}$$

Now, slightly modifying the proof of Theorem 14.2 in [26] we get the following result which characterizes the topological dual of $E_\Psi(X)$.

Theorem 5.1. *For every continuous linear form T on $E_\Psi(X)$, endowed with the Luxemburg norm, there exists a unique function $g \in L_\Phi(X)$ such that for all $f \in E_\Psi(X)$*

$$T(f) = \int_X V(fg)\, dr. \tag{5.8}$$

Moreover:

(a) $\|T\| := \sup_{f \in {}^E B_{(\Psi)}} |T(f)| = \|g\|_\Phi$.

(b) *If $T \geq 0$ (i.e., $T(f) \geq 0$ for all $f \in E^+_\Psi(X)$) then $g \in L^+_\Phi(X)$.*

5.2. The Martin-Orlicz Capacity. We call *Martin-Orlicz capacity* the set function c_Ψ defined for every Borel subset E of Y by

$$c_\Psi(E) := \sup\left\{\nu(Y) : \nu \in \mathcal{M}^+(E), \|P\nu\|_\Psi \leq 1\right\}$$

and extended to any (arbitrary) subset F of Y by

$$c_\Psi(F) = \inf\{c_\Psi(E) : E \supset F,\ E \text{ Borel}\}.$$

Then c_Ψ is a capacity in the terminology of N. G. Meyers [32]. In other words, $c_\Psi(\emptyset) = 0$ and for any sequence (F_n) of subsets of Y the following properties hold:

$$F_1 \subset F_2 \Rightarrow c_\Psi(F_1) \leq c_\Psi(F_2), \tag{5.9}$$

$$c_\Psi\left(\textstyle\bigcup_{n=1}^\infty F_n\right) \leq \textstyle\sum_{n=1}^\infty c_\Psi(F_n). \tag{5.10}$$

A set $F \subset Y$ will be called *c_Ψ-polar* if $c_\Psi(F) = 0$, and we shall say that a property $\mathcal{P}$ holds *c_Ψ-quasi-everywhere* (abb., *c_Ψ-q.e*) provided $\mathcal{P}$ is valid on $Y \backslash F$ for some c_Ψ-polar subset $F \subset Y$. From (5.9) it follows that every subset of a c_Ψ-polar set is also c_Ψ-polar, and by (5.10) it is clear that the union of any countable family of c_Ψ-polar sets of Y is again c_Ψ-polar.

Using the fact that $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ for any Borel subset E of Y and any $\mu \in \mathcal{M}^+(Y)$, we easily obtain the following proposition.

Proposition 5.2. *For every Borel set $E \subset X$ we have*

$$c_\Psi(E) = \sup\{c_\Psi(K) : K \subset E,\ K \text{ compact}\}. \tag{5.11}$$

For $f \in \mathcal{B}(X)$ we consider the function $\check{P}f$ defined at every $y \in Y$ by

$$\check{P}f(y) = \int_X V(P_y f)\, dr$$

provided the integral possesses a sense. Recall that $P_y = P(\cdot, y)$ is the (Martin) function given by (J.1). If $f \in \mathcal{B}^+(X)$ and $\nu \in \mathcal{M}^+(Y)$, it is obvious that

$$\int_Y \check{P}f\, d\nu = \int_X V(fP\nu)\, dr. \tag{5.12}$$

Proposition 5.3. *For every compact subset K of Y we have*

$$c_\Psi(K) = \inf\left\{\|f\|_{(\Phi)} : f \in E_\Phi^+(X) \text{ and } \check{P}f \geq 1 \text{ on } K\right\}. \tag{5.13}$$

Moreover, (5.13) *holds also true if $E_\Phi^+(X)$ is replaced by $L_\Phi^+(X)$.*

Proof. Let K be a compact subset of Y and denote by α the right side in (5.13)[1]. Let $\mathcal{W} := \{\nu \in \mathcal{M}^+(K) : \nu(Y) = 1\}$ and endow it with the weak* topology. Then $\mathcal{W}$ is a compact Hausdorff space. On the other hand, by (J.2) the mapping $\nu \mapsto P\nu(x)$ is continuous on $\mathcal{W}$ for any fixed $x \in X$. Consequently the function $\nu \mapsto \int_Y \check{P}f\, d\nu$ is lower semicontinuous on $\mathcal{W}$ for every fixed function $f \in {}^E B_{(\Phi)}^+$. Then, in view of (5.7) and (5.12), the minimax theorem (see, e.g., [1]) yields that

$$\inf_{\nu \in \mathcal{W}} \|P\nu\|_\Psi = \sup_{f \in {}^E B_{(\Phi)}^+} \inf_{\nu \in \mathcal{W}} \int_Y \check{P}f\, d\nu = \sup_{f \in {}^E B_{(\Phi)}^+} \inf_{y \in K} \check{P}f(y). \tag{5.14}$$

Remark first that by the definition of $c_\Psi(K)$ it is not difficult to obtain (5.13) in the case of $\{\alpha, c_\Psi(K)\} \cap \{0, \infty\} \neq \emptyset$. So suppose that $0 < c_\Psi(K), \alpha < \infty$. Then

$$\begin{aligned} \frac{1}{c_\Psi(K)} &= \inf\left\{\frac{1}{\nu(K)} : \nu \in \mathcal{M}^+(K), \nu \neq 0, \|P\nu\|_\Psi \leq 1\right\} \\ &= \inf\left\{\frac{\|P\nu\|_\Psi}{\nu(K)} : \nu \in \mathcal{M}^+(K), \nu \neq 0\right\} \\ &= \inf_{\nu \in \mathcal{W}} \|P\nu\|_\Psi, \end{aligned}$$

[1] If there is no $f \in E_\Phi^+(X)$ such that $\check{P}f \geq 1$ on K then, by convention, $\alpha = \infty$.

and

$$\begin{aligned}
\frac{1}{\alpha} &= \sup\left\{\frac{1}{\|f\|_{(\Phi)}} : f \in E_\Phi^+(X), f \not\equiv 0, \check{P}f \geq 1 \text{ on } K\right\} \\
&= \sup\left\{\frac{\inf_{y\in K} \check{P}f(y)}{\|f\|_{(\Phi)}} : f \in E_\Phi^+(X), f \not\equiv 0\right\} \\
&= \sup_{f\in {}^E B^+_{(\Phi)}} \inf_{y\in K} \check{P}f(y).
\end{aligned}$$

So the proof of equality (5.13) is finished in view of (5.14). Finally, using (5.1) instead of (5.7), the second statement of the proposition can be shown by the same reasoning. □

5.3. Sufficient Condition for ν to be in $\mathcal{Q}_\Psi(Y)$. In addition to the fact that Ψ is a function in $\mathbb{Y}(X)$, we also suppose in the present subsection that:

(†) Ψ has the doubling property, and

(‡) (V, r) is an admissible pair (see subsection 2.5).

Let us consider the duality $\langle \cdot, \cdot \rangle$ between $E_\Phi(X)$ and $L_\Psi(X)$ given by

$$\langle f, g\rangle = \int_X V(fg)\, dr$$

for every $f \in E_\Phi(X)$ and $g \in L_\Psi(X)$. If $\mathcal{F} \subset E_\Phi(X)$, we denote by $\mathcal{F}^\perp$ the (closed) subspace of $L_\Psi(X)$ consisting of all $g \in L_\Psi(X)$ such that $\langle f, g\rangle = 0$ for all $f \in \mathcal{F}$. For a set $\mathcal{G} \subset L_\Psi(X)$, $\mathcal{G}^\perp$ is the subspace of $E_\Phi(X)$ defined is the same way.

Let $\mathcal{H}_\Psi^+(X) := \mathcal{H}_r^+(X) \cap L_\Psi(X)$, $\mathcal{H}_\Psi(X) := \mathcal{H}_\Psi^+(X) - \mathcal{H}_\Psi^+(X)$ and

$$\mathcal{M}_\Psi(Y) := \{\nu \in \mathcal{M}(Y) : P\nu \in \mathcal{H}_\Psi(X)\}.$$

By Theorems 4.2.b and 3.10.b we have $\mathcal{M}_\Psi(Y) \subset \mathcal{Q}_\Psi(Y)$. (Notice that assumption (†) above implies that $E_\Psi(X) = \mathcal{L}_\Psi(X) = L_\Psi(X)$)

Lemma 5.4. *Let $f \in E_\Phi(X)$ and $E \subset Y$ be a Borel set. The following holds:*

(a) *E is c_Ψ-polar if and only if $\nu(E) = 0$ for all $\nu \in \mathcal{M}_\Psi^+(Y)$.*

(b) *$\mathcal{H}_\Psi(X)^\perp = \{f \in E_\Phi(X) : \check{P}f = 0\ c_\Psi - q.e \text{ on } Y\}$*

(c) *$\mathcal{H}(X) \cap L_\Psi(X)$ is a closed subspace of $L_\Psi(X)$.*

Proof. (a) Trivial.

(b) This follows from (5.12) and assertion (a).

(c) Let K be a compact subset of X and choose $\Omega \in \mathcal{O}$, $c > 0$ as in (2.4). Applying the Hölder inequality we obtain that

$$\sup_K |h| \leq c \int_X V|h1_\Omega|\, dr \leq c\|1_\Omega\|_{(\Phi)}\|h\|_\Psi$$

for every $h \in \mathcal{H}(X)$. Therefore, any sequence in $\mathcal{H}(X) \cap L_\Psi(X)$ converges locally uniformly on X whenever it converges in $L_\Psi(X)$ relative to the Orlicz norm. This finishes the proof of (c). □

Remark 5.5. As consequence of (7.4) and Lemma 7.2 we remark that the set $\{\check{P}|f| = \infty\}$ is c_Ψ-polar for every $f \in E_\Phi(X)$. Furthermore, by the same arguments, every sequence $(f_n) \subset E_\Phi(X)$ convergent (in norm) to some function f admits a subsequence (g_n) with the property that $(\check{P}g_n)$ converges c_Ψ-q.e to $\check{P}f$.

Remark 5.6. If $f \in \mathcal{C}(X)$ such that $\int_X V(fg)\,dr \geq 0$ for all $g \in \mathcal{B}^+_{bc}(X)$, then $f(x) \geq 0$ for all $x \in X$. In fact, it suffices to remark that the measure m defined for every Borel subset $A \subset X$ by $m(A) = \int_X V1_A\,dr$ charges all open nonempty subsets of X. To see this, let $D \in \mathcal{O}$ and suppose that $m(D) = 0$. Seeing that $\{V1_D = 0\}$ is an absorbing set (see, [6, Satz 1.4.1]) and recalling the definition of a reference measure (see Subsection 2.5) we conclude that $V1_D$ is identically zero on X. Consequently, $D = \emptyset$ by (AP1).

Theorem 5.7. *Every $\nu \in \mathcal{M}(Y)$ which does not charge any compact c_Ψ-polar subset of Y is a trace of some moderate $\mathcal{U}$-function on X.*

Proof. In virtue of Theorem 4.2.b we consider only the case when ν is positive. Let $\nu \in \mathcal{M}^+(Y)$ not charging compact c_Ψ-polar subsets of Y and define for every $f \in E_\Phi(X)$

$$\Lambda(f) := \int_Y [\check{P}f]^+\,d\nu.$$

Then Λ is a positively homogenous subadditive map from $E_\Phi(X)$ into $\overline{\mathbb{R}}_+$. Furthermore, Λ is lower semicontinuous on $E_\Phi(X)$ (see Remark 5.5) and thereby

$$\operatorname{epi}\Lambda := \{(f,t) \in E_\Phi(X) \times \mathbb{R} : \Lambda(f) \leq t\}$$

is a closed convex cone of $E_\Phi(X) \times \mathbb{R}$ (see, e.g., [11]). Considering $\varphi := \sum_{n=1}^\infty \alpha_n 1_{\Omega_n}$, where

$$\alpha_n = \frac{2^{-n}}{(1 + \langle 1_{\Omega_n}, h\rangle)(1 + \|1_{\Omega_n}\|_\Phi)},$$

it is not difficult to see that $\varphi \geq \alpha_n > 0$ on Ω_n, $\varphi \in E^+_\Phi(X)$, and $\Lambda(\varphi) < \infty$. Then, Theorem 5.1 and the Hahn-Banach theorem (see, e.g., [11, Théorème I.7]) imply that there exist $g_n \in L_\Psi(X)$ and $a_n \in \mathbb{R}$ such that

$$\langle \varphi, g_n\rangle \;>\; a_n(\Lambda(\varphi) - 1/n) \tag{5.15}$$

and

$$\langle f, g_n\rangle \;\leq\; a_n t \quad \text{for all } (f,t) \in \operatorname{epi}\Lambda. \tag{5.16}$$

Taking $f = 0$ and $t = 1$ in (5.16) we get that $a_n \geq 0$. Assuming that $a_n = 0$ we obtain that $\langle \varphi, g_n \rangle > 0$ by (5.15), and $\langle \varphi, g_n \rangle \leq 0$ by (5.16), which yields a contradiction. So we suppose without loss of generality that $a_n = 1$ (otherwise we replace g_n by $\alpha_n^{-1} g_n$).

We claim that $g_n \in \mathcal{H}^+(X)$. In fact, using the characterization of $\mathcal{H}_\Psi(X)^\perp$ given by Lemma 5.4.b, we deduce from (5.16) that $g_n \in (\mathcal{H}_\Psi(X)^\perp)^\perp$. On the other hand, Lemma 5.4.c and [11, Proposition II.12] prove that

$$(\mathcal{H}_\Psi(X)^\perp)^\perp \subset L_\Psi(X) \cap \mathcal{H}(X).$$

Now, applying (5.16) to $(-f, 0)$ we get that $\langle f, g_n \rangle \geq 0$ for every $f \in \mathcal{B}_{bc}^+(X)$, which implies that $g_n(x) \geq 0$ for all $x \in X$ (see Remark 5.6 above). The claim is proved.

Put $h = P\nu$ and apply again (5.16) for $f \in \mathcal{B}_{bc}^+(X)$ and $t = \Lambda(f)$, we obtain in view of (5.12) that $\int_X V(f(h - g_n))\, dr \geq 0$ for every $f \in \mathcal{B}_{bc}^+(X)$, which yields that $h \geq g$ on X. Define now

$$h_n = \lim_{k \to \infty} H_{\Omega_k} \sup_{1 \leq i \leq n} g_i,$$

i.e., h_n is the least harmonic majorant of $\{g_i : 1 \leq i \leq n\}$. Then (h_n) is an increasing sequence of positive harmonic functions on X satisfying

$$\int_X V(\varphi(h - h_n))\, dr \leq \frac{1}{n} \quad (n \geq 1). \tag{5.17}$$

Recalling that $\varphi > 0$ on X we conclude from (5.17) that $h = \sup_{n \geq 1} h_n$, and consequently

$$\nu = \sup_{n \geq 1} \nu_n$$

where $\nu_n \in \mathcal{M}^+(Y)$ satisfying $P\nu_n = h_n$ for all $n \geq 1$. The fact that $h_n \leq \sum_{i=1}^n g_i$ and $g_i \in \mathcal{H}_\Psi^+(X)$ for all $i \geq 1$, prove that all measures ν_n belong to the class $\mathcal{Q}_\Psi^+(Y)$. Whence, $\nu \in \mathcal{Q}_\Psi^+(Y)$ by Theorem 4.2.c. □

We notice that, in general, the converse statement in the above theorem does not hold. A counterexample will be given in subsection 6.6.

6 Applications to semilinear PDEs

We call Greenian domain every open and connected set $D \subset \mathbb{R}^d$ which has a Green function G_D $(-\Delta G_D(\cdot, \zeta) = \delta_\zeta$ for every $\zeta \in D)$. As usual, Δ denotes the Laplace operator on $\mathbb{R}^d$, $d \geq 2$. Let X be a Greenian domain of $\mathbb{R}^d$ and let $\mathcal{H}$ be the classical sheaf of harmonic functions on X. Fix a point x_0 in X and consider, as reference measure on X, the Dirac measure $r = \delta_{x_0}$ concentrated at the point x_0 (here X and the empty set are the only absorbing subsets of X; see, e.g., [6]). So, trivially

$$\mathcal{H}_r(X) = \mathcal{H}^+(X) - \mathcal{H}^+(X).$$

We choose Y and P so that Y is the set of all minimal Martin boundary points of X and P is the Martin kernel satisfying $P(x_0, y) = 1$ for every $y \in Y$.

Let $\Psi \in \mathbb{Y}(X)$ and denote by Φ the function Ψ^*. Consider also a local Kato measure γ on X, i.e., $V = V_X^\gamma$ given by (2.3) is a potential kernel on X. Then it is not difficult to see that, for every $D \in \mathcal{O}$, the kernel V_D is given by the formula

$$V_D f = \int_D G_D(\cdot, \zeta) f(\zeta)\, d\gamma(\zeta).$$

Our goal here is to apply the general study presented in the preceeding sections in order to investigate the boundary value problem:

$$\begin{aligned} \Delta u &= \Psi(\cdot, u)\gamma \quad \text{in } X, \\ u &= \nu \quad \text{on } Y, \end{aligned} \tag{6.1}$$

where ν is a signed Borel measure with bounded variation on Y.

6.1. Continuous Solutions to (6.2).

A solution to the equation

$$\Delta u = \Psi(\cdot, u)\gamma \tag{6.2}$$

on an open subset $\Omega \subset X$ has to be understood as a continuous function u on Ω which satisfies (6.2) in the distributional sense, i.e.,

$$\int_\Omega u(x) \Delta\varphi(x)\, dx = \int_\Omega \Psi(x, u(x))\varphi(x)\, d\gamma(x) \tag{6.3}$$

for every φ in the space $\mathcal{C}_c^\infty(\Omega)$ of all infinitely differentiable functions on Ω with compact support in Ω.

Proposition 6.1. *Let Ω be an open subset of X and let $u \in \mathcal{C}(\Omega)$. Then u is a solution to* (6.2) *in Ω if and only if u is a $\mathcal{U}$-function on Ω.*

Proof. Suppose first that u is a $\mathcal{U}$-function on Ω. Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and choose $D \in \mathcal{O}$ such that $\operatorname{supp}(\varphi) \subset \overline{D} \subset \Omega$. By Theorem 3.5, the function

$$h := u + \int_D G_D(\cdot, \zeta)\Psi(\zeta, u(\zeta))\, d\gamma(\zeta) \tag{6.4}$$

is harmonic and bounded on D. Therefore, multiplying (6.4) by $\Delta\varphi$ and integrating, we obtain (6.3) which means that u is a solution to (6.2) in Ω.

Conversely, assume that (6.3) holds true for every $\varphi \in \mathcal{C}_c^\infty(\Omega)$. A similar computation proves that for any $D \in \mathcal{O}$ with $\overline{D} \subset \Omega$, the function h given by (6.4) is harmonic on D. So, again by Theorem 3.5, this yields that $U_D u = u$ for all $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. Whence $u \in \mathcal{U}(\Omega)$. □

6.2. EXAMPLES OF Ψ. The class $\mathbb{Y}(X)$ contains every function of the form

$$\Psi(x,t) = \xi(x)\mathrm{sgn}(t)M(t)$$

where M is a Young function (see Subsection 2.6) and ξ is a Borel measurable positive function on X such that ξ and $1/\xi$ are locally bounded on X. Furthermore, Ψ has the doubling property if and only if M possesses the same property.

We quote as first example the function

$$\Psi(x,t) = t|t|^{\alpha-1}, \; x \in X, \; t \in \mathbb{R}, \tag{6.5}$$

where α is a real > 1. In this case, $L_\Psi(X)$ is the classical Lebesgue space $L^\alpha(X,m)$ where $m = G_X(x_0,\cdot)\gamma$, and trivially $L_\Phi(X) = L^{\alpha'}(X,m)$ for $\alpha' := \alpha/(\alpha-1)$. In this example, clearly both functions Ψ and Φ possess the doubling property.

As second example of Ψ, we consider

$$\Psi(x,t) = \mathrm{sgn}(t)[-|t| + (1+|t|)\ln(1+|t|)], \; x \in X, \; t \in \mathbb{R}. \tag{6.6}$$

In this example, the function Ψ has the doubling property but it is not the case for Φ. In fact, by elementary calculations we may show that $\Phi(x,t) = \mathrm{sgn}(t)[-1-|t|+\exp|t|]$.

The reader has certainly noticed that our results (especially Theorem 5.7) hold without assuming that Φ possesses the doubling property.

6.3. EXAMPLES OF γ. Obviously the d-dimensional Lebesgue measure λ and any Radon measure on X with a locally bounded density with respect to λ are local Kato measures on X. A further example of γ can be constructed as follows: Suppose that

$$X = B := B(0,1)$$

is the open unit ball of $\mathbb{R}^d$ and let $x_0 = 0$. From the definition of the Green function G_B (see [16]) we know that for every $0 < \rho < 1$ there exists $a_\rho > 0$ such that

$$\{\zeta \in B : G_B(0,\zeta) > a_\rho\} = B_\rho := B(0,\rho).$$

Denote by σ_ρ the normalized surface area measure on ∂B_ρ and let I be the set of all rational numbers $0 < \rho < 1$. For each $\rho \in I$ choose $\eta_\rho > 0$ so that $\sum_{\rho\in I} \eta_\rho a_\rho < \infty$ and define

$$\gamma := \sum_{\rho\in I} \eta_\rho \, \sigma_\rho. \tag{6.7}$$

Therefore, γ is a (local) Kato measure on B which is singular with respect to λ and it charges all nonempty open subsets of B.

Proposition 6.2. *For $r = \delta_{x_0}$, the pair (γ, r) is admissible in each of the following cases:*

(a) *γ is the restriction of the Lebesgue measure λ to X.*

(b) *γ is given by* (6.7) *(where $X = B$ and $x_0 = 0$).*

Proof. In both cases the measure γ charge all nonempty open subsets of X. So it only remains to prove that (AP2) is satisfied. Let K be a compact subset of X.

(a) Take $\Omega, D \in \mathcal{O}$ such that $K \cup \{x_0\} \subset D \subset \overline{D} \subset \Omega$ and let $h \in \mathcal{H}_b(\Omega)$. From the mean-value property of h it follows that

$$\sup_K |h| \leq a \int_D |h| d\lambda$$

where a is a strictly positive constant not depending on h. Consequently, remarking that $\inf_{\zeta \in D} G_\Omega(x_0, \zeta) := \alpha > 0$ we obtain that

$$V_\Omega |h|(x_0) \geq \int_D G_\Omega(x_0, \zeta) |h(\zeta)| \, d\lambda(\zeta) \geq \alpha \int_D |h| \, d\lambda \geq \frac{\alpha}{a} \sup_K |h|.$$

This finishes the proof in the case of $\gamma = \lambda|_X$.

(b) Let $\rho \in I$ such that $K \cup \{0\} \subset B_\rho$. Seeing that $\sigma_\rho = \mu_0^{B_\rho}$, it follows from the Harnack inequality that there exists a constant $a > 0$ such that the inequality

$$\mu_x^{B_\rho} \leq a \, \sigma_\rho$$

is valid for all $x \in K$. Choose $\tau \in I$ such that $\tau > \rho$ and put $\alpha := \inf_{\zeta \in \partial B_\rho} G_{B_\tau}(0, \zeta)$. Since $\alpha > 0$ we get that

$$|h(x)| \leq \int_{\partial B_\rho} |h| \, d\mu_x^{B_\rho} \leq a \int_{\partial B_\rho} |h| \, d\sigma_\rho \leq \frac{a}{\alpha \eta_\rho} V_{B_\tau} |h|(0)$$

for every $x \in K$ and every $h \in \mathcal{H}_b(B_\tau)$. Thus, the proof is complete. □

6.4. Removable Singularities. We suppose in this subsection that X is a bounded Lipschitz domain. Consequently, the boundary Harnack principle holds for X and we may choose Y to be the Euclidean boundary ∂X of X (see, e.g., [4, Sect. 8.7]).

Given $u \in \mathcal{B}^+(X)$, $u = 0$ on $\Gamma \subset \partial X$ will mean that $\lim_{X \ni x \to z} u(x) = 0$ for all $z \in \Gamma$.

Proposition 6.3. *Let $E \subset \partial X$ be a Borel set. The following statements are equivalent:*

(a) *E is a removable set.*

(b) *Equation* (6.2) *has no nontrivial continuous solution u in X such that*

$$u = 0 \ \text{ on } \partial X \backslash E \ \text{ and } 0 \leq u \leq g \ \text{ for some } g \in \mathcal{H}^+(X).$$

Proof. Take u as in (b). By Lemma 4.1,

$$h := u + \int_X G_X(\cdot,\zeta)\Psi(\zeta,u(\zeta))\,d\gamma(\zeta)$$

is a harmonic function on X. Moreover, $u = L\mu$ where μ is the measure in $\mathcal{M}^+(\partial X)$ satisfying $h = P\mu$. We claim that μ is supported by E. Indeed, let O be a relatively open subset of ∂X such that $E \subset O$ and let ν be the restriction of μ to $\partial X\backslash O$. Then, in view of the boundary Harnack principle, we see that $P\nu$ vanishes on O and thereby $L\nu = 0$ on O. On the other hand, since $L\nu \le L\mu = u$ it follows that $L\nu = 0$ on $\partial X\backslash E$. Therefore, $L\nu \equiv 0$ on X which in turn implies that $\nu = Q\nu = 0$. Notice that $\nu \in \mathcal{Q}^+_\Psi(\partial X)$ by Theorem 4.2.a. We then conclude that $\mu(O) = \mu(\partial X)$ for every open subset O of ∂X containing E which means that $\mu \in \mathcal{M}^+(E)$.

(a)⇒(b) If E is removable then $Q\mu = 0$ and thereby $u \equiv 0$ on X.

(b)⇒(a) Suppose that E is not removable. By Proposition 4.4, there exists a compact subset $K \subset E$ which is not removable. Therefore, we may find a measure $\tau \in \mathcal{M}^+(K)$ such that $u := L\tau$ is not identically zero on X. This contradicts (b). □

Remark 6.4. Assume that the positive solutions to the equation (6.2) are locally uniformly bounded. (For instance, in the case of $\gamma = \lambda_X$ and $\Psi(x,t) \ge t^\alpha$ for some $\alpha > 1$; see [9].) Then, a compact set $K \subset \partial X$ is removable if and only if every positive solution to (6.2) vanishing on $\partial X\backslash K$ belongs to $\mathcal{L}_\Psi(X)$. In fact, in this setting, ϑ_K is a non-moderate solution to (6.2) in X satisfying $\vartheta_K = 0$ on $\partial X\backslash K$.

6.5. A Semilinear Dirichlet Problem. Suppose that $\Psi \in \mathbb{Y}(\mathbb{R}^d)$ and γ is a local Kato measure on $\mathbb{R}^d$. Consider the case when $X = B$ is an open ball of $\mathbb{R}^d$, Y is the sphere ∂B and the formula (3.18) is the Poisson integral. According to Theorem 3.3, for every $f \in \mathcal{C}(\partial B)$ the semilinear Dirichlet problem

$$\begin{aligned} \Delta u &= \Psi(\cdot,u)\gamma \quad \text{in } B,\\ u &= f \quad \text{on } \partial B \end{aligned} \tag{6.8}$$

has a unique continuous solution u. It is the only continuous extension of f to $\bar{B}$ which belongs to $\mathcal{U}(B)$. Furthermore, u is a solution to (6.8) if and only if u solves the following integral equation:

$$u + \int_B G_B(\cdot,\zeta)\Psi(\zeta,u(\zeta))\,d\gamma(\zeta) = \int_{\partial B} P(\cdot,y)f(y)\,d\sigma(y), \tag{6.9}$$

where σ denotes the surface area measure on ∂B. Here, P is chosen so that $P\sigma \equiv 1$.

6.6. Solutions to Problem (6.1). The boundary value problem (6.1) is interpreted as the natural generalization of (6.8). In other words, a continuous function u on X is a solution to (6.1) means that $|u|$ is dominated by some harmonic function on X and that

$$u + \int_X G_X(\cdot, \zeta)\Psi(\zeta, u(\zeta))\, d\gamma(\zeta) = \int_Y P(\cdot, y)\, d\nu(y). \tag{6.10}$$

So the class $\mathcal{Q}_\Psi(Y)$ is the set of all $\nu \in \mathcal{M}(Y)$ for which (6.1) has a solution. In particular, by Proposition 4.4,

(NC) $|\nu|(E) = 0$ for every removable set $E \subset Y$

whenever (6.1) has a solution, and if Ψ possesses the doubling property then Theorem 5.7 assures that the condition

(SC) $|\nu|(\Gamma) = 0$ for every compact c_Ψ-polar set $\Gamma \subset Y$

is sufficient for (6.1) to be solvable.

Let $\gamma = \lambda$ and Ψ as in (6.5). For $1 < \alpha \leq 2$ and if X is bounded and sufficiently smooth, Dynkin and Kuznetsov [18, 17] (see also Le Gall [28] for $\alpha = 2$) showed using probabilistic methods that removable sets are the c_Ψ-polar sets. Consequently, (6.1) is solvable if and only if ν does not charge any c_Ψ-polar set. Similar results are given by Marcus and Véron [29, 30] for $\alpha > 2$. Analogous parabolic problems were also investigated by similar techniques (see [27, 31]).

Remark 6.5. In virtue of Theorem 3.10.b, if Ψ has the doubling property then all removable sets are c_Ψ-polar. However, in general a c_Ψ-polar subset of Y is not necessarily removable. In fact, let again X, Y, P be as in Subsection 6.5 and suppose that $\gamma = \lambda_X$. Take a ball B' internally tangent to ∂B at a point $z \in \partial B$. Then $A := B \backslash B'$ is minimal thin at z (see, e.g., [16]). Put $h = P\delta_z$. Choose $1 < \alpha < (d+1)/(d-1)$ and a locally bounded Borel measurable function $\theta \geq 1$ on B such that

$$\int_A G_B(x_0, \zeta)[h(\zeta)]^\alpha \theta(\zeta)\, d\zeta = \infty. \tag{6.11}$$

Let $\Psi(x,t) = [1_{B'}(x) + \theta(x) 1_A(x)]\, t|t|^{\alpha-1}$ for all $(x,t) \in B \times \mathbb{R}$. Seeing that

$$\int_{B'} G_B(x_0, \zeta)\Psi(\zeta, h(\zeta))\, d\zeta < \infty$$

and applying [20, Theorem 5.1] we conclude that the problem (6.1) is solvable for $\nu = \delta_z$. This implies that the set $\{z\}$ is not removable. However, by (6.11) it is clear that $\{z\}$ is a c_Ψ-polar subset of ∂B.

Remark 6.6. Let X_0 be an open subset of $\mathbb{R}^d$, $d \geq 3$, and consider a uniformly elliptic second order differential operator of the kind

$$\mathcal{L}u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} \tag{6.12}$$

where a_{ij} are Borel measurable bounded functions on X_0 and b_i are in the Lebesgue space $L^p(X_0, \lambda)$ for some $p > d$. If X is an $\mathcal{L}$-adapted domain of X_0 in the sense of R. M. and M. Hervé [24], we get the same results replacing the Laplacian by the operator $\mathcal{L}$.

6.7. PARABOLIC SETTING. As application of our abstract study we may suppose that the harmonic space $(X, \mathcal{H})$ is given by a domain X of $\mathbb{R}^d \times \mathbb{R}$, $d \geq 1$, endowed with the sheaf $\mathcal{H}$ of the solutions to the heat equation on X[2]. Consider the semilinear problem

$$\begin{aligned} \Delta u - \frac{\partial u}{\partial t} &= \Psi(\cdot, u)\gamma \ \text{ in } X, && (6.13)\\ u &= \nu \qquad\quad \text{on } Y, && (6.14)\end{aligned}$$

where $\nu \in \mathcal{M}(Y)$, (γ, r) is an admissible pair, and $\Psi \in \mathbb{Y}(X)$ admitting the doubling property. Similar to the previous elliptic case, $\mathcal{U}(X)$ coincides with the set of all continuous solutions (in the distributional sense) to (6.13). Therefore, for any $\nu \in \mathcal{M}(Y)$

$$(\mathbf{SC}) \ \Rightarrow \ (6.13)\text{-}(6.14) \text{ has a solution in } \mathcal{U}_r(X) \ \Rightarrow \ (\mathbf{SN}).$$

7 Appendix

I) *Proof of Theorem 3.3.* (c.f. [5]) We only have to prove the existence of u. In fact, the uniqueness of u satisfying (3.4) is assured by the comparison principle.

Take $\Omega \in \mathcal{O}$, $f \in \mathcal{B}_b(\Omega)$ and let $a = \sup_{\partial\Omega}|f|$. The function Ψ_a defined on $X \times \mathbb{R}$ by

$$\Psi_a(x,t) = \operatorname{sgn}(t)\Psi(x, \min(|t|, a))$$

satisfies the assumptions (A_1) and (A_2). For every $v \in \mathcal{B}_b(\Omega)$ consider

$$\Lambda(v) := H_\Omega f - V_\Omega^{\Psi_a} v.$$

It is easy verified that $V_\Omega^{\Psi_a}(\mathcal{B}_b(\Omega))$ is a bounded subset of $\mathcal{B}_b(\Omega)$. So, since V_Ω is a compact operator on $\mathcal{B}_b(\Omega)$ (see [22, Proposition 3.1]), it follows from Schauder's fixed point theorem that $\Lambda(u) = u$ for some $u \in \mathcal{B}_b(\Omega)$. Remark now that $|u| \leq a$ by Proposition 3.1, which yields that $V_\Omega^{\Psi_a} u = V_\Omega^{\Psi} u$. Consequently, $\Lambda(u) = u$ and the proof is finished. □

[2]Since in this case there are nontrivial absorbing subsets of X, we cannot choose r to be a Dirac measure.

Proof of Theorem 3.5. (c.f. [20]) Let $u \in \mathcal{B}_b(\Omega)$ and let $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. From (3.2) and (3.4) we get that

$$\begin{aligned} u + V_\Omega^\Psi u - H_D V_\Omega^\Psi u &= u + V_D^\Psi u, \\ H_D(u + V_\Omega^\Psi u) - H_D V_\Omega^\Psi u &= U_D u + V_D^\Psi U_D u. \end{aligned}$$

Therefore Proposition 3.1 completes the proof. □

Proof of Theorem 3.7. (c.f. [20]) Take $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. For every $n \geq 1$ let

$$h_n = u_n + V_D^\Psi u_n. \tag{7.1}$$

(a) Since (h_n) is an increasing sequence of harmonic functions on D and is uniformly bounded, we conclude that $h = \sup_{n\geq 1} h_n$ is harmonic on D. Passing to the limit in (7.1) we obtain that $u + V_D^\Psi u = h$. So, by Theorem 2.3, statement (a) is proved.

(b) Let $K \subset D$ be a compact subset and choose a subsequence (h_{n_k}) of (h_n) which converges uniformly on K. Since the family

$$\left\{ V_D^\Psi u_{n_k} : k \geq 1 \right\}$$

is equicontinuous [22, Proposition 3.1], by Ascoli's theorem there exists a subsequence (v_k) of (u_{n_k}) such that $(V_D^\Psi v_k)$ converges uniformly on K. Consequently, (v_k) is uniformly convergent on K. Now, in order to show the first statement of (b) it will be enough to use an exhaustion (Ω_n) of X and apply the diagonal procedure. The second statement in (b) is obvious. □

Proof of Proposition 3.14. (c.f., [20]) Let $g, h \in \mathcal{H}^+(X)$, $u_n = U_{\Omega_n}(Qg)$, $v_n = U_{\Omega_n}(Qh)$ and $w_n = U_{\Omega_n}(Qg + Qh)$. By Lemma 3.11, we have $w_n \leq u_n + v_n$ and hence

$$0 \leq \Psi(\cdot, w_n) \leq \Psi(\cdot, u_n + v_n) \leq \kappa(\Psi(\cdot, u_n) + \Psi(\cdot, v_n)) := \phi_n$$

where κ is the constant given in (2.6). On the other hand, the continuity of $\Psi(x, \cdot)$ and statement (e) of Theorem 3.10 imply that

$$\begin{aligned} \lim_{n\to\infty} \phi_n &= \kappa(\Psi(\cdot, Lg) + \Psi(\cdot, Lh)) := \phi, \quad \text{and} \\ \lim_{n\to\infty} V_{\Omega_n} \phi_n &= V\phi = \kappa(V^\Psi Lg + V^\Psi Lh) < \infty. \end{aligned}$$

Then Lemma 3.8.b shows that $(V_{\Omega_n}^\Psi w_n)$ converges to $V^\Psi L(Qg + Qh)$. So, letting n tend to infinity in the formula $w_n + V_{\Omega_n}^\Psi w_n = Qg + Qh$ we obtain that

$$L(Qg + Qh) + V^\Psi L(Qg + Qh) = Qg + Qh.$$

This means that $Q(Qg+Qh) = Qg+Qh$ and consequently $Qg+Qh \le Q(g+h)$ by monotonicity of Q on $\mathcal{H}^+(X)$. Therefore, according to Corollary 3.12.a we get that

$$Q(g+h) = Qg + Qh. \tag{7.2}$$

Finally, (7.2), Corollary 3.13, and the density of $\mathbb{Q}_+$ in $\mathbb{R}_+$ show that Q is positively homogeneous on $\mathcal{H}^+(X)$. □

II) Let $\Psi \in \mathbb{Y}(X)$ and put $\Phi = \Psi^*$. For every subset F of Y we define

$$C_\Phi(F) := \inf\left\{\|f\|_{(\Phi)} : f \in L^+_\Phi(X),\ \check{P}f(y) \ge 1 \text{ for all } y \in F\right\}, \tag{7.3}$$

and $C'_\Phi(F)$ by the same formula (7.3) where $L^+_\Phi(X)$ is replaced by $E^+_\Phi(X)$. It is not difficult to see that for any arbitrary subset F of Y

$$c_\Psi(F) \le C_\Phi(F) \le C'_\Phi(F). \tag{7.4}$$

We have already proved in Proposition 5.3 that c_Ψ, C_Φ, and C'_Φ coincide on compact subsets of Y. So, according to Choquet's Theorem [13], one immediately concludes that

$$c_\Psi(E) = C_\Phi(E) = C'_\Phi(E)$$

for every $\mathcal{K}$-Suslin subset E of Y (see [12]) provided C'_Φ defines a capacity in the sense of G. Choquet [13] (see also, [8, p. 27]).

Assumption: We suppose that both functions Ψ and Φ possess the doubling property. (Then trivially $C_\Phi = C'_\Phi$)

Using the same techniques as in Chapter 2 of [1] (see also [3]) we obtain the following properties of C_Φ:

1. C_Φ is a capacity on Y (in the sense of Section 5).
2. C_Φ is an outer capacity, that is, for every $F \subset Y$, $C_\Phi(F) = \inf C_\Phi(O)$ where the infimum is taken over all open subsets O containing E.
3. $C_\Phi(\cap_{n=1}^\infty \Gamma_n) = \inf_{n\ge1} C_\Phi(\Gamma_n)$ for every decreasing sequence (Γ_n) of compact subsets of Y. (This is a consequence of the previous property.)

We notice that properties (1)-(3) hold, for every function $\Phi \in \mathbb{Y}(X)$, even if both functions Φ and Ψ do not satisfy the Δ_2-condition.

Proposition 7.1. *C_Φ is a Choquet capacity.*

To prove the proposition we shall proceed as in the proof of[2, Théorème 2]. Let us first note that for every subset $E \subset Y$,

$$C_\Phi(E) = \inf_{f \in \mathcal{F}_E} \|f\|_{(\Phi)} \quad \text{where} \quad \mathcal{F}_E := \{f \in L^+_\Phi(X) : \check{P}f \geq 1 \; C_\Phi - q.e \text{ on } E\}.$$

Lemma 7.2. *Let $f, f_n \in L_\Phi(X)$ such that (f_n) converges (in norm) to f.*

(a) *The set $\{\check{P}|f| = \infty\}$ is C_Φ-polar.*

(b) *There exists a subsequence (g_n) of (f_n) such that $(\check{P}g_n)$ converges C_Φ-q.e to $\check{P}f$.*

Proof. (a) For every $j \geq 1$,

$$C_\Phi\{\check{P}|f| = \infty\} \leq C_\Phi\{\check{P}|f| \geq j\} \leq j^{-1}\|f\|_{(\Phi)}.$$

(b) Choose a subsequence (g_j) of (f_n) such that $\|f - g_j\|_\Phi \leq 2^{-j}/j$ for every $j \geq 1$, and let $E_j = \{j\check{P}|f - g_j| > 1\}$, $F_j = \cup_{n \geq j} E_n$, and $E = \cap_{j \geq 1} F_j$. Then

$$C_\Phi(E) \leq C_\Phi(F_j) \leq \sum_{n=j}^{\infty} C_\Phi(E_n) \leq 2^{1-j}$$

which yields that E is C_Φ-polar. Thus the proof of (b) is finished seeing that $\check{P}g_j(y)$ converges to $\check{P}f(y)$ for every $y \in Y \backslash E$. □

Proof of Proposition 7.1. By Theorem 5.1, $L_\Phi(X)^* = L_\Psi(X)$ and $L_\Psi(X)^* = L_\Phi(X)$ which implies, in particular, that $L_\Phi(X)$ is reflexive. Let (E_n) be an increasing sequence of subsets of Y and let $E = \cup_{n=1}^\infty E_n$. We claim that

$$C_\Phi(E) = \sup_{n \geq 1} C_\Phi(E_n). \tag{7.5}$$

To prove (7.5) it is sufficient to check that $\alpha := \sup_{n \geq 1} C_\Phi(E_n) \geq C_\Phi(E)$. So, without loss of generality we assume that $\alpha < \infty$. Fix $\varepsilon > 0$. Then the convex subset

$$\mathcal{A}_n := \{f \in \mathcal{F}_{E_n} : \|f\|_{(\Phi)} \leq \alpha + \varepsilon\}$$

is nonempty for every $n \geq 1$. Besides, by statement (b) of the above lemma, $\mathcal{A}_n$ is closed in $L_\Phi(X)$. So, $\mathcal{A}_n$ is compact with respect to the topology $\sigma(L_\Phi(X), L_\Psi(X))$ (see, e.g., [11]). Therefore, since $(\mathcal{A}_n)$ is decreasing we deduce that there exists $f \in \cap_{n=1}^\infty \mathcal{A}_n$. Now, seeing that $f \in \mathcal{F}_E$ and $\|f\|_{(\Phi)} \leq \alpha + \varepsilon$ it follows that $C_\Phi(E) \leq \alpha + \varepsilon$ for every $\varepsilon > 0$. Whence $C_\Phi(E) \leq \alpha$. □

Corollary 7.3. *C_Φ and c_Ψ coincide on $\mathcal{K}$-Suslin subsets of Y. In particular, if the Borel subsets of Y are $\mathcal{K}$-Suslin (for instance, if Y is locally compact) then $c_\Psi(F) = C_\Phi(F)$ for every subset F of Y.*

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