Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection

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Abstract

We consider the law ν of the 3-d Bessel Bridge on the convex set K_0 of continuous non-negative paths on $[0,1]$. We prove an integration by parts formula on K_0 w.r.t. to ν, where an integral with respect to an explicit infinite-dimensional boundary measure σ appears. We apply this to the solution (u,η) of the reflected stochastic partial differential equations studied by Nualart and Pardoux, where $u : [0,\infty) \times [0,1] \mapsto \mathbb{R}$ is a random function and η is a random positive measure on $[0,\infty) \times (0,1)$. First, we prove that u can be realized as the radial part of the solution to a $\mathbb{R}^3$-valued stochastic heat equation. Then we prove that η has the following structure: $s \mapsto 2\eta([0,s],(0,1))$ is the Additive Functional of u with Revuz measure σ; for $\eta(ds,(0,1))$-a.e. s, there exists a unique $r(s) \in (0,1)$ s.t. $u(s,r(s)) = 0$, and $\eta(ds,d\xi) = \delta_{r(s)}(d\xi)\,\eta(ds,(0,1))$, where δ_a is the Dirac mass at $a \in (0,1)$. This gives a complete description of (u,η) as solution to a Skorokhod Problem in an infinite-dimensional non-smooth convex domain.

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1 Introduction

Consider the Brownian Bridge $(\beta_\tau)_{\tau\in[0,1]}$ and the 3-dimensional Bessel Bridge $(e_\tau)_{\tau\in[0,1]}$, and call μ and ν, respectively, their laws on the space $C_0(0,1)$ of continuous $x:[0,1]\mapsto\mathbb{R}$ with $x(0)=x(1)=0$.

The aim of this paper is to prove the following infinite-dimensional integration by parts formulae with respect to μ and ν on the convex sets of paths $K_\alpha := \{x:[0,1]\mapsto\mathbb{R} : x\geq -\alpha\}$, $\alpha\geq 0$:

$$\int_{K_\alpha}\langle\nabla\varphi,h\rangle\,d\mu \;=\; -\int_{K_\alpha}\varphi(x)\,\langle x,h''\rangle\,d\mu - \int_0^1 dr\;\; h(r)\int\varphi(x)\,\sigma_\alpha(r,dx) \quad (1)$$

$$\int_{K_0}\langle\nabla\varphi,h\rangle\,d\nu \;=\; -\int_{K_0}\varphi(x)\,\langle x,h''\rangle\,d\nu \;-\;\int_0^1 dr\;\, h(r)\int\varphi(x)\,\sigma_0(r,dx). \quad (2)$$

In (1) and (2), $\langle\,\cdot\,,\,\cdot\,\rangle$ is the scalar product in $H := L^2(0,1)$, $\varphi : H\mapsto\mathbb{R}$ is bounded and Fréchet differentiable with bounded gradient $\nabla\varphi : H\mapsto H$, $h\in W^{2,2}\cap W^{1,2}_0(0,1)\subset H$ and $h''\in H$ is the second derivative of h. Moreover we set for $\alpha>0$, $r\in\,]0,1[$:

$$\int\varphi(x)\,\sigma_\alpha(r,dx) \;:=\; \frac{\sqrt{2}\,\alpha^2\,e^{-\alpha^2/(2r(1-r))}}{\sqrt{\pi r^3(1-r)^3}}\,\mathbb{E}\left[\varphi\left(e^r_{0,\alpha}\oplus_r\hat{e}^{1-r}_{0,\alpha}-\alpha\right)\right] \qquad (3)$$

$$\int\varphi(x)\,\sigma_0(r,dx) \;:=\; \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\,\mathbb{E}\left[\varphi\left(e^r_{0,0}\oplus_r\hat{e}^{1-r}_{0,0}\right)\right] \qquad (4)$$

where $e^r_{0,a}$, $\hat{e}^r_{0,a}$ are two independent copies of the 3-d Bessel Bridge on $[0,r]$ between 0 and $a\geq 0$, and for $(y,z)\in L^2(0,r)\times L^2(0,1-r)$:

$$y\oplus_r z\in H,\quad [y\oplus_r z](\tau) := y(r-\tau)\,1_{[0,r]} + z(\tau-r)\,1_{(r,1]}. \qquad (5)$$

Formulae (1) and (2) provide examples of infinite-dimensional Caccioppoli sets, for which boundary measures and outer normal vectors can be explicitely computed: we refer to [Gi 84] for the classical theory. Recall that, by the Divergence Theorem in finite dimension, we have:

$$\int_O(\partial_h\varphi)\,\rho\,dx \;=\; -\int_O\varphi\,(\partial_h\log\rho)\,\rho\,dx \;-\;\int_{\partial O}\varphi\,\langle n,h\rangle\,\rho\,d\sigma \qquad (6)$$

where O is a regular bounded open subset of $\mathbb{R}^d$, $h\in\mathbb{R}^d$, $\varphi,\rho\in C^1_b(O)$, $0<\lambda\leq\rho\leq\Lambda<\infty$, n is the inward-pointing normal vector to the boundary ∂O and σ is the surface measure.

Since μ is equal to the Gaussian measure $\mathcal{N}(0,(-2A)^{-1})$, where $2A := \partial^2/\partial\xi^2$ with Dirichlet Boundary Condition on $[0,1]$, the Cameron-Martin Theorem gives:

$$\int_H \partial_h \varphi \, d\mu \;=\; -\int_H \varphi(x)\, \langle x, h''\rangle \, d\mu.$$

Therefore, the first term in the right-hand side of (1) comes from the well-known fact that the measure μ admits as logarithmic derivative the map $x \mapsto x''$.

On the other hand, the second term in the right-hand side of (1) is essentially of a different type, and can be interpreted as a boundary term: indeed, it is concentrated on the set $\{x \in C_0(0,1) : \inf x = -\alpha\}$, i.e. the topological boundary of $K_\alpha \cap C_0(0,1)$ in the sup-norm, which has zero μ-measure.

Recall that a.s. the Brownian Bridge β attains its minimum on $[0,1]$ at an unique time ζ, and ζ is uniformly distributed on $[0,1]$: a trajectory $x(\cdot) \in K_\alpha$ of β lies on the boundary of K_α if and only if $x(\zeta(x)) = -\alpha$. We define for all $r \in (0,1)$:

$$\begin{aligned} \partial_r^* K_\alpha \;\; := \;\; & \{x : [0,1] \mapsto [-\alpha, \infty) \text{ continuous} : \\ & \qquad x(0) = x(1) = 0,\ x(\xi) = -\alpha \Longleftrightarrow \xi = r\,\}, \end{aligned}$$

and $\partial^* K_\alpha := \bigcup_{r\in(0,1)} \partial_r^* K_\alpha$. Then $\partial_r^* K_\alpha$, $r \in (0,1)$, are the faces with lowest co-dimension in $\partial^* K_\alpha$. Moreover, the factor $h(r) = \langle \delta_r, h\rangle$ corresponds in the finite-dimensional case (6) to the scalar product $\langle n, h\rangle$, where n is the inward-pointing normal vector to the boundary: this suggests that the inward-pointing normal vector to $\partial^* K_\alpha$ is equal to the Dirac mass δ_r at r, on each face $\partial_r^* K_\alpha$, $r \in (0,1)$. Notice that $\delta_r \notin H$, which is related to the fact that K_α is not a C^1 domain in H.

Following De Giorgi, we say that $\partial^* K_\alpha$ is the *μ-reduced boundary* of K_α. This terminology is justified, since $\partial^* K_\alpha$ is smaller than the boundary of K_α in any reasonable topology: see [Gi 84].

Formulae (1) and (2) find also applications in the study of the stochastic partial differential equation with reflection of Nualart and Pardoux [NP 92]:

$$\begin{cases} \dfrac{\partial u_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\alpha}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta_\alpha(t,\xi) \\[2ex] u_\alpha(0,\xi) = x(\xi),\ u_\alpha(t,0) = u_\alpha(t,1) = 0 \\[2ex] u_\alpha + \alpha \geq 0,\ \int (u_\alpha + \alpha)\, d\eta_\alpha = 0 \end{cases} \tag{7}$$

where u_α is a continuous function of $(t,\xi) \in \overline{\mathcal{O}} := [0,+\infty) \times [0,1]$, η_α a positive measure on $\mathcal{O} := [0,+\infty) \times (0,1)$, $x \in C_0(0,1) \cap K_\alpha$, $\alpha \geq 0$ and $\{W(t,\xi) : \ (t,\xi) \in \overline{\mathcal{O}}\}$ is a Brownian sheet.

In [Za 00] we proved that the process $C_0(0,1) \cap K_\alpha \ni x \mapsto u_\alpha(t,\cdot)$ is symmetric with respect to $\nu_\alpha := \mu(\,\cdot\,|\,K_\alpha)$ if $\alpha > 0$ and to $\nu_0 := \nu$ if $\alpha = 0$. The same results were stated in [Ot 98] for $\alpha > 0$ and [FO 00] for $\alpha = 0$.

Formulae (1) and (2) allow to prove that u_α is associated with a gradient-type Dirichlet Form on the space (K_α, ν_α) and that:

1. For all Borel set $I \subseteq (0,1)$, the process $t \mapsto \eta_\alpha([0,t] \times I)$ is an Additive Functional of u_α, with Revuz-measure $\frac{1}{2}\int_I dr\, \sigma_\alpha(r,\cdot)$
2. There exists a Borel set $S \subseteq \mathbb{R}^+$ and a map $r : S \mapsto (0,1)$, such that $\eta_\alpha((\mathbb{R}^+ \backslash S) \times (0,1)) = 0$, and for all $s \in S$, $u_\alpha(s,\cdot) \in \partial_{r(s)} K_\alpha$, i.e.
$$u_\alpha(s, r(s)) \,=\, 0, \quad u_\alpha(s,\xi) \,>\, 0 \quad \forall\, \xi \in (0,1) \backslash \{r(s)\}.$$
3. The measure η_α admits the decomposition:
$$\eta_\alpha(ds, d\xi) \,=\, \delta_{r(s)}(d\xi)\, \eta_\alpha(ds, (0,1)). \tag{8}$$

In particular, we can provide a full interpretation of (7) as an infinite-dimensional Skorokhod problem, writing (7) in the following way:

$$du \,=\, \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\, dt \,+\, dW \,+\, \frac{1}{2}\, n(u) \cdot dL \tag{9}$$

where n is the inward-pointing normal vector to the boundary, i.e. $n(x) = \delta_r$ if $x \in \partial_r^* K_0$, and $L_t := 2\,\eta([0,t] \times (0,1))$ is the Additive Functional associated with the boundary measure $\int dr\, \sigma_0(r,\cdot)$. For the finite-dimensional theory of Skorokhod Problems, see e.g. [Ta 67], [LS 84], [BH 90] and the references therein.

In order to apply the Theory of Additive Functionals to u_α, we first have to prove that, for all $\alpha \geq 0$, the symmetric bilinear form

$$E^\alpha(\varphi,\psi) \,:=\, \int_{K_\alpha} \langle \nabla\varphi, \nabla\psi \rangle \, d\nu_\alpha, \quad \varphi,\psi \in C_b^1(H)$$

is closable, the closure $(\mathcal{E}^\alpha, D(\mathcal{E}^\alpha))$ is a Dirichlet form and that the process $C_0(0,1) \cap K_\alpha \ni x \mapsto u_\alpha(t,\cdot)$ is the diffusion associated with $\mathcal{E}^\alpha$.

In the case $\alpha = 0$, this allows to prove that $(u_0(t,\cdot))_{t\geq 0}$ is the radial part of the solution of a $\mathbb{R}^3$-valued linear SPDE with additive white-noise. More precisely, we denote by $(Z_3(t,\cdot,\overline{x}))_{t\geq 0}$ the Gaussian process with values in $H^3 := L^2(0,1;\mathbb{R}^3)$, solving the following $\mathbb{R}^3$-valued linear SPDE with additive white-noise:

$$\begin{cases} \dfrac{\partial Z_3}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Z_3}{\partial \xi^2} + \dfrac{\partial^2 \overline{W}}{\partial t \partial \xi} \\ \\ Z_3(t,\overline{x})(0) = Z_3(t,\overline{x})(1) = 0 \\ \\ Z_3(0,\overline{x}) \ = \overline{x} \in H^3 \end{cases} \tag{10}$$

where $\overline{x} \in H^3$ and $\overline{W} := (W_1, W_2, W_3) \mapsto \mathbb{R}^3$, and (W_i) are three independent copies of W. It is well known that Z_3 is the Markov process associated with the Dirichlet Form:

$$\Lambda^3(F,G) := \int_{H^3} \langle \overline{\nabla} F, \overline{\nabla} G\rangle_{H^3}\, d\mu^{\otimes 3}, \quad F, G \in W^{1,2}(H^3, \mu^{\otimes 3}),$$

where $\overline{\nabla} F : H^3 \mapsto H^3$ is the gradient of F in H^3. If we set $\Phi : H^3 = L^2(0,1;\mathbb{R}^3) \mapsto K_0$, $\Phi(y)(\tau) := |y(\tau)|_{\mathbb{R}^3}$, then $(\mathcal{E}^\alpha, D(\mathcal{E}^\alpha))$ is the image Dirichlet Form of Λ^3 under Φ, i.e. $D(\mathcal{E}^\alpha) = \{\varphi \in L^2(\nu) : \varphi \circ \Phi \in W^{1,2}(H^3, \mu^{\otimes 3})\}$ and:

$$\mathcal{E}^0(\varphi,\psi) \ = \ \Lambda^3(\varphi\circ\Phi, \psi\circ\Phi), \quad \varphi,\psi \in C^1_b(H).$$

Recall that M. Fukushima has given in [Fu 00] a general theory of stochastic equations in domains with reflecting boundary on an abstract Wiener Space. However, the results presented here are not covered by this theory: indeed, in the Abstract Wiener Space setting one would consider, in our notations, the triple $(H, D((-A)^{1/2}), \mu)$, where A is the second derivative operator on $L^2(0,1)$ with Dirichlet boundary conditions, and the Dirichlet Form:

$$(\varphi,\psi) \ \mapsto \int_{K_\alpha} \langle (-A)^{-1/2}\nabla\varphi, (-A)^{-1/2}\nabla\psi\rangle\, d\mu,$$

which is different from $\mathcal{E}^\alpha$, $\alpha > 0$. In particular, only SDEs with values in H but no SPDEs arise in general from the Abstract Wiener Space analysis. Furthermore, $\mathcal{E}^0$ has a μ-negligible set as state space.

Moreover, recall that in [DP 93] existence of a solution (v,η) was proved, for a semilinear reflected SPDE and with a reaction-diffusion type nonlinearity f and a non-constant diffusion coefficient σ. Then, in [DP 97], under suitable smoothness assumption on f and σ, it was proved that for all $t>0$, $\xi \in (0,1)$, the law of $v(t,\xi)$ is absolutely continuous w.r.t. the Lebesgue measure on $(0,\infty)$. A Strong Feller property of $x \mapsto u_0(t,\cdot)$ in $L^2(0,1)$ and the explicit knowledge of the invariant measure ν of (7) allow to prove that, if $\sigma \equiv 1$, then for all $t>0$, $\xi \in (0,1)$, the law of $v(t,\xi)$ is absolutely continuous w.r.t. the measure $y^2\,dy$ on the whole of $[0,\infty)$.

Finally, recall that Funaki and Olla proved in [FO 00], that the fluctuations around the hydrodynamical limit of a $\nabla\phi$ interface model on a hard wall converge in law to the solution of a Nualart-Pardoux equation. Therefore, we hope that the results of this paper can also be applied to these problems.

2 Definitions and setting of the problem

We introduce the following notations: $(t,\xi) \in \mathcal{O} := [0,+\infty) \times [0,1]$, $H := L^2(0,1)$ with the canonical scalar product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$,

$$\langle h,k\rangle := \int_0^1 h(\xi)k(\xi)\,d\xi, \qquad \|h\|^2 := \langle h,h\rangle,$$

$$C_0 := C_0(0,1) := \{c : [0,1] \mapsto \mathbb{R} \ \text{ continuous}, \ c(0)=c(1)=0\},$$

$$A : D(A) \subset H \mapsto H, \quad D(A) := W^{2,2} \cap W_0^{1,2}(0,1), \quad A := \frac{1}{2}\frac{\partial^2}{\partial \xi^2}.$$

We denote by $C_c^k(0,1)$, $k \in \mathbb{N}\cup\{\infty\}$, the subset of $C_0(0,1)$ of all C^k functions with support being compact in $(0,1)$.

We set $K_\alpha := \{h \in H : \ h \geq -\alpha\}$ with $\alpha \geq 0$, and we denote by $\Pi_{K_\alpha} : H \mapsto K_\alpha$ the projection from H onto the closed convex set $K_\alpha \subset H$. Recall that Π_{K_α} is 1-Lipschitz continuous. We introduce the following function spaces:

- If $D \subseteq H$, we denote by $C_b(D)$ the space of all $\varphi : D \mapsto \mathbb{R}$ being bounded and uniformly continuous in the norm of H. If $D \subseteq H$ and $\varphi \in C_b(D)$, we denote the modulus of continuity of φ by $\omega_\varphi : [0,\infty) \mapsto [0,1]$:

$$\omega_\varphi(r) := \sup\{|\varphi(x)-\varphi(x')| \wedge 1 : x,x' \in D, \ \|x-x'\| \leq r\}.$$

We let $\|\varphi\|_\infty := \sup|\varphi|$. Then $(C_b(D),\|\cdot\|_\infty)$ is a Banach space.

- For all $\alpha \geq 0$, we identify $C_b(K_\alpha)$ with a subspace of $C_b(H)$ by means of the injection: $C_b(K_\alpha) \ni \varphi \mapsto \varphi \circ \Pi_{K_\alpha} \in C_b(H)$. If $0 \leq \alpha \leq \beta$, then $C_b(K_\alpha) \subseteq C_b(K_\beta)$.

- We denote by $\mathrm{Exp}_A(H)$ the linear span of $\{1, \cos(\langle \cdot, h \rangle), \sin(\langle \cdot, h \rangle) : h \in D(A)\}$; $\mathrm{Exp}_A(K_\alpha)$ is equal to the restrictions of $\mathrm{Exp}_A(H)$ to K_α.

- If $D \subseteq H$, the space $\mathrm{Lip}(D)$ is the set of all $\varphi \in C_b(D)$ such that:

$$\|\varphi\|_{\mathrm{Lip}} := \|\varphi\|_\infty + \sup_{r>0} \frac{\omega_\varphi(r)}{r} < \infty.$$

- The space $C_b^1(H)$ is defined as the set of all Fréchet-differentiable $\varphi \in C_b(H)$, with continuous gradient $\nabla\varphi : H \mapsto H$; finally, $C_b^1(K_\alpha) \subset C_b(K_\alpha)$ is equal to the set of all φ such that:

  1. For all $x \in K_\alpha$, there exists a vector $\nabla\varphi(x) \in H$ such that for all $h \in K_0$, we have:

  $$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x + t\, h) - \varphi(x)) = \langle \nabla\varphi(x), h \rangle.$$

  2. $K_\alpha \ni x \mapsto \nabla\varphi(x) \in H$ is continuous and bounded.

  For all $\varphi \in C_b^1(K_\alpha)$ we call $\nabla\varphi : H \mapsto H$ the gradient of φ.

If $\{m_n\}_n \cup \{m\}$ is a sequence of probability measures on $(H, \mathcal{B}(H))$, where $\mathcal{B}(H)$ is the Borel σ-field of H, we say that m_n converges weakly to m, if $\lim_{n\to\infty} \int_H \varphi\, dm_n = \int_H \varphi\, dm$, $\forall \varphi \in C_b(H)$. Recall the following lemma:

Lemma 1 *Let T be a Polish metric space, and let $\{m_n\}_n \cup \{m\}$, respectively $\{\varphi_n\}_n$, a sequence of probability measures, resp. of real-valued continuous functions, on T, satisfying:*

- *m_n converges weakly to m.*

- *The family $\{\varphi_n\}_n$ is uniformly bounded and equicontinuous on T.*

- *$\varphi_n(x)$ has a limit $\varphi(x)$ as $n \to \infty$ $\forall x \in S$, with $S \subseteq T$ Borel and $m(S) = 1$.*

Then: $\lim_{n\to\infty} \int_T \varphi_n\, dm_n = \int_S \varphi\, dm$.

Given a Markov process $\{Y(t,x) : t \geq 0, x \in D\}$ on $D \subseteq H$, we say that a probability measure m on D is symmetrizing for Y, or that Y is symmetric w.r.t. m, if, setting for all $\varphi \in C_b(D)$: $P_t^Y \varphi(x) := \mathbb{E}[\varphi(Y(t,x))]$, $x \in D$, we have:

$$\int_D \varphi\, P_t^Y \psi\, dm \;=\; \int_D \psi\, P_t^Y \varphi\, dm \qquad \forall \varphi, \psi \in C_b(D).$$

A symmetrizing measure is in particular invariant, i.e.:

$$\int_D P_t^Y \varphi\, dm \;=\; \int_D \varphi\, dm, \qquad \forall \varphi \in C_b(D).$$

We denote by $1_D(\cdot)$ the characteristic function of a set D. We sometimes write: $m(\varphi)$ for $\int_H \varphi\, dm$, $\varphi \in C_b(H)$.

By $W = \{W(t,\xi) : (t,\xi) \in \mathcal{O}\}$ we denote a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. W is a Gaussian process with zero mean and covariance function

$$\mathbb{E}[W(t,\xi)W(t',\xi')] = (t \wedge t')(\xi \wedge \xi'), \qquad (t,\xi), (t',\xi') \in \mathcal{O}.$$

We denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,\xi) : (s,\xi) \in [0,t] \times [0,1]\}$.

Let $(B_t)_{t \geq 0}$ a linear Brownian Motion, and $(B_t^3)_{t \geq 0}$ a $\mathbb{R}^3$-valued BM. We set the following notations:

- μ is the law on $L^2(0,1)$ of a Brownian Bridge β between 0 and 0 on $[0,1]$, i.e. the law of $(B_\tau)_{\tau \in [0,1]}$ conditioned on $\{B_1 = 0\}$.

- $e_{0,a}^r$, $r \in (0,1)$, $a \geq 0$, is a 3-d Bessel Bridge on $[0,r]$ between 0 and a: i.e. the process $\{e_{0,a}^r(\tau)\}_{\tau \in [0,1]}$ has the law of the modulus of $(B_t^3)_{t \in [0,1]}$, conditioned on $\{|B_1^3| = a\}$.

- ν is the law on $L^2(0,1)$ of $e := e_{0,0}^1$, i.e. of a 3-d Bessel Bridge between 0 and 0 on $[0,1]$.

We recall the following Proposition, see e.g. [DPZ 96], Chap. 8.

Proposition 1

- *The positive symmetric bilinear form:*

$$\varphi, \psi \in C_b^1(H) \mapsto \frac{1}{2} \int_H \langle \nabla\varphi, \nabla\psi \rangle d\mu$$

  is closable in $L^2(H,\mu)$. *We denote by* $(\Lambda, W^{1,2}(H,\mu))$ *its closure.*

- *The semigroup associated with* Λ *is the Ornstein-Uhlenbeck semigroup* $(\Pi_t)_{t\geq 0}$, *given by the Mehler Formula:*

$$\Pi_t\varphi(z) := \int_H \varphi(y)\, \mathcal{N}(e^{tA}z, Q_t)(dy), \qquad \forall \varphi \in C_b(H),\ z \in H, \quad (11)$$

  where $Q_t := \int_0^t e^{2sA}\, ds$. *The infinitesimal generator* $(\mathcal{M}, D(\mathcal{M}))$ *of* $(\Pi_t)_{t\geq 0}$ *is the closure in* $L^2(H,\mu)$ *of the Ornstein-Uhlenbeck operator:*

$$M\varphi(x) := \frac{1}{2}\mathrm{Tr}\left[D^2\varphi(x)\right] + \langle x, A\nabla\varphi(x)\rangle, \quad \varphi \in \mathrm{Exp}_A(H). \quad (12)$$

- $(\Pi_t)_{t\geq 0}$ *is the transition semigroup of the Markov process* $\{Z(t,x) : t \geq 0, x \in H\}$ *in* H, *satisfying the linear SPDE:*

$$\begin{cases} \dfrac{\partial Z}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 Z}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi}, \\ Z(t,x) \in C_0, \quad t > 0 \\ Z(0,x) = x \in H \end{cases} \quad (13)$$

- *For all* $t > 0$ *and* $\varphi \in L^2(H,\mu)$, *we have* $\Pi_t\varphi \in W^{1,2}(H,\mu)$ *and:*

$$W^{1,2}(H,\mu) = \left\{\varphi \in L^2(H,\mu) : \sup_{t>0} \Lambda(\Pi_t\varphi, \Pi_t\varphi) < \infty\right\}. \quad (14)$$

- *For all* $t > 0$ *and* $\varphi \in L^\infty(H,\mu)$, *we have* $\Pi_t\varphi \in C_b^1(H)$: *in particular,* Π *is Strong Feller. Moreover, for all* $\varphi \in C_b(H)$ *and* $x \in H$, *the map* $0 \leq t \mapsto \Pi_t\varphi(x)$ *is continuous, i.e.* Π *is weakly continuous: see [Ce 94].*

- $\mathrm{Lip}(H) \subset W^{1,2}(H,\mu)$ *with continuous inclusion.*

The last assertion in Proposition 1 follows from (14), since

$$\begin{aligned}
\|\nabla \Pi_t \varphi(x)\| &= \sup_{\|h\|\le 1} |\langle \nabla \Pi_t \varphi(x), h\rangle| \\
&= \sup_{\|h\|\le 1} \lim_{s\downarrow 0} \frac{1}{s} \left| \int \left(\varphi\left(e^{tA}(x+sh)+y\right) - \varphi\left(e^{tA}x+y\right)\right) \mathcal{N}(0,Q_t)(dy) \right| \\
&\le \|\varphi\|_{\mathrm{Lip}}
\end{aligned}$$

We study the following SPDE with reflection:

$$\begin{cases} \dfrac{\partial u_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\alpha}{\partial \xi^2} - f(u_\alpha) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta_\alpha(t,\xi) \\ u_\alpha(0,\xi) = x(\xi),\ u_\alpha(t,0) = u_\alpha(t,1) = 0 \\ u_\alpha + \alpha \ge 0,\ \int (u_\alpha + \alpha)\, d\eta_\alpha = 0 \end{cases} \tag{15}$$

where $f(u_\alpha) := f(\cdot, u_\alpha(t,\cdot))$ and we assume that:

(H1) $f : [0,1]\times\mathbb{R} \mapsto \mathbb{R}$ is measurable.

(H2) $f(\xi,\cdot)$ is continuously differentiable for all $\xi \in [0,1]$, and there exists $c > 0$ such that

$$|f| \le c, \quad |\partial_y f(\xi,y)| \le c,\ \forall \xi \in [0,1],\ y \in \mathbb{R}.$$

(H3) There exists $C \ge 0$ such that for all $\xi \in [0,1]$:

$$\left| \int_0^t f(\xi,u)\, du \right| \le C, \qquad \forall t \ge 0.$$

Hypothesis (H1)-(H3) do not aim at the greatest generality, and we think that the results of this paper can be proved also for more general f.

Following [NP 92], we set the:

Definition 1 *A pair (u_α, η_α) is said to be a solution of equation (15) with reflection in $-\alpha \le 0$ and initial value $x \in K_\alpha \cap C_0(0,1)$, if:*

(i) $\{u_\alpha(t,\xi) : (t,\xi) \in \overline{\mathcal{O}}\}$ *is a continuous and adapted process, i.e.* $u_\alpha(t,\xi)$ *is* $\mathcal{F}_t$*-measurable for all* $(t,\xi) \in \overline{\mathcal{O}}$*, a.s.* $u_\alpha(\cdot,\cdot)$ *is continuous on* $\overline{\mathcal{O}}$*,* $u_\alpha(t,\cdot) \in K_\alpha \cap C_0(0,1)$ *for all* $t \geq 0$*, and* $u_\alpha(0,\cdot) = x$*.*

(ii) $\eta_\alpha(dt,d\xi)$ *is a random positive measure on* $\mathcal{O}$ *such that* $\eta_\alpha([0,T] \times [\delta, 1-\delta]) < +\infty$ *for all* $T, \delta > 0$*, and* η_α *is adapted, i.e.* $\eta_\alpha(B)$ *is* $\mathcal{F}_t$*-measurable for every Borel set* $B \subset [0,t] \times (0,1)$*.*

(iii) For all $t \geq 0$ *and* $l \in C_c^\infty(0,1)$

$$\langle u_\alpha(t,\,\cdot\,), l\rangle - \langle x, l\rangle - \int_0^t \left[\langle u_\alpha(s,\,\cdot\,), Al\rangle - \langle f(\cdot, u_\alpha(s,\,\cdot\,)), l\rangle\right] ds$$

$$= \int_0^t \int_0^1 l(\xi)\, W(ds,d\xi) + \int_0^t \int_0^1 l(\xi)\, \eta_\alpha(ds,d\xi), \qquad \text{a.s.} \qquad (16)$$

(iv) $\int_{\mathcal{O}} (u_\alpha + \alpha) d\eta_\alpha \;=\; 0.$

In [NP 92], the following theorem is proved:

Theorem 1 *For all* $x \in K_\alpha \cap C_0(0,1)$*, there exists a unique solution* (u_α, η_α) *of equation (15) with reflection at* $-\alpha$ *and initial value* x*.*

3 Integration by parts formulae

In this section we prove formulae (1) and (2). The main tools are the following:

Theorem 2 (Biane, [Bi 86]) *Let* $(e_\tau)_{\tau\in[0,1]}$ *be a 3-d Bessel Bridge, and let* ζ *be a random variable with uniform distribution on* $[0,1]$ *and independent of* e*. Then the process:*

$$(\beta_\tau)_{\tau\in[0,1]}, \qquad \beta_\tau := e_{\tau\oplus\zeta} - e_\zeta,$$

where $\oplus$ *denotes the sum* mod 1*, is a Brownian Bridge.*

Theorem 3 *For all continuous* $\varphi : H \mapsto \mathbb{R}$ *such that for some* $\omega < \pi^2$*,* $|\varphi(x)| \leq e^{\omega_0 \|x\|^2}$ *for all* $x \in H$*, we have:* $\int \varphi\, d\nu_\alpha \to \int \varphi\, d\nu_0$ *as* $\alpha \downarrow 0$*. In particular,* ν_α *converge weakly as* $\alpha \downarrow 0$ *to* $\nu_0 = \nu$*.*

Theorem 3 was proved in [DIM 77]: in [Za 00], we gave a different proof, based on Theorem 2. The key observation there was that $\beta = e_{\cdot \oplus \zeta} - e_\zeta \in K_\alpha$ if and only if $e_\zeta \leq \alpha$: a remarkable semplification, which reduced an infinite-dimensional information, namely that $\beta(\tau) \geq -\alpha$ for all $\tau \in [0,1]$, to an information on two independent real valued random variables, namely ζ and e_r, $r \in]0,1[$. Now formula (1) says in particular that $\beta = e_{\cdot \oplus \zeta} - e_\zeta$ is in the boundary of K_α if and only if $e_\zeta = \alpha$: the proof of (1) is formalization of this intuitive fact.

Proof of (1) and (2)—Recall the notations given in (3), (4) and (5). For $x \in H$, we set $x^+ \in H$, $x^+(\tau) := \sup\{x(\tau), 0\}$, $x^- := x^+ - x$. Notice that $h_\lambda := \lambda(\lambda - A)^{-1} h$ converges to h in $D(A)$ as $\lambda \to \infty$. Moreover, we have $h_\lambda = (h^+)_\lambda - (h^-)_\lambda$, with $(h^+)_\lambda, (h^-)_\lambda \in D(A)$, $(h^+)_\lambda, (h^-)_\lambda \geq 0$ and:

$$\partial_{h_\lambda} \varphi \;=\; \langle \nabla \varphi, h_\lambda \rangle \;=\; \langle \nabla \varphi(x), (h^+)_\lambda \rangle \;-\; \langle \nabla \varphi(x), (h^-)_\lambda \rangle,$$

Then, we can suppose that $h \geq 0$, so that $K_\alpha \subseteq K_\alpha - th$, $t \geq 0$. Moreover, since φ is bounded and $\nabla(\varphi - \inf \varphi) = \nabla \varphi$, we can suppose $\varphi \geq 0$. Recall that $\partial_h \varphi(x) = \lim_{t \downarrow 0} (\varphi(x) - \varphi(x - th))/t$. By the Cameron Martin Theorem:

$$\begin{aligned} \frac{1}{t} \int_{K_\alpha} (\varphi(x) - \varphi(x - th))\, \mu(dx) \;&=\; -\frac{1}{t} \int_{(K_\alpha - th) \setminus K_\alpha} \varphi(x)\, \mu(dx) \\ &+\; \frac{1}{t} \int_{K_\alpha - th} \varphi(x) \, \left(1 - \exp\left(\frac{1}{2} \|th'\|^2 + t\langle x, h'' \rangle \right) \right) \, \mu(dx). \end{aligned} \tag{17}$$

Let $n \in \mathbb{N}$, $c_n \geq c_{n-1} \geq \cdots \geq c_1 \geq c_0 := 0$, $\{I_1, \ldots, I_n\}$ a Borel partition of $[0,1]$ and $I_0 := \emptyset$, and set:

$$h_i := \sum_{j=1}^{n} (c_j \wedge c_i)\, 1_{I_j}, \qquad i = 1, \ldots, n.$$

The key point is the following: for $i = 1, \ldots, n$, since $h_i \geq h_{i-1}$, and $h_i = h_{i-1}$ on $\bigcup_{j=1}^{i-1} I_j$, then for all $r \in (0,1)$

$$e_{\cdot \oplus r} - e_r \;\in\; (K_\alpha - th_i) \setminus (K_\alpha - th_{i-1}) \quad \Longleftrightarrow$$

$$e_{\cdot \oplus r} - e_r \;\in\; K_\alpha - th_i, \; 1 - r \in \bigcup_{j=i}^{n} I_i \;\text{ and } e_r \in [\alpha + tc_{i-1}, \alpha + tc_i).$$

Indeed, recall that $e_{\cdot\oplus r} - e_r$ attains its minimum $-e_r$ only at time $1-r$. Applying Theorem 2 we obtain for all $t \geq 0$ and $i = 1, \ldots, n$:

$$\int_{(K_\alpha - th_i)\setminus K_\alpha} \varphi(x)\,\mu(dx) \;=\; \int_0^1 \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)\setminus K_\alpha}\ (e_{\cdot\oplus r} - e_r)\right]\,dr$$

$$= \int_0^1 \mathbb{E}\left[\varphi \cdot \left[1_{(K_\alpha - th_{i-1})\setminus K_\alpha} + 1_{(K_\alpha - th_i)\setminus(K_\alpha - th_{i-1})}\right](e_{\cdot\oplus r} - e_r)\right]\,dr$$

$$= \int_0^1 \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_{i-1})\setminus K_\alpha}\,(e_{\cdot\oplus r} - e_r)\right]\,dr$$

$$+ \int_{1-\cup_{j=i}^n I_j} \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\,1_{[\alpha + tc_{i-1}, \alpha + tc_i[}(e_r)\right]\,dr,$$

where $1 - I := \{1 - \tau : \tau \in I\}$. Proceeding by induction on n we obtain:

$$\int_{(K_\alpha - th_n)\setminus K_\alpha} \varphi(x)\,\mu(dx)$$

$$= \sum_{i=1}^n \int_{1-\cup_{j=i}^n I_j} \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\,1_{[\alpha + tc_{i-1}, \alpha + tc_i[}(e_r)\right]\,dr$$

$$= \sum_{i=1}^n \sum_{j=i}^n \int_{1-I_j} dr \int_{\alpha + tc_{i-1}}^{\alpha + tc_i} da\ \lambda(r,a)\ \mathbb{E}\left[\varphi \cdot 1_{(K_\alpha - th_i)}\,(e_{\cdot\oplus r} - e_r)\ \Big|\, e_r = a\right]$$

where $\lambda(r,a)da$, law of e_r, is defined by

$$\lambda(r,a) \;:=\; \sqrt{\frac{2}{\pi r^3 (1-r)^3}}\ a^2 \exp\left(-\frac{a^2}{2r(1-r)}\right), \qquad r \in [0,1],\ a \geq 0,$$

and for all bounded Borel $\psi : H \mapsto \mathbb{R}$ and $a \geq 0$:

$$\mathbb{E}\left[\psi\,(e_{\cdot\oplus r} - e_r)\ \Big|\, e_r = a\right] \;:=\; \mathbb{E}\left[\psi\left(e_{0,a}^r \oplus_r \hat{e}_{0,a}^{1-r} - a\right)\right]. \tag{18}$$

The measure defined by (18) depends continuously on $a \geq 0$. Then we obtain, since $\lambda(1-r, a) = \lambda(r, a)$:

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{(K_\alpha - th_n)\setminus K_\alpha} \varphi(x)\,\mu(dx)$$

$$= \sum_{i=1}^{n}\sum_{j=i}^{n}(c_i - c_{i-1})\int_{I_j}\lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e_{\cdot\oplus(1-r)} - \alpha\right)\,\Big|\, e_{1-r} = \alpha\right]\,dr$$

$$= \sum_{j=1}^{n}\int_{I_j} c_j\,\lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e_{\cdot\oplus(1-r)} - \alpha\right)\,\Big|\, e_{1-r} = \alpha\right]\,dr$$

$$= \int_0^1 dr\, h_n(r)\,\lambda(r,\alpha)\,\mathbb{E}\left[\varphi\left(e^r_{0,\alpha}\oplus_r \hat{e}^{1-r}_{0,\alpha} - \alpha\right)\right]$$

$$= \int_0^1 dr\, h_n(r)\int\varphi(z)\,\sigma_\alpha(r,dz).$$

Set now $I_i := h^{-1}([(i-1)/n, i/n))$, $i \in \mathbb{N}$,

$$f_n := \sum_{i=1}^{\infty}\frac{i-1}{n}\,1_{I_i}, \quad g_n := \sum_{i=1}^{\infty}\frac{i}{n}\,1_{I_i},$$

where both sums are finite, since h is bounded. Then $f_n \le h \le g_n$, f_n and g_n converge uniformly on $[0,1]$ to h as $n\to\infty$ and: $K_\alpha - tf_n \subseteq K_\alpha - th \subseteq K_\alpha - tg_n$, $t\ge 0$. Therefore we have, since $\varphi\ge 0$,

$$\int_0^1 dr\, f_n(r)\int\varphi(z)\,\sigma_\alpha(r,dz) \le \liminf_{t\downarrow 0}\frac{1}{t}\int_{(K_\alpha - th)\setminus K_\alpha}\varphi(x)\,\mu(dx)$$

$$\le \limsup_{t\downarrow 0}\frac{1}{t}\int_{(K_\alpha - th)\setminus K_\alpha}\varphi(x)\,\mu(dx) \le \int_0^1 dr\, g_n(r)\int\varphi(z),\,\sigma_\alpha(r,dz)$$

and by (17) : $$\int_{K_\alpha}\partial_h\varphi\,d\mu = \lim_{t\downarrow 0}\frac{1}{t}\int_{K_\alpha}(\varphi(x)-\varphi(x-th))\mu(dx)$$

$$= -\int_{K_\alpha}\varphi(x)\langle x, h''\rangle d\mu - \int_0^1 dr\, h(r)\int\varphi(z)\,\sigma_\alpha(r,dz)$$

so that (1) is proved. In order to prove (2), we recall that $\mu(K_\alpha) = 1 - \exp(-2\alpha^2)$. We divide (1) by $\mu(K_\alpha)$ and let $\alpha\downarrow 0$: in the second term of the right-hand side, we have for all $r\in\,]0,1[$:

$$\lim_{\alpha\downarrow 0}\frac{1}{2\alpha^2}\lambda(r,\alpha) = \lambda(r,0) := \frac{1}{\sqrt{2\pi r^3(1-r)^3}},\quad \frac{|h(r)|}{2\alpha^2}\lambda(r,\alpha) \le |h(r)|\lambda(r,0),$$

which is integrable, since $h \in W^{2,2} \cap W_0^{1,2}(0,1)$ implies $|h(r)| \le C\, r(1-r)$, $r \in [0,1]$, for some $C \ge 0$. Moreover, the laws of $e^r_{0,\alpha}$ are continuous in $\alpha \ge 0$. Then we apply Theorem 3 to the first and second term in (1) and the proof of (2) is complete. $\square$

Corollary 1 *For all $\varphi \in \mathrm{Lip}(K_0)$ and $h \in K_0$ there exists the limit in $L^2(\nu)$:*

$$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x + t\,h) - \varphi(x)) \;=:\; \langle \nabla\varphi(x), h\rangle \qquad x \in K_0. \tag{19}$$

We call $\nabla\varphi \in L^\infty(K_0, \nu; H)$ the generalized gradient of φ. Then (2) holds for all $\varphi \in \mathrm{Lip}(K_0)$, setting $\partial_h\varphi := \langle \nabla\varphi, h\rangle$.

Proof–The family $\{(\psi(\cdot + t\,h) - \psi))/t\}_{t>0}$ is bounded in $L^2(\nu)$. For all $\varphi \in \mathrm{Exp}_A(H)$:

$$\begin{aligned}\lim_{t\downarrow 0} \int_{K_0} \frac{1}{t}(\psi(\cdot + t\,h) - \psi)\,\varphi\,d\nu \;=\; -\int_{K_0} \psi\,\langle \nabla\varphi, h\rangle\,d\nu \\ - \int_{K_0} \varphi(x)\,\psi(x)\langle x, h''\rangle\,\nu(dx) + \int_0^1 h(r) \int_{K_0} \varphi(x)\,\psi(x)\,\sigma_0(r, dx).\end{aligned} \tag{20}$$

Indeed, (20) holds for all $\psi \in C^1_b(H)$; moreover, the family of functionals

$$C^1_b(H) \ni \psi \mapsto \int_{K_0} \frac{1}{t}(\psi(\cdot + t\,h) - \psi)\,\varphi\,d\nu, \quad t > 0,$$

is uniformly bounded in the sup-norm, by (2). By the density of $C^1_b(H)$ in $C_b(H)$ in the sup-norm, we obtain (20) for all $\psi \in C_b(H)$. Then, (20) allows to identify any limit point in the weak topology of $L^2(\nu)$ of $(\psi(\cdot + t\,h) - \psi))/t$ as $t \downarrow 0$. $\square$

Corollary 2 *For all $\varphi \in C_b(H)$, $\alpha > 0$, $h \in W^{2,2} \cap W_0^{1,2}(0,1)$:*

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_H \varphi(x)\,\langle (x+\alpha)^-, h\rangle\, \exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\,\mu(dx) \\ &= \frac{1}{2}\int_0^1 dr\,h(r) \int \varphi(z)\,\sigma_\alpha(r, dz).\end{aligned}$$

Proof—We can suppose $h \geq 0$. If $\varphi \in C_b^1(H)$, then by (1):

$$\frac{1}{\varepsilon}\int_H \varphi(x)\langle h,(x+\alpha)^-\rangle \exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\,\mu(dx) \tag{21}$$

$$= -\frac{1}{2}\int_H \left(\langle\nabla\varphi(x),h\rangle + \langle x,h''\rangle\,\varphi(x)\right)\exp\left(-\frac{\|(x+\alpha)^-\|^2}{\varepsilon}\right)\,\mu(dx)$$

$$\to -\frac{1}{2}\int_{K_\alpha}\left(\langle\nabla\varphi(x),h\rangle + \langle x,h''\rangle\,\varphi(x)\right)\,\mu(dx) \quad \text{as } \varepsilon\downarrow 0. \tag{22}$$

Setting $\varphi \equiv 1$, we see that the family of finite measures on H defined by (21) have equibounded mass by (22). Since $C_b^1(H)$ is dense in $C_b(H)$ in the uniform norm, the thesis follows for all $\varphi \in C_b(H)$. $\square$

Corollary 3 *For all $\psi \in C_b^1(H)$, $\varphi(x) \in \mathrm{Exp}_A(H)$,*

$$\frac{1}{2}\int_{K_\alpha}\langle\nabla\varphi,\nabla\psi\rangle\,d\mu = -\int_{K_\alpha}\psi\,M\varphi\,d\mu - \frac{1}{2}\int_0^1 dr\int\langle\nabla\varphi,\delta_r\rangle\,\psi\,d\sigma_\alpha(r,\cdot) \tag{23}$$

$$\frac{1}{2}\int_{K_0}\langle\nabla\varphi,\nabla\psi\rangle\,d\nu = -\int_{K_0}\psi\,M\varphi\,d\nu - \frac{1}{2}\int_0^1 dr\int\langle\nabla\varphi,\delta_r\rangle\,\psi\,d\sigma_0(r,\cdot), \tag{24}$$

where M is the Ornstein-Uhlenbeck operator, defined in (12), and denoting by i the imaginary unit, for $\varphi = \exp(i\langle h,\cdot\rangle)$, $h \in D(A)$:

$$\langle\nabla\varphi(x),\delta_r\rangle := i\,h(r)\,\exp(i\langle h,x\rangle), \qquad x \in H. \tag{25}$$

4 The process X_α, $\alpha \geq 0$

We introduce the following problem:

$$\begin{cases} \dfrac{\partial u_\alpha^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\alpha^\varepsilon}{\partial \xi^2} - f(\cdot,u_\alpha^\varepsilon(t,\cdot)) + \dfrac{\partial^2 W}{\partial t\partial\xi} + \dfrac{(\alpha+u_\alpha^\varepsilon)^-}{\varepsilon} \\ \\ u_\alpha^\varepsilon(0,\cdot) = x \in H, \quad u_\alpha^\varepsilon(t,0) = u_\alpha^\varepsilon(t,1) = 0, \ \forall t \geq 0. \end{cases} \tag{26}$$

with $\varepsilon > 0$, $(r)^- := \sup\{-r,0\}$ and $\alpha \geq 0$. This is a SPDE in $L^2(0,1)$ with additive noise and monotone or Lipschitz-continuous drift terms, for which existence and uniqueness of a solution are well known: see e.g. [DPZ 92]. We write for all $\alpha \geq 0$, $\varepsilon > 0$:

$$X_\alpha^\varepsilon(t,x) \;:=\; u_\alpha^\varepsilon(t,\cdot) \;\in\; C_0(0,1) \quad t \geq 0,\; x \in H.$$

Let $\alpha, \varepsilon > 0$, and set:

$$F : H \mapsto \mathbb{R}, \qquad F(x) \;:=\; \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds. \tag{27}$$

By (H1)-(H3), $F \in C_b^1(H)$ and $\nabla F(x) = f(\cdot, x(\cdot))$ for all $x \in H$. Set also:

$$\mu_\alpha^\varepsilon(dx) \;:=\; \frac{1}{Z_\alpha^\varepsilon} \exp\left(- 2F(x) \;-\; \frac{\|(x+\alpha)^-\|^2}{\varepsilon} \right)\; \mu(dx), \quad x \in H,$$

$$E^{\alpha,\varepsilon}(\varphi,\psi) \;:=\; \frac{1}{2}\int_H \langle \nabla\varphi, \nabla\psi\rangle\, d\mu_\alpha^\varepsilon, \qquad \forall \varphi, \psi \in C_b^1(H),$$

$$L\varphi(x) \;:=\; M\varphi(x) \;-\; \langle \nabla F(x), \nabla\varphi(x)\rangle, \quad \varphi \in \mathrm{Exp}_A(H),$$

$$L_\alpha^\varepsilon\varphi(x) \;:=\; L\varphi(x) \;+\; \frac{1}{\varepsilon}\langle (x+\alpha)^-, \nabla\varphi(x)\rangle, \quad \varphi \in \mathrm{Exp}_A(H),$$

where Z_α^ε is a normalization constant such that $\mu_\alpha^\varepsilon(H) = 1$ and M is the Ornstein-Uhlenbeck operator defined in (12). Finally, set for all $\varphi \in C_b(H)$:

$$R_\alpha^\varepsilon(\lambda)\varphi(x) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[\varphi(X_\alpha^\varepsilon(t,x))\right] dt, \qquad x \in H,\; \lambda > 0.$$

Then we have the following:

Theorem 4

1. *$(E^{\alpha,\varepsilon}, \mathrm{Exp}_A(H))$ is closable in $L^2(\mu_\alpha^\varepsilon)$: we denote by $(\mathcal{E}^{\alpha,\varepsilon}, D(\mathcal{E}^{\alpha,\varepsilon}))$ the closure. We have $W^{1,2}(H,\mu) \subseteq D(\mathcal{E}^{\alpha,\varepsilon})$ with continuous immersion.*

2. *$(L_\alpha^\varepsilon, \mathrm{Exp}_A(H))$ is essentially self-adjoint in $L^2(\mu_\alpha^\varepsilon)$: we denote the closure by $(\mathcal{L}_\alpha^\varepsilon, D(\mathcal{L}_\alpha^\varepsilon))$. We have that $D(\mathcal{E}^{\alpha,\varepsilon}) = D((-\mathcal{L}_\alpha^\varepsilon)^{1/2})$ and:*

$$\int_H \varphi\, \mathcal{L}_\alpha^\varepsilon\psi\, d\mu_\alpha^\varepsilon \;=\; -\,\mathcal{E}^{\alpha,\varepsilon}(\varphi,\psi), \quad \forall \varphi \in D(\mathcal{E}^{\alpha,\varepsilon}),\; \psi \in D(\mathcal{L}_\alpha^\varepsilon). \tag{28}$$

3. *The process $(X^\varepsilon_\alpha(t,x))_{t\geq 0, x\in H}$ is the diffusion generated by $\mathcal{E}^{\alpha,\varepsilon}$, i.e. for all $\lambda > 0$ and $\varphi \in C_b(H)$, $R^\varepsilon_\alpha(\lambda)\varphi \in D(\mathcal{E}^{\alpha,\varepsilon})$ and:*

$$\lambda \int_H R^\varepsilon_\alpha(\lambda)\varphi\,\psi\,d\mu^\varepsilon_\alpha + \mathcal{E}^{\alpha,\varepsilon}(R^\varepsilon_\alpha(\lambda)\varphi,\psi) \;=\; \int_H \varphi\,\psi\,d\mu^\varepsilon_\alpha, \quad \forall \psi \in D(\mathcal{L}^\varepsilon_\alpha).$$

4. *μ^ε_α is the unique invariant probability measure of X^ε_α. Moreover, X^ε_α is symmetric with respect to μ^ε_α.*

Proof–This result is well known: see [MR 92], [DP 98], [DP 00]. $\square$

The existence statement in Theorem 1 is proved in the following way: let u^ε_α be the solution of (26). Then:

(a) $u^\varepsilon_\alpha(t,\cdot) \in C_0(0,1)$ for all $t \geq 0$, and u^ε_α is continuous on $\mathcal{O}$.

(b) The map $0 < \varepsilon \mapsto u^\varepsilon_\alpha(t,\xi)$ is non-increasing for all $(t,\xi) \in \mathcal{O}$. The limit $\lim_{\varepsilon\downarrow 0} u^\varepsilon_\alpha(t,\xi) = \sup_{\varepsilon>0} u^\varepsilon_\alpha(t,\xi) =: u_\alpha(t,\xi)$ is finite for all $(t,\xi) \in \mathcal{O}$, $u_\alpha(t,\cdot) \in K_\alpha \cap C_0(0,1)$ for all $t \geq 0$, and u_α is continuous on $\mathcal{O}$.

(c) The measure on $\mathcal{O}$, $\eta^\varepsilon_\alpha(dt,d\xi) := (1/\varepsilon)(\alpha + u^\varepsilon_\alpha)^- dt d\xi$, converges distributionally as $\varepsilon \downarrow 0$ to a Radon measure $\eta_\alpha(dt,d\xi)$ on $\mathcal{O}$.

(d) The pair (u_α,η_α) is the solution of (15) with reflection in $-\alpha$ and initial value $x \in K_\alpha \cap C_0(0,1)$.

We set for all $\alpha \geq 0$:

$$X_\alpha(t,x) := u_\alpha(t,\cdot) \in C_0(0,1), \quad t \geq 0,\ x \in K_\alpha \cap C_0(0,1).$$

Since X_α, $\alpha \geq 0$ satisfies for all $x, x' \in K_\alpha \cap C_0(0,1)$: $\|X_\alpha(t,x) - X_\alpha(t,x')\| \leq C\|x - x'\|$, we can uniquely extend $X_\alpha(t,\cdot)$ to a map from K_α to K_α, that we denote by the same symbol. In particular we can also define η_α for $x \in K_\alpha$, using (16). Then we can set:

- $P^\varepsilon_\alpha(t)\varphi : H \mapsto \mathbb{R}$, $P^\varepsilon_\alpha(t)\varphi(x) := \mathbb{E}[\varphi(X^\varepsilon_\alpha(t,x))]$, $x \in H$,
- $P_\alpha(t)\varphi : K_\alpha \mapsto \mathbb{R}$, $P_\alpha(t)\varphi(x) := \mathbb{E}[\varphi(X_\alpha(t,x))]$, $x \in K_\alpha$.

In [Za 00] it was proved that:

- $\lim_{\varepsilon\downarrow 0} P^\varepsilon_\alpha(t)\varphi(x) = P_\alpha(t)\varphi(x)$, $\forall x \in K_\alpha$,

- If $\varphi \in C_b(H)$, then $\{P^\varepsilon_\alpha(t)\varphi,\ P_\alpha(t)\varphi : \alpha \geq 0, \varepsilon > 0\}$ is an equibounded and equicontinuous family.
- $(P_\alpha(t))_{t\geq 0}$ is a Markov semigroup acting on $C_b(K_\alpha)$.
- For all $\varphi \in C_b(H)$, $\lim_{\alpha\downarrow 0} P_\alpha(t)\varphi(x) = P_0(t)\varphi(x)$, $t \geq 0$, $x \in K_0$.
- P_α is symmetric with respect to ν^F_α, $\alpha \geq 0$, where: $\nu_\alpha := \mu(\,\cdot\,|K_\alpha)$ for $\alpha > 0$, $\nu_0 := \nu$,

$$d\nu^F_\alpha \;:=\; \frac{1}{\nu_\alpha(e^{-2F})}\,\exp(-2F(x))\,\nu_\alpha(dx), \tag{29}$$

where F is defined as in (27). Moreover, we have:

- ν^F_α is the unique invariant probability measure of X_α.

Indeed, let m^1 and m^2 be two invariant probability measures for X_α and let q^1 and q^2 be K_α-valued random variables, such that the law of q^i is m^i and $\{q^1, q^2, W\}$ is an independent family. Setting $b := \|X^\varepsilon_\alpha(t,q^1) - X^\varepsilon_\alpha(t,q^2)\|$ we have:

$$\frac{d}{dt}b^2 \;\leq\; -\pi^2 b^2 \;+\; cb \;\leq\; -\frac{\pi^2}{2}b^2 + C^2,$$

$$\|X^\varepsilon_\alpha(t,q^1) - X^\varepsilon_\alpha(t,q^2)\| \;\leq\; C\,e^{-\pi^2 t/4}\|q^1 - q^2\|, \quad \forall t \geq 0.$$

Then for all $t \geq 0$:

$$\|X_\alpha(t,p^1) - X_\alpha(t,p^2)\| \;=\; \lim_{\varepsilon\downarrow 0}\|X^\varepsilon_\alpha(t,p^1) - X^\varepsilon_\alpha(t,p^2)\| \;\leq\; C\,e^{-\pi^2/4t}\|p^1 - p^2\|.$$

Since the law of $X(t,q^i)$ is equal to m^i for all $t \geq 0$, this implies $m^1 = m^2$. In particular, X_α is ν^F_α-ergodic.

Finally, we have:

Proposition 2 *For all $\alpha \geq 0$ and $\varphi : H \mapsto \mathbb{R}$ bounded and Borel we have:*

$$|P_\alpha(t)\varphi(x) - P_\alpha(t)\varphi(y)| \;\leq\; C\,\|\varphi\|_\infty (1\wedge t)^{-\frac{1}{2}}\|x-y\|, \;\; x,y \in K_\alpha,\;\; t > 0. \tag{30}$$

In particular, the process X_α is Strong Feller.

Proof—Fix $\varepsilon > 0$, $\alpha \geq 0$ and set for $\gamma > 0$:

$$s_\gamma : \mathbb{R} \mapsto \mathbb{R}, \qquad s_\gamma(r) := \begin{cases} [(r)^-]^{1+\gamma}, & r \leq (1+\gamma)^{-1/\gamma} \\ \\ r - \gamma(1+\gamma)^{-1-1/\gamma}, & r \geq (1+\gamma)^{-1/\gamma} \end{cases}$$

Then s_γ is $C^1(\mathbb{R})$, monotone non-decreasing, and for all $r \in \mathbb{R}$, $s_\gamma(r) \uparrow (r)^-$ as $\gamma \downarrow 0$. Consider the following equation:

$$\begin{cases} \dfrac{\partial \tilde{u}_\gamma}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \tilde{u}_\gamma}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} - f(\tilde{u}_\gamma) + \dfrac{s_\gamma(\alpha + \tilde{u}_\gamma)}{\varepsilon} \\ \\ \tilde{u}_\gamma(0,\cdot) = x \in H, \quad \tilde{u}_\gamma(t,0) = \tilde{u}_\gamma(t,1) = 0, \ \forall t \geq 0. \end{cases} \tag{31}$$

We set $\tilde{X}_\gamma(t,x) := \tilde{u}_\gamma(t,\cdot)$. Equation (31) is a white-noise driven SPDE with differentiable non-linearity of Nemytskii type, satisfying the hypothesis of Proposition 8.3.3 of [Ce 99]. Then, we have for $\varphi \in C_b(H)$, $x, y \in H$:

$$|\mathbb{E}[\varphi(\tilde{X}_\gamma(t,x))] - \mathbb{E}[\varphi(\tilde{X}_\gamma(t,y))]| \ \leq \ C\,\|\varphi\|_\infty (1 \wedge t)^{-\frac{1}{2}} |x-y|. \tag{32}$$

By the monotonicity properties of s_γ and the uniqueness of solutions of (26), we have that $\tilde{u}_\gamma \uparrow u^\varepsilon_\alpha$ as $\gamma \downarrow 0$. Then letting $\gamma \downarrow 0$ in (32), we obtain:

$$|P^\varepsilon_\alpha \varphi(x) - P^\varepsilon_\alpha \varphi(y)| \ \leq \ C\,\|\varphi\|_\infty (1 \wedge t)^{-\frac{1}{2}} |x-y|.$$

The thesis follows letting $\varepsilon \downarrow 0$ and using the Monotone Class Theorem. $\square$

Recall that Donati-Martin and Pardoux proved in [DP 93] the existence of a minimal solution (v, θ) of the following semilinear SPDE with reflection at 0:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \ \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \xi^2} - f(v(t,\xi)) + \sigma(v(t,\xi))\,\dfrac{\partial^2 W}{\partial t \partial \xi} + \theta(t,\xi) \\ \\ v(0,\xi) = x(\xi), \ v(t,0) = v(t,1) = 0 \\ \\ v \geq 0, \ d\theta \geq 0 \ \int_{\mathcal{O}} v \, d\eta = 0. \end{cases} \tag{33}$$

and in [DP 97], under the assumptions that f, σ are differentiable on $\mathbb{R}$ with bounded derivative, that for all $t > 0$, $\xi \in (0,1)$, the law of $v(t,\xi)$ is absolutely continuous w.r.t. the Lebesgue measure dy on $(0,\infty)$. If $\sigma \equiv 1$, we can improve this result. Indeed, we have:

Corollary 4 *For all $t > 0$, $x \in H$, the law of $X_\alpha(t,x)$ is absolutely continuous with respect to ν_α. In particular, for all $t > 0$ and $\xi \in]0,1[$, the law of $u_0(t,\xi)$ is absolutely continuous w.r.t. $y^2\,dy$ on $[0,\infty)$.*

5 The Dirichlet Form E^α, $\alpha \geq 0$

The aim of this section is to apply (1) and (2) to the symmetric bilinear forms

$$C_b^1(H) \ni \varphi, \psi \mapsto E^\alpha(\varphi,\psi) := \frac{1}{2}\int \langle \nabla\varphi, \nabla\psi\rangle \, d\nu_\alpha^F, \quad \alpha \geq 0.$$

The main result is that E^α is closable in $L^2(\nu_\alpha)$ for all $\alpha \geq 0$, and X_α is the associated diffusion. We refer to [FOT 94] and [MR 92] for all basic definitions. We set for all $\varphi \in C_b(H)$:

$$R_\alpha(\lambda)\varphi(x) := \int_0^\infty e^{-t}\mathbb{E}[\varphi(X_\alpha(t,x))]\,dt, \qquad x \in K_\alpha, \ \lambda > 0.$$

Since ν_α^F is invariant for X_α, $R_\alpha(\lambda)$ extends to a bounded linear operator in $L^2(\nu_\alpha^F)$ for all $\lambda > 0$: we denote also such extension by $R_\alpha(\lambda)$. We also set:

$$\sigma_\alpha^F(r,dx) := \frac{1}{\nu_\alpha(e^{-2F})}\exp(-2F(x))\,\sigma_\alpha(r,dx), \qquad \alpha \geq 0, \ r \in (0,1). \quad (34)$$

Theorem 5 *Let $\alpha > 0$. Set for all $\varphi, \psi \in W^{1,2}(H,\mu)$:*

$$E^\alpha(\varphi,\psi) := \frac{1}{2}\int_{K_\alpha} \langle \nabla\varphi, \nabla\psi\rangle \, d\nu_\alpha^F.$$

Then the positive symmetric bilinear form

$$C_b^1(K_\alpha) \ni \varphi, \psi \mapsto E^\alpha(\varphi,\psi)$$

is closable in $L^2(\nu_\alpha^F)$. We denote by $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$ the closure. The family $(R_\alpha(\lambda))_{\lambda>0}$ on $L^2(\nu_\alpha^F)$ is the strongly continuous resolvent associated with $\mathcal{E}^\alpha$, $\mathrm{Lip}(K_\alpha) \subseteq W^{1,2}(\nu_\alpha^F)$ and $\mathrm{Exp}_A(K_\alpha)$ is a core for $\mathcal{E}^\alpha$.

Proof–Let $\varphi \in \mathrm{Exp}_A(H)$, $\psi \in C_b(H)$. Set $V_\alpha^\varepsilon := R_\alpha^\varepsilon(1)\psi$, $V_\alpha := R_\alpha(1)\psi$. Then, by Lemma 1, Proposition 2 and Corollary 2:

$$\begin{aligned}
\mathcal{E}^{\alpha,\varepsilon}(V_\alpha^\varepsilon, \varphi) \;&=\; -\int_H V_\alpha^\varepsilon \, \mathcal{L}_\alpha^\varepsilon \varphi \; d\mu_\alpha^\varepsilon \\
&= -\int_H V_\alpha^\varepsilon \; L\varphi \; d\mu_\alpha^\varepsilon \;-\; \int_H V_\alpha^\varepsilon(x) \; \frac{1}{\varepsilon}\langle (x+\alpha)^-, \nabla\varphi(x)\rangle \; \mu_\alpha^\varepsilon(dx) \\
&\to -\int_{K_\alpha} V_\alpha \; L\varphi \; d\nu_\alpha^F \;-\; \frac{1}{2}\int_0^1 dr \int V_\alpha(z)\,\langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz),
\end{aligned}$$

as $\varepsilon \downarrow 0$. On the other hand, we have

$$\int V_\alpha^\varepsilon \; \varphi \; d\mu_\alpha^\varepsilon \;+\; \mathcal{E}^{\alpha,\varepsilon}(V_\alpha^\varepsilon, \varphi) \;=\; \int \psi \; \varphi \; d\mu_\alpha^\varepsilon, \quad \text{so that :}$$

$$\int_{K_\alpha} (V_\alpha - \psi)\,\varphi \, d\nu_\alpha^F = \int_{K_\alpha} V_\alpha \, L\varphi \, d\nu_\alpha^F + \frac{1}{2}\int_0^1 dr \int V_\alpha \, \langle \nabla\varphi, \delta_r\rangle \, \sigma_\alpha^F(r,dz). \tag{35}$$

Notice that $V_\alpha \circ \Pi_{K_\alpha}$ is Lipschitz on H: therefore it is in $W^{1,2}(H,\mu)$. Set $\{\gamma_n\}_n := \Pi_{1/n}(V_\alpha \circ \Pi_{K_\alpha})$, where $(\Pi_t)_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup defined in (11). Then $\{\gamma_n\} \subseteq C_b^1(H)$, $\sup_n \|\gamma_n\|_\infty < \infty$ and γ_n converges to $V_\alpha \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$ and pointwise. By (23) and (35):

$$\begin{aligned}
-\,E^\alpha(V_\alpha, \varphi) \;&=\; -\,\lim_n \, E^\alpha(\gamma_n, \varphi) \\
&= \lim_n \left(\int_{K_\alpha} \gamma_n \, L\varphi \; d\nu_\alpha^F \;+\; \frac{1}{2}\int_0^1 dr \int \gamma_n(z)\,\langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz) \right) \\
&= \int_{K_\alpha} V_\alpha \, L\varphi \; d\nu_\alpha^F \;+\; \frac{1}{2}\int_0^1 dr \int V_\alpha(z)\,\langle \nabla\varphi(z), \delta_r\rangle \, \sigma_\alpha^F(r,dz) \\
&= \int_{K_\alpha} (V_\alpha \;-\; \psi)\,\varphi \, d\nu_\alpha^F.
\end{aligned}$$

Let now $\psi \in \mathrm{Lip}(K_\alpha)$: then $\psi \circ \Pi_{K_\alpha} \in \mathrm{Lip}(H)$, and by Proposition 1 we can find a sequence $\varphi_n \in \mathrm{Exp}_A(H)$ converging to $\psi \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$. Then we obtain

$$E^\alpha(V_\alpha, \varphi) \;=\; \int_{K_\alpha} (\psi \;-\; V_\alpha)\,\varphi \; d\nu_\alpha^F \qquad \forall \psi \in \mathrm{Lip}(K_\alpha),$$

and analogously, $R_\alpha(\lambda)\psi \circ \Pi_{K_\alpha} \in W^{1,2}(H,\mu)$ and for all $\lambda > 0$:

$$E^\alpha(R_\alpha(\lambda)\psi, \varphi) \;=\; \int_{K_\alpha} (\psi \,-\, \lambda R_\alpha(\lambda)\psi)\, \varphi \, d\nu_\alpha^F \quad \forall \varphi \in \mathrm{Lip}(K_\alpha). \tag{36}$$

Since $(R_\alpha(\lambda))_{\lambda>0}$ is a strongly-continuous resolvent in $L^2(\nu_\alpha^F)$, then there exists a Dirichlet Form $(\tilde{E}^\alpha, D(\tilde{E}^\alpha))$ with $D(\tilde{E}^\alpha)$ dense in $L^2(\nu_\alpha^F)$, associated with $(R_\alpha(\lambda))_{\lambda>0}$. Consider $\psi \in \mathrm{Lip}(K_\alpha)$: by the general theory of Dirichlet Forms,

$$\psi \in D(\tilde{E}^\alpha) \iff \sup_{\lambda>0} \int_{K_\alpha} \lambda\, (\psi - \lambda R_\alpha(\lambda)\psi)\; \psi \; d\nu_\alpha^F \;<\; \infty \tag{37}$$

By (30) and (36), we have:

$$\int_{K_\alpha} \lambda\, (\psi - \lambda R_\alpha(\lambda)\psi)\; \psi\; d\nu_\alpha^F \;=\; E^\alpha(\lambda R_\alpha(\lambda)\psi, \psi) \;\le\; C\, \|\psi\|^2_{\mathrm{Lip}(K_\alpha)},$$

for some $C \geq 0$, so that $\mathrm{Lip}(K_\alpha) \subseteq D(\tilde{E}^\alpha)$. Then, by (36), E^α is closable on $R_\alpha(1)(\mathrm{Lip}(K_\alpha))$, the closure $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$ coincides with $(\tilde{E}^\alpha, D(\tilde{E}^\alpha))$ and $(R_\alpha(\lambda))_{\lambda>0}$ is the resolvent associated with $(\mathcal{E}^\alpha, W^{1,2}(\nu_\alpha^F))$.

Finally, since for all $\psi \in \mathrm{Lip}(K_\alpha)$ there exists a sequence $\varphi_n \in \mathrm{Exp}_A(H)$ converging to $\psi \circ \Pi_{K_\alpha}$ in $W^{1,2}(H,\mu)$, then we have that $\mathrm{Exp}_A(H)$ is a core for $\mathcal{E}^\alpha$, and the Theorem is proved. $\square$

We turn now to the case $\alpha = 0$. We have the following:

Theorem 6 *Set for all $\varphi, \psi \in \mathrm{Lip}(K_0)$:*

$$E^0(\varphi, \psi) \;:=\; \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\nu_0^F,$$

where $\nabla\varphi$ and $\nabla\psi$ are defined by (19). Then the positive symmetric bilinear form $(E^0, \mathrm{Lip}(K_0))$ is closable in $L^2(\nu_0^F)$. We denote the closure of $(E^0, \mathrm{Lip}(K_0))$ by $(\mathcal{E}^0, W^{1,2}(\nu_0^F))$. The family $(R_0(\lambda))_{\lambda>0}$ on $L^2(\nu_0^F)$ is the strongly continuous resolvent associated with $\mathcal{E}^0$ and $\mathrm{Exp}_A(K_0)$ is a core for $\mathcal{E}^0$.

We set $H^3 = \oplus_{i=1}^3 H = L^2(0,1;\mathbb{R}^3)$, $\Phi_3 : H^3 \mapsto K_0$, $\Phi_3(y)(\tau) := |y(\tau)|_{\mathbb{R}^3}$, $\tau \in (0,1)$. We denote by $(\Lambda^3, W^{1,2}(\mu^{\otimes 3}))$ the closure of the symmetric bilinear form:

$$C_b^1(H^3) \;\ni\; G_1, G_2 \;\mapsto\; \frac{1}{2} \int_{H^3} \langle \overline{\nabla}\, G_1, \overline{\nabla}\, G_2 \rangle_{H^3} d\mu^{\otimes 3}, \tag{38}$$

where $\overline{\nabla} G \in C_b(H^3; H^3)$ is the usual gradient of G. If $G \in W^{1,2}(\mu^{\otimes 3})$, then we denote the generalized gradient of G by $\overline{\nabla} G \in L^2(H^3, \mu^{\otimes 3}; H^3)$. Moreover, if $\varphi \in \mathrm{Lip}(K_0)$, then $\varphi \circ \Phi_3 \in \mathrm{Lip}(H^3) \subseteq W^{1,2}(\mu^{\otimes 3})$.

Proof of Theorem 6.—Since the image measure of $\mu^{\otimes 3}$ under Φ_3 is ν, there exists a measurable set $\Omega_0 \subseteq H^3$ with $\mu^{\otimes 3}(\Omega_0) = 1$, such that for all $y \in \Omega_0$, $|y| > 0$ on $(0,1)$. Then, for all $h \in K_0$ the following map is well-defined:

$$\Omega_0 \ni y \mapsto h \frac{y}{|y|} \in C\left([0,1]; \mathbb{R}^3\right)$$

Notice that an analogue of Proposition 1 also holds for the Gaussian space $(H^3, \mu^{\otimes 3})$: in particular, for all $G \in \mathrm{Lip}(H^3)$ there exists a sequence $\{G_n\} \subset C_b^1(H^3)$, such that

$$\|G_n\|_{\mathrm{Lip}(H^3)} \leq \|G\|_{\mathrm{Lip}(H^3)}, \quad G_n \to G \quad \text{in } W^{1,2}(H^3, \mu^{\otimes 3}).$$

Then, by a density argument, for all $G \in \mathrm{Lip}(H^3)$:

$$\lim_{t \downarrow 0} \frac{1}{t} \left[G\left(y + t\, h \frac{y}{|y|} \right) - G(y) \right] = \langle \overline{\nabla}\, G(y), h \frac{y}{|y|} \rangle_{H^3} \qquad \text{in } L^2(\mu^{\otimes 3}).$$

Then, for $h \in C_0(0,1)$ and $G := \varphi \circ \Phi_3$ with $\varphi \in \mathrm{Lip}(H)$:

$$\begin{aligned}
\langle \nabla \varphi(|y|), h \rangle &:= \lim_{t \downarrow 0} \frac{1}{t} \left(\varphi(|y| + t\, h) - \varphi(|y|) \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left[[\varphi \circ \Phi_3] \left(y + t\, h \frac{y}{|y|} \right) - [\varphi \circ \Phi_3](y) \right] \\
&= \langle \overline{\nabla}\, [\varphi \circ \Phi_3](y), h \frac{y}{|y|} \rangle_{H^3} \qquad \text{in } L^2(\mu^{\otimes 3}).
\end{aligned}$$

For all $\varphi, \psi \in \mathrm{Lip}(H)$, it follows that:

$$D^3(\varphi, \psi) := \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\nu = \Lambda^3 \left(\varphi \circ \Phi_3, \psi \circ \Phi_3 \right). \tag{39}$$

Since Λ^3 is a closed form, $(D^3, \mathrm{Lip}(H))$ is closable: we denote the closure by $(\mathcal{D}^3, W^{1,2}(\nu))$. Then (39) holds for all $\varphi, \psi \in W^{1,2}(\nu)$.

By (H1)-(H3), we have that $0 < e^{-2C} \leq \exp(-2F) \leq e^{2C} < \infty$, so that $L^2(\nu) = L^2(\nu_0^F)$ with equivalence of norms, and D^3 in (39) is equivalent to E^0

on $\mathrm{Lip}(K_0)$. This implies closability of $(E^0, \mathrm{Lip}(K_0))$ in $L^2(\nu_0^F)$: we denote the closure by $(\mathcal{E}^0, W^{1,2}(\nu_0^F))$.

As in the proof of Theorem 5, let $\varphi(x) \in \mathrm{Exp}_A(H)$ and $\psi \in C_b(H)$. Set $V^\varepsilon := R_0^\varepsilon(1)\psi$, $V := R_\alpha(1)\psi$. Then, by Lemma 1 and Proposition 2, letting $\varepsilon \downarrow 0$ in (35) we obtain:

$$\int_{K_0} (V - \psi)\, \varphi \, d\nu_0^F \;=\; \int_{K_0} V \, L\varphi \, d\nu_0^F + \frac{1}{2} \int_0^1 dr \int V \, \langle \nabla \varphi, \delta_r \rangle \, \sigma_0^F(r, dz). \quad (40)$$

By Corollary 3, for all $\gamma \in C_b^1(H)$ and $h \in D(A)$, denoting the imaginary unit by i and setting $\varphi_h := \exp(i\langle h, x\rangle)$:

$$\begin{aligned}
T_t\gamma \;:=&\; \frac{i}{2} \int_{K_0} \frac{1}{t}(\gamma(x + t\,h) - \gamma(x))\, \varphi_h(x)\, \nu_0^F(dx) \\
=&\; \frac{1}{2} \int_0^1 ds \int_{K_0} \langle \nabla \gamma(x + s\,t\,h), i\,h\rangle \, \varphi_h \, \nu_0^F(dx) \\
=&\; \frac{1}{2} \int_0^1 ds \int_{K_0} \langle \nabla \gamma(x + s\,t\,h), \nabla \varphi_h\rangle \, \nu_0^F(dx) \\
=&\; - \int_0^1 ds \left[\int_{K_0} \gamma(\,\cdot\, + s\,t\,h)\, M\varphi_h \, d\nu_0^F \right. \\
&\quad \left. + \frac{1}{2} \int_0^1 dr \int \gamma(\,\cdot\, + st\,h)\, \langle \nabla \varphi_h, \delta_r\rangle \, \sigma_0^F(r, dz) \right],
\end{aligned}$$

$$|T_t\gamma| \;\le\; C\, \|\gamma\|_\infty,$$

with $C > 0$ independent of $\gamma \in C_b^1(H)$. By the density of $C_b^1(H)$ in $C_b(H)$ in the sup-norm, by (40) and (19), we obtain:

$$E^0(V, \varphi_h) \;=\; \lim_{t \downarrow 0} \frac{i}{2} \int_{K_0} \frac{1}{t}(V(\cdot + t\,h) - V(x))\, \varphi_h \, d\nu_0^F \;=\; \int_{K_0} (\psi - V)\, \varphi_h \, d\nu_0^F,$$

and for all $\varphi \in \mathrm{Exp}_A(H)$:

$$E^0(V, \varphi) \;=\; \int_{K_0} (\psi - V)\, \varphi \, d\nu_0^F,$$

and analogously for all $\lambda > 0$, $\varphi \in \mathrm{Exp}_A(H)$:

$$E^0(R_0(\lambda)\psi, \varphi) \;=\; \int_{K_0} (\psi \,-\, \lambda R_0(\lambda)\psi)\, \varphi \, d\nu_0^F. \quad (41)$$

For all $\varphi \in \mathrm{Lip}(H)$, by a standard approximation argument, we can find a net $\{\varphi_i\}_{i\in I} \subset \mathrm{Exp}_A(H)$, such that: $\sup_{i,x} |\varphi_i(x)| < \infty$, $\lim_i \varphi_i(x) = \varphi(x)$ $\forall x \in H$, and $\varphi_i \to \varphi$ weakly in $W^{1,2}(\nu_0^F)$. Therefore (41) holds for all $\varphi \in \mathrm{Lip}(H)$ and moreover $\mathrm{Exp}_A(H)$ is dense in $\mathrm{Lip}(H)$ with respect to the weak topology of $W^{1,2}(\nu_0^F)$: by Hahn-Banach Theorem, this implies that $\mathrm{Exp}_A(H)$ is a core in $W^{1,2}(\nu_0^F)$. $\square$

Corollary 5 *Formulae (1) and (2) hold for all $\varphi \in \mathrm{Lip}(H)$, where for fixed $\alpha \geq 0$, $\partial_h\varphi := \langle \nabla\varphi, h\rangle \in L^2(\nu_\alpha)$.*

For all $\alpha \geq 0$, the Dirichlet Form $\mathcal{E}^\alpha$ enjoys the following properties:

(i) $\mathrm{Lip}(K_\alpha)$ is dense in $W^{1,2}(\nu_\alpha^F)$.

(ii) $\mathrm{Exp}_A(K_\alpha)$ separates the points of K_α and is contained in $W^{1,2}(\nu_\alpha^F)$.

By Definition IV.3.1 in [MR 92], $\mathcal{E}^\alpha$ is **quasi-regular** if moreover:

(iii) There exists a sequence of compact sets F_k in K_α, such that the set:

$$\bigcup_k \left\{\varphi \in W^{1,2}(\nu_\alpha^F) : \varphi = 0 \ \nu_\alpha^F - \text{a.e. on } K_\alpha \backslash F_k\right\}$$

is dense in $W^{1,2}(\nu_\alpha^F)$.

On the other hand, by Nualart-Pardoux's Theorem 1, the process X_α is continuous, with infinite life-time and Strong Markov. Therefore X_α is a Hunt process on K_α, properly associated with $\mathcal{E}^\alpha$, see Chapter IV in [MR 92]: indeed, for all Borel bounded $\varphi : K_\alpha \mapsto \mathbb{R}$ and $t > 0$, $P_\alpha(t)\varphi \in C_b(K_\alpha)$, and by Theorems 5-6, P_α is the semigroup associated with $\mathcal{E}^\alpha$. Then we have:

Theorem 7 *Let $\alpha \geq 0$. The process $\{X_\alpha(\,\cdot\,, x)\}_x$ is a continuous Hunt process on K_α with infinite life-time, properly associated with the Dirichlet form $\mathcal{E}^\alpha$. In particular, $\mathcal{E}^\alpha$ is quasi-regular.*

The last assertion in Theorem 7 is a consequence of Theorem IV.5.1 in [MR 92], which states the necessity of quasi regularity for a Dirichlet Form to be properly associated with a nice Markov process. Theorem 7 plays a crucial role in the next section.

Corollary 6 *The Log-Sobolev and the Poincaré inequalities hold for the Nualart-Pardoux equation (15) for all* $\alpha \geq 0$*, i.e. there exists* $C > 0$ *such that for all* $\varphi \in W^{1,2}(\nu_\alpha^F)$*:*

$$\int_{K_\alpha} \left|\varphi - \nu_\alpha^F(\varphi)\right|^2 \, d\nu_\alpha^F \;\leq\; C \int_{K_\alpha} \|\nabla\varphi\|^2 \, d\nu_\alpha^F,$$

$$\int_{K_\alpha} \varphi^2 \log(\varphi^2) \, d\nu_\alpha^F \;\leq\; C \int_{K_\alpha} \|\nabla\varphi\|^2 \, d\nu_\alpha^F \;+\; \|\varphi\|^2_{L^2(\nu_\alpha^F)} \log(\|\varphi\|^2_{L^2(\nu_\alpha^F)}).$$

For the proof, see e.g. [St 93], [DPDG 00] and [DP 01]. Finally, in the case $f \equiv 0$, we also have the following:

Theorem 8 *The Dirichlet Form* $\mathcal{D}^3$*:*

$$W^{1,2}(\nu) \ni \varphi, \psi \;\mapsto\; \mathcal{D}^3(\varphi, \psi) \;:=\; \frac{1}{2} \int_{K_0} \langle \nabla\varphi, \nabla\psi \rangle \, d\nu$$

is the image of Λ^3 *under the map* Φ_3*, i.e.*

$$W^{1,2}(\nu) = \{\varphi \in L^2(\nu) : \varphi \in \Phi_3 \in W^{1,2}(\mu^{\otimes 3})\},$$

$$\mathcal{D}^3(\varphi, \psi) \;=\; \Lambda^3(\varphi \circ \Phi_3, \psi \circ \Phi_3) \quad \forall\, \varphi, \psi \in W^{1,2}(\nu).$$

Proof–In the whole proof, we consider the case $f \equiv 0$ in equation (15). Recall (39). It remains to prove only that if $\varphi \in L^2(\nu)$ satisfies $\varphi \circ \Phi_3 \in W^{1,2}(\mu^{\otimes 3})$, then $\varphi \in W^{1,2}(\nu)$. Set $\mathcal{Y}_3 := \{\varphi \circ \Phi_3 : \varphi \in \cap W^{1,2}(\nu)\}$. By (39), $\mathcal{Y}_3$ is a closed subspace of $W^{1,2}(\mu^{\otimes 3})$: we denote by $\Gamma_3 : W^{1,2}(\mu^{\otimes 3}) \mapsto \mathcal{Y}_3$ the symmetric projection operator w.r.t. the scalar product $\Lambda_1^3 := (\cdot, \cdot)_{L^2(\mu^{\otimes 3})} + \Lambda^3$, i.e. for all $G \in W^{1,2}(\mu^{\otimes 3})$, $\Gamma_3 G \in \mathcal{Y}_3$ is characterized by:

$$\Lambda_1^3(G - \Gamma_3 G, G - \Gamma_3 G) \;\leq\; \Lambda_1^3(G - G', G - G') \quad \forall G' \in \mathcal{Y}_3.$$

Since $G \in \mathcal{Y}_3$ implies $(G \vee 0) \wedge 1 \in \mathcal{Y}_3$, we have:

$$\|\Gamma_3 G\|_{L^\infty(\mu^{\otimes 3})} \;\leq\; \|G\|_{L^\infty(\mu^{\otimes 3})}, \;\; \forall G \in W^{1,2}(\mu^{\otimes 3}) \cap L^\infty(\mu^{\otimes 3}).$$

By Theorem 6, $\mathcal{D}^3$ is a quasi-regular symmetric Dirichlet form: see [MR 92]. Then, for all $h \in D(A)$, $\varphi \in W^{1,2}(\nu) \cap L^\infty(\nu)$:

$$\int_K \langle \varphi, h \rangle \, d\nu = -\int_K \varphi^*(x) \left(\langle x, h'' \rangle + \kappa(3) \langle x^{-3}, h \rangle \right) \nu(dx),$$

where φ^* is a $\mathcal{D}^3$-quasi-continuous ν-version of φ. For all $\psi \in \mathrm{Lip}(H)$ we have:

$$\Lambda_1^3(G, [R_0(1)\psi] \circ \Phi_3) \;=\; \Lambda_1^3(\Gamma_3 G, [R_0(1)\psi] \circ \Phi_3) \;=\; \int_{H^3} (\Gamma_3 G)^* \, \psi \circ \Phi_3 \, d\mu^{\otimes 3}$$

for all $G \in W^{1,2}(\mu^{\otimes 3}) \cap L^\infty(\mu^{\otimes 3})$. Then there exists $C_\psi \geq 0$ such that:

$$|\Lambda_1^3(G, [R_0(1)\psi] \circ \Phi_3)| \;\leq\; C_\psi \, \|G\|_\infty \quad \forall G \in W^{1,2}(\mu^{\otimes 3}) \cap L^\infty(\mu^{\otimes 3}),$$

and by Theorem 4.2 in [Fu 99], there exists a finite signed measure Σ_ψ on H^3, charging no Λ^3-exceptional set, such that for all $G \in W^{1,2}(\mu^{\otimes 3}) \cap L^\infty(\mu^{\otimes 3})$:

$$\Lambda^3(G, [R_0(1)\psi] \circ \Phi_3) = -\int_{H^3} G^* \, d\Sigma_\psi, \tag{42}$$

where G^* is a Λ^3-quasi-continuous $\mu^{\otimes 3}$-version of G, and for all $\varphi \in C_b(H)$:

$$\int_{H^3} \varphi \circ \Phi_3 \, d\Sigma_\psi \;=\; \int_{H^3} \varphi \circ \Phi_3 \,\cdot\, \psi \circ \Phi_3 \, d\mu^{\otimes 3}.$$

Now, to complete the proof, it is enough to prove that $\{[R_0(1)\psi] \circ \Phi_3 : \psi \in \mathrm{Lip}(H)\}$ is dense in $\{\varphi \circ \Phi_3 : \varphi \in L^2(\nu)\} \cap W^{1,2}(\mu^{\otimes 3})$ w.r.t. Λ_1^3. Suppose that $\varphi \in L^2(\nu)$, $\varphi \circ \Phi_3 \in W^{1,2}(\mu^{\otimes 3})$, and:

$$\Lambda_1^3(\varphi \circ \Phi_3, [R_0(1)\psi] \circ \Phi_3) \;=\; 0 \qquad \forall \psi \in \mathrm{Lip}(H),$$

We set $\varphi_m := (\varphi^* \wedge m) \vee (-m)$, $m \in \mathbb{N}$, and

$$G_{n,m}(y) := \varphi_m \circ \Phi_3(Z_3(1/n, y)), \quad y \in H^3.$$

where Z_3 is the solution of (10). Then $(G_{n,m}) \subset C_b^1(H^3)$, $|G_{n,m}| \leq m$, $G_{n,m} \to \varphi_m \circ \Phi_3$ Λ^3-quasi everywhere as $n \to \infty$ and in $W^{1,2}(\mu^{\otimes 3})$. Moreover:

$$\Lambda_1^3(G_{n,m}, [R_0(1)\psi] \circ \Phi_3) \;=\; -\int G_{n,m} \, d\Sigma_\psi,$$

and passing to the limit in $n \to \infty$ and $m \to \infty$, we obtain for all $\psi \in \mathrm{Lip}(H)$:

$$\begin{aligned} 0 \;&=\; \Lambda_1^3(\varphi \circ \Phi_3, [R_0(1)\psi] \circ \Phi_3) \;=\; -\int [\varphi \circ \Phi_3]^* \, d\Sigma_\psi \\ &=\; -\int_{K_0} [\varphi \circ \Phi_3]^* \,\cdot\, \psi \circ \Phi_3 \, d\mu^{\otimes 3}, \end{aligned}$$

which implies $\varphi \equiv 0$. $\square$

6 The Revuz-measure of η

The aim of this section is to characterize η_α as a family of Positive Continuous Additive Functionals of X_α and to prove the decomposition formula (8). Notice that Theorem 7 has the following important consequences: by the transfer method of Chapter VI in [MR 92], several statements of the theory of Dirichlet Form can rephrased from the classical locally-compact case into our setting. In particular, we can apply the results of Chapter 5 in [FOT 94]. We refer to [MR 92] and [FOT 94] for all basic definitions.

Let now $E := C([0,\infty); H)$ and define $X_t : E \mapsto H$, $t \geq 0$, $X_t(e) := e(t)$,

$$\mathcal{N}^0_\infty \ := \ \sigma\{X_s,\, s \in [0,\infty)\}, \quad \mathcal{N}^0_t \ := \ \sigma\{X_s,\, s \in [0,t]\}.$$

Fix $\alpha \geq 0$. For all $x \in K_\alpha$, we denote by $\mathbb{P}_x$ the law of $X_\alpha(\cdot, x)$ on $(E, \mathcal{N}^0_\infty)$, and for all probability measure λ on K_α, we define the probability measure $\mathbb{P}_\lambda$ on $(E, \mathcal{N}^0_\infty)$:

$$\mathcal{N}^0_\infty \ni \Lambda \mapsto \mathbb{P}_\lambda := \int_{K_\alpha} \mathbb{P}_x(\Lambda)\, \lambda(dx).$$

Then we denote by $\mathcal{N}^\lambda_\infty$ (resp. $\mathcal{N}^\lambda_t$) the completion of $\mathcal{N}^0_\infty$ (resp. completion of $\mathcal{N}^0_t$ in $\mathcal{N}^\lambda_\infty$) with respect to $\mathbb{P}_\lambda$. We also set $\mathcal{N}_\infty := \cap_{\lambda \in \mathcal{P}(K_\alpha)} \mathcal{N}^\lambda_\infty$, $\mathcal{N}_t := \cap_{\lambda \in \mathcal{P}(K_\alpha)} \mathcal{N}^\lambda_t$, where $\mathcal{P}(K_\alpha)$ denotes the set of probability measures on K_α. By an Additive Functional (**AF**) of X_α, we mean a family of functions $A(t) : E \mapsto \mathbb{R}^+$, $t \geq 0$, such that:

(A.1) $A(t)$ is $(\mathcal{N}_t)$-adapted

(A.2) There exist a set $\Lambda \in \mathcal{N}_\infty$ and a $\mathcal{E}^\alpha$-exceptional set $V \subset K_\alpha$, such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in K_\alpha \backslash V$, $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \geq 0$, and for all $\omega \in \Lambda$: $A(\cdot)(\omega)$ is continuous, $A(0)(\omega) = 0$ and for all $t, s \geq 0$:

$$A(t)(\omega) \ < \ \infty, \quad A(t+s)(\omega) = A(s)(\omega) + A(t)(\theta_t \omega). \qquad (43)$$

where $(\theta_s)_{s \geq 0}$ is the time-translation semigroup on $C([0,\infty); H)$. We say that an AF A is a Positive Continuous Additive Functional (**PCAF**) is A satisfies moreover:

(A.3) For all $\omega \in \Lambda$: $A(\cdot)(\omega)$ is non-decreasing.

Two AFs A_1 and A_2 are said to be equivalent if for each $t > 0$, $\mathbb{P}_x(A_1(t) = A_2(t)) = 1$, for $\mathcal{E}^\alpha$-q.e. x. Moreover, we say that A is a **PCAF in the strict sense** if one can choose $V = \emptyset$ in (A.1). Recall that X_α is Strong Feller and Corollary 4 implies the "absolute continuity condition" of [FOT 94]. This condition often allows to avoid the restriction: $x \in K_\alpha \backslash V$ of (A.2) above: see e.g. Theorems 5.1.6 and 5.1.7.

In the sequel, when it is necessary to stress the dependence of η_α on the initial datum x and the Brownian sheet W, we write η_α^x or $\eta_\alpha^{x,W}$. By the uniqueness statement of Theorem 1, we have a.s. for all $t \geq 0$:

$$\eta_\alpha^{x,W}([0,t+s],l) \;=\; \eta_\alpha^{x,W}([0,t],l) \;+\; \eta_\alpha^{X_\alpha(t,x),W^t}([0,s],l) \tag{44}$$

where $W^t := W(\,\cdot\, + t, \,\cdot\,) - W(t, \,\cdot\,)$ is a Brownian sheet, independent of $\mathcal{F}_t$. Notice that Formula (44) is reminiscent of (43). However, it is not clear whether η_α is a PCAF of X_α: in fact, η_α is adapted to the filtration of the noise W, but a priori not to the natural filtration of X_α.

Recall that, since X_α is conservative with unique invariant measure ν_α^F, the Revuz-measure of a AF $(A(t))_{t\geq 0}$ is defined as:

$$C_b(K_\alpha) \ni \varphi \mapsto \int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x))\, dA^x(t)\right] \nu_\alpha^F(dx),$$
$$\text{where}: \quad A^x(t) := A(t)\,(X_\alpha(\cdot,x))\,, \qquad t \geq 0.$$

In the proof of Theorem 10 below, we use several results of the Theory Additive Functionals. In the next theorem we collect the results we need, stating them in our setting in order to avoid notational confusion. For the general statements and the proofs, we refer to Theorem 2.4 in [MR 92] and Theorem 4.2 in [Fu 99].

Theorem 9 *Let $\alpha \geq 0$. For $\varphi \in W^{1,2}(\nu_\alpha^F)$, the next three conditions are equivalent:*

(i) For some constant $C > 0$ we have:

$$|\mathcal{E}^\alpha(\varphi,\psi)| \;\leq\; C\|\psi\|_\infty, \qquad \forall \psi \in W^{1,2}(\nu_\alpha^F) \cap L^\infty(\nu_\alpha^F). \tag{45}$$

(ii) There exists a finite signed measure m charging no $\mathcal{E}^\alpha$-exceptional set such that:

$$\mathcal{E}_1^\alpha(\varphi,\psi) \;=\; \int \psi\, dm, \qquad \forall \psi \in W^{1,2}(\nu_\alpha^F) \cap C_b(K_\alpha), \tag{46}$$

where $\mathcal{E}_1^\alpha := (\cdot,\cdot)_{L^2(\nu_\alpha^F)} + \mathcal{E}^\alpha$. *We say that* m *is a* $\mathcal{E}^\alpha$*-smooth measure with 1-potential* φ.

(iii) There exists a AF $(A(t))_t$ *of* X_α*, unique up to equivalence, such that* $\varphi(x) = \mathbb{E}_x[\int_0^\infty e^{-t} dA(t)]$ *for* $\mathcal{E}^\alpha$*-q.e.* x.

If (i)-(iii) hold, then m *in (ii) is the Revuz-measure of* A *in (iii).*

Moreover, we shall use that for all AF A with Revuz-measure m and for all φ bounded and Borel, we have that $(f \cdot A)(t) := \int_0^t f(X_t)\, dA(t)$, $t \geq 0$, is a AF with Revuz-measure $f \cdot dm$: see e.g. Lemma 5.1.3 in [FOT 94].

Theorem 10

1. *Let* $\alpha > 0$, $x \in K_\alpha \cap C_0$. *Almost surely, there exist a measurable random set* $S_\alpha \subseteq \mathbb{R}^+$ *with* $\eta_\alpha^x(\mathbb{R}^+ \backslash S_\alpha, (0,1)) = 0$*, and a measurable map* $r_\alpha : S_\alpha \mapsto (0,1)$*, such that:*

$$\forall t \in S_\alpha,\ u_\alpha(t, r_\alpha(t)) = -\alpha, \ \textit{and} \quad u_\alpha(t,\xi) > -\alpha \quad \forall \xi \in (0,1)\backslash\{r_\alpha(t)\}.$$

Almost surely, for all continuous l *with compact support in* $[0,\infty) \times (0,1)$*, we have:*

$$\int_{\mathcal{O}} l \, d\eta_\alpha^x \ = \ \int_0^\infty l(t, r_\alpha(t))\, \eta_\alpha^x(dt, (0,1)), \tag{47}$$

$$\textit{i.e.} \qquad \eta_\alpha^x(dt, d\xi) = \delta_{r_\alpha(t)}(d\xi)\, \eta_\alpha^x(dt,(0,1)) \quad \textit{on } \mathcal{O}.$$

Finally, $t \mapsto \eta_\alpha([0,t]\times(0,1))$ *is a PCAF in the strict sense of* X_α *with Revuz measure given by* $\frac{1}{2}\sigma_\alpha^F((0,1),\cdot)$*: i.e. there exists a PCAF in the strict sense of* X_α, $(A_{(0,1)}(t))_{t\geq 0}$*, such that*

$$\eta_\alpha^x([0,t]\times(0,1)) \ = \ A_{(0,1)}(t)(X_\alpha(\cdot,x)) \qquad \forall\ t \geq 0,\ x \in K_\alpha,$$

$$\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x))\, dA^x_{(0,1)}(t)\right] \nu_\alpha^F(dx) = \frac{1}{2}\int_0^1 dr \int \varphi(z)\, \sigma_\alpha^F(r,dz).$$

2. *Let* $\alpha = 0$, $x \in K_0 \cap C_0$. *Almost surely, there exist a measurable random set* $S_0 \subseteq \mathbb{R}^+$ *with* $\eta_0^x(\mathbb{R}^+ \backslash S_0, (0,1)) = 0$*, and a measurable map* $r_0 : S_0 \mapsto (0,1)$*, such that:*

$$\forall t \in S_0, \quad u_0(t, r_0(t)) = 0, \quad \textit{and} \quad u_0(t,\xi) > 0 \quad \forall \xi \in (0,1)\backslash\{r_0(t)\}.$$

Almost surely, for any $\delta \in (0,1/2)$ and for all continuous l with compact support in $\mathbb{R}^+ \times [\delta, 1-\delta]$, we have:

$$\begin{aligned} \int_{\mathcal{O}} l \, d\eta_0^x &= \int_0^\infty l(t, r_0(t)) \, \eta_0^x(dt, (0,1)) \qquad (48) \\ &:= \int_0^\infty l(t, r_0(t)) \, \eta_0^x(dt, [\delta, 1-\delta]), \end{aligned}$$

$$i.e. \qquad \eta_0^x(dt, d\xi) = \delta_{r_0(t)}(d\xi) \, \eta_0^x(dt, (0,1)) \quad on \ \mathcal{O}.$$

Finally, for all $\delta \in (0,1/2)$, there exists a PCAF in the strict sense $(A_{[\delta,1-\delta]}(t))_{t\geq 0}$ of X_0 with Revuz measure given by $\frac{1}{2}\sigma_0^F([\delta, 1-\delta], \cdot)$ such that:

$$\eta_0^x([0,t] \times [\delta, 1-\delta]) \;=\; A_{[\delta,1-\delta]}(t)(X_0(\cdot, x)) \qquad \forall \; t \geq 0, \;\; x \in K_0,$$

$$\int \mathbb{E}\left[\int_0^1 \varphi(X_0(t,x)) \, dA^x_{[\delta,1-\delta]}(t)\right] \nu_0^F(dx) \;=\; \frac{1}{2} \int_\delta^{1-\delta} dr \int \varphi \, \sigma_0^F(r, dz),$$

The family $(A_{[\delta,1-\delta]})_{\delta \in (0,1/2)}$ satisfies the consistency condition:

$$A_{[\delta',1-\delta']}(t) \;=\; \int_0^t 1_{[\delta',1-\delta']}(r_0(s)) \, dA_{[\delta,1-\delta]}(s), \qquad \forall \; 0 < \delta \leq \delta' < \frac{1}{2}.$$

Proof–We divide the proof into two steps.

Step 1. Let $\alpha \geq 0$ and $h \in C_c^2(0,1)$, $h \geq 0$. We claim that there exists a PCAF $(A_h(t))_{t\geq 0}$ of X_α with Revuz measure:

$$\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi(X_\alpha(t,x)) \, A_h^x(dt)\right] \nu_\alpha^F(dx) \;=\; \frac{1}{2} \int_0^1 dr \, h(r) \int \varphi \, \sigma_\alpha^F(r, dz)$$

and such that for $\mathbb{P}$-a.e. ω:

$$\int_0^1 h(\xi) \, \eta_\alpha^x([0,t], d\xi)(\omega) \;=: \; \eta_\alpha^x([0,t], h)(\omega) \;=\; A_h(t) \left(X_\alpha(\cdot, x)(\omega)\right).$$

In particular, $t \mapsto \eta_\alpha([0,t], h)$ is adapted to the filtration of X_α.

We can restrict to a dense countable family $\{h_n\} \subset D(A)$. We set for all $x \in H$:

$$U^{\varepsilon,\alpha}(x) := \mathbb{E}\left[\int_0^\infty e^{-t} \frac{1}{\varepsilon} \langle h, (X_\alpha^\varepsilon(t,x) + \alpha)^- \rangle dt\right],$$

$$U^\alpha(x) := \mathbb{E}\left[\int_0^\infty e^{-t}\eta_\alpha^x(dt,h)\right].$$

Then we have for all $\varphi \in \mathrm{Exp}_A(H)$:

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_H \varphi(x)\langle h, (x+\alpha)^-\rangle\, \mu_\alpha^\varepsilon(dx) \\
&= \int U^{\varepsilon,\alpha}\,\varphi\, d\mu_\alpha^\varepsilon \,+\, E^{\varepsilon,\alpha}(U^{\varepsilon,\alpha},\varphi) \,=\, \int U^{\varepsilon,\alpha}(\varphi \,-\, L_\alpha^\varepsilon\varphi)\, d\mu_\alpha^\varepsilon
\end{aligned}$$

For $\alpha > 0$, letting $\varepsilon \downarrow 0$, we find by Corollary 2:

$$\begin{aligned}
&\int U^\alpha(\varphi \,-\, L\varphi)\, d\nu_\alpha^F \,-\, \frac{1}{2}\int_0^1 dr \int \langle \nabla\varphi(z), \delta_r\rangle\, U^\alpha(z)\, \sigma_\alpha^F(r,dz) \\
&= \frac{1}{2}\int_0^1 dr\, h(r)\int \varphi(z)\, \sigma_\alpha^F(r,dz),
\end{aligned} \tag{49}$$

Notice that we have for all $\alpha \geq 0$:

$$\begin{aligned}
\int_0^\infty e^{-t}\eta_\alpha^x(dt,h) \;&=\; -\langle x,h\rangle \,+\, \int_0^\infty e^{-t}\langle X_\alpha(t,x), h - Ah\rangle\, dt \\
&\quad + \int_0^\infty e^{-t}\langle f(X_\alpha(t,x)), h\rangle\, dt - \int_0^\infty e^{-t} h(\xi)\, W(dt,d\xi),
\end{aligned}$$

$$U^\alpha(x) \,=\, -\langle x,h\rangle \,+\, \int_0^\infty e^{-t}\,\mathbb{E}\left[\langle X_\alpha(t,x), h - Ah\rangle \,+\, \langle f(X_\alpha(t,x)), h\rangle\right] dt.$$

Then $U^\alpha(x) \to U^0(x)$ as $\alpha \downarrow 0$ for all $x \in K_\alpha$, and by Proposition 2 $\{U^\alpha\}_{\alpha\geq 0}$ is an equi-Lipschitz family. Moreover, $0 \leq U^\alpha \circ \Pi_{K_\alpha} \leq C(1 + |\cdot|)$. Then Lemma 1 and Theorem 3 yield (49) for every $\alpha \geq 0$. Moreover, $U^\alpha \in \mathrm{Lip}(K_\alpha) \subseteq D(\mathcal{E}^\alpha)$. By (49), Corollary 5 and the density of $\mathrm{Exp}_A(K_\alpha)$ in $D(\mathcal{E}^\alpha)$, we obtain for all $\varphi \in D(\mathcal{E}^\alpha) \cap C_b(K_\alpha)$, $\alpha \geq 0$:

$$\mathcal{E}_1^\alpha(U^\alpha,\varphi) \,=\, \frac{1}{2}\int_0^1 h(r)\int \varphi(z)\,\sigma_\alpha^F(r,dz) \,=:\, \frac{1}{2}\int \varphi(z)\,\sigma_\alpha^F(h,dz),$$

$$|\mathcal{E}_1^\alpha(U^\alpha,\varphi)| \,\leq\, \sigma_\alpha^F(h,K_\alpha)\,\|\varphi\|_\infty, \quad \sigma_\alpha^F(h,K_\alpha) \,<\, \infty, \tag{50}$$

where $\mathcal{E}_1^\alpha := (\cdot,\cdot)_{L^2(\nu_\alpha^F)} + \mathcal{E}^\alpha$. If $\varphi \in D(\mathcal{E}^\alpha) \cap L^\infty(\nu_\alpha^F)$, we set $\varphi_n := P_\alpha(1/n)\varphi$. Since P_α is Strong Feller, letting $n \to \infty$, we obtain that (50) holds for all

$\varphi \in D(\mathcal{E}^\alpha) \cap L^\infty(\nu^F_\alpha)$. Now we can apply Theorem 9: U^α satisfies (i), $\sigma^F_\alpha(h,\cdot)$ is equal to the measure given by (ii), so that by (iii) there exists a PCAF $(A_h(t))_{t\geq 0}$ with Revuz measure $\sigma^F_\alpha(h,\cdot)$ and α-potential equal to U^α: in particular, we have $U^\alpha(x) = \mathbb{E}\left[\int_0^\infty e^{-t}\, dA^x_h(t)\right]$ for all $x \in K_\alpha \backslash V_\alpha$, for some $\mathcal{E}^\alpha$-properly exceptional set V_α.

Since U^α is continuous and therefore locally bounded on K_α, we can repeat the proof of Theorem 5.1.6 in [FOT 94], and extend $(A_h(t))_{t\geq 0}$ to a PCAF in the strict sense, which we still denote by $(A_h(t))_{t\geq 0}$. In particular, $U^\alpha(x) = \mathbb{E}\left[\int_0^\infty e^{-t}\, dA^x_h(t)\right]$ for all $x \in K_\alpha$. Now we can mimic the proof of Theorem 5.1.2 in [FOT 94]: by (44), we have for all $x \in K_\alpha$:

$$\mathbb{E}\left[\left(\int_0^\infty e^{-t}\, dA^x_h(t)\right)^2\right] = 2\,\mathbb{E}\left[\int_0^\infty e^{-2t} U^\alpha(X_\alpha(t,x))\, dA^x_h(t)\right]$$

$$= 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_t^\infty e^{-u}\eta^x_\alpha(du,h)\right)\, dA^x_h(t)\right],$$

$$\mathbb{E}\left[\left(\int_0^\infty e^{-t}\eta^x_\alpha(dt,h)\right)^2\right] = 2\,\mathbb{E}\left[\int_0^\infty e^{-2t} U^\alpha(X_\alpha(t,x))\, \eta^x_\alpha(dt,h)\right]$$

$$= 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_t^\infty e^{-u} dA^x_h(u)\right)\, \eta^x_\alpha(dt,h)\right]$$

$$= 2\,\mathbb{E}\left[\int_0^\infty e^{-t}\left(\int_0^t e^{-u}\eta^x_\alpha(du,h)\right)\, dA^x_h(t)\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-t}\eta^x_\alpha(dt,h)\int_0^\infty e^{-t}\, dA^x_h(t)\right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-t}\left(\left[\int_0^t + \int_t^\infty\right] e^{-u}\eta^x_\alpha(du,h)\right)\, dA^x_h(t)\right],$$

so that : $\mathbb{E}\left[\left(\int_0^\infty e^{-t}\eta^x_\alpha(dt,h) - \int_0^\infty e^{-t}\, dA^x_h(t)\right)^2\right] = 0$, and analogously :

$$\mathbb{E}\left[\left(\int_0^\infty e^{-\lambda t}\eta^x_\alpha(dt,h) - \int_0^\infty e^{-\lambda t}\, dA^x_h(t)\right)^2\right] = 0, \quad \forall \lambda > 0,$$

which implies $\eta^x_\alpha([0,t],h) = A^x_h(t)$ for all x, t, a.s.

Step 2. Let $\alpha > 0$, and $I \subseteq (0,1)$ be an interval. Denote by ψ_I the indicator function of the Borel set $\{x \in K_\alpha \cap C_0 : x \cdot 1_{(0,1)\setminus I} > -\alpha\}$. The key point is that the following holds:

$$\int_0^1 dr \int \varphi\, \psi_I(z)\, \sigma_\alpha^F(r,dz) \;=\; \int_I dr \int \varphi(z)\, \sigma_\alpha^F(r,dz), \qquad \forall \varphi \in C_b(K_\alpha). \tag{51}$$

Set now:

$$A_I^x(t) := \int_0^t \psi_I(X_\alpha(s,x))\, \eta_\alpha^x(ds,(0,1)), \quad t \geq 0,\ x \in H.$$

By Step 1, we have that A_I is a PCAF of X_α with Revuz measure equal to $\psi_I(z) \cdot \sigma_\alpha^F((0,1),dz)$. In particular, by (51):

$$\begin{aligned}
&\int_{K_\alpha} \mathbb{E}\left[\int_0^1 \varphi\psi_I(X_\alpha(s,x))\, \eta_\alpha^x(ds,(0,1))\right] \nu_\alpha(dx) \\
&= \frac{1}{2}\int_0^1 dr \int \varphi\, \psi_I(z)\, \sigma_\alpha^F(r,dz) \;=\; \frac{1}{2}\int_I dr \int \varphi(z)\, \sigma_\alpha^F(r,dz)
\end{aligned}$$

which is the Revuz measure of $t \mapsto \eta_\alpha^x([0,t],I)$. By (iii) in Theorem 9, A_I and $\eta_\alpha(\,\cdot\,,I)$ are equivalent, i.e. there exists a $\mathcal{E}^\alpha$-properly exceptional set V_α, such that for all $x \in K_\alpha \setminus V_\alpha$, and for every interval $I \subseteq (0,1)$ with rational extremes, we have

$$\eta_\alpha^x([0,T],I) \;=\; \int_0^T \psi_I(X_\alpha(s,x))\, \eta_\alpha^x(ds,(0,1)) \quad \forall T \geq 0,\ \text{a.s.}. \tag{52}$$

We claim that (52) holds for all $x \in K_\alpha \cap C_0$. First, $\eta_\alpha^z(\{0\},(0,1)) = 0$ for all $z \in K_\alpha$. Moreover, by Corollary 4, if $t > 0$ then $\mathbb{P}(X_\alpha(t,x) \in V_\alpha) = 0$. Then, for all $x \in K_\alpha \cap C_0$, $t > 0$, a.s. $X_\alpha(t,x) \in K_\alpha \setminus V_\alpha$ and by (52):

$$\begin{aligned}
\eta_\alpha^x([t,T],I) &= \eta_\alpha^{X_\alpha(t,x),W^t}([0,T-t],I) \\
&= \int_{]0,T-t]} \psi_I(X_\alpha(s+t,x))\, \eta_\alpha^{X_\alpha(t,x),W^t}(ds,(0,1)) \\
&= \int_t^T \psi_I(X_\alpha(s,x))\, \eta_\alpha^x(ds,(0,1)), \quad \forall T \geq t > 0,\ \text{a.s.}
\end{aligned}$$

and the claim is proved. Now, fix $x \in K_\alpha \cap C_0$ and consider a regular conditional distribution of η on $[0,\infty)\times(0,1)$, w.r.t. the Borel map $(t,\xi)\mapsto t$: i.e., a measurable kernel $(t,J)\mapsto\gamma(t,J)$, where $t\geq 0$, $J\subseteq(0,1)$ Borel, such that

$$\eta^x_\alpha([t,T],J) \;=\; \int_t^T \gamma(s,J)\,\eta^x_\alpha(ds,(0,1)), \tag{53}$$

for all $0\leq t\leq T$, $J\subseteq(0,1)$ Borel. By (52) and (53) we obtain that a.s. and for $\eta^x_\alpha(ds,(0,1))$-a.e. s:

$$\gamma(s,[a_n,b_n]) \;=\; \psi_{[a_n,b_n]}(X_\alpha(s,x)), \quad \forall a_n,b_n\in\mathbb{Q}\cap(0,1). \tag{54}$$

Notice that, since ψ_I is an indicator function, the right hand side of (54) assumes only the values 0 and 1. Therefore the measure $I\mapsto\gamma(s,I)$ takes only the values 0 and 1 on all intervals I with rational extremes in $(0,1)$, and the value 1 is assumed, since $\psi_{(0,1)} = 1_{K_\alpha\cap C_0}$. Then $\gamma(s,\cdot)$ is a Dirac mass at some point $r_\alpha(s)\in(0,1)$.

Consider $s\in S$, $q_n,p_n\in\mathbb{Q}$, $q_n\uparrow r_\alpha(s)$, $p_n\downarrow r(s)$, and set $I_n := [q_n,p_n]$: then $1=\gamma(s,I_n)=\psi_{I_n}(X_\alpha(s,x))$, which means $u_\alpha(s,\xi)>-\alpha$ for all $\xi\in(0,1)\backslash I_n$. Therefore, $r_\alpha(s)$ is the unique $\xi\in(0,1)$ such that $X_\alpha(s,x)(\xi)=u_\alpha(s,\xi)=-\alpha$.

Let now $\alpha=0$, and for all interval $I\subset(0,1)$ define ψ_I as the indicator function of the Borel set $\{x\in K_0\cap C_0 : x(\xi)>0,\ \forall\xi\in(0,1)\backslash I\}$. Notice that in this case it is not known whether $\eta^x_0([0,T],(0,1))$ is finite or not. However, $\eta^x_0([0,T],(\delta,1-\delta))<\infty$ for all $\delta>0$. Therefore, the proof can proceed as in the case of $\alpha>0$, provided one replaces $\eta^x_0(dt,(0,1))$ with $\eta^x_0(dt,(1/n,1-1/n))$ and then let $n\to\infty$. $\square$

To our knowledge, it is still unknown, whether $\eta_0([0,t]\times[0,1])$ is finite or infinite for $t>0$. In [NP 92] Nualart and Pardoux proved the estimate:

$$\int_0^T\int_0^1 \xi(1-\xi)\,\eta_0(dt,d\xi) \;<\; \infty, \qquad \forall T\geq 0.$$

Recall the definition (4) of σ_0, and in particular the factor $(r(1-r))^{-3/2}$, $r\in(0,1)$: Theorem 10 allows to improve the Nualart-Pardoux estimate, and obtain the following

Corollary 7 *For all Borel $\rho:(0,1)\mapsto\mathbb{R}^+$:*

$$\int_0^1 \frac{\rho(\xi)}{[\xi(1-\xi)]^{3/2}}\,d\xi \;<\; \infty \Longrightarrow \int_0^T\int_0^1 \rho(\xi)\,\eta_0(dt,d\xi) \;<\; \infty, \qquad \forall T\geq 0.$$

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