Scaling limit of stochastic dynamics in classical continuous systems

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Abstract

We investigate a scaling limit of gradient stochastic dynamics associated to Gibbs states in classical continuous systems on $\mathbb{R}^d$, $d \geq 1$. For these dynamics several scalings have already been studied, see e.g. [Bro80]. The aim is to derive macroscopic quantities from a given micro- or mesoscopic system. The scaling we consider has been investigated in [Bro80] and [Ros81]. Assuming that the underlying potential is smooth, compactly supported and positive, convergence of the generators of the scaled stochastic dynamics, averaged with respect to time, has been analyzed in [Spo86]. Another approach has been proposed in [GP85], where the idea has been to prove convergence of the corresponding resolvents. We prove that the Dirichlet forms of the scaled stochastic dynamics converge on a core of functions to the Dirichlet form of a generalized Ornstein–Uhlenbeck process. The proof is based on the analysis and geometry on the configuration space which was developed in [AKR98a], [AKR98b], and works for general Gibbs measures of Ruelle type. Hence, the underlying potential may have a singularity at the origin, only has to be bounded from below, and may not be compactly supported. Therefore, singular interactions of physical interest are covered, as e.g. the one given by the Lennard–Jones potential, which is studied in the theory of fluids. Furthermore, using the Lyons–Zheng decomposition we give a simple proof for the tightness of the scaled processes. We also prove that the corresponding generators, however, do not converge in the L^2-sense. This settles a conjecture formulated in [Bro80], [Ros81], [Spo86].

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Contents

1 Introduction

The stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$ of a classical continuous system is an infinite dimensional diffusion process having a Gibbs measure μ, e.g. of the type studied by Ruelle in [Rue69], as an invariant measure. Physically, it describes the stochastic dynamics of Brownian particles which are interacting via the gradient of a pair-potential ϕ. Since each particle can move through each position in space, the system is called continuous and is used for modelling gas and fluid. For realistic models which can be described by these stochastic dynamics, e.g. suspensions, we refer to [Spo86].

Since these dynamics are stochastic, they have to be interpreted as mesoscopic processes. The aim of analyzing scaling limits, in general, is to derive from micro- or mesoscopic systems macroscopic statements and quantities. The kind of scaling to study depends on which features of a given system one is interested in, see e.g. [Bro80], [KL99], [Spo91].

The scaling we consider in this paper has been investigated in [Bro80] and [Ros81]. In his Doctor-thesis, [Bro80], T. Brox has given some heuristic arguments for non-convergence in law of the scaled process and has conjectured that there is no limiting Markov process. However, assuming the convergence of the generators of the scaled stochastic dynamics averaged over time, cf. Conjecture 6.5 below, H. Rost has given some heuristic arguments in [Ros81] for the existence of a limiting generalized Ornstein–Uhlenbeck process, which, of course, contradicts the statement of Brox. Assuming that the underlying potential is smooth, compactly supported and positive, in [Spo86] H. Spohn presents an approach to prove Conjecture 6.5 within the proof of his main theorem. In a remark to the latter theorem the author, however, states himself that his proof is incomplete. Another approach has been proposed in [GP85]. The idea of M. Z. Guo and G. Papanicolaou has been to prove convergence of the corresponding resolvent. As remarked by themself, at that time the authors did not have an appropriate infinite dimensional analysis and geometry at their disposal, and therefore their considerations have been on a non-rigorous level.

After these contributions, for a long time there has been no progress in this problem. Recently, however, techniques considerably improved and in [AKR98a], [AKR98b] an infinite dimensional analysis and geometry on the configuration space was developed. In this paper we shall make use of these concepts in order to tackle the problem described above again.

The stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$ of a classical continuous system takes values in the configuration space

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\},$$

and informally solves the following infinite system of stochastic differential equations:

$$\begin{aligned} dx(t) &= -\beta \sum_{\substack{y(t)\in \mathbf{X}(t)\\ y(t)\neq x(t)}} \nabla\phi(x(t)-y(t))\,dt + \sqrt{2}\,dB^x(t), \qquad x(t)\in \mathbf{X}(t),\\ \mathbf{X}(0) &= \gamma, \qquad\qquad \gamma \in \Gamma, \end{aligned} \tag{1}$$

where $(B^x)_{x\in\gamma}$ is a sequence of independent Brownian motions. The study of such diffusions has been initiated by R. Lang [Lan77], who considered the case $\phi \in C_0^3(\mathbb{R}^d)$ using finite dimensional approximations and stochastic differential equations. More singular ϕ, which are of particular interest in physics, as e.g. the Lennard–Jones potential, have been treated by H. Osada and also M. Yoshida, see [Osa96], [Yos96]. The latter two authors were the first to use Dirichlet forms for the construction of such processes. But, they were not able to write down the corresponding generators or martingale problems explicitly, hence could not prove that their processes actually solve (1) weakly. This, however, was proved in [AKR98b] and thus the latter work became the starting point of this paper. In [AKR98b] the authors also used Dirichlet forms and their construction also works for singular potentials of the above mentioned type, see Theorem 3.2 below. Additionally, and this is essential for our considerations, an explicit expression for the corresponding generator and martingale problem was provided, that shows that the process in [AKR98b] indeed solves (1) in the weak sense.

The scaled process $(\mathbf{X}_\epsilon(t))_{t\geq 0}$ studied in this paper is defined by

$$\mathbf{X}_\epsilon(t) := S_{out,\epsilon}(S_{in,\epsilon}(\mathbf{X}(\epsilon^{-2}t))), \qquad t\geq 0, \quad \epsilon > 0,$$

and we are interested in the scaling limit for $\epsilon \to 0$. The first scaling $S_{in,\epsilon}$ scales the position of the particles inside the configuration space as follows:

$$\Gamma \ni \gamma \mapsto S_{in,\epsilon}(\gamma) := \{\epsilon\, x \,|\, x\in\gamma\} \in \Gamma, \qquad \epsilon > 0.$$

Hence, for small $\epsilon > 0$ this scaling concentrates the particles towards the origin. The second scaling $S_{out,\epsilon}$ leads us out of the configuration space and is given by

$$\Gamma \ni \gamma \mapsto S_{out,\epsilon}(\gamma) := \epsilon^{d/2}\Big(\gamma - \rho^{(1)}_{\tilde{\mu}_\epsilon}\,dx\Big) \in \mathcal{D}',$$

where $\mathcal{D}'$ is the space of Schwartz-distributions. In the second scaling we first center the configuration γ by subtracting the first correlation measure $\rho^{(1)}_{\tilde{\mu}_\epsilon}\,dx$ of the Gibbs measure $\tilde{\mu}_\epsilon := S^*_{in,\epsilon}\mu$. Furthermore, we scale the mass of the particles by $\epsilon^{d/2}$ to avoid divergence of the total mass at the origin when $\epsilon \to 0$.

We start with constructing the Dirichlet form $\mathcal{E}_\epsilon$, the generator H_ϵ and the semi-group $(T_{\epsilon,t})_{t\geq 0}$ associated to $(\mathbf{X}_\epsilon(t))_{t\geq 0}$. These objects are images of the Dirichlet form, generator, and semi-group, respectively, which are associated to the original stochastic dynamics $(\mathbf{X}(t))_{t\geq 0}$, see Theorem 4.1 below.

The first convergence we show is the following, see Theorem 5.3. We prove that

$$\lim_{\epsilon\to 0}\mathcal{E}_\epsilon(F,G) = \mathcal{E}_{\nu_\mu}(F,G), \tag{2}$$

for all smooth cylinder functions $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$. The limit Dirichlet form $\mathcal{E}_{\nu_\mu}$ is defined on $L^2(\mathcal{D}',\nu_\mu)$ with ν_μ being white noise, and associated to a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ solving the stochastic differential equation

$$d\mathbf{X}(t,x) = \frac{\rho^{(1)}_\phi(\beta,1)}{\chi_\phi(\beta)}\Delta\mathbf{X}(t,x)\,dt + \sqrt{2\,\rho^{(1)}_\phi(\beta,1)}\,d\mathbf{W}(t,x), \tag{3}$$

where $(\mathbf{W}(t))_{t\geq 0}$ is a Brownian motion in $\mathcal{D}'$ with covariance operator $-\Delta$. The coefficient $\rho^{(1)}_\phi(\beta,1)/\chi_\phi(\beta)$ is called the bulk diffusion coefficient and β is the inverse temperature. The convergence (2) determines the limit process uniquely, see Remark 5.4(i), and requires only very weak assumptions. The interaction potential ϕ only has to be stable (S) and we have to assume the LA-HT (low activity high temperature) regime. A basic ingredient in the proof is the convergence of the image measures $\mu_\epsilon := S^*_{out,\epsilon}S^*_{in,\epsilon}\mu$ to the Gaussian white noise measure ν_μ as $\epsilon \to 0$, see Theorem 5.1. The latter fact has been also proved by T. Brox in his already mentioned Doctor-thesis [Bro80].

The convergence in terms of the Dirichlet forms, however, so far has no probabilistic interpretation. Hence, we also study convergence in law of the scaled processes. By $\mathbf{P}^\epsilon$ we denote the laws of the scaled equilibrium processes, i.e., the law of the scaled process starting with a distribution equal to the equilibrium measure μ_ϵ. Then, in Theorem 6.1 we prove that the family $(\mathbf{P}^\epsilon)_{\epsilon>0}$ is tight. The proof, again, works under quite weak assumptions on the potential. We only need conditions which ensure the existence of the original stochastic process and have to assume the LA-HT regime. In the proof we use the well-known Lyons-Zheng decomposition, [LZ88] , [LZ94], of the scaled process and the Burkholder–Davies–Gundy inequalities in order to establish the required estimate of the increments. Since the state space of the scaled process is a space of distributions, we first prove tightness in a weak sense. Then, via some Hilbert–Schmidt embeddings, we find a negative, weighted Sobolev spaces $\mathcal{H}_{-m}$ as state space such that the family $(\mathbf{P}^\epsilon)_{\epsilon>0}$ is tight on $C([0,\infty),\mathcal{H}_{-m})$.

Tightness implies the existence of accumulation points. However, we do not know a priori whether there is more than one accumulation point and whether the generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ is one of them. A well-known method to identify the limit is based on considering the associated martingale problem. More precisely, if we could prove that all accumulation points of $(\mathbf{P}^\epsilon)_{\epsilon>0}$ satisfy the martingale problem for the generator H associated to equation (3) with initial condition ν_μ, then we could use (a slight modification of) a result of R. Holley and D. Stroock [HS78] which states that, by uniqueness, all these accumulation points coincide.

The obvious first idea to prove that all limit points solve the martingale problem for H is to try to prove strong convergence of $H_\epsilon \to H$ as $\epsilon \to 0$. In [Bro80], [Ros81], and [Spo86] it has, however, been conjectured that, in general, the difference

$$\| (H - H_\epsilon)F \|_{L^2(\mu_\epsilon)}, \qquad F \in \mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}'),$$

does not tend to zero as $\epsilon \to 0$. In Theorem 6.3 we prove that this conjecture is indeed true. The proof is a quite elaborate task and is done via a mathematically rigorous high temperature expansion. Basic tools for this are provided by Theorem A.4, where we derive explicit formulas for the derivative of the correlation functions with respect to the inverse temperature β, and by Theorem B.1, where we prove a coercivity identity for Gibbs measures.

It turns out that for the above described identification of the accumulation points of $(\mathbf{P}^\epsilon)_{\epsilon>0}$, however, a weaker convergence of the generators is sufficient. In Theorem 6.7 we prove convergence in law under the assumption that Conjecture 6.5 is true, i.e., under the assumption that the generators converge in time average.

To complete the program also from a purely probabilistic point of view, it remains to prove Conjecture 6.5 in physically relevant models. The first essential ingredient has already been established in [AKR98b], namely the (time) ergodicity of the solution to equation (1) with initial (hence stationary) distribution being equal to the Gibbs measure μ, provided we are in the LA-HT regime, cf. Remark 6.6(iii). This remaining steps will be the subject of future work.

2 Gibbs states of classical continuous systems

2.1 Configuration space and Poisson measure

$\mathcal{O}(\mathbb{R}^d)$ is defined as the family of all open sets of $\mathbb{R}^d, d \geq 1$, with norm $|\cdot|_{\mathbb{R}^d}$ given by the Euclidean scalar product $(\cdot,\cdot)_{\mathbb{R}^d}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the corresponding Borel σ-algebra. $\mathcal{O}_c(\mathbb{R}^d)$ denotes the system of all elements in $\mathcal{O}(\mathbb{R}^d)$, which have compact closure. The Lebesgue measure on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ we denote by dx.

A *configuration space* Γ over $\mathbb{R}^d$ is defined as the set of all locally finite subsets (configurations) in $\mathbb{R}^d$:

$$\Gamma := \{\gamma \subset \mathbb{R}^d \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

Here $|A|$ denotes the cardinality of a set A. Via the identification of $\gamma \in \Gamma$ with

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_p(\mathbb{R}^d),$$

where ε_x denotes the Dirac measure in $x \in \mathbb{R}^d$, Γ can be considered as a subset of the set $\mathcal{M}_p(\mathbb{R}^d)$ of all positive Radon measures on $\mathbb{R}^d$. Hence Γ can be topologized by the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\, d\gamma(x) = \sum_{x\in\gamma} f(x)$$

are continuous. Here $f \in C_0(\mathbb{R}^d)$, the set of continuous functions on $\mathbb{R}^d$ with compact support. We denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-algebra.

For a given $z > 0$ (activity parameter), let π_z denote the Poisson measure on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure $z\, dx$. This measure is characterized via its Fourier transform

$$\int_\Gamma \exp(i\langle f,\gamma\rangle)\,d\pi_z(\gamma)=\exp\Big(z\int_{\mathbb{R}^d}(\exp(if(x))-1)\,dx\Big),\qquad f\in\mathcal{D},$$

where $\mathcal{D}:=C_0^\infty(\mathbb{R}^d)$, the set of smooth functions on $\mathbb{R}^d$ with compact support. For a construction of this measure as a measure on the configuration space we refer to e.g. [AKR98a].

2.2 Gibbs measures

Let ϕ be a symmetric pair potential, i.e., a measurable function $\phi:\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ such that $\phi(x)=\phi(-x)$. For $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\phi:\Gamma\to\mathbb{R}\cup\{\infty\}$ is defined by

$$E_\Lambda^\phi(\gamma):=\begin{cases}\displaystyle\sum_{\{x,y\}\subset\gamma_\Lambda}\phi(x-y)+\sum_{x\in\gamma_\Lambda,\,y\in\gamma_{\mathbb{R}^d\setminus\Lambda}}\phi(x-y), & \\ \qquad\text{if}\displaystyle\sum_{\{x,y\}\subset\gamma_\Lambda}|\phi(x-y)|+\sum_{x\in\gamma_\Lambda,\,y\in\gamma_{\mathbb{R}^d\setminus\Lambda}}|\phi(x-y)|<\infty, & \\ +\infty, \qquad\text{otherwise}, &\end{cases}$$

where $\gamma_\Lambda:=\gamma\cap\Lambda$, and the sum over the empty set is defined to be zero. Furthermore, given $\Lambda\in\mathcal{O}_c(\mathbb{R}^d),\beta\geq0,z>0$, we define for $\gamma\in\Gamma$, $\Delta\in\mathcal{B}(\Gamma)$ the *specification*, see [Föl75], [Pre76],

$$\begin{aligned}\Pi_\Lambda^{\beta\phi,z}(\gamma,\Delta):=&\,\mathbf{1}_{\{Z_\Lambda^{\beta\phi,z}<\infty\}}(\gamma)\,(Z_\Lambda^{\beta\phi,z}(\gamma))^{-1}\\ &\times\int_\Gamma\mathbf{1}_\Delta(\gamma_{\mathbb{R}^d\setminus\Lambda}+\gamma'_\Lambda)\exp\big(-\beta E_\Lambda^\phi(\gamma_{\mathbb{R}^d\setminus\Lambda}+\gamma'_\Lambda)\big)\,\pi_z(d\gamma'),\end{aligned}\quad(4)$$

where

$$Z_\Lambda^{\beta\phi,z}(\gamma):=\int_\Gamma\exp\big(-\beta E_\Lambda^\phi(\gamma_{\mathbb{R}^d\setminus\Lambda}+\gamma'_\Lambda)\big)\,\pi_z(d\gamma').$$

A probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ is called a *grand canonical Gibbs measure* with interaction potential ϕ, inverse temperature β and activity z if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\mu\Pi_\Lambda^{\beta\phi,z}=\mu\qquad\text{for all }\Lambda\in\mathcal{O}_c(\mathbb{R}^d).$$

Let $\mathcal{G}(\beta\phi, z)$ denote the set of all such probability measures μ. It can be shown, see [Geo79], that the unique grand canonical Gibbs measure corresponding to the free case, $\phi \equiv 0$, is the Poisson measure π_z.

Next, let us describe a class of Gibbs measures which is considered in classical statistical mechanics of continuous systems, see [Rue69], [Rue70]. Rewrite the conditional energy E_Λ^ϕ in the following form

$$E_\Lambda^\phi(\gamma) = E_\Lambda^\phi(\gamma_\Lambda) + W^\phi(\gamma_\Lambda \mid \gamma_{\mathbb{R}^d\setminus\Lambda}),$$

where the term

$$W^\phi(\gamma_\Lambda \mid \gamma_{\mathbb{R}^d\setminus\Lambda}) := \begin{cases} \sum\limits_{x\in\gamma_\Lambda, y\in\gamma_{\mathbb{R}^d\setminus\Lambda}} \phi(x-y) & \text{if } \sum\limits_{x\in\gamma_\Lambda, y\in\gamma_{\mathbb{R}^d\setminus\Lambda}} |\phi(x-y)| < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

describes the interaction energy between γ_Λ and $\gamma_{\mathbb{R}^d\setminus\Lambda}$.

For every $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ we define a cube

$$Q_r = \left\{ x \in \mathbb{R}^d \,\middle|\, r_i - 1/2 \le x_i < r_i + 1/2 \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$ we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. Additionally, we introduce for $n \in \mathbb{N}$ a cube Λ_n with side length $2n-1$ centered at the origin in $\mathbb{R}^d$.

From now on we assume that the potential fulfills the following conditions:

(SS) (*superstability*) There exist $A(\phi) > 0$, $B(\phi) \ge 0$ such that, if $\gamma = \gamma_{\Lambda_n}$ for some $n \in \mathbb{N}$, then

$$E_{\Lambda_n}^\phi(\gamma) \ \ge \ \sum_{r\in\mathbb{Z}^d} \Big(A(\phi)|\gamma_r|^2 - B(\phi)|\gamma_r| \Big).$$

(SS) obviously implies:

(S) (*stability*) For any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma$ we have

$$E_\Lambda^\phi(\gamma_\Lambda) \ \ge \ -B(\phi)|\gamma_\Lambda|.$$

A consequence of (S), in turn, is, of course, that ϕ is bounded from below. For $\beta \ge 0, z > 0$, let us define

$$C(\beta\phi, z) := \exp(2\beta B(\phi)) \int_{\mathbb{R}^d} |\exp(-\beta\phi(x)) - 1| \, z \, dx.$$

We also need

(UI) (*uniform integrability*) We have:

$$C(\beta\phi, z) < \exp(-1).$$

(UI) is stronger than (I) (*integrability*), i.e., $C(\beta\phi, z) < \infty$, which is also called *regularity*, see e.g. [Rue69].

(LR) (*lower regularity*) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\| r \|) < \infty$$

and for any Λ', Λ'' which are finite unions of cubes of the form Q_r and disjoint,

$$W^\phi(\gamma' \mid \gamma'') \geq - \sum_{r', r'' \in \mathbb{Z}^d} a(\| r' - r'' \|) \, |\gamma'_{r'}| \, |\gamma''_{r''}|,$$

provided $\gamma' = \gamma'_{\Lambda'}$, $\gamma'' = \gamma''_{\Lambda''}$. Here W^ϕ is the interaction energy, see (2.2), extended to arbitrary disjoint configurations and $\| \cdot \|$ denotes the maximum norm on $\mathbb{R}^d$.

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by

$$S_\infty := \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n := \left\{ \gamma \in \Gamma \,\middle|\, \sum_{r \in \Lambda_N \cap \mathbb{Z}^d} |\gamma_r|^2 \leq n^2 |\Lambda_N \cap \mathbb{Z}^d|, \, \forall N \in \mathbb{N} \right\}.$$

By $\mathcal{G}^t(\beta\phi, z) \subset \mathcal{G}(\beta\phi, z)$, $\beta \geq 0, z > 0$, we denote the set of all tempered grand canonical Gibbs measures (Ruelle measures for short). Due to [Rue70] the set $\mathcal{G}^t(\beta\phi, z)$ is non-empty for all $\beta \geq 0, z > 0$ and any potential ϕ satisfying conditions (SS), (LR), and (I). Condition (UI) is equivalent to the fact that $\mu \in \mathcal{G}(\beta\phi, z)$ is in the LA-HT (low activity high temperature) regime, see e.g. [Rue63] and [Min67]. Furthermore, the set $\mathcal{G}(\beta\phi, z)$ is not empty for potentials satisfying (S) and (UI), or, equivalently, in the LA-HT regime with potentials satisfying (S), see e.g. [Rue63], [Min67], [MM91] and [Kun99], Theorem 4.6.10 together with Remark 4.6.1.

2.3 K-transform and correlation functions

Next, we recall the definition of correlation functions using the concept of the so-called K-transform, see e.g. [KK99], [Len73], [Len75a], [Len75b].

Denote by Γ_0 the space of finite configurations over $\mathbb{R}^d$:

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}, \quad \Gamma_0^{(0)} := \{\varnothing\}, \quad \Gamma_0^{(n)} := \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}, \quad n \in \mathbb{N}.$$

Let

$$\widetilde{\mathbb{R}^{d\times n}} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^{d\times n} \mid x_i \neq x_j \quad \text{for} \quad i \neq j \right\}$$

and let S^n denote the group of all permutations of $\{1, \ldots, n\}$. Through the natural bijection

$$\widetilde{\mathbb{R}^{d\times n}}/S^n \longleftrightarrow \Gamma_0^{(n)} \tag{5}$$

one defines a topology on $\Gamma_0^{(n)}$. The space Γ_0 is equipped then with the topology of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denote the Borel σ-algebra on Γ_0.

A $\mathcal{B}(\Gamma_0)$-measurable function $G\colon \Gamma_0 \to \mathbb{R}$ is said to have bounded support if there exist $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $\operatorname{supp}(G) \subset \bigsqcup_{n=0}^{N} \Gamma_{0,\Lambda}^{(n)}$, where $\Gamma_{0,\Lambda}^{(n)} = \{\eta \subset \Lambda \mid |\eta| = n\}$. The space of measurable bounded functions on Γ_0 with bounded support is denoted by $B_{\mathrm{bs}}(\Gamma_0)$.

For any $\gamma \in \Gamma$ let $\sum_{\eta \Subset \gamma}$ denote the summation over all $\eta \subset \gamma$ such that $|\eta| < \infty$. For a function $G : \Gamma_0 \to \mathbb{R}$, the K-transform of G is defined by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \tag{6}$$

for each $\gamma \in \Gamma$ such that at least one of the series $\sum_{\eta \Subset \gamma} G^+(\eta)$ or $\sum_{\eta \Subset \gamma} G^-(\eta)$ converges, where $G^+ := \max\{0, G\}$ and $G^- := -\min\{0, G\}$. For each $G \in B_{\mathrm{bs}}(\Gamma_0)$ and each $\gamma \in \Gamma$, the series $\sum_{\eta \Subset \gamma} G(\eta)$ is always finite, and moreover, KG is a $\mathcal{B}(\Gamma)$-measurable function on Γ, see [KK99], Proposition 3.5.

Let μ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. The correlation measure corresponding to μ is defined by

$$\rho_\mu(A) := \int_\Gamma (K1_A)(\gamma)\, d\mu(\gamma), \qquad A \in \mathcal{B}(\Gamma_0).$$

Obviously, ρ_μ is a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$.

Let $G \in L^1(\Gamma_0, \mathcal{B}(\Gamma_0), \rho_\mu)$, then

$$\| KG \|_{L^1(\mu)} \leq \| K|G| \|_{L^1(\mu)} = \| G \|_{L^1(\rho_\mu)}, \tag{7}$$

hence $KG \in L^1(\Gamma, \mathcal{B}(\Gamma), \mu)$ and $KG(\gamma)$ is for μ-a.e. $\gamma \in \Gamma$ absolutely convergent. Moreover, then obviously

$$\int_{\Gamma_0} G(\eta)\, d\rho_\mu(\eta) = \int_\Gamma (KG)(\gamma)\, d\mu(\gamma), \tag{8}$$

see [KK99], [Len75a], [Len75b].

The Lebesgue–Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ with activity parameter $z > 0$ is defined by

$$\lambda_z := \delta_\emptyset + \sum_{n=1}^\infty \frac{z^n}{n!}\, dx^{\otimes n},$$

where $dx^{\otimes n}$ is defined via the bijection (5).

For a tempered Gibbs measure $\mu \in \mathcal{G}^t(\beta\phi, z)$, $\beta \geq 0, z > 0$, with ϕ satisfying (SS), (LR), and (I), or a Gibbs measure $\mu \in \mathcal{G}(\beta\phi, z)$, $\beta \geq 0, z > 0$, in the LA-HT regime with ϕ satisfying (S), the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue-Poisson measure, see e.g. [Rue70], Proposition 5.2 and Corollary 5.3., or e.g. [Rue63] and [Min67], respectively. Its Radon-Nikodym derivative

$$\rho_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_z}(\eta), \qquad \eta \in \Gamma_0,$$

w.r.t. λ_z we denote by the same symbol and the functions

$$\rho_\mu^{(n)}(x_1, \ldots, x_n) := \rho_\mu(\{x_1, \ldots, x_n\}), \quad x_1, \ldots, x_n \in \mathbb{R}^d,\ x_i \neq x_j \text{ if } i \neq j, \tag{9}$$

are called the n-th order correlation functions of the measure μ.

If we assume (UI), i.e., the LA-HT regime, (S) and that the Gibbs measure μ is constructed via finite volume Gibbs measures corresponding to empty boundary conditions, then the correlation functions can be expressed as functions of the underlying potential ϕ, inverse temperature β and activity z, i.e.,

$$\rho_\mu(\eta) = \rho_\phi(\beta, z, \eta), \qquad \eta \in \Gamma_0,$$

see e.g. [Rue63], [Min67]. Hence, due to the translation invariance of the pair interaction, the correlation functions as well as the Gibbs measure μ are translation invariant. In particular, $\rho_\phi^{(1)}(\beta, z)$ does not depend on $x_1 \in \mathbb{R}^d$.

Additionally, for these functions the so-called Ruelle bound holds: for fixed $\beta \geq 0, z > 0$, there exists a constant $\xi > 0$ such that for all n and $x_1, \dots, x_n \in \mathbb{R}^d$, $x_i \neq x_j$ for $i \neq j$, we have

$$\rho_\phi^{(n)}(\beta, z, x_1, \dots, x_n) \leq \xi^n, \tag{10}$$

see [Rue63]. Using this bound one, in particular, gets that all local moments of μ are finite:

$$\int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < \infty \qquad \forall n \in \mathbb{N}, \qquad \Lambda \in \mathcal{O}_c(\mathbb{R}^d). \tag{11}$$

3 Dirichlet forms, their generators, and corresponding stochastic dynamics

Here we recall the analysis and geometry on configuration space developed in [AKR98a] and [AKR98b].

Let $T_x(\mathbb{R}^d) = \mathbb{R}^d$ denote the tangent space to $\mathbb{R}^d$ at a point $x \in \mathbb{R}^d$. The tangent space to Γ at a point $\gamma \in \Gamma$ is defined as the Hilbert space

$$T_\gamma(\Gamma) := L^2(\mathbb{R}^d \to T\mathbb{R}^d, \gamma) = \bigoplus_{x \in \gamma} T_x(\mathbb{R}^d).$$

Thus, each $V(\gamma) \in T_\gamma(\Gamma)$ has the form $V(\gamma) = (V(\gamma, x))_{x \in \gamma}$, where $V(\gamma, x) \in T_x(\mathbb{R}^d)$, and

$$\|V(\gamma)\|^2_{T_\gamma(\Gamma)} = \sum_{x \in \gamma} \|V(\gamma, x)\|^2_{T_x(\mathbb{R}^d)} = \sum_{x \in \gamma} \|V(\gamma, x)\|^2_{\mathbb{R}^d}.$$

Let $\gamma \in \Gamma$ and $x \in \gamma$. We denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \varnothing$. Now, for a function $F \colon \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$, and $x \in \gamma$, we define a function $F_x(\gamma, \cdot) \colon \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by

$$\mathcal{O}_{\gamma,x} \ni y \mapsto F_x(\gamma, y) := F(\gamma - \varepsilon_x + \varepsilon_y) \in \mathbb{R}.$$

We say that a function $F \colon \Gamma \to \mathbb{R}$ is differentiable at $\gamma \in \Gamma$ if, for each $x \in \gamma$, the function $F_x(\gamma, \cdot)$ is differentiable at x and

$$\nabla^\Gamma F(\gamma) := (\nabla_x F(\gamma))_{x \in \gamma} \in T_\gamma(\Gamma),$$

where

$$\nabla_x F(\gamma) := \nabla_y F_x(\gamma, y)\big|_{y=x}.$$

Evidently, this definition is independent of the choice of the set $\mathcal{O}_{\gamma,x}$. We call $\nabla^\Gamma F(\gamma)$ the gradient of F at $\gamma \in \Gamma$.

We define a set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as the set of all functions on Γ of the form

$$\gamma \mapsto F(\gamma) = g_F(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle), \tag{12}$$

where $f_1, \ldots, f_N \in \mathcal{D}$ and $g_F \in C_b^\infty(\mathbb{R}^N)$. Clearly, $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is dense in $L^p(\mu), p \geq 1$. Any function F of the form (12) is differentiable at each point $\gamma \in \Gamma$, and its gradient is given by

$$(\nabla^\Gamma F)(\gamma, x) = \sum_{j=1}^N \partial_j g_F(\langle f_1, \gamma\rangle, \ldots, \langle f_N, \gamma\rangle) \nabla f_j(x), \quad \gamma \in \Gamma,\, x \in \gamma, \tag{13}$$

where ∂_j denotes the partial derivative w.r.t. the j-th variable. For $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we define

$$\mathcal{E}_\mu^\Gamma(F, G) := \int_\Gamma (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma(\Gamma)}\, d\mu(\gamma).$$

Since the Gibbs measure μ corresponding to the construction with empty boundary conditions has all local moments finite, see (11), with the help of (13) we have $(\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T_\gamma(\Gamma)} \in L^1(\mu)$. Furthermore, the gradient respects μ-classes $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)^\mu$ determined by $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, see e.g. [Röc98], [MR00]. Hence, $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ is a densely defined, positive definite, symmetric bilinear form on $L^2(\mu)$.

To ensure closability of this bilinear form we have to assume further properties of the potential ϕ :

(D) (*differentiability*) The function $\exp(-\phi)$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$ and the weak gradient $\nabla\phi$ (which is a locally dx-integrable function on $\mathbb{R}^d \setminus \{0\}$), considered as a dx-a.e. defined function on $\mathbb{R}^d$, satisfies

$$\nabla\phi \in L^1(\mathbb{R}^d, \exp(-\phi)\, dx) \;\cap\; L^2(\mathbb{R}^d, \exp(-\phi)\, dx).$$

Note that, for many typical potentials in statistical physics, we have $\phi \in C^\infty(\mathbb{R}^d \backslash \{0\})$. For such "outside the origin regular" potentials, condition (D) nevertheless does not exclude a singularity at the point $0 \in \mathbb{R}^d$.

(LS) (*local summability*) For all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and all $\gamma \in S_\infty$

$$\lim_{n\to\infty} \sum_{y \in \gamma_{\Lambda_n \backslash \Lambda}} \nabla\phi(\cdot - y)$$

exists in $L^1_{\mathrm{loc}}(\Lambda, dx)$.

For each $\gamma \in \Gamma$, consider the triple

$$T_{\gamma,\infty}(\Gamma) \supset T_\gamma(\Gamma) \supset T_{\gamma,0}(\Gamma). \tag{14}$$

Here, $T_{\gamma,0}(\Gamma)$ consists of all finite sequences from $T_\gamma(\Gamma)$, and $T_{\gamma,\infty}(\Gamma) := (T_{\gamma,0}(\Gamma))'$ is the dual space consisting of all sequences $V(\gamma) = (V(\gamma,x))_{x\in\gamma}$, where $V(\gamma,x) \in T_x(\mathbb{R}^d)$. The pairing between any $V(\gamma) \in T_{\gamma,\infty}(\Gamma)$ and $v(\gamma) \in T_{\gamma,0}(\Gamma)$ with respect to the zero space $T_\gamma(\Gamma)$ is given by

$$(V(\gamma), v(\gamma))_{T_\gamma(\Gamma)} := \sum_{x\in\gamma} (V(\gamma,x), v(\gamma,x))_{T_x(\mathbb{R}^d)}.$$

This series is, in fact, finite.

For $\gamma \in \Gamma$, we define $B_\mu(\gamma) = (B_\mu(\gamma,x))_{x\in\gamma} \in T_{\gamma,\infty}(\Gamma)$ by

$$B_\mu(\gamma,x) := -\beta \sum_{y\in\gamma\backslash\{x\}} \nabla\phi(x-y), \qquad x \in \gamma. \tag{15}$$

As follows from the proof of Lemma 4.1 in [AKR98b], for μ-a.e. $\gamma \in \Gamma$ the series on the right hand side of (15) converges absolutely in $\mathbb{R}^d$, provided $\mu \in \mathcal{G}^t(\beta\phi, z)$ is the Gibbs measure corresponding to the construction with empty boundary conditions and ϕ satisfies (SS), (UI), (LR), and (D). We call B_μ the logarithmic derivative of the measure μ. Assuming additionally (LS) one can prove an integration by parts formula for the gradient ∇^Γ, see [AKR98b], Theorem 4.3. Utilizing this formula we obtain for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$:

$$\mathcal{E}_\mu^\Gamma(F,G) = \int_\Gamma H_\mu^\Gamma F G \, d\mu, \tag{16}$$

where

$$H_\mu^\Gamma F(\gamma) := -\Delta^\Gamma F(\gamma) - (B_\mu(\gamma), \nabla^\Gamma F(\gamma))_{T_\gamma(\Gamma)}, \tag{17}$$
$$\Delta^\Gamma F(\gamma) := \sum_{x\in\gamma} \Delta_x F(\gamma), \quad \Delta_x F(\gamma) := \Delta_y F_x(x,y)\big|_{y=x},$$

Δ denoting the Laplacian on $\mathbb{R}^d$. For $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ as in (12), we have

$$\begin{aligned} H_\mu^\Gamma F(\gamma) = &- \sum_{i,j=1}^N \partial_i\partial_j g_F(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle)\langle(\nabla f_i,\nabla f_j)_{\mathbb{R}^d},\gamma\rangle \\ &- \sum_{j=1}^N \partial_j g_F(\langle f_1,\gamma\rangle,\dots,\langle f_N,\gamma\rangle) \\ &\times \Big(\langle\Delta f_j,\gamma\rangle - \beta \sum_{\{x,y\}\subset\gamma} \Big(\nabla\phi(x-y), \nabla f_j(x) - \nabla f_j(y)\Big)_{\mathbb{R}^d}\Big), \quad \mu\text{-a.e.} \quad \gamma\in\Gamma. \end{aligned} \tag{18}$$

Moreover, $H_\mu^\Gamma F \in L^2(\mu)$ for each $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, see [AKR98b], Lemma 4.1. Utilizing (16) in [AKR98b], Proposition 5.1, the following statement has been proven.

Proposition 3.1 *Assume that the potential ϕ fulfills conditions* (SS), (UI), (LR), (D), (LS) *and let $\mu \in \mathcal{G}^t(\beta\phi, z)$, $\beta \geq 0$, $z > 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then the bilinear form $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ is closable on $L^2(\mu)$ and its closure $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is a symmetric Dirichlet form which is conservative. Its generator is the Friedrichs extension of H_μ^Γ, which will be denoted by the same symbol.*

Of course, H_μ^Γ generates a strongly continuous contraction semi-group

$$T_t^\mu := \exp(-tH_\mu^\Gamma), \qquad t \geq 0.$$

The existence of the diffusion process corresponding to $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ was shown in [AKR98b], Theorem 5.2, and [MR00], Theorem 4.13. For all $d \geq 1$ it lives on the bigger state space $\ddot{\Gamma}$ consisting of all integer-valued Radon measures on $\mathbb{R}^d$, see e.g. [Kal75]. For $d \geq 2$ in [RS98], Corollary 1, the authors have proven that the set $\ddot{\Gamma}/\Gamma$ is $\mathcal{E}_\mu^\Gamma$-exceptional. Thus, the associated diffusion process can be restricted to a process on Γ. For simplicity of notations, we exclude the case $d = 1$ in what follows. However, all our further considerations do also work in that case.

Theorem 3.2 *Let the potential ϕ have the same properties as in Proposition 3.1 and let $\mu \in \mathcal{G}^t(\beta\phi, z)$, $\beta \geq 0$, $z > 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then:*
(i) *There exists a conservative diffusion process (i.e., a conservative strong Markov process with continuous sample paths)*

$$\mathbf{M} = (\Omega, \hat{\mathbf{F}}, (\hat{\mathbf{F}}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma})$$

on Γ which is properly associated with $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, i.e., for all ($\mu$-versions) of $F \in L^2(\Gamma, \mu)$ and all $t > 0$ the function

$$\gamma \mapsto p(t, F)(\gamma) := \int_\Omega F(\mathbf{X}(t))\, d\mathbf{P}_\gamma, \qquad \gamma \in \Gamma,$$

is an $\mathcal{E}_\mu^\Gamma$-quasi-continuous version of $T_t^\mu F$. The process $\mathbf{M}$ is up to μ-equivalence unique, has μ as an invariant measure and is called microscopic stochastic dynamics.
(ii) *The diffusion process $\mathbf{M}$ is up to μ-equivalence the unique diffusion process having μ as invariant measure and solving the martingale problem for $(-H_\mu^\Gamma, D(H_\mu^\Gamma))$, i.e., for all $G \in D(H_\mu^\Gamma)$*

$$G(\mathbf{X}(t)) - G(\mathbf{X}(0)) + \int_0^t H_\mu^\Gamma G(\mathbf{X}(s))\, ds, \qquad t \geq 0,$$

is an $\hat{\mathbf{F}}_t$-martingale under $\mathbf{P}_\gamma$ (hence starting in γ) for $\mathcal{E}_\mu^\Gamma$-q.a. $\gamma \in \Gamma$.

In the above theorem $\mathbf{M}$ is canonical, i.e., $\Omega = C([0,\infty) \to \Gamma)$, $\mathbf{X}(t)(\xi) = \xi(t), \xi \in \Omega$. The filtration $(\hat{\mathbf{F}}_t)_{t\geq 0}$ is the natural "minimum completed admissible filtration", cf. [FOT94], Chap. A.2, or [MR92], Chap. IV, obtained from

$$\sigma\{\langle f, \mathbf{X}(s)\rangle \,|\, 0 \leq s \leq t,\ f \in \mathcal{D}\}, \quad t \geq 0,$$

and

$$\hat{\mathbf{F}} := \hat{\mathbf{F}}_\infty := \bigvee_{t\in[0,\infty)} \hat{\mathbf{F}}_t$$

is the smallest σ-algebra containing all $\hat{\mathbf{F}}_t$, and $(\Theta_t)_{t\geq 0}$ are the corresponding natural time shifts. For a detailed discussions of these objects and the notion of quasi-continuity we refer to [MR92]. The second part of the above theorem was proved in [AKR98b], Theorem 5.3.

Remark 3.3 Let us consider the diffusion process $(\mathbf{X}(t))_{t\geq 0}$ provided by Theorem 3.2. In (18) we have an explicit formula for the action of the associated generator $-H_\mu^\Gamma$ on smooth cylinder functions. Utilizing an extension of Itô's formula to this infinite dimensional situation on a heuristic level we find the associated infinite system of stochastic differential equations:

$$
\begin{aligned}
dx(t) &= -\beta \sum_{\substack{y(t)\in\mathbf{X}(t)\\ y(t)\neq x(t)}} \nabla\phi(x(t)-y(t))\,dt + \sqrt{2}\,dB^x(t), \quad && x(t)\in\mathbf{X}(t),\\
\mathbf{X}(0) &= \gamma, && \gamma\in\Gamma, \qquad (19)
\end{aligned}
$$

where $(B^x)_{x=x(0)\in\mathbf{X}(0)}$ is a sequence of independent Brownian motions. Theorem 3.2(ii) implies that the process $((\mathbf{X}(t))_{t\geq 0}, \mathbf{P}_\gamma)$ solves the infinite system (19) in the sense of the associated martingale problem for $\mathcal{E}_\mu^\Gamma$-q.a. $\gamma\in\Gamma$ as a starting point.

4 Scaling of stochastic dynamics and associated Dirichlet form

We perform the scaling of the process $(\mathbf{X}(t))_{t\geq 0}$ in two steps.

First scaling: We scale the position of the particles inside the configuration space as follows:

$$\Gamma \ni \gamma \mapsto S_{in,\epsilon}(\gamma) := \{\epsilon\, x \,|\, x\in\gamma\} \in \Gamma, \qquad \epsilon > 0,$$

i.e., for $f\in\mathcal{D}$, the scaling is given through

$$\langle f, S_{in,\epsilon}(\gamma)\rangle = \sum_{x\in\gamma} f(\epsilon\, x).$$

Obviously, $S_{in,\epsilon}$ is a homeomorphism on Γ. From now on we assume that $\mu\in\mathcal{G}(\beta\phi,1)$, $\beta\geq 0$. Let us define the image measure

$$\tilde{\mu}_\epsilon := S^*_{in,\epsilon}\mu.$$

This measure is also defined on $(\Gamma,\mathcal{B}(\Gamma))$ and is an element of $\mathcal{G}(\beta\phi_\epsilon,\epsilon^{-d})$, $\beta\geq 0$, where $\phi_\epsilon := \phi(\epsilon^{-1}\cdot)$. The first statement is obvious, while the second is a consequence of the following equation for the corresponding specifications, see (4):

$$\Pi_{\Lambda}^{\beta\phi_\epsilon,\epsilon^{-d}}(\gamma,\Delta)=\Pi_{\epsilon^{-1}\Lambda}^{\beta\phi,1}((S_{in,\epsilon})^{-1}(\gamma),(S_{in,\epsilon})^{-1}\Delta),\qquad \Lambda\in\mathcal{O}_c(\mathbb{R}^d).$$

Of course, temperedness is preserved. Furthermore, since $C(\beta\phi_\epsilon,\epsilon^{-d}) = C(\beta\phi,1)$ the measure $\tilde{\mu}_\epsilon$ is in the LA-HT regime if and only if this is true for μ.

Second scaling: This scaling leads us out of the configuration space and is given by

$$\Gamma\ni\gamma\mapsto S_{out,\epsilon}(\gamma):=\epsilon^{d/2}\Big(\gamma-\rho_{\phi_\epsilon}^{(1)}(\beta,\epsilon^{-d})\,\epsilon^{-d}\,dx\Big)\in\Gamma_\epsilon$$

where $\Gamma_\epsilon := S_{out,\epsilon}(\Gamma)\subset\mathcal{D}'$, $\epsilon>0$, $\mathcal{D}'$ is the topological dual of $\mathcal{D}$ (where both $\mathcal{D}$ and $\mathcal{D}'$ are equipped with their respective usual locally convex topology). We consider Γ_ϵ as a topological subspace of $\mathcal{D}'$, thus Γ_ϵ is equipped with the corresponding Borel σ-algebra. Obviously, $S_{out,\epsilon}:\Gamma\to\Gamma_\epsilon$ is continuous, hence Borel-measurable. Since it is also one-to-one and since both Γ and $\mathcal{D}'$ are standard measurable spaces, it follows by [Par67], Chap. V, Theorem 2.4, that Γ_ϵ is a Borel subset of $\mathcal{D}'$ and that $S_{out,\epsilon}^{-1}:\Gamma_\epsilon\to\Gamma$ is also Borel-measurable. The function $\rho_{\phi_\epsilon}^{(1)}(\beta,\epsilon^{-d})$ is the first correlation function corresponding to the Gibbs measure $\tilde{\mu}_\epsilon$, i.e.,

$$\int_{\mathbb{R}^d} f(x)\,\rho_{\phi_\epsilon}^{(1)}(\beta,\epsilon^{-d})\,\epsilon^{-d}\,dx=\int_\Gamma\langle f,\gamma\rangle\,d\tilde{\mu}_\epsilon(\gamma),\qquad \forall f\in C_0(\mathbb{R}^d).$$

Applied to a test function $f\in\mathcal{D}$, the second scaling gives

$$\langle f,S_{out,\epsilon}(\gamma)\rangle=\epsilon^{d/2}\Big(\sum_{x\in\gamma}f(x)-\rho_{\phi_\epsilon}^{(1)}(\beta,\epsilon^{-d})\,\epsilon^{-d}\int f(x)\,dx\Big),\tag{20}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual paring between $\mathcal{D}$ and $\mathcal{D}'$. Here we assume the LA-HT regime. So, as mentioned before $\rho_\phi^{(1)}(\beta,1)$ is a constant, and thus by definition of $\tilde{\mu}_\epsilon$ also $\rho_{\phi_\epsilon}^{(1)}(\beta,\epsilon^{-d})$ is a constant, see Subsection 2.3. Obviously, the random variable (20) is centered w.r.t. the measure $\tilde{\mu}_\epsilon$.

Scaled process: The scaled process of our interest is

$$\mathbf{X}_\epsilon(t):=S_{out,\epsilon}(S_{in,\epsilon}(\mathbf{X}(\epsilon^{-2}t))),\qquad t\ge 0,\quad \epsilon>0.$$

Associated Dirichlet form:

Next for each $\epsilon > 0$ we construct a Dirichlet form $\mathcal{E}_\epsilon$ such that $(\mathbf{X}_\epsilon(t))_{t\geq 0}$ is the unique process which is properly associated to $\mathcal{E}_\epsilon$.

Let $\mu_\epsilon := S^*_{out,\epsilon} S^*_{in,\epsilon}\mu = S^*_{out,\epsilon}\tilde{\mu}_\epsilon$. Then we define a unitary mapping $\mathcal{S}_{out,\epsilon}$: $L^2(\Gamma_\epsilon, \mu_\epsilon) \to L^2(\Gamma, \tilde{\mu}_\epsilon)$ by defining $\mathcal{S}_{out,\epsilon} F$ to be the $\tilde{\mu}_\epsilon$-class represented by $\tilde{F} \circ S_{out,\epsilon}$ for any μ_ϵ-version $\tilde{F}$ of $F \in L^2(\Gamma_\epsilon, \mu_\epsilon)$. Using this mapping we define a bilinear form $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$ as the image bilinear form of $(\mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}, D(\mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}))$ under the mapping $\mathcal{S}_{out,\epsilon}$:

$$\mathcal{E}_\epsilon(F,G) := \mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}(\mathcal{S}_{out,\epsilon}F, \mathcal{S}_{out,\epsilon}G), \qquad F, G \in D(\mathcal{E}_\epsilon), \tag{21}$$

where

$$D(\mathcal{E}_\epsilon) := \mathcal{S}^{-1}_{out,\epsilon} D(\mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}).$$

Let $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_\epsilon)$ be defined analogously to the space $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. Then obviously, $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_\epsilon) \subset D(\mathcal{E}_\epsilon)$, hence $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$ is densely defined. It follows by [MR92], Chapter VI, Exercise 1.1, that $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$ is a Dirichlet form. It is called the image Dirichlet form of $(\mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}, D(\mathcal{E}^\Gamma_{\tilde{\mu}_\epsilon}))$ under the mapping $S_{out,\epsilon}$. Its generator $(-H_\epsilon, D(H_\epsilon))$ is given by

$$H_\epsilon = \mathcal{S}^{-1}_{out,\epsilon} H^\Gamma_{\tilde{\mu}_\epsilon} \mathcal{S}_{out,\epsilon}, \qquad D(H_\epsilon) = \mathcal{S}^{-1}_{out,\epsilon} D(H^\Gamma_{\tilde{\mu}_\epsilon}), \tag{22}$$

i.e.,

$$\mathcal{E}_\epsilon(F,G) = \int_{\Gamma_\epsilon} H_\epsilon F(\omega) G(\omega)\, d\mu_\epsilon(\omega), \qquad F \in D(H_\epsilon),\, G \in D(\mathcal{E}_\epsilon).$$

Then for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_\epsilon) \subset D(H_\epsilon)$ we have

$$H_\epsilon F(\omega) = -\sum_{i,j=1}^N \frac{\partial^2 g_F}{\partial s_i \partial s_j}(\langle f_1, \omega\rangle, \ldots, \langle f_N, \omega\rangle) \tag{23}$$

$$\times \Big\langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \epsilon^{d/2}\omega + \rho^{(1)}_{\phi_\epsilon}(\beta, \epsilon^{-d})\, dx \Big\rangle - \sum_{j=1}^N \partial_j g_F(\langle f_1, \omega\rangle, \ldots, \langle f_N, \omega\rangle)$$

$$\times \Big(\langle \Delta f_j, \omega\rangle - \epsilon^{d/2}\beta \sum_{\{x,y\}\subset S^{-1}_{out,\epsilon}\omega} \Big(\nabla\phi_\epsilon(x-y), \nabla f_j(x) - \nabla f_j(y) \Big)_{\mathbb{R}^d} \Big),$$

where F is of the form (12) and the variable ω is running through Γ_ϵ. Note that the last term is well-defined for μ_ϵ-a.e. $\omega \in \Gamma_\epsilon$.

Theorem 4.1 *Let the potential ϕ fulfill conditions* (SS), (UI), (LR), (D), (LS) *and let $\mu \in \mathcal{G}^t(\beta\phi, 1)$, $\beta \geq 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then for all (μ_ϵ-versions) of $F \in L^2(\Gamma_\epsilon, \mu_\epsilon)$ and all $t > 0$ the function*

$$\omega \mapsto p_\epsilon(t, F)(\omega) := \int_\Omega F(\mathbf{X}_\epsilon(t))\, d\mathbf{P}_{S_{in,\epsilon}^{-1} S_{out,\epsilon}^{-1}\omega}, \qquad \omega \in \Gamma_\epsilon,$$

is a μ_ϵ-version of $T_{\epsilon,t}F := \exp(-tH_\epsilon)F$. For $\mathbf{Q}_\omega := \mathbf{P}_{S_{in,\epsilon}^{-1} S_{out,\epsilon}^{-1}\omega}$, $\omega \in \Gamma_\epsilon$, the process $\mathbf{M}^\epsilon = (\Omega, \hat{\mathbf{F}}, (\hat{\mathbf{F}}_{t/\epsilon^2})_{t\geq 0}, (\Theta_{t/\epsilon^2})_{t\geq 0}, (\mathbf{X}_\epsilon(t))_{t\geq 0}, (\mathbf{Q}_\omega)_{\omega\in\Gamma_\epsilon})$ is a diffusion process and thus up to μ_ϵ-equivalence the unique process in this class which is properly associated with $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$ and has μ_ϵ as an invariant measure.

Proof: For $F \in L^2(\Gamma_\epsilon, \mu_\epsilon)$ we have

$$F(\mathbf{X}_\epsilon(t)) = (\mathcal{S}_{in,\epsilon}\mathcal{S}_{out,\epsilon}F)(\mathbf{X}(\epsilon^{-2}t)), \qquad t \geq 0,$$

where $\mathcal{S}_{in,\epsilon}F := F \circ S_{in,\epsilon}$. By Theorem 3.2 we have

$$\begin{aligned}(\mathcal{S}_{out,\epsilon}^{-1}&\mathcal{S}_{in,\epsilon}^{-1} \exp(-t\epsilon^{-2}H_\mu^\Gamma)\mathcal{S}_{in,\epsilon}\mathcal{S}_{out,\epsilon}F)(\omega) \\ &= \int_\Omega \mathcal{S}_{out,\epsilon}\mathcal{S}_{in,\epsilon}F(\mathbf{X}(\epsilon^{-2}t))\, d\mathbf{P}_{S_{in,\epsilon}^{-1} S_{out,\epsilon}^{-1}\omega} = \int_\Omega F(\mathbf{X}_\epsilon(t))\, d\mathbf{P}_{S_{in,\epsilon}^{-1} S_{out,\epsilon}^{-1}\omega}, \end{aligned} \tag{24}$$

for μ_ϵ almost all $\omega \in \Gamma_\epsilon$. We note that $(\mathcal{E}_{\tilde{\mu}_\epsilon}^\Gamma, D(\mathcal{E}_{\tilde{\mu}_\epsilon}^\Gamma))$ is obviously the image Dirichlet form under the map $\mathcal{S}_{in,\epsilon}$ of $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ times ϵ^{-2}. Hence we have for the corresponding generator $(H_{\tilde{\mu}_\epsilon}^\Gamma, D(H_{\tilde{\mu}_\epsilon}^\Gamma))$

$$H_{\tilde{\mu}_\epsilon}^\Gamma = \mathcal{S}_{in,\epsilon}^{-1}\epsilon^{-2}H_\mu^\Gamma\mathcal{S}_{in,\epsilon}, \quad D(H_{\tilde{\mu}_\epsilon}^\Gamma) = \mathcal{S}_{in,\epsilon}^{-1}(D(H_\mu^\Gamma)). \tag{25}$$

Using the Hille-Yosida theorem (via resolvent) and (22), (25), we can conclude that

$$\begin{aligned}\mathcal{S}_{out,\epsilon}^{-1}\mathcal{S}_{in,\epsilon}^{-1} &\exp(-t\epsilon^{-2}H_\mu^\Gamma)\mathcal{S}_{in,\epsilon}\mathcal{S}_{out,\epsilon} \\ &= \exp(-t\mathcal{S}_{out,\epsilon}^{-1}\mathcal{S}_{in,\epsilon}^{-1}\epsilon^{-2}H_\mu^\Gamma\mathcal{S}_{in,\epsilon}\mathcal{S}_{out,\epsilon}) = \exp(-tH_\epsilon) \end{aligned} \tag{26}$$

on $L^2(\Gamma_\epsilon, \mu_\epsilon)$. Thus, by (24) and (26) the first statement of the theorem is proved. The fact that $\mathbf{M}^\epsilon$ is a diffusion is straightforward to check. In particular, it then follows by [MR92], Chap. IV, Theorem 3.5, that $\mathbf{M}^\epsilon$ is properly associated with $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$. ■

Remark 4.2 Consider the scaled process $(\tilde{\mathbf{X}}_\epsilon(t))_{t\geq 0}$ obtained from $(\mathbf{X}(t))_{t\geq 0}$ by the first scaling only, i.e.,

$$\tilde{\mathbf{X}}_\epsilon(t) := (S_{in,\epsilon}\mathbf{X})(\epsilon^{-2}t), \qquad t \geq 0.$$

For $(\tilde{\mathbf{X}}_\epsilon(t))_{t\geq 0}$ one has an analogous statement as in Theorem 4.1 for the process $(\mathbf{X}_\epsilon(t))_{t\geq 0}$. The generator corresponding to $(\tilde{\mathbf{X}}_\epsilon(t))_{t\geq 0}$ is $H^\Gamma_{\tilde{\mu}_\epsilon}$, and therefore $(\tilde{\mathbf{X}}_\epsilon(t))_{t\geq 0}$ solves (19) with the potential ϕ_ϵ.

5 Convergence of Dirichlet forms

Our aim is to show convergence of $(\mathbf{X}_\epsilon(t))_{t\geq 0}$ to a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\geq 0}$ as $\epsilon \to 0$. In this section we prove this in terms of the corresponding Dirichlet forms.

It will turn out that the limit Dirichlet form is defined in $L^2(\mathcal{D}', \nu_\mu)$, where ν_μ is the Gaussian white noise measure on $\mathcal{D}'$ with covariance operator $\chi_\phi(\beta)\,\mathrm{Id}$ and

$$\chi_\phi(\beta) := \rho^{(1)}_\phi(\beta, 1) + \int_{\mathbb{R}^d} u^{(2)}_\phi(\beta, 1, x, 0)\, dx$$

is the *compressibility* of the Gibbs state μ, see (42) below for the definition of the Ursell function $u^{(2)}_\phi$ and Proposition A.3 for the existence of the integral. The measure ν_μ exists due to the Bochner–Minlos theorem via its characteristic function given by

$$\int_{\mathcal{D}'} \exp(i\langle f, \omega\rangle)\, d\nu_\mu(\omega) = \exp\Big(- \frac{\chi_\phi(\beta)}{2} \int_{\mathbb{R}^d} (f(x))^2\, dx\Big), \quad f \in \mathcal{D}.$$

For $n \in \mathbb{Z}$ we define a weighted Sobolev spaces $\mathcal{H}_n$ as the closure of $\mathcal{D}$ w.r.t. the Hilbert norm

$$\| f \|_n^2 = \langle f, f\rangle_n := \int_{\mathbb{R}^d} A^n f(x) f(x)\, dx, \qquad f \in \mathcal{D},$$

where

$$Af(x) = -\Delta f(x) + |x|^2 f(x), \qquad x \in \mathbb{R}^d,$$

is the Hamilton operator of the harmonic oscillator with ground state eigenvalue d. We identify $\mathcal{H}_0 = L^2(\mathbb{R}^d, dx)$ with its dual and obtain

$$\mathcal{D} \subset S(\mathbb{R}^d) \subset \mathcal{H}_n \subset L^2(\mathbb{R}^d, dx) \subset \mathcal{H}_{-n} \subset S'(\mathbb{R}^d) \subset \mathcal{D}', \quad n \in \mathbb{N}.$$

Here as usual $S'(\mathbb{R}^d)$ denotes the space of tempered distributions which is the topological dual of $S(\mathbb{R}^d)$, the Schwartz space of smooth functions on $\mathbb{R}^d$ decaying faster than any polynomial. Of course, $\mathcal{H}_{-n}$ is the topological dual of $\mathcal{H}_n$ w.r.t. $\mathcal{H}_0$. The dual paring between these spaces we denote by $\langle \cdot, \cdot \rangle$. Since the embeddings $\mathcal{H}_n \subset \mathcal{H}_{n-d}$ are Hilbert–Schmidt for all $n \in \mathbb{Z}$, it follows by the Bochner–Minlos theorem that $\nu_\mu(\mathcal{H}_{-d}) = 1$.

The first part of the following theorem is an easy generalization of Proposition 3.9 in [Bro80]. The second and third part have been proved in [Bro80], Proposition 5.4 and Theorem 6.5, respectively.

Theorem 5.1 *Let us assume that the potential ϕ fulfills* (S), (UI) *and let $\mu \in \mathcal{G}(\beta\phi, 1)$, $\beta \geq 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then:*
(i) *There exists $C^{(1)} \in (0, \infty)$ such that*

$$\int_{\mathcal{D}'} \| \omega \|^2_{-(d+1)} \; d\mu_\epsilon(\omega) \leq C^{(1)}$$

uniformly in $\epsilon \in (0, 1]$ and, in particular,

$$\mu_\epsilon(\mathcal{H}_{-(d+1)}) = 1.$$

(ii) *For each $f \in \mathcal{D}$*

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mu_\epsilon}[\langle f, \cdot \rangle^2] = \mathbb{E}_{\nu_\mu}[\langle f, \cdot \rangle^2].$$

(iii) *The family of measures $(\mu_\epsilon)_{\epsilon > 0}$ converges weakly on $\mathcal{H}_{-(d+1)}$ to the Gaussian measure ν_μ as $\epsilon \to 0$.*

We shall also use the following lemma, which is easy to derive by using the properties of the correlation functions, see Section 2.3, and recalling that $\tilde{\mu}_\epsilon = S^*_{in,\epsilon}\mu \in \mathcal{G}(\beta\phi_\epsilon, \epsilon^{-d})$.

Lemma 5.2 *Let the conditions of Theorem 5.1 hold. Then we have:*

$$\rho^{(1)}_{\phi_\epsilon}(\beta, \epsilon^{-d}) = \rho^{(1)}_{\phi}(\beta, 1),$$
$$\rho^{(2)}_{\phi_\epsilon}(\beta, \epsilon^{-d}, x, y) = \rho^{(2)}_{\phi}\Big(\beta, 1, \frac{x-y}{\epsilon}, 0\Big).$$

We define the Dirichlet form $(\mathcal{E}_{\nu_\mu}, D(\mathcal{E}_{\nu_\mu}))$ as the closure of the bilinear form

$$\begin{aligned}\mathcal{E}_{\nu_\mu}(F,G) &= -\rho_\phi^{(1)}(\beta,1)\int_{\mathcal{D}'}\int_{\mathbb{R}^d}\partial_x F(\omega)\Delta\partial_x G(\omega)\,dx\,d\nu_\mu(\omega)\\ &= \rho_\phi^{(1)}(\beta,1)\int_{\mathcal{D}'}\int_{\mathbb{R}^d}(\nabla\partial_x F(\omega),\nabla\partial_x G(\omega))_{\mathbb{R}^d}\,dx\,d\nu_\mu(\omega),\end{aligned}$$

where $F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$ and the space $\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$ is defined analogously to $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. Here $\partial_x F$ denotes the derivative of $F=g_F(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)$ $\in\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$ in direction ε_x, $x\in\mathbb{R}^d$, i.e.,

$$\begin{aligned}\partial_x F(\omega) &= \frac{d}{dt}F(\omega+t\varepsilon_x)\Big|_{t=0}\\ &= \sum_{j=1}^N \partial_j g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)f_j(x), \qquad \omega\in\mathcal{D}',\end{aligned}$$

where $N\in\mathbb{N}$ and $f_1,\dots,f_N\in\mathcal{D}$.

Integrating by parts in the Gaussian space, see e.g. [BK95], Theorem 6.1.2 and 6.1.3, we obtain

$$\mathcal{E}_{\nu_\mu}(F,G)=\int_{\mathcal{D}'}HF(\omega)G(\omega)\,d\nu_\mu(\omega), \qquad F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}'),$$

where

$$\begin{aligned}HF = &-\rho_\phi^{(1)}(\beta,1)\sum_{i,j=1}^N\partial_i\partial_j g_F(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)\int_{\mathbb{R}^d}(\nabla f_i(x),\nabla f_j(x))_{\mathbb{R}^d}\,dx\\ &-\frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)}\sum_{j=1}^N\partial_j g_F(\langle f_1,\cdot\rangle,\dots,\langle f_N,\cdot\rangle)\langle\Delta f_j,\cdot\rangle. \qquad (27)\end{aligned}$$

It is well-known, see e.g. [BK95], Theorem 6.1.4, that the operator H is essentially self-adjoint on $\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$. We preserve the same notation for its closure. The operator H generates an infinite dimensional Ornstein–Uhlenbeck semi-group

$$T_t := \exp(-tH), \qquad t\ge 0,$$

in $L^2(\nu_\mu)$. This semi-group is associated to a generalized Ornstein–Uhlenbeck process $(\mathbf{X}(t))_{t\ge0}$ on $\mathcal{D}'$, see [BK95], Chapter 6, Section 1.5.

Theorem 5.3 *Suppose the potential ϕ satisfies the conditions* (S), (UI) *and let $\mu \in \mathcal{G}(\beta\phi, 1)$, $\beta \geq 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \mathcal{D}')$ we have*

$$\lim_{\epsilon \to 0} \mathcal{E}_\epsilon(F, G) = \mathcal{E}_{\nu_\mu}(F, G). \tag{28}$$

Remark 5.4 (i) The process $(\mathbf{X}(t))_{t \geq 0}$ is the unique process associated to the closure of the pre-Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty(\mathcal{D}, \mathcal{D}'))$ on $L^2(\mathcal{D}', \nu_\mu)$. In this sense the convergence of bilinear forms proven in Theorem 5.3 determines uniquely the limit process $(\mathbf{X}(t))_{t \geq 0}$.
(ii) The generator H corresponds to the following stochastic differential equation:

$$d\mathbf{X}(t, x) = \frac{\rho_\phi^{(1)}(\beta, 1)}{\chi_\phi(\beta)} \Delta \mathbf{X}(t, x)\, dt + \sqrt{2\, \rho_\phi^{(1)}(\beta, 1)}\, d\mathbf{W}(t, x),$$

where $(\mathbf{W}(t))_{t \geq 0}$ is a Brownian motion in $\mathcal{D}'$ with covariance operator $-\Delta$, and the coefficient $\rho_\phi^{(1)}(\beta, 1)/\chi_\phi(\beta)$ is called *bulk diffusion coefficient.*

Proof: We first note that each function $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \mathcal{D}')$, when restricted to Γ_ϵ, belongs to $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_\epsilon) \subset D(\mathcal{E}_\epsilon)$. Furthermore, since $\mathcal{B}(\mathcal{D}') \cap \Gamma_\epsilon = \mathcal{B}(\Gamma_\epsilon)$, the measure μ_ϵ can be considered as a measure on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$. By the polarization identity, it is sufficient to prove (28) for the case $G = F = g_F(\langle f_1, \omega \rangle, \dots, \langle f_N, \omega \rangle)$. Evaluating (21) and applying Lemma 5.2 we obtain

$$\begin{aligned}
\mathcal{E}_\epsilon(F, F) &= \epsilon^d \sum_{i,j=1}^N \int_\Gamma \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma \rangle \\
&\times \partial_i g_F(\langle f_1, \epsilon^{d/2}(\gamma - \epsilon^{-d} \rho_{\phi_\epsilon}^{(1)}(\beta, \epsilon^{-d})\, dx) \rangle, \dots, \langle f_N, \epsilon^{d/2}(\gamma - \epsilon^{-d} \rho_{\phi_\epsilon}^{(1)}(\beta, \epsilon^{-d})\, dx) \rangle) \\
&\times \partial_j g_F(\langle f_1, \epsilon^{d/2}(\gamma - \epsilon^{-d} \rho_{\phi_\epsilon}^{(1)}(\beta, \epsilon^{-d})\, dx) \rangle, \\
&\dots, \langle f_N, \epsilon^{d/2}(\gamma - \epsilon^{-d} \rho_{\phi_\epsilon}^{(1)}(\beta, \epsilon^{-d})\, dx) \rangle)\, d\tilde{\mu}_\epsilon(\gamma) \\
&= \epsilon^d \sum_{i,j=1}^N \int_{\Gamma_\epsilon} \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \epsilon^{-d/2}\omega + \epsilon^{-d} \rho_{\phi_\epsilon}^{(1)}(\beta, \epsilon^{-d})\, dx) \rangle \\
&\times \partial_i g_F(\langle f_1, \omega \rangle, \dots, \langle f_N, \omega \rangle) \partial_j g_F(\langle f_1, \omega \rangle, \dots, \langle f_N, \omega \rangle)\, d\mu_\epsilon(\omega) \\
&= \epsilon^{d/2} \sum_{i,j=1}^N \int_{\mathcal{D}'} \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \omega \rangle
\end{aligned}$$

$$\times\, \partial_i g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\, d\mu_\epsilon(\omega)$$
$$+\,\rho_\phi^{(1)}(\beta,1)\sum_{i,j=1}^{N}\int_{\mathbb{R}^d}(\nabla f_i(x),\nabla f_j(x))_{\mathbb{R}^d}\,dx$$
$$\times \int_{\mathcal{D}'}\partial_i g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\, d\mu_\epsilon(\omega). \quad (29)$$

By Theorem 5.1(iii) we get

$$\lim_{\epsilon\to 0}\int_{\mathcal{D}'}\partial_i g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\, d\mu_\epsilon(\omega)$$
$$=\int_{\mathcal{D}'}\partial_i g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\partial_j g_F(\langle f_1,\omega\rangle,\dots,\langle f_N,\omega\rangle)\nu_\mu(\omega),$$

hence, the second term in (29) converges to $\mathcal{E}_{\nu_\mu}(F,F)$ and it only remains to show that first term in (29) converges to zero as $\epsilon\to 0$. But this is obvious from Theorem 5.1(i), because $F\in\mathcal{F}C_b^\infty(\mathcal{D},\mathcal{D}')$. ∎

6 Convergence in law

The convergence in terms of the Dirichlet forms admits no probabilistic interpretation. Hence, next we study convergence in law of the scaled processes.

The laws of the scaled equilibrium processes

$$\mathbf{P}^\epsilon := \mathbf{Q}_{\mu_\epsilon}\circ\mathbf{X}_\epsilon^{-1} (= \mathbf{P}_\mu\circ\mathbf{X}_\epsilon^{-1}),$$

are probability measures on $C([0,\infty),\Gamma_\epsilon)$, where

$$\mathbf{Q}_{\mu_\epsilon} := \int_{\Gamma_\epsilon}\mathbf{Q}_\omega\, d\mu_\epsilon(\omega) \quad\text{and}\quad \mathbf{P}_\mu := \int_\Gamma \mathbf{P}_\gamma\, d\mu(\gamma),$$

cf. Theorem 4.1. Since $C([0,\infty),\Gamma_\epsilon)$ is a Borel subset of $C([0,\infty),\mathcal{D}')$ (under the natural embedding) with compatible measurable structures we can consider $\mathbf{P}^\epsilon$ as a measure on $C([0,\infty),\mathcal{D}')$ and by using Theorem 3.2(ii) we find that the process $(\mathbf{X}(t))_{t\geq 0}$ corresponding to $\mathbf{P}^\epsilon$, i.e., the realization of $(\mathbf{X}_\epsilon(t))_{t\geq 0}$ as a coordinate process in $C([0,\infty),\mathcal{D}')$, solves the martingale problem for $(-H_\epsilon, D(H_\mu^\Gamma))$ w.r.t. the corresponding minimum completed admissible filtration $(\mathbf{F}_t)_{t\geq 0}$ for all $\epsilon>0$.

6.1 Tightness

Theorem 6.1 *Let the potential ϕ fulfill conditions* (SS), (UI), (LR), (D), (LS) *and let $\mu \in \mathcal{G}^t(\beta\phi, 1)$, $\beta \geq 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then there exists $m \in \mathbb{N}$, $m \geq d+1$, such that the family of probability measures $(\mathbf{P}^\epsilon)_{\epsilon>0}$ can be restricted to the space $C([0,\infty), \mathcal{H}_{-m})$. Furthermore, $(\mathbf{P}^\epsilon)_{\epsilon>0}$ is tight on $C([0,\infty), \mathcal{H}_{-m})$.*

Remark 6.2 Note that the Gibbs measure corresponding to the construction with empty boundary conditions and a potential fulfilling (SS), (UI) is tempered, see [Rue70], Sec. 5.

Proof: Let $f \in \mathcal{D}$. By Theorem 5.1(i) we know, in particular, that the functions $\langle f, \cdot\rangle$, $\langle \nabla f, \cdot \rangle \in L^2(\mu_\epsilon)$. Hence it is easy to show by approximation that $\langle f, \cdot \rangle \in D(\mathcal{E}_\epsilon)$. Consider the conservative diffusion process $\mathbf{M}^\epsilon$ on Γ_ϵ associated with $(\mathcal{E}_\epsilon, D(\mathcal{E}_\epsilon))$ according to Theorem 4.1. We may regard $\mathbf{M}^\epsilon$ on the state space $\mathcal{D}'$ (common to all $\mathbf{M}^\epsilon, \epsilon > 0$). Considering its distribution on $C([0,\infty), \mathcal{D}')$ we may regard its canonical realization $\mathbf{M}^\epsilon = (\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\Theta_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{Q}^\epsilon_\omega)_{\omega \in \mathcal{D}'})$. So, in particular, $\Omega = C([0,\infty), \mathcal{D}')$, $X(t)(\omega) = \omega(t), t \geq 0$, $\theta_t(\omega) = \omega(t + \cdot)$, and $\mathbf{P}^\epsilon = \int_{\Gamma_\epsilon} \mathbf{Q}^\epsilon_\omega \, d\mu_\epsilon(\omega)$. Fix $T > 0$. Below we canonically project the process onto $\Omega_T := C([0,T], \mathcal{D}')$ without expressing this explicitly. We define the time reversal $r_T(\omega) := \omega(T - \cdot)$, $\omega \in \Omega_T$. Now, by the well-known Lyons-Zheng decomposition, cf. [LZ88], [FOT94], and also [LZ94] for its infinite dimensional variant, we have for all $T > 0$:

$$\begin{aligned}&\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(0)\rangle \\ &= \frac{1}{2}\mathbf{M}_t(\epsilon, f) + \frac{1}{2}\Big(\mathbf{M}_{T-t}(\epsilon, f)(r_T) - \mathbf{M}_T(\epsilon, f)(r_T)\Big), \quad 0 \leq t \leq T, \quad \mathbf{P}^\epsilon\text{-a.e.},\end{aligned}$$

where $(\mathbf{M}_t(\epsilon, f))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^\epsilon, (\mathbf{F}_t)_{0\leq t\leq T})$-martingale and $(\mathbf{M}_t(\epsilon, f)(r_T))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^\epsilon, (r_T^{-1}(\mathbf{F}_t))_{0\leq t\leq T})$-martingale. (We note that $\mathbf{P}^\epsilon \circ r_T^{-1} = \mathbf{P}^\epsilon$ because $(T_{\epsilon,t})_{t\geq 0}$ is symmetric on $L^2(\mu_\epsilon)$.) Moreover, by (29) the bracket of $\mathbf{M}(\epsilon, f)$ is given by

$$\langle \mathbf{M}(\epsilon, f)\rangle_t = 2\int_0^t \epsilon^{d/2}\langle |\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(u)\rangle + \rho^{(1)}_\phi(\beta, 1)\int_{\mathbb{R}^d} |\nabla f(x)|^2_{\mathbb{R}^d} \, dx \, du,$$

as e.g. directly follows from [FOT94], Theorem 5.2.3 and Theorem 5.1.3(i). We note here that both theorems in [FOT94] are formulated and proved for locally compact separable metric spaces, while $\mathcal{D}'$ is not of this type. However, both theorems carry over to general state spaces by virtue of the local compactification and regularization procedure developed in [MR92], Chap. VI.2, which is easy to see to be applicable in our case, see e.g. [MR92], Chap. VI, Theorem 2.4, in regard to [FOT94], Theorem 5.1.3(i). Hence by the Burkholder–Davies–Gundy inequalities and since $\mathbf{P}^\epsilon \circ r_T^{-1} = \mathbf{P}^\epsilon$ we can find $C^{(3)} \in (0,\infty)$ such that for all $f \in \mathcal{D}$, $0 < \epsilon \leq 1$, $0 \leq s \leq t \leq T$,

$$\begin{aligned}
&\mathbb{E}_{\mathbf{P}^\epsilon}[|\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(s)\rangle|^4] \\
&\leq \mathbb{E}_{\mathbf{P}^\epsilon}[|\mathbf{M}_t(\epsilon, f) - \mathbf{M}_s(\epsilon, f)|^4] + \mathbb{E}_{\mathbf{P}^\epsilon}[|\mathbf{M}_{T-t}(\epsilon, f)(r_T) - \mathbf{M}_{T-s}(\epsilon, f)(r_T)|^4] \\
&\leq C^{(3)} \Bigg(\mathbb{E}_{\mathbf{P}^\epsilon}\Big[\Big(\int_s^t \Big(\epsilon^{d/2}\langle|\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(u)\rangle + \rho_\phi^{(1)}(\beta,1)\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\Big)\,du\Big)^2\Big] \\
&+\mathbb{E}_{\mathbf{P}^\epsilon}\Big[\Big(\int_{T-t}^{T-s} \Big(\epsilon^{d/2}\langle|\nabla f|^2_{\mathbb{R}^d}, \mathbf{X}(T-u)\rangle + \rho_\phi^{(1)}(\beta,1)\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\Big)\,du\Big)^2\Big]\Bigg) \\
&\leq 4\,C^{(3)}\,(t-s)^2\Bigg(\epsilon^d \int_{\Gamma_\epsilon}\langle|\nabla f|^2_{\mathbb{R}^d}, \omega\rangle^2\,d\mu_\epsilon(\omega) + \rho_\phi^{(1)}(\beta,1)^2\Big(\int_{\mathbb{R}^d}|\nabla f(x)|^2_{\mathbb{R}^d}\,dx\Big)^2\Bigg) \\
&\qquad\qquad\leq C^{(4)}\,(t-s)^2(\|\,|\nabla f|^2_{\mathbb{R}^d}\,\|^2_{d+1} + \|\,|\nabla f|_{\mathbb{R}^d}\,\|^4_0), \quad (30)
\end{aligned}$$

where $C^{(4)} := 4\,C^{(3)}\max(C^{(1)}), \rho_\phi^{(1)}(\beta,1)^2)$ and $C^{(1)}$ as in Theorem 5.1(i).

Now we can use (30) to define $\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(s)\rangle$ for $f \in S(\mathbb{R}^d)$ via an approximation as an element in $L^4(\Omega, \mathbf{P}^\epsilon)$. Then, of course, the estimate (30) is also true for $f \in S(\mathbb{R}^d)$.

Let $m \in \mathbb{N}$ and let $(e_i)_{i\in\mathbb{N}}$ be the sequence of Hermite functions, forming an orthonormal system in $\mathcal{H}_{m-2d}$. Then $(a_i^{m-d}e_i)_{i\in\mathbb{N}}$, where $(a_i)_{i\in\mathbb{N}}$ are the eigenvalues of A w.r.t. the Hermite functions, forms an orthonormal system in $\mathcal{H}_{-m}$. Since the mappings $f \mapsto \|\,|\nabla f|^2_{\mathbb{R}^d}\,\|^2_{d+1}$ and $f \mapsto \|\,|\nabla f|_{\mathbb{R}^d}\,\|^4_0$ are continuous on $S(\mathbb{R}^d)$, we can choose $\alpha > 0$ and $m \in \mathbb{N}$ large enough so that

$$\|\,|\nabla f|^2_{\mathbb{R}^d}\,\|^2_{d+1} + \|\,|\nabla f|_{\mathbb{R}^d}\,\|^4_0 \leq \alpha\,\|\,f\,\|^4_{m-2d}, \qquad \forall f \in S(\mathbb{R}^d).$$

In particular, for all $i \in \mathbb{N}$

$$\|\,|\nabla e_i|^2_{\mathbb{R}^d}\,\|^2_{d+1} + \|\,|\nabla e_i|_{\mathbb{R}^d}\,\|^4_0 \leq \alpha.$$

Hence, by the above we can estimate

$$
\begin{aligned}
&\Big(\mathbb{E}_{\mathbf{P}^\epsilon}[\| \mathbf{X}(t) - \mathbf{X}(s) \|_{-m}^4]\Big)^{1/2} \\
&\quad = \Big(\mathbb{E}_{\mathbf{P}^\epsilon}\Big[\Big(\sum_{i=0}^{\infty} a_i^{2m-2d}\Big((e_i, \mathbf{X}(t))_{-m} - (e_i, \mathbf{X}(s))_{-m}\Big)^2\Big)^2\Big]\Big)^{1/2} \\
&\quad = \Big(\mathbb{E}_{\mathbf{P}^\epsilon}\Big[\Big(\sum_{i=0}^{\infty} a_i^{-2d}\Big(\langle e_i, \mathbf{X}(t)\rangle - \langle e_i, \mathbf{X}(s)\rangle\Big)^2\Big)^2\Big]\Big)^{1/2} \\
&\quad \le \sum_{i=0}^{\infty} a_i^{-2d}\Big(\mathbb{E}_{\mathbf{P}^\epsilon}[|\langle e_i, \mathbf{X}(t)\rangle - \langle e_i, \mathbf{X}(s)\rangle|^4]\Big)^{1/2} \le C^{(5)}\,(t-s), \qquad (31)
\end{aligned}
$$

where the constant $C^{(5)} := (\alpha C^{(4)})^{1/2} \sum_{i=0}^{\infty} a_i^{-2d}$ is finite, because A^{-d} is a Hilbert-Schmidt operator.

Inequality (31) together with the Kolmogorov-Chentsov criterium (see e.g. [SV79], Corollary 2.5.1, whose proof directly extends from $\mathbb{R}^d$ to Banach valued process) implies that the family of measures $(\mathbf{P}^\epsilon)_{\epsilon>0}$ can be restricted to $C([0,T], \mathcal{H}_{-m})$. Then, from Theorem 5.1(ii) and (31) together with [EK86], Chap. 3, Theorem 7.2, it follows that $(\mathbf{P}^\epsilon)_{\epsilon>0}$ is tight on $C([0,T], \mathcal{H}_{-m})$ for all $T > 0$ with the same $m \in \mathbb{N}$, hence tight on $C([0,\infty), \mathcal{H}_{-m})$. ■

6.2 Identification of the limit via the associated martingale problem

In order to identify the limit by Theorem 6.7 below it would be sufficient to show that each accumulation point $\mathbf{P}$ of $(\mathbf{P}^\epsilon)_{\epsilon>0}$ solves the martingale problem for $(-H,\ D_0)$, where $D_0 := \{G(\langle f, \cdot\rangle)\,|\,G \in C_b^2(\mathbb{R}),\, f \in S(\mathbb{R}^d)\}$, with initial distribution ν_μ, i.e.,

$$
G(\langle f, \mathbf{X}(t)\rangle) - G(\langle f, \mathbf{X}(0)\rangle) + \int_0^t HG(\langle f, \cdot\rangle)(\mathbf{X}(s))\,ds, \qquad t \ge 0,
$$

is an $\mathbf{F}_t$-martingale under $\mathbf{P}$ and $\mathbf{P} \circ \mathbf{X}(0)^{-1} = \nu_\mu$. One well-known way to establish this property is to prove convergence of the generators H_ϵ to the generator H as $\epsilon \to 0$. Thus, first we study the difference

$$
\| \,(H - H_\epsilon) G(\langle f, \cdot\rangle)\, \|_{L^2(\mu_\epsilon)} \qquad (32)
$$

for $\epsilon \to 0$. In order see that $HG(\langle f, \cdot\rangle) \in L^2(\mu_\epsilon)$ we use representation (27).

6.2.1 Non-convergence of generators

Using (23) and (27) again, by an approximation argument it is easy to show that $\langle f, \cdot \rangle$, $f \in \mathcal{D}$, is an element of $D(H_\epsilon)$ and $D(H)$. As we shall prove now at least on such functions the above convergence does not hold if we have non-trivial interactions. For the proof of the following Theorem we refer to Appendix C.

Theorem 6.3 *Let the potential ϕ be isotropic, i.e., $\phi(x) = V(r)$, $r = |x|_{\mathbb{R}^d}$, $x \in \mathbb{R}^d$. Furthermore, let $x^k x^l \partial_i \partial_j \phi \in L^1(\mathbb{R}^d, dx)$ and $x^i \partial_i \phi \in L^2(\mathbb{R}^d, dx)$. Additionally, let the assumptions required in Theorem A.4 and Theorem B.1 below hold and let $\mu \in \mathcal{G}^t(\beta\phi, 1)$, where $\beta \in [0, \beta_0]$ and $\beta_0 > 0$ is as in Theorem A.4, be the Gibbs measure corresponding to the construction with empty boundary conditions. Then there exists a function $[0, \beta_0] \ni \beta \mapsto R_\phi(\beta) \in \mathbb{R}_+$ such that*

$$\lim_{\epsilon \to 0} \| (H - H_\epsilon)\langle f, \cdot \rangle \|_{L^2(\mu_\epsilon)} = R_\phi(\beta) \| \Delta f \|^2_{L^2(dx)}, \qquad \forall f \in \mathcal{D}.$$

Furthermore, if $\mu \neq \pi_1$, then there exist $\beta_1(\phi) \in (0, \beta_0]$ such that $R_\phi(\beta) > 0$ for all $\beta \in (0, \beta_1]$.

Remark 6.4 (i) Theorem 6.3 states that for high temperatures (small inverse temperature) and sufficiently smooth isotropic potentials the generators do not converge in the L^2-sense. Analyzing the proof of Theorem 6.3 one finds that for a suitable choice of $f \in \mathcal{D}$ one also has non-convergence in the non-isotropic case.
(ii) The statement of Theorem 6.3 obviously applies to compactly supported potentials $\phi \in C_0^2(\mathbb{R}^d)$ and has been conjectured in [Bro80], [Ros81], and [Spo86].

6.2.2 A conditional theorem on convergence in law

In order to identify the limit the following weaker type of convergence is sufficient.

Conjecture 6.5 *Let the potential ϕ fulfill conditions* (SS), (UI), (LR), (D), (LS) *and let $\mu \in \mathcal{G}^t(\beta\phi, 1)$, $\beta \geq 0$, be the Gibbs measure corresponding to the construction with empty boundary conditions. Furthermore, for $G \in C_b^2(\mathbb{R})$, $f \in \mathcal{D}$, and $t, s \geq 0$, define*

$$V_\epsilon(f, t, s) \ := \ \int_t^{t+s} G'(\langle f, \mathbf{X}(u) \rangle)(H - H_\epsilon)\langle f, \cdot \rangle(\mathbf{X}(u))\, du.$$

Then

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbf{P}^\epsilon}[|V_\epsilon(f,t,s)|] = 0.$$

Remark 6.6 (i) By the same arguments as in the proof of Theorem 5.3 one can check that

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbf{P}^\epsilon}[V_\epsilon(f,t,s)] = 0.$$

Furthermore, in the proof of Theorem 6.7 below we really only use a weaker version of Conjecture 6.5, namely that

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbf{P}^\epsilon}[F\, V_\epsilon(f,t,s)] = 0$$

for all bounded, continuous, $\mathbf{F}_t$-measurable $F \in C([0,\infty), \mathcal{H}_{-m})$.
(ii) Conjecture 6.5 states that the scaled generators converge in time average, whereas Theorem 6.3 concerns convergence of the scaled generators at an arbitrary fixed time. Conjecture 6.5 first has been formulated in [Ros81]. In [Spo86] the author presents an approach to prove it for positive, smooth, compactly supported potentials within the proof of the main theorem. As mentioned by the author himself in the remark to the latter theorem, the proof of this theorem is, however, incomplete.
(iii) The ideas for proving Conjecture 6.5 are based on the ergodic behavior of the semi-group $(T_t^\mu)_{t\geq 0}$ of the microscopic stochastic dynamics. Indeed, due to the fact that $\mathbf{P}^\epsilon = \mathbf{P}_\mu \circ \mathbf{X}_\epsilon^{-1}$ we have that

$$\begin{aligned}\mathbb{E}_{\mathbf{P}^\epsilon}[|\, V_\epsilon(f,t,s)|] = \mathbb{E}_{\mathbf{P}_\mu}\Big[\Big|\, \epsilon^2 \int_{t/\epsilon^2}^{(t+s)/\epsilon^2} & G'(\langle f, S_{out,\epsilon} S_{in,\epsilon}(\mathbf{X}(u))\rangle) \\ & \times (H - H_\epsilon)\langle f, \cdot\rangle (S_{out,\epsilon} S_{in,\epsilon}(\mathbf{X}(u)))\, du\Big|\Big]. \end{aligned} \quad (33)$$

However, one has ergodicity only if the Gibbs measure μ is an extreme element of $\mathcal{G}_t(\beta\phi, 1)$, $\beta \geq 0$. This is one of the main results in [AKR98b], see Theorem 6.1 therein. If we are in the LA-HT regime, i.e., if (UI) holds, the Gibbs measure corresponding to the construction with empty boundary conditions is the unique element of $\mathcal{G}_t(\beta\phi, 1)$ satisfying the Ruelle bounds (10) for a given $\xi > 0$, see [Rue70]. Then since $\beta \mapsto \int |\exp(\beta\phi(x)) - 1|\, dx$ is continuous, it is easy to see that the Gibbs measure is indeed an extreme point in $\mathcal{G}_t(\beta\phi, 1)$. (For a recent uniqueness proof see also [PZ99].) So, one has ergodicity for μ by [AKR98b] and, therefore, for any $F \in L^2(\mu)$

$$\frac{1}{T}\int_0^T F(\mathbf{X}(u))\,du \to \int_\Gamma F\,d\mu,$$

in $L^2(\mathbf{P}_\mu)$ as $T \to \infty$. Deducing the convergence to zero of the right hand side of (33) from this, however, turns out to be a very hard problem. For the general class of two-body potential ϕ that we consider here we were not able to do this so far, but will keep it as a subject of further study.

Theorem 6.7 *Let the potential* ϕ *fulfill conditions* (SS), (UI), (LR), (D), (LS) *and let* $\mu \in \mathcal{G}^t(\beta\phi, 1)$, $\beta \geq 0$, *be the Gibbs measure corresponding to the construction with empty boundary conditions and assume Conjecture* 6.5. *Additionally, let* $\mathbf{P}$ *be an accumulation point of* $(\mathbf{P}^\epsilon)_{\epsilon>0}$ *on* $C([0,\infty), \mathcal{H}{-}m)$ *with* $m \in \mathbb{N}$ *as in Theorem 6.1. Then* $\mathbf{P}$ *solves the martingale problem for* $(-H,\, D_0)$ *with initial distribution* ν_μ, *i.e., for all* $G \in C_b^2(\mathbb{R})$, $f \in S(\mathbb{R}^d)$,

$$G(\langle f, \mathbf{X}(t)\rangle) - G(\langle f, \mathbf{X}(0)\rangle) + \int_0^t HG(\langle f, \cdot\rangle)(\mathbf{X}(s))\,ds, \qquad t \geq 0, \tag{34}$$

is an $\mathbf{F}_t$*-martingale under* $\mathbf{P}$ *and*

$$\mathbf{P} \circ \mathbf{X}(0)^{-1} = \nu_\mu. \tag{35}$$

The measure $\mathbf{P}$ *is uniquely determined by these properties, in particular, all such* $\mathbf{P}$ *coincide. Hence* $\mathbf{P}_\epsilon \to \mathbf{P}$ *weakly as* $\epsilon \to 0$.

Proof: Let $f \in \mathcal{D}$, $t, s \geq 0$, and define the following random variables on $C([0,\infty),\, \mathcal{H}_{-m})$:

$$U_\epsilon(f,t,s) := G(\langle f, \mathbf{X}(t)\rangle) - G(\langle f, \mathbf{X}(s)\rangle) + \int_t^{t+s} H_\epsilon G(\langle f, \cdot\rangle)(\mathbf{X}(u))\,du,$$

$$U(f,t,s) := G(\langle f, \mathbf{X}(t)\rangle) - G(\langle f, \mathbf{X}(s)\rangle) + \int_t^{t+s} HG(\langle f, \cdot\rangle)(\mathbf{X}(u))\,du,$$

$$S_\epsilon(f,t,s) := \epsilon^{d/2}\int_t^{t+s} G''(\langle f, \mathbf{X}(u)\rangle)\mathbf{X}(|\nabla f|^2_{\mathbb{R}^d})(u)\,du.$$

Utilizing Theorem 5.1(i) it follows that

$$\lim_{\epsilon\to 0}\mathbb{E}_{\mathbf{P}^\epsilon}[|S_\epsilon(f,t,s)|] = 0. \tag{36}$$

The trace filtration obtained by restricting $(\mathbf{F}_t)_{t\geq 0}$ to $C([0,\infty),\mathcal{H}_{-m})$ coincides with the natural filtration of $C([0,\infty),\mathcal{H}_{-m})$, which we also denote by $(\mathbf{F}_t)_{t\geq 0}$. Since $\mathbf{P}^\epsilon$ solves the martingale problem for $(-H_\epsilon, D_0)$ w.r.t. $(\mathbf{F}_t)_{t\geq 0}$ we have, for all $\mathbf{F}_t$-measurable bounded, continuous, $F_t : C([0,\infty),\mathcal{H}_{-m}) \to \mathbb{R}$, and $\epsilon > 0$:

$$\mathbb{E}_{\mathbf{P}^\epsilon}[F_t U_\epsilon(f,t,s)] = 0.$$

Thus, together with Conjecture 6.5 and (36), it follows that

$$\lim_{\epsilon\to 0}\mathbb{E}_{\mathbf{P}^\epsilon}[F_t U(f,t,s)] = \lim_{\epsilon\to 0}\mathbb{E}_{\mathbf{P}^\epsilon}[F_t(U_\epsilon(f,t,s)+V_\epsilon(f,t,s)+S_\epsilon(f,t,s))] = 0. \quad (37)$$

Let $\mathbf{P}$ be an accumulation point of $(\mathbf{P}^\epsilon)_{\epsilon>0}$ on $C([0,\infty),\mathcal{H}_{-m})$, i.e., $\mathbf{P}^{\epsilon_n} \to \mathbf{P}$ weakly for some subsequence $\epsilon_n \to 0$ for $n\to\infty$. Obviously, by Theorem 5.1(iii) we have $\mathbf{P}\circ\mathbf{X}(t)^{-1} = \nu_\mu$ for all $t\geq 0$, in particular (35) holds. By (37) it remains to show that

$$\lim_{n\to\infty}\mathbb{E}_{\mathbf{P}^{\epsilon_n}}[F_t U(f,t,s)] = \mathbb{E}_{\mathbf{P}}[F_t U(f,t,s)]. \quad (38)$$

Obviously, we only have to prove (38) with $U(f,t,s)$ replaced by the last summand in its definition, because for the first two summands convergence is clear. In order to do this we set

$$h := HG(\langle f,\cdot\rangle) = -\rho_\phi^{(1)}(\beta,1)G''(\langle f,\cdot\rangle) \parallel |\nabla f|_{\mathbb{R}^d} \parallel_0^2 \\ - \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)} G'(\langle f,\cdot\rangle)\langle \Delta f,\cdot\rangle.$$

Then

$$\Big|\mathbb{E}_{\mathbf{P}}\Big[F_t\int_t^{t+s} HG(\langle f,\cdot\rangle)(\mathbf{X}(u))\,du\Big] - \mathbb{E}_{\mathbf{P}^{\epsilon_n}}\Big[F_t\int_t^{t+s} HG(\langle f,\cdot\rangle)(\mathbf{X}(u))\,du\Big]\Big| \\ \leq \int_t^{t+s} |\mathbb{E}_{\mathbf{P}}[F_t h(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\epsilon_n}}[F_t h(\mathbf{X}(u))]|\,du$$

and for $K_r := \{\omega\in\mathcal{H}_{-m} \mid \parallel \omega \parallel_{-m} \leq r\}$, $r>0$, we have both for the positive and negative parts h^+, h^- of h and $u\in[t,t+s]$, setting $h_r^\pm := h^\pm \wedge \sup_{K_r}|h|$,

$$|\mathbb{E}_{\mathbf{P}}[F_t h^\pm(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\epsilon_n}}[F_t h^\pm(\mathbf{X}(u))]|$$

$$
\begin{aligned}
&\leq \Big| \int_{\{\mathbf{X}(u)\in K_r\}} |F_t| h_r^{\pm}(\mathbf{X}(u))\, d\mathbf{P} - \int_{\{\mathbf{X}(u)\in K_r\}} |F_t| h_r^{\pm}(\mathbf{X}(u))\, d\mathbf{P}^{\epsilon_n} \Big| \\
&+ \int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P} + \int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P}^{\epsilon_n} \\
&\qquad \leq |\mathbb{E}_{\mathbf{P}}[|F_t| h_r^{\pm}(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\epsilon_n}}[|F_t| h_r^{\pm}(\mathbf{X}(u))]| \\
&+ 2\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P} + 2\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P}^{\epsilon_n}
\end{aligned}
$$

But for all $r > 0$

$$
\begin{aligned}
&\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P}^{\epsilon_n} \leq \| F_t \|_\infty \int_{\mathcal{H}_{-m}\setminus K_r} |h|\, d\mu_\epsilon \\
&\leq \rho_\phi^{(1)}(\beta,1) \| F_t \|_\infty \| G'' \|_\infty \| |\nabla f|_{\mathbb{R}^d} \|_0^2 \frac{1}{r} \int_{\mathcal{H}_{-m}\setminus K_r} \| \omega \|_{-m}\; d\mu_\epsilon \\
&\quad + \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)} \| F_t \|_\infty \| G' \|_\infty \| \Delta f \|_m \int_{\mathcal{H}_{-m}\setminus K_r} \| \omega \|_{-m}\; d\mu_\epsilon(\omega) \\
&\leq \rho_\phi^{(1)}(\beta,1) \| F_t \|_\infty \| G'' \|_\infty \| |\nabla f|_{\mathbb{R}^d} \|_0^2 \frac{1}{r^2} \int_{\mathcal{H}_{-m}} \| \omega \|_{-m}^2\; d\mu_\epsilon(\omega) \\
&\quad + \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)} \| F_t \|_\infty \| G' \|_\infty \| \Delta f \|_m \frac{1}{r} \int_{\mathcal{H}_{-m}} \| \omega \|_{-m}^2\; d\mu_\epsilon(\omega) \\
&\leq \frac{1}{r}\rho_\phi^{(1)}(\beta,1)\, C^{(1)} \| F_t \|_\infty \Big(\frac{\| G'' \|_\infty}{r} \| |\nabla f|_{\mathbb{R}^d} \|_0^2 + \frac{\| G' \|_\infty}{\chi_\phi(\beta)} \| \Delta f \|_m \Big),
\end{aligned}
$$

where $C^{(1)}$ is as in Theorem 5.1(i). Similarly,

$$
\begin{aligned}
&\int_{\{\mathbf{X}(u)\in \mathcal{H}_{-m}\setminus K_r\}} |F_t||h|(\mathbf{X}(u))\, d\mathbf{P} \\
&\quad \leq \rho_\phi^{(1)}(\beta,1) \| F_t \|_\infty \| G'' \|_\infty \| |\nabla f|_{\mathbb{R}^d} \|_0^2 \frac{1}{r^2} \int_{\mathcal{H}_{-m}} \| \omega \|_{-m}^2\; d\nu_\mu(\omega) \\
&\qquad + \frac{\rho_\phi^{(1)}(\beta,1)}{\chi_\phi(\beta)} \| F_t \|_\infty \| G' \|_\infty \| \Delta f \|_m \frac{1}{r} \int_{\mathcal{H}_{-m}} \| \omega \|_{-m}^2\; d\nu_\mu(\omega),
\end{aligned}
$$

and since the Gaussian measure ν_μ has measure 1 on $\mathcal{H}_{-m}$ there exists a constant $C^{(6)} \in (0,\infty)$ such that

$$
\int_{\mathcal{H}_{-m}} \| \omega \|_{-m}^2\; d\nu_\mu(\omega) \leq C^{(6)}.
$$

Hence by the weak convergence of $\mathbf{P}^{\epsilon_n} \to \mathbf{P}$ as $n \to \infty$ and Lebesgue's dominated convergence theorem

$$
\begin{aligned}
\limsup_{n\to\infty} \int_t^{t+s} &\left| \mathbb{E}_{\mathbf{P}}[F_t h^{\pm}(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}^{\epsilon_n}}[F_t h^{\pm}(\mathbf{X}(u))] \right| du \\
&\leq \frac{2s}{r} \rho_\phi^{(1)}(\beta, 1) \max\{C^{(1)}, C^{(6)}\} \parallel F_t \parallel_\infty \\
&\times \left(\frac{\parallel G'' \parallel_\infty}{r} \parallel |\nabla f|_{\mathbb{R}^d} \parallel_0^2 + \frac{\parallel G' \parallel_\infty}{\chi_\phi(\beta)} \parallel \Delta f \parallel_m \right), \quad \forall r > 0.
\end{aligned}
$$

Letting $r \to \infty$ equality (38) follows and therefore

$$\mathbb{E}_{\mathbf{P}}[F_t U(f,t,s)] = 0, \quad \forall f \in \mathcal{D}. \tag{39}$$

But by an approximation (39) is also true for all $f \in S(\mathbb{R}^d)$.

Now it remains to show that $\mathbf{P}$ is uniquely determined by (34). But this follows by an easy generalization of Theorem 1.4 in [HS78]. All the assumptions required there are fulfilled in our situation except for the assumption on the operator B. This operator B in our case is $\sqrt{-\Delta}$, which is not bounded as required in [HS78]. Analyzing the proof, however, one finds that continuity and boundedness of the function

$$[0,\infty) \ni t \mapsto \langle B \exp(t\Delta) f, B \exp(t\Delta) f \rangle \in [0, \infty)$$

for a fixed $f \in S(\mathbb{R}^d)$ is sufficient, which in our case is obviously true. ■

A Inverse temperature derivative of correlation functions

First, we have to define the finite volume correlation functions

$$
\begin{aligned}
\rho_{\phi,\Lambda}(\beta, z, \eta) &:= Z_{\phi,\Lambda}^{-1}(\beta, z) \int_{\Gamma_{0,\Lambda}} \exp(-\beta E_\Lambda^\phi(\eta \cup \xi))\, d\lambda_z(\xi), \quad \beta \geq 0,\ z > 0, \\
Z_{\phi,\Lambda}(\beta, z) &:= \int_{\Gamma_{0,\Lambda}} \exp(-\beta E_\Lambda^\phi(\xi))\, d\lambda_z(\xi), \quad \eta \in \Gamma_{0,\Lambda},\ \Lambda \in \mathcal{O}_c(\mathbb{R}^d),
\end{aligned}
$$

where we restricted the Lebesgue-Poisson measure to $\Gamma_{0,\Lambda} := \bigsqcup_{n=0}^\infty \Gamma_{0,\Lambda}^{(n)}$, see Section 2.3.

Lemma A.1 *Let $\beta_0, z > 0$, and ϕ satisfy conditions* (S), (UI). *Furthermore, let ϕ fulfill the condition*

$$0 < \int_{\mathbb{R}^d \setminus \Lambda_0} (\exp(\beta_0 |\phi(x)|) - 1)\, dx < \infty \tag{40}$$

for some $\Lambda_0 \in \mathcal{O}_c(\mathbb{R}^d)$. Then

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \rho^{(n)}_{\phi,\Lambda}(\beta, z, x_1, \ldots, x_n) = \rho^{(n)}_{\phi}(\beta, z, x_1, \ldots, x_n) \tag{41}$$

for all $z > 0$ and uniformly in $\beta, x_1, \ldots x_n$ on any set $[0, \beta_0] \times (\Lambda')^n$, where $\Lambda' \in \mathcal{O}_c(\mathbb{R}^d)$.

Remark A.2 Condition (40) is obviously fulfilled for smooth, compactly supported potentials ϕ. Or, if $\phi \in L^1(\mathbb{R}^d \setminus \Lambda_0)$ and bounded on $\mathbb{R}^d \setminus \Lambda_0$ for some $\Lambda_0 \in \mathcal{O}_c(\mathbb{R}^d)$, and not dx-a.e. zero on $\mathbb{R}^d \setminus \Lambda_0$, then condition (40) is also fulfilled.

Proof: Analyzing the proof of [Kun99], Theorem 3.3.18, we see that it suffices to show that the function

$$F_\Lambda(\beta) := \frac{\int_{\mathbb{R}^d \setminus \Lambda} |\exp(-\beta\phi(x)) - 1|\, dx}{\int_{\mathbb{R}^d} |\exp(-\beta\phi(x)) - 1|\, dx}, \qquad \beta \in (0, \beta_0],$$

uniformly converges to zero as $\Lambda \nearrow \mathbb{R}^d$.

We have for each $\beta \in (0, \beta_0]$:

$$F_\Lambda(\beta) \leq \frac{\int_{\mathbb{R}^d \setminus \Lambda} |\exp(-\beta\phi(x)) - 1|\, dx}{\int_{\mathbb{R}^d \setminus \Lambda_0} |\exp(-\beta\phi(x)) - 1|\, dx} = \frac{\int_{\mathbb{R}^d \setminus \Lambda} |\sum_{n=1}^\infty (-\beta)^{n-1} \frac{\phi(x)^n}{n!}|\, dx}{\int_{\mathbb{R}^d \setminus \Lambda_0} |\sum_{n=1}^\infty (-\beta)^{n-1} \frac{\phi(x)^n}{n!}|\, dx}.$$

Next let us define the function

$$[0, \beta_0] \ni \beta \mapsto D(\beta) := \int_{\mathbb{R}^d \setminus \Lambda_0} \Big| \sum_{n=1}^\infty (-\beta)^{n-1} \frac{\phi(x)^n}{n!} \Big|\, dx.$$

Since

$$\Big| \sum_{n=1}^\infty (-\beta)^{n-1} \frac{\phi(x)^n}{n!} \Big| \leq \sum_{n=1}^\infty \beta_0^{n-1} \frac{|\phi(x)|^n}{n!} = \frac{1}{\beta_0} (\exp(\beta_0 |\phi(x)|) - 1),$$

for all $\beta \in [0, \beta_0]$, by (40) and Lebesgue's dominated convergence theorem, D is a continuous function on $[0, \beta_0]$. Moreover, by (40) again we obtain

$$D(0) = \int_{\mathbb{R}^d \setminus \Lambda_0} |\phi(x)| \, dx > 0$$

$$D(\beta) = \int_{\mathbb{R}^d \setminus \Lambda_0} \frac{1}{\beta} |\exp(\beta\phi(x)) - 1| \, dx > 0, \quad \forall \beta \in (0, \beta_0].$$

Thus, D is a positive, continuous function on $[0, \beta_0]$ and therefore

$$C^{(7)} := \inf\{D(\beta) \,|\, \beta \in [0, \beta_0]\} > 0.$$

Hence, we have

$$F_\Lambda(\beta) \le \frac{1}{C^{(7)}} \int_{\mathbb{R}^d \setminus \Lambda} \sum_{n=1}^{\infty} \beta_0^{n-1} \frac{|\phi(x)|^n}{n!} \, dx$$
$$= \frac{1}{C^{(7)}\beta_0} \int_{\mathbb{R}^d \setminus \Lambda} (\exp(\beta_0|\phi(x)|) - 1) \, dx \to 0$$

as $\Lambda \nearrow \mathbb{R}^d$ by using (40) again. ■

Via a recursion formula one can transform the correlation functions $\rho^{(n)}_{\phi,\Lambda}$ into the so-called *Ursell functions* $u^{(n)}_{\phi,\Lambda}$ and vice versa, see e.g. [MM91], [Rue69]. Their relation is given by

$$\rho_{\phi,\Lambda}(\beta, z, \eta) = \sum_{\substack{\eta_1 \cup \ldots \cup \eta_j = \eta \\ \eta_k \cap \eta_l = \emptyset, k \neq l, j \in \mathbb{N}}} u_{\phi,\Lambda}(\beta, z, \eta_1) \cdots u_{\phi,\Lambda}(\beta, z, \eta_j), \qquad \eta \in \Gamma_0, \quad (42)$$

where $u^{(n)}_{\phi,\Lambda}$ is related to $u_{\phi,\Lambda}$ analogously to (9). Correspondingly, $u^{(n)}_{\phi}$ and u_ϕ are defined with ρ_ϕ replacing $\rho_{\phi,\Lambda}$. Due to the translation invariance of the correlation functions, Ursell functions are also translation invariant. Furthermore, by an easy generalization of Theorem 4.5 in [Bro80], see also [Rue69], Chapter 4, we obtain the following integrability property.

Proposition A.3 *Let $\beta \ge 0, z > 0$, and ϕ satisfy conditions* (S), (UI). *Then for each $n \ge 1$, there exists a non-negative measurable function $U^{(n+1)}_{\phi,\beta,z}$: $\mathbb{R}^{d \times n} \to \mathbb{R}_+$, such that*

$$|u^{(n+1)}_{\phi,\Lambda}(\beta, z, \cdot, 0)| \le U^{(n+1)}_{\phi,\beta,z}, \quad \forall \Lambda \in \mathcal{O}_c(\mathbb{R}^d),$$

and

$$\int_{\mathbb{R}^{d\times n}} |U^{(n+1)}_{\phi,\beta,z}(x_1,\dots,x_n)f(x_1,\dots,x_n)|\,dx_1\dots dx_n$$
$$\le \exp(2n\beta B(\phi))\Big(\sum_{m=0}^{\infty}\frac{1}{m!}(n+m+1)^{n+m-1}C(\beta\phi,z)^m\Big)$$
$$\times \sup_{x_n\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\exp(-\beta\phi(x_n-y_n))-1|\sup_{x_{n-1}\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\exp(-\beta\phi(x_{n-1}-y_{n-1}))-1|$$
$$\cdots\sup_{x_1\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\exp(-\beta\phi(x_1-y_1))-1||f(y_1,\dots,y_n)|\,dy_1\dots dy_n,$$

for all measurable functions $f:\mathbb{R}^{d\times n}\to\mathbb{R}$.

Theorem A.4 *Let* $\beta_0, z>0$, ϕ *satisfy conditions* (S), (UI), *and let either* $\phi\equiv 0$ *or* $\phi\in L^1(\mathbb{R}^d,dx)\cap L^2(\mathbb{R}^d,dx)$ *and condition (40) hold. Then* $\rho_\phi\in C^2([0,\beta_0])$ *and for* λ_z*-a.e.* $\eta\in\Gamma_0$ *we have*

$$\frac{\partial\rho_\phi}{\partial\beta}(\beta,z,\eta)=-E^\phi(\eta)\rho_\phi(\beta,z,\eta)-\int_{\mathbb{R}^d}W^\phi(\eta\mid x)\rho_\phi(\beta,z,\eta\cup\{x\})\,z\,dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^{d\times 2}}\phi(x-y)\Big(\rho_\phi(\beta,z,\eta\cup\{x,y\})-\rho_\phi(\beta,z,\eta)\rho^{(2)}_\phi(\beta,z,x,y)\Big)\,z^2\,dx\,dy, \tag{43}$$

where $E^\phi(\eta):=\lim_{\Lambda\nearrow\mathbb{R}^d}E^\phi_\Lambda(\eta)$.

Remark A.5 We conjecture that under the same assumptions as in Theorem A.4 plus the condition $\phi\in L^1(\mathbb{R}^d,dx)\cap L^p(\mathbb{R}^d,dx), p\in\mathbb{N}$, it can be proved that $\rho_\phi\in C^p([0,\beta_0])$. Calculating higher order derivatives of (43) the question arises whether the obtained integrals are well-defined. Till now we only have checked this up to the second order derivative of ρ_ϕ. Furthermore, if these integrals are well-defined, then we do not need condition (40) to prove Theorem A.4.

Proof: First, we note that the expression on the r.h.s. of (43) is well-defined and finite. Indeed, since $\phi\in L^1(\mathbb{R}^d)$ and the correlation functions are bounded, see (10), the first integral in this expression is finite. Using

(42) and Proposition A.3, one finds that the second integral is also finite. Indeed, for $\eta = \{v_1, \ldots, v_n\}$, $n \in \mathbb{N}$, by (42) the terms occurring in the second integral are of the form:

$$\Big(\rho_\phi^{(1)}(\beta, z)\Big)^n \int_{\mathbb{R}^{d\times 2}} \phi(x-y)\Big(u_\phi^{(2)}(\beta, z, x, v_1) + u_\phi^{(2)}(\beta, z, y, v_1) + \cdots + u_\phi^{(2)}(\beta, z, x, v_n) + u_\phi^{(2)}(\beta, z, y, v_n)\Big)\, z^2\, dx\, dy, \quad (44)$$

or

$$\Big(\rho_\phi^{(1)}(\beta, z)\Big)^{n-1} \int_{\mathbb{R}^{d\times 2}} \phi(x-y)\Big(u_\phi^{(3)}(\beta, z, x, y, v_1) + \cdots + u_\phi^{(3)}(\beta, z, x, y, v_n) + u_\phi^{(3)}(\beta, z, x, v_1, v_2) + u_\phi^{(3)}(\beta, z, y, v_1, v_2) + \cdots + u_\phi^{(3)}(\beta, z, x, v_{n-1}, v_n) + u_\phi^{(3)}(\beta, z, y, v_{n-1}, v_n)\Big)\, z^2\, dx\, dy, \quad (45)$$

and so forth. All the other, non-integrable, terms cancel each another. Using $\phi \in L^1(\mathbb{R}^d)$, the remaining terms (44), (45) turn out to be well-defined and finite due to Proposition A.3.

Using the definition of the Lebesgue-Poisson measure we find for $\eta \in \Gamma_{0,\Lambda}$:

$$\frac{\partial \rho_{\phi,\Lambda}}{\partial \beta}(\beta, z, \eta) = -E_\Lambda^\phi(\eta)\rho_{\phi,\Lambda}(\beta, z, \eta) - \int_\Lambda W^\phi(\eta \mid x)\rho_{\phi,\Lambda}(\beta, z, \eta \cup \{x\})\, z\, dx$$
$$-\frac{1}{2}\int_{\Lambda^2} \phi(x-y)\Big(\rho_{\phi,\Lambda}(\beta, z, \eta \cup \{x, y\}) - \rho_{\phi,\Lambda}(\beta, z, \eta)\rho_{\phi,\Lambda}^{(2)}(\beta, z, x, y)\Big)\, z^2\, dx\, dy.$$

By (41) and Proposition A.3 it follows from the dominated convergence theorem that

$$\lim_{\Lambda \nearrow \mathbb{R}^d} \phi(\cdot - \cdot) \times u_{\phi,\Lambda}^{(m)}(\beta, z, \cdot, v_1, \ldots, v_{m-1}) = \phi(\cdot - \cdot) \times u_\phi^{(m)}(\beta, z, \cdot, v_1, \ldots v_{m-1}), \quad m \geq 2,$$
$$\lim_{\Lambda \nearrow \mathbb{R}^d} \phi(\cdot - \cdot) \times u_{\phi,\Lambda}^{(m)}(\beta, z, \cdot, \cdot, v_1, \ldots, v_{m-2}) = \phi(\cdot - \cdot) \times u_\phi^{(m)}(\beta, z, \cdot, \cdot, v_1, \ldots v_{m-2}), \quad m \geq 3,$$

in $L^1(\mathbb{R}^{d\times 2})$ for dx-a.e. $(v_1, \ldots, v_{m-1}) \in \mathbb{R}^{d\times(m-1)}$ and $(v_1, \ldots, v_{m-2}) \in \mathbb{R}^{d\times(m-2)}$, respectively. The bound (10) also holds for the finite volume correlation functions, uniformly in $\Lambda \subset \mathbb{R}^d$. Using these facts, again (41) and dominated convergence imply

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\Big(\frac{\partial\rho_{\phi,\Lambda}}{\partial\beta}\Big)(\beta,z,\eta)=-E^{\phi}(\eta)\rho_\phi(\beta,z,\eta)-\int_{\mathbb{R}^d}W^{\phi}(\eta\mid x)\rho_\phi(\beta,z,\eta\cup\{x\})z\,dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^{d\times 2}}\phi(x-y)\Big(\rho_\phi(\beta,z,\eta\cup\{x,y\})-\rho_\phi(\beta,z,\eta)\rho_\phi^{(2)}(\beta,z,x,y)\Big)\,z^2\,dx\,dy,$$

for λ_z-a.e. $\eta\in\Gamma_0$. It remains to show that derivative and the infinite volume limit can be interchanged. We evidently have to show this only for potentials which are not identically equal to zero. By using Lemma A.1 and Proposition A.3, we see that for $z>0$, $\eta\in\Gamma_0$, fixed the function $\frac{\partial\rho_{\phi,\Lambda}}{\partial\beta}(\beta,z,\eta)$ converges uniformly on $[0,\beta_0]$ as $\Lambda\nearrow\mathbb{R}^d$ and

$$\frac{\partial\rho_\phi}{\partial\beta}(\beta,z,\eta)=\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{\partial\rho_{\phi,\Lambda}}{\partial\beta}(\beta,z,\eta).$$

The second order derivative can be derived analogously. The only difference is that in the second order derivative the potential ϕ appears in its second power. Hence, for $\phi\in L^1(\mathbb{R}^d,dx)\cap L^2(\mathbb{R}^d,dx)$ we obtain that $\rho_\phi\in C^2([0,\beta_0])$. ∎

B Coercivity identity for Gibbs measures

Here we derive an analog of the usual coercivity identity on $L^2(\mathbb{R}^d,g\,dx)$ for $L^2(\Gamma,\mu)$, where μ is a Ruelle measure on Γ, whose potential satisfies some weak additional conditions.

Let $A(\gamma)\in(T_{\gamma,\infty}(\Gamma))^{\otimes 2}$, cf. (14), so that $A(\gamma)=(A(\gamma,x,y))_{x,y\in\gamma}$, where $A(\gamma,x,y)\in T_y(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)$. We realize $A(\gamma)$ as a linear operator acting from $T_{\gamma,0}(\Gamma)$ into $T_{\gamma,\infty}(\Gamma)$ setting

$$T_{\gamma,0}(\Gamma)\ni V(\gamma)\mapsto A(\gamma)V(\gamma):=\Big(\sum_{x\in\gamma}(A(\gamma,x,y),V(\gamma,x))_{T_x(\mathbb{R}^d)}\Big)_{y\in\gamma}\in T_{\gamma,\infty}(\Gamma).$$

Evidently, if $A(\gamma)\in(T_{\gamma,0}(\Gamma))^{\otimes 2}$, then $A(\gamma)$ defines a linear continuous operator in $T_\gamma(\Gamma)$. We denote by $A(\gamma)^*$ its adjoint operator.

For a vector field $\Gamma\ni\gamma\mapsto W(\gamma)\in T_{\gamma,\infty}(\Gamma)$, we define its derivative $\nabla^\Gamma W(\gamma)$ as a mapping

$$\Gamma\ni\gamma\mapsto\nabla^\Gamma W(\gamma)=(\nabla^\Gamma W(\gamma,x,y))_{x,y\in\gamma}\in(T_{\gamma,\infty}(\Gamma))^{\otimes 2}$$

such that

$$\nabla^\Gamma W(\gamma,x,y){:=}\nabla_y W(\gamma,x)=\begin{cases}\nabla_z W(\gamma-\varepsilon_y+\varepsilon_z,x)\big|_{z=y}, & \text{if } x\neq y,\\ \nabla_z W(\gamma-\varepsilon_y+\varepsilon_z,z)\big|_{z=y}, & \text{if } x=y,\end{cases}$$

if all derivatives $\nabla_y W(\gamma,x)$, $x,y\in\gamma$, exist. For a function $F\colon\Gamma\to\mathbb{R}^d$, we denote $F''{:=}\nabla^\Gamma\nabla^\Gamma F$, if it exists.

Theorem B.1 (coercivity identity) *Let the potential* ϕ *satisfy* (SS), (I), (LR), *and the three following conditions:*
(i) $\phi\in C^2(\mathbb{R}^d\setminus\{0\})$, $e^{-\phi}$ *is continuous on* $\mathbb{R}^d$, *and* $e^{-\phi}\nabla\phi$ *can be extended to a continuous, vector-valued function on* $\mathbb{R}^d$*;*
(ii) *for each* $\gamma\in S_\infty$, *the three series* $\sum_{x\in\gamma}\phi(\cdot-x)$, $\sum_{x\in\gamma}\nabla\phi(\cdot-x)$, *and* $\sum_{x\in\gamma}\nabla^2\phi(\cdot-x)$ *converge locally uniformly on* $X\setminus\gamma$*;*
(iii) *we have*

$$\nabla\phi\in L^1(\mathbb{R}^d,\exp(-\phi(x))\,dx)\cap L^2(\mathbb{R}^d,\exp(-\phi(x))\,dx),$$
$$\nabla^2\phi\in L^1(\mathbb{R}^d,\exp(-\phi(x))\,dx).$$

Furthermore, let $\mu\in\mathcal{G}^t(\beta\phi,z)$, $\beta\geq 0, z>0$*. Then, for any* $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned}\|H_\mu^\Gamma F\|^2_{L^2(\mu)}&=\int_\Gamma \operatorname{Tr}_{T_\gamma(\Gamma)}F''(\gamma)F''(\gamma)^*\,d\mu(\gamma)\\&\qquad-\int_\Gamma(\nabla^\Gamma F(\gamma),\nabla^\Gamma B_\mu(\gamma)\nabla^\Gamma F(\gamma))_{T_\gamma(\Gamma)}\,d\mu(\gamma)\\&=\int_\Gamma \operatorname{Tr}_{T_\gamma(\Gamma)}F''(\gamma)F''(\gamma)^*\,d\mu(\gamma)+\beta\int_\Gamma\sum_{\{x,y\}\subset\gamma}\big((\nabla^\Gamma F(\gamma,x)-\nabla^\Gamma F(\gamma,y)),\\&\qquad\nabla^2\phi(x-y)(\nabla^\Gamma F(\gamma,x)-\nabla^\Gamma F(\gamma,y))\big)_{\mathbb{R}^d}\,d\mu(\gamma).\end{aligned}\tag{46}$$

Remark B.2 As easily seen, conditions (i)–(iii) of the above theorem imply (D) and (LS).

Remark B.3 As will be seen from the proof of Theorem B.1, the coercivity identity (46) holds for each monomial $F=\langle f,\cdot\rangle^n$, where $f\in\mathcal{D}$ and $n\in\mathbb{N}$.

Proof: Let $G:\Gamma\times\mathbb{R}^d\to\mathbb{R}_+$ be measurable, then, by [NZ79], we have due to condition (ii):

$$\int_\Gamma\sum_{x\in\gamma}G(\gamma,x)\,d\mu(\gamma)=\int_\Gamma\int_{\mathbb{R}^d}z\,\exp\Big(-\beta\sum_{y\in\gamma}\phi(x-y)\Big)G(\gamma+\varepsilon_x,x)\,dx\,d\mu(\gamma).\tag{47}$$

Let $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$. By (17) and (47), we get

$$\begin{aligned}
\|H_\mu^\Gamma F\|^2_{L^2(\mu)} &= \int_\Gamma \sum_{x\in\gamma} \big(\Delta_x F(\gamma) + (B_\mu(\gamma,x), \nabla^\Gamma F(\gamma,x))_{T_x(\mathbb{R}^d)}\big)^2\, d\mu(\gamma)\\
&\quad + \int_\Gamma \sum_{x,y\in\gamma,\, x\neq y} \big(\Delta_x F(\gamma) + (B_\mu(\gamma,x), \nabla^\Gamma F(\gamma,x))_{T_x(\mathbb{R}^d)}\big)\\
&\quad \times \big(\Delta_y F(\gamma) + (B_\mu(\gamma,y), \nabla^\Gamma F(\gamma,y))_{T_y(\mathbb{R}^d)}\big)\, d\mu(\gamma)\\
&= \int_\Gamma \int_{\mathbb{R}^d} z \exp\Big(-\beta \sum_{y\in\gamma} \phi(x-y)\Big)\\
&\quad \times \big(\Delta_x F(\gamma+\varepsilon_x) + (B_\mu(\gamma+\varepsilon_x,x), \nabla_x F(\gamma+\varepsilon_x))_{T_x(\mathbb{R}^d)}\big)^2\, dx\, d\mu(\gamma)\\
&\quad + \int_\Gamma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} z^2 \exp\Big(-\beta\Big(\sum_{y_1\in\gamma}\phi(x_1-y_1) + \sum_{y_2\in\gamma\cup\{x_1\}}\phi(x_2-y_2)\Big)\Big)\\
&\quad \times \big(\Delta_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) + (B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_1), \nabla_{x_1} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}))_{T_{x_1}(\mathbb{R}^d)}\big)\\
&\quad \times \big(\Delta_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}) + (B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_2),\\
&\qquad \nabla_{x_2} F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}))_{T_{x_2}(\mathbb{R}^d)}\big)\, dx_1\, dx_2\, d\mu(\gamma). \qquad (48)
\end{aligned}$$

By conditions (i) and (ii), we conclude that, for each fixed $\gamma \in S_\infty$, the function

$$g_\gamma(x) := \exp\Big(-\beta \sum_{y\in\gamma}\phi(x-y)\Big)$$

is continuous on $\mathbb{R}^d$, two times continuously differentiable on $\mathbb{R}^d \setminus \gamma$, and ∇g_γ extends to a continuous function on $\mathbb{R}^d$. Moreover, by (15), $B_\mu(\gamma+\varepsilon_x, x)$ is the logarithmic derivative of the measure $\nu_\gamma := g_\gamma\, dx$. Finally, it is easy to see from (i)–(iii) that the function

$$g_\gamma(x)\big(\log g_\gamma(x)\big)'' = \exp\Big(-\beta\sum_{y\in\gamma}\phi(x-y)\Big)\beta\sum_{y\in\gamma}\nabla^2\phi(x-y)$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Thus, the usual coercivity identity on the space of square-integrable functions $L^2(\mathbb{R}^d, d\nu_\gamma)$ implies that

$$\int_{\mathbb{R}^d} \exp\Big(-\beta\sum_{y\in\gamma}\phi(x-y)\Big)$$

$$\times\left(\Delta_x F(\gamma+\varepsilon_x)+(B_\mu(\gamma+\varepsilon_x,x),\nabla_x F(\gamma+\varepsilon_x))_{T_x(\mathbb{R}^d)}\right)^2 dx$$
$$= \int_{\mathbb{R}^d} \exp\Big(-\beta\sum_{y\in\gamma}\phi(x-y)\Big)\big(\mathrm{Tr}_{T_x(\mathbb{R}^d)}\nabla_x\nabla_x F(\gamma+\varepsilon_x)(\nabla_x\nabla_x F(\gamma+\varepsilon_x))^*$$
$$-(\nabla_x F(\gamma+\varepsilon_x),\nabla_x B_\mu(\gamma+\varepsilon_x,x)\nabla_x F(\gamma+\varepsilon_x))_{T_x(\mathbb{R}^d)}\big)\,dx. \quad (49)$$

Absolute analogously, a slight modification of the proof of the coercivity identity on $\mathbb{R}^d$ implies that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\Big(-\beta\sum_{y_1\in\gamma}\phi(x_1-y_1)-\beta\sum_{y_2\in\gamma}\phi(x_2-y_2)-\beta\phi(x_1-x_2)\Big)$$
$$\times\big(\Delta_{x_1}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})$$
$$+(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_1),\nabla_{x_1}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}))_{T_{x_1}(\mathbb{R}^d)}\big)$$
$$\times\big(\Delta_{x_2}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})$$
$$+(B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_2),\nabla_{x_2}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}))_{T_{x_2}(\mathbb{R}^d)}\big)\,dx_1\,dx_2$$
$$=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\Big(-\beta\sum_{y_1\in\gamma}\phi(x_1-y_1)-\beta\sum_{y_2\in\gamma}\phi(x_2-y_2)-\beta\phi(x_1-x_2)\Big)$$
$$\big(\|\nabla_{x_2}\nabla_{x_1}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\|^2_{T_{x_2}(\mathbb{R}^d)\otimes T_{x_1}(\mathbb{R}^d)}-(\nabla_{x_2}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}),$$
$$\nabla_{x_2}B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_1)\nabla_{x_1}F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2}))_{T_{x_2}(\mathbb{R}^d)}\big)\,dx_1\,dx_2. \quad (50)$$

Next, by (15), (i), and (ii), we get for any $\gamma\in S_\infty$:

$$\nabla_y B_\mu(\gamma,x)=\begin{cases}-\beta\sum_{z\in\gamma\setminus\{x\}}\nabla^2\phi(x-z), & \text{if } x=y,\\ \beta\nabla^2\phi(x-y), & \text{otherwise.}\end{cases} \quad (51)$$

By (47), (51), condition (iii), and estimate (4.29) in [AKR98b], we have for any $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$:

$$\int_\Gamma\int_\Lambda z\,\exp\Big(-\beta\sum_{y\in\gamma}\phi(x-y)\Big)\big(1+\|\nabla_x B_\mu(\gamma+\varepsilon_x,x)\|_{T_x(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\big)\,dx\,d\mu(\gamma)$$
$$=\int_\Gamma\sum_{x\in\gamma_\Lambda}\big(1+\|\nabla_x B_\mu(\gamma,x)\|_{T_x(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\big)\,d\mu(\gamma)$$
$$\le\int_\Gamma\sum_{x\in\gamma_\Lambda}\Big(1+\beta\sum_{y\in\gamma\setminus\{x\}}\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\Big)\,d\mu(\gamma)$$

$$
\begin{aligned}
&= \int_\Lambda \rho_\mu^{(1)}(x)\,dx + \int_\Lambda \int_{\mathbb{R}^d} \rho_\mu^{(2)}(x,y)\,\beta\ \|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\,dy\,dx \\
&\le \int_\Lambda \rho_\mu^{(1)}(x)\,dx + C^{(8)}\int_\Lambda \int_{\mathbb{R}^d} \beta\,\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\,e^{-\beta\phi(x-y)}\,dy\,dx < \infty,
\end{aligned}
\tag{52}
$$

where $C^{(8)} \in (0,\infty)$ is a constant, and analogously

$$
\begin{aligned}
&\int_\Gamma \int_\Lambda \int_\Lambda z^2 \exp\Big(-\beta \sum_{y_1\in\gamma} \phi(x_1-y_1) - \beta \sum_{y_2\in\gamma} \phi(x_2-y_2) - \beta\phi(x_1-x_2)\Big) \\
&\quad \times \big(1 + \|\nabla_{x_2} B_\mu(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2},x_1)\|_{T_{x_2}(\mathbb{R}^d)\otimes T_{x_1}(\mathbb{R}^d)}\big)\,dx_1\,dx_2\,d\mu(\gamma) \\
&\qquad = \int_\Gamma \sum_{x,y\in\gamma_\Lambda,\,x\neq y} \big(1 + \|\nabla_y B_\mu(\gamma,x)\|_{T_y(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)}\big)\,d\mu(\gamma) \\
&\qquad = \int_\Gamma \sum_{x,y\in\gamma_\Lambda,\,x\neq y} \big(1 + \beta\,\|\nabla^2\phi(x-y)\|_{\mathbb{R}^d\otimes\mathbb{R}^d}\big)\,d\mu(\gamma) < \infty.
\end{aligned}
\tag{53}
$$

Now, by (47)–(50), (52), and (53),

$$
\begin{aligned}
&\|H_\mu^\Gamma F\|^2_{L^2(\mu)} = \int_\Gamma \sum_{x\in\gamma} \big(\mathrm{Tr}_{T_x(\mathbb{R}^d)}\,\nabla_x\nabla_x F(\gamma)(\nabla_x\nabla_x F(\gamma))^* \\
&- (\nabla_x F(\gamma), \nabla_x B_\mu(\gamma,x)\nabla_x F(\gamma))_{T_x(\mathbb{R}^d)}\big) + \sum_{x,y\in\gamma,\,x\neq y} \Big(\|\nabla_y\nabla_x F(\gamma)\|^2_{T_y(\mathbb{R}^d)\otimes T_x(\mathbb{R}^d)} \\
&\qquad - (\nabla_y F(\gamma), \nabla_y B_\mu(\gamma,x)\nabla_x F(\gamma))_{T_y(\mathbb{R}^d)}\Big)\,d\mu(\gamma) \\
&= \int_\Gamma \mathrm{Tr}_{T_\gamma(\Gamma)}\,F''(\gamma)F''(\gamma)^*\,d\mu(\gamma) - \int_\Gamma (\nabla^\Gamma F(\gamma), \nabla^\Gamma B_\mu(\gamma)\nabla^\Gamma F(\gamma))_{T_\gamma(\Gamma)}\,d\mu(\gamma).
\end{aligned}
\tag{54}
$$

Finally, from (51) and (54) we get the second equality in (46). ■

C Proof for non-convergence of generators

Proof of Theorem 6.3: We have

$$
\|\,(H-H_\epsilon)\langle f,\cdot\rangle\,\|^2_{L^2(\mu_\epsilon)} = \int_{\Gamma_\epsilon} H\langle f,\omega\rangle H\langle f,\omega\rangle\,d\mu_\epsilon(\omega)
$$

$$- 2 \int_{\Gamma_\epsilon} H\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle \, d\mu_\epsilon(\omega) + \int_{\Gamma_\epsilon} H_\epsilon\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle \, d\mu_\epsilon(\omega). \quad (55)$$

A direct consequence of Theorem 5.1(ii) is that

$$\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} H\langle f,\omega\rangle H\langle f,\omega\rangle \, d\mu_\epsilon(\omega) = \frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} \parallel \Delta f \parallel^2_{L^2(dx)} .$$

Furthermore, we have

$$\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} H\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle \, d\mu_\epsilon(\omega) = \frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} \parallel \Delta f \parallel^2_{L^2(dx)}, \qquad (56)$$

where (56) can be shown in the same way as the convergence of Dirichlet forms in Theorem 5.3, the argument is even more simple. Showing convergence of the third term in (55) is a quite elaborate task. Using (22), the coercivity identity provided in Theorem B.1, (8) and Lemma 5.2 we obtain

$$\begin{aligned}
\int_{\Gamma_\epsilon} H_\epsilon\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle \, d\mu_\epsilon(\omega) &= \epsilon^d \int_{\Gamma} H_{\tilde\mu_\epsilon}\langle f,\gamma\rangle H_{\tilde\mu_\epsilon}\langle f,\gamma\rangle \, d\tilde\mu_\epsilon(\gamma) \\
&= \rho^{(1)}_{\phi_\epsilon}(\beta,\epsilon^{-d}) \parallel \Delta f \parallel^2_{L^2(dx)} + \frac{\epsilon^{-d}}{2}\beta \int_{\mathbb{R}^{2d}} \Big(\nabla f(x) - \nabla f(y), \\
&\quad \nabla^2\phi_\epsilon(x-y)(\nabla f(x) - \nabla f(y))\Big)_{\mathbb{R}^d} \rho^{(2)}_{\phi_\epsilon}(\beta,\epsilon^{-d},x,y)\, dx\, dy \\
&= \rho^{(1)}_{\phi}(\beta,1) \parallel \Delta f \parallel^2_{L^2(dx)} + \frac{\epsilon^{-(d+2)}}{2}\beta \int_{\mathbb{R}^{2d}} \Big(\nabla f(x) - \nabla f(y), \\
&\qquad \nabla^2\phi\Big(\frac{x-y}{\epsilon}\Big)(\nabla f(x) - \nabla f(y))\Big)_{\mathbb{R}^d} \rho^{(2)}_{\phi}\Big(\beta,1,\frac{x-y}{\epsilon},0\Big)\, dx\, dy.
\end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned}
&\frac{\epsilon^{-(d+2)}}{2}\Big(\nabla f(x) - \nabla f(y), \nabla^2\phi\Big(\frac{x-y}{\epsilon}\Big)(\nabla f(x) - \nabla f(y))\Big)_{\mathbb{R}^d} \\
&\qquad = \frac{\epsilon^{-d}}{2}\int_0^1\int_0^1 \Big(\nabla^2 f(y+q_1(x-y))\frac{x-y}{\epsilon}, \\
&\qquad\qquad \nabla^2\phi\Big(\frac{x-y}{\epsilon}\Big)\nabla^2 f(y+q_2(x-y))\frac{x-y}{\epsilon}\Big)_{\mathbb{R}^d} dq_1\, dq_2
\end{aligned}$$

Thus, we obtain an approximate identity and

$$\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} H_\epsilon\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle\, d\mu_\epsilon(\omega) = \rho_\phi^{(1)}(\beta,1) \parallel \Delta f \parallel^2_{L^2(dx)}$$

$$+\frac{1}{2}\sum_{i,j,k,l=1}^{d}\int_{\mathbb{R}^d}\beta\, x^k x^l\, \partial_i\partial_j\phi(x)\,\rho_\phi^{(2)}(\beta,1,x,0)\,dx\int_{\mathbb{R}^d}\partial_i\partial_k f(y)\,\partial_j\partial_l f(y)\,dy,$$

where x^i is the i-th component of $x\in\mathbb{R}^d$. Set

$$D_\phi(\beta,i,j,k,l) := \int_{\mathbb{R}^d}\beta\, x^k x^l\,\partial_i\partial_j\phi(x)\,\rho_\phi^{(2)}(\beta,1,x,0)\,dx. \tag{57}$$

For isotropic potentials the corresponding second correlation function is also isotropic and the coefficient (57) turns out to be:

$$\int_{\mathbb{R}^d}\beta\Big(\frac{x^ix^j}{r^3}(rV''(r)-V'(r))+\frac{V'(r)}{r}\delta_{i,j}\Big)\,x^kx^l\,\tilde{\rho}_\phi^{(2)}(\beta,1,r)\,dx,$$

here $\delta_{i,j}$ is the Kronecker delta. Hence, for isotropic potentials the coefficient $D_\phi(\beta,i,j,k,l)$ is only different from zero if each index in the set $\{i,j,k,l\}$ at least occurs twice. Utilizing polar coordinates, the identity

$$\int_0^{2\pi}\sin^4(\theta)\,d\theta = 3\int_0^{2\pi}\sin^2(\theta)\cos^2(\theta)\,d\theta$$

and the symmetry of

$$\int_{\mathbb{R}^d}\partial_i\partial_k f(y)\,\partial_j\partial_l f(y)\,dy$$

in all its indexes, we find

$$\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} H_\epsilon\langle f,\omega\rangle H_\epsilon\langle f,\omega\rangle\, d\mu_\epsilon(\omega) = D_\phi(\beta) \parallel \Delta f \parallel^2_{L^2(dx)}$$

where

$$D_\phi(\beta) = \rho_\phi^{(1)}(\beta,1)+\frac{1}{2}\int_{\mathbb{R}^d}\beta\, x^1x^1\,\partial_1\partial_1\phi(x)\,\rho_\phi^{(2)}(\beta,1,x,0)\,dx.$$

Next, we compare the coefficients $D_\phi(\beta)$ and $(\rho_\phi^{(1)}(\beta,1))^2/\chi_\phi(\beta)$ in terms of a high temperature expansion. The latter coefficient is the *isothermal* *compressibility* of the fluid or gas characterized by μ. Applying Theorem A.4, we obtain

$$
\begin{aligned}
D_\phi(\beta) &= 1 + \beta^2 \frac{\partial^2}{\partial \beta^2} D_\phi(0) + o(\beta^2), \\
\frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} &= 1 + \beta^2 \frac{\partial^2}{\partial \beta^2} \frac{(\rho_\phi^{(1)}(\cdot,1))^2}{\chi_\phi}(0) + o(\beta^2), \qquad \beta \in [0,\beta_0],
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial^2}{\partial \beta^2} D_\phi(0) &= -\Big(\int_{\mathbb{R}^d} \phi(x)\, dx \Big)^2 + \int_{\mathbb{R}^d} (x^1 \partial_1 \phi(x))^2\, dx, \\
\frac{\partial^2}{\partial \beta^2} \frac{(\rho_\phi^{(1)}(\cdot,1))^2}{\chi_\phi}(0) &= -\Big(\int_{\mathbb{R}^d} \phi(x)\, dx \Big)^2.
\end{aligned}
$$

Thus, the remainder function is given by

$$
R_\phi(\beta) = D_\phi(\beta) - \frac{(\rho_\phi^{(1)}(\beta,1))^2}{\chi_\phi(\beta)} = \beta^2 \int_{\mathbb{R}^d} (x^1 \partial_1 \phi(x))^2\, dx + o(\beta^2),\ \beta \in [0,\beta_0]. \tag{58}
$$

Hence, $R_\phi \equiv 0$ on $[0,\beta_0]$ is equivalent to $\int_{\mathbb{R}^d} (x^1 \partial_1 \phi(x))^2\, dx = 0$. This, in turn, is equivalent to

$$
\partial_1 \phi(x) = 0 \quad \text{for} \quad dx\text{-a.e.} \quad x \in \mathbb{R}^d. \tag{59}
$$

Since the potential is isotropic, the only potential in consideration which fulfills (59) is $\phi \equiv 0$. Hence, by (58) for $\mu \neq \pi_1$ there exist $\beta_1 \in (0,\beta_0]$ such that $R_\phi(\beta) > 0$ for all $\beta \in (0,\beta_1]$. ∎

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References

[AKR98a] S. Albeverio, Yu.G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154:444–500, 1998.

[AKR98b] S. Albeverio, Yu.G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: The Gibbsian case. *J. Funct. Anal.*, 157:242–291, 1998.

[BK95] Yu.M. Berezansky and Yu.G. Kondratiev. *Spectral Methods in Infinite-Dimensional Analysis.* Kluwer Academic Publishers, Dordrecht, 1995. Originally in Russian, Naukova Dumka, Kiev, 1988.

[Bro80] T. Brox. *Gibbsgleichgewichtsfluktuationen für einige Potentiallimiten.* PhD thesis, Universität Heidelberg, 1980.

[EK86] S. Ethier and T. Kurtz. *Markov Processes.* Wiley & Sons, New York, 1986.

[Föl75] H. Föllmer. *Phase Transition and Martin Boundary*, volume 465 of *Lecture Notes in Mathematics, Séminaire de Probabilités XI.* Springer, Berlin, Heidelberg, New York, 1975.

[FOT94] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes.* W. de Gruyter, Berlin, New York, 1994.

[Geo79] H.O. Georgii. *Canonical Gibbs Measures*, volume 760 of *Lecture Notes in Mathematics.* Springer, Berlin, Heidelberg, New York, 1979.

[GP85] M. Z. Guo and G. Papanicolaou. Bulk diffusion for interacting Brownian particles. In *Statistical physics and dynamical systems (Köszeg, 1984)*, pages 41–48. Birkhäuser Boston, Boston, Mass., 1985.

[HS78] R.A. Holley and D.W Stroock. Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motion. *Publ. RIMS, Kyoto Univ.*, 14:741–788, 1978.

[Kal75] O. Kallenberg. *Random Measures.* Academic Press, San Diego, 1975.

[KK99] Yu.G. Kondratiev and T. Kuna. Harmonic analysis on configuration space I. General theory. SFB 256 report 626, Universität Bonn, 1999. Submitted to Infinite Dim. Anal. Quantum Prob.

[KL99] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems.* Springer, Heidelberg, Berlin, New York, 1999.

[Kun99] T. Kuna. *Studies in configuration space analysis and applications.* PhD thesis, Universität Bonn, 1999.

[Lan77] R. Lang. Unendlichdimensionale Wienerprozesse mit Wechselwirkung II. *Z. Wahrsch. verw. Gebiete*, 39:277–299, 1977.

[Len73] A. Lenard. Correlation functions and the uniqueness of the state in classical statistical mechanics. *Comm. Math. Phys.*, 30:35–44, 1973.

[Len75a] A. Lenard. States of classical statistical mechanical systems of infinitely many particles. I. *Arch. Rational Mech. Anal.*, 59:219–239, 1975.

[Len75b] A. Lenard. States of classical statistical mechanical systems of infinitely many particles. II. *Arch. Rational Mech. Anal.*, 59:241–256, 1975.

[LZ88] T.J. Lyons and W.A. Zheng. A crossing estimate for the canonical process on a Dirichlet space and a tightness result. *Colloque Paul Lévy sur les processes stochastique, Asterisque*, 157–158:249–272, 1988.

[LZ94] T.J. Lyons and T.S. Zhang. Decomposition of Dirichlet processes and its applications. *Ann. Prob.*, 22(1):494–524, 1994.

[Min67] R.A. Minlos. Limiting Gibbs distribution. *Funktsional'nyj Analiz i Ego Prilozhenija*, 1(2):60–73, 1967.

[MM91] V.A. Malyshev and R.A. Minlos. *Gibbs Random Fields: Cluster Expansion.* Kluwer Academic Publisher, Dordrecht, Boston, London, 1991.

[MR92] Z.-M. Ma and M. Röckner. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Springer, Berlin, New York, 1992.

[MR00] Z.-M. Ma and M. Röckner. Construction of diffusions on configuration spaces. *Osaka J. Math.*, 37(2):273–314, 2000.

[NZ79] X.X. Nguyen and H. Zessin. Integral and differentiable characterizations of the Gibbs process. *Math. Nachr.*, 88:105–115, 1979.

[Osa96] H. Osada. Dirichlet form approach to infinite-dimensional Wiener process with singular interactions. *Comm. Math. Phys.*, 176:117–131, 1996.

[Par67] K.R. Parthasarathy. *Probability measures on metric spaces.* Academic Press, New York, 1967.

[Pre76] C.J. Preston. *Random Fields*, volume 534 of *Lecture Notes in Mathematics.* Springer, Berlin, Heidelberg, New York, 1976.

[PZ99] E. Pechersky and Yu. Zhukov. Uniqueness of Gibbs state for nonideal gas in R^d: The case of pair potentials. *J. Stat. Phys.*, 97(1–2):145–172, 1999.

[Röc98] M. Röckner. Stochastic analysis on configuration spaces: Basic ideas and recent results. In J. Jost and al., editors, *New Directions in Dirichlet Forms*, volume 8 of *Studies in Advanced Mathematics*, pages 157–232. Amer. Math. Soc., Rhode Island, 1998.

[Ros81] H. Rost. Hydrodynamik gekoppelter Diffusionen: Fluktuationen im Gleichgewicht. In Ph. Blanchard and L. Streit, editors, *Dynamics and Pocesses*, pages 97–107, Berlin, Heidelberg, New York, 1981. Springer. Lecture Notes in Mathematics 1031.

[RS98] M. Röckner and B. Schmuland. A support property for infinite-dimensional interacting diffusion processes. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(3):359–364, 1998.

[Rue63] D. Ruelle. Correlation functions of classical gases. *Ann. Phys.*, 25:109–120, 1963.

[Rue69] D. Ruelle. *Statistical Mechanics. Rigorous Results.* Benjamins, Amsterdam, 1969.

[Rue70] D. Ruelle. Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.*, 18:127–159, 1970.

[Spo86] H. Spohn. Equilibrium fluctuations for interacting Brownian particles. *Comm. Math. Phys.*, 103:1–33, 1986.

[Spo91] H. Spohn. *Large Scale Dynamics of Interacting Particles.* Text and Monographs in Physics. Springer, Heidelberg, Berlin, New York, 1991.

[SV79] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes.* Springer, Berlin, Heidelberg, New York, 1979.

[Yos96] M.W. Yoshida. Construction of infinite-dimensional interacting diffusion process through Dirichlet forms. *Prob. Theory Related Fields*, 106:265–297, 1996.