Integration by parts on the δ-d Bessel Bridge, $\delta > 3$, and related SPDEs

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Abstract

We study a white-noise driven semilinear partial differential equation on the spatial interval $[0,1]$ with Dirichlet boundary condition and with a singular drift of the form $c\,u^{-3}$, $c > 0$. We prove existence and uniqueness of a non-negative continuous adapted solution u on $[0,\infty) \times [0,1]$ for every non-negative continuous initial datum x, satisfying $x(0) = x(1) = 0$. We prove that the law π_δ of the Bessel Bridge on $[0,1]$ of dimension $\delta > 3$ is the unique invariant probability measure of the process $x \mapsto u$, with $c = (\delta-1)(\delta-3)/8$. An explicit integration by parts formula w.r.t. π_δ is given for all $\delta > 3$.

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1 Introduction

We are concerned with the following white-noise driven SPDE on the spatial interval $[0,1]$:

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\delta}{\partial \xi^2} + \dfrac{(\delta-1)(\delta-3)}{8\,(u_\delta)^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\ u_\delta(t,0) = u_\delta(t,1) = 0, \quad t \geq 0 \\ u_\delta(0,\xi) = x(\xi), \quad \xi \in [0,1] \end{cases} \tag{1}$$

where $x : [0,1] \mapsto [0,\infty)$ is continuous and satisfies $x(0) = x(1) = 0$, W is a Brownian sheet and $\delta > 3$.

In this paper we prove first that, for all $\delta > 3$, there exists a unique continuous non-negative solution u_δ of (1) on $[0,\infty)\times[0,1]$ such that $(u_\delta)^{-3} \in L^1_{loc}([0,\infty)\times(0,1))$, and that u_δ is adapted. Notice that the non-linearity in (1) is singular enough to make the standard techniques non-effective.

Secondly, we study the ergodicity of the solution of (1): we prove that the process $x \mapsto u_\delta$ is symmetric w.r.t. the law π_δ of the δ-dimensional Bessel Bridge on $[0,1]$ and that π_δ is the unique invariant probability measure of $x \mapsto u_\delta$.

One of the main tools is the following integration by parts formula w.r.t. the probability measure π_δ, $\delta > 3$:

$$\int_{K_0} \partial_h \varphi \, d\pi_\delta = -\int_{K_0} \varphi(x) \left(\langle x, h'' \rangle + \frac{(\delta-1)(\delta-3)}{4} \langle x^{-3}, h \rangle \right) \pi_\delta(dx). \qquad (2)$$

where $\varphi : L^2(0,1) \mapsto \mathbb{R}$ is Fréchet differentiable with bounded gradient, $h : [0,1] \mapsto \mathbb{R}$ is twice continuously differentiable with compact support in $(0,1)$ and h'' is the second derivative of h, $\partial_h\varphi$ is the directional derivative of φ along $h \in L^2(0,1)$ and $\langle \cdot,\cdot \rangle$ is the canonical scalar product in $L^2(0,1)$. This result allows to prove also that $x \mapsto u_\delta$ is a gradient system, i.e. it is the diffusion associated with the symmetric Dirichlet Form with state space $K_0 := \{x \in L^2(0,1),\ x \geq 0\}$:

$$W^{1,2}(\pi_\delta) \ni \varphi, \psi \ \mapsto \ \frac{1}{2} \int_{K_0} \langle \nabla\varphi, \nabla\psi \rangle \, d\pi_\delta$$

where ∇ denotes the gradient in the Hilbert space $H := L^2(0,1)$.

Finally, if $\delta \in \mathbb{N} \cap [4,\infty)$, we prove that the process $x \mapsto u_\delta$ is the radial part of the Gaussian process Z_δ, solution of the $\mathbb{R}^\delta$-valued linear SPDE:

$$\begin{cases} \dfrac{\partial Z_\delta}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 Z_\delta}{\partial \xi^2} + \dfrac{\partial^2 \overline{W}}{\partial t \partial \xi} \\[2ex] Z_\delta(t,\overline{x})(0) = Z_\delta(t,\overline{x})(1) = 0, \quad t \geq 0 \\[2ex] Z_\delta(0,\overline{x}) \ = \ \overline{x} \end{cases} \qquad (3)$$

where $\overline{x} \in L^2(0,1;\mathbb{R}^\delta)$, $\overline{W} := (W_1, W_2, \ldots, W_\delta) \mapsto \mathbb{R}^\delta$, and $\{W_i\}_{i=1,\ldots,\delta}$ are independent copies of W. By this, we mean the following: it is well known that

Z_δ is associated with the Dirichlet Form $(\Lambda^\delta, W^{1,2}(\mu^{\otimes\delta}))$ on $H^\delta = L^2(0,1;\mathbb{R}^\delta)$:

$$W^{1,2}(\mu^{\otimes\delta}) \ni F, G \mapsto \Lambda^\delta(F,G) := \frac{1}{2} \int_{H^\delta} \langle \overline{\nabla} F, \overline{\nabla} G \rangle_{H^\delta} \, d\mu^{\otimes\delta}$$

where μ is the law on $L^2(0,1)$ of a Brownian Bridge on $[0,1]$, $F, G : H^\delta \mapsto \mathbb{R}$ and $\overline{\nabla} F : H^\delta \mapsto H^\delta$ is the gradient of F in H^δ. We set:

$$\Phi_\delta : H^\delta \mapsto K_0, \quad \Phi_\delta(y)(\tau) := |y(\tau)|_{\mathbb{R}^\delta}, \quad \tau \in [0,1].$$

Then we prove that $\mathcal{D}^\delta$ is the image of Λ^δ under the map Φ_δ, i.e.

$$W^{1,2}(\pi_\delta) = \{\varphi \in L^2(\pi_\delta) : \varphi \circ \Phi_\delta \in W^{1,2}(\mu^{\otimes\delta})\},$$

$$\mathcal{D}^\delta(\varphi, \psi) = \Lambda^\delta(\varphi \circ \Phi_\delta, \psi \circ \Phi_\delta), \quad \forall \, \varphi, \psi \in W^{1,2}(\pi_\delta).$$

In [NP 92], Nualart and Pardoux proved existence and uniqueness of a pair (u_3, η), where u_3 is a continuous function of $(t,\xi) \in \mathcal{O} := [0,+\infty) \times [0,1]$ and η a measure on $\mathcal{O}$, solving the SPDE with reflection:

$$\begin{cases} \dfrac{\partial u_3}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u_3}{\partial \xi^2} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta \\[2ex] u_3(0,\cdot) = x, \ u_3(t,0) = u_3(t,1) = 0 \\[2ex] u_3 \geq 0, \ d\eta \geq 0, \ \int_{\mathcal{O}} u_3 \, d\eta = 0, \end{cases} \tag{4}$$

see Section 3 below. In [Za 00] and [Za 01], we proved that the process $x \mapsto u_3$ is symmetric w.r.t. the law π_3 of the 3-dimensional Bessel Bridge on $[0,1]$, π_3 is the unique invariant probability measure of $x \mapsto u_3$, and $x \mapsto u_3$ is the diffusion associated with the Dirichlet Form $(\mathcal{D}^3, W^{1,2}(\pi_3))$:

$$W^{1,2}(\pi_3) \ni \varphi, \psi \mapsto \mathcal{D}^3(\varphi, \psi) := \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\pi_3$$

where ∇ denotes the gradient in H. One of the key tools was the following integration by parts formula w.r.t. the probability measure π_3 on $L^2(0,1)$:

$$\int_{K_0} \partial_h \varphi \, d\pi_3 = - \int_{K_0} \varphi(x) \, \langle x, h'' \rangle \, d\pi_3 - \int_0^1 dr \ h(r) \int_{K_0} \varphi(x) \, \sigma_0(r, dx), \tag{5}$$

where the measure $\sigma_0(r, \cdot)$ is explicitly defined in terms of two independent 3-d Bessel Bridges, respectively on $[0, r]$ and on $[0, 1-r]$, glued at $r \in (0, 1)$: see (15) below. The last term of (5) was interpreted as a boundary term and applied to characterize η as a family of additive functionals of u_3. Finally, we proved that $x \mapsto u_\delta$ is the radial part of the Gaussian process Z_3, solution of the $\mathbb{R}^3$-valued SPDE (3) above with $\delta = 3$.

Mueller in [Mu 98] and Mueller and Pardoux in [MP 99] considered the following SPDE with periodic boundary condition:

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial \hat{u}}{\partial t} = \dfrac{\partial^2 \hat{u}}{\partial \xi^2} + \hat{u}^{-\alpha} + g(\hat{u})\,\dfrac{\partial^2 W}{\partial t \partial \xi}, \quad t \geq 0,\ \xi \in \mathbb{S}^1 := \mathbb{R}/\mathbb{Z} \\ \\ \hat{u}(0, \cdot) = \hat{x} \end{cases}$$

where $\alpha > 0$, $\hat{x} : \mathbb{S}^1 \mapsto \mathbb{R}$ is continuous, $\inf \hat{x} > 0$ and g satisfies suitable growth conditions, and proved that $\alpha = 3$ is the critical exponent for $\hat{u}$ to hit zero in finite time. More precisely, the following was proved:

1. If $\alpha > 3$, then a.s. $\hat{u}(t, \xi) > 0$ for all $t \geq 0$, $\xi \in \mathbb{S}^1$.
2. If $\alpha < 3$, then with positive probability, there exist $t > 0$, $\xi \in \mathbb{S}^1$, such that $\hat{u}(t, \xi) = 0$.

The results presented above allow to prove that for all continuous $x : [0, 1] \mapsto [0, \infty)$ with $x(0) = x(1) = 0$, for all $\alpha \geq 3$ and $C > 0$ the following SPDE admits a unique continuous non-negative adapted solution $\hat{u}_\alpha$, being well-defined for all $t \geq 0$:

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \xi^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\ \\ \hat{u}_\alpha(t, 0) = \hat{u}_\alpha(t, 1) = 0, \quad t \geq 0 \\ \\ \hat{u}_\alpha(0, \cdot) = x, \end{cases} \tag{6}$$

while for all $0 \leq \alpha < 3$ and $C \geq 0$ the following SPDE of Nualart-Pardoux type admits a unique solution $(\hat{u}_\alpha, \hat{\eta}_\alpha)$:

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \xi^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \hat{\eta}_\alpha \\ \\ \hat{u}_\alpha(0,\cdot) = x, \ \hat{u}_\alpha(t,0) = \hat{u}_\alpha(t,1) = 0, \quad t \geq 0 \\ \\ \hat{u}_\alpha \geq 0, \ d\hat{\eta}_\alpha \geq 0, \ \int_{\mathcal{O}} \hat{u}_\alpha \, d\hat{\eta}_\alpha = 0 \end{cases} \tag{7}$$

and $\hat{\eta}_\alpha \neq 0$. Notice that the nonlinearity $f(y) = C\,y^{-\alpha}$, unbounded in a neighbourhood of 0, makes the equation with Dirichlet boundary condition more delicate to handle, then the one with periodic condition.

The family $(u_\delta)_{\delta \geq 3}$, defined by (1) and (4), reveal several analogies with the family of Bessel Processes $(\rho_\delta)_{\delta \geq 1}$. Indeed, recall that:

1. If $(B_t)_{t\geq 0}$ is a linear BM and $x \geq 0$, then for all $\delta > 1$ there exists a unique solution $(\rho_\delta(t,x))_{t\geq 0}$ of the SDE:

$$d\rho_\delta = \frac{\delta - 1}{2\rho_\delta}\, dt + dB, \quad t \geq 0, \quad \rho_\delta(0,x) = x, \tag{8}$$

   and, for $\delta = 1$, there exists a unique pair (ρ_1, L), with $t \mapsto \rho_1(t,x)$ continuous and positive and $t \mapsto L(t,x)$ continuous and monotone non-decreasing, satisfying:

$$d\rho_1 = dL + dB, \quad t \geq 0, \quad \rho_1(0,x) = x, \ L(0,x) = 0. \tag{9}$$

   For all $\delta \geq 1$, $\rho_\delta = (\rho_\delta(t,x))_{t\geq 0, x\geq 0}$, is called the δ-dimensional Bessel process.

2. The process ρ_δ is the diffusion associated with the Dirichlet Form:

$$W^{1,2}([0,\infty), x^{\delta-1}\, dx) \ni f, g \mapsto \gamma^\delta(f,g) := \frac{\omega_\delta}{2}\int_0^\infty f'(x)\, g'(x)\, x^{\delta-1}\, dx,$$

   where $\omega_\delta := \pi^{\delta/2}/\Gamma(1+\delta/2)$.

3. If $\delta \in \mathbb{N} \cap [1,\infty)$ and $(B_\delta(t))_{t\geq 0}$ is a δ-dimensional Brownian Motion, then ρ_δ is characterized as the radial part of B_δ, i.e. the Dirichlet form γ^δ, generating ρ_δ, is the image of

$$W^{1,2}(\mathbb{R}^\delta) \ni F, G \mapsto \frac{1}{2}\int_{\mathbb{R}^\delta} \langle \nabla F, \nabla G \rangle \, dx$$

under the map $\mathbb{R}^\delta \ni y \mapsto |y| \in [0,\infty)$. Notice that, in this case, it is even true that the law of ρ_δ is equal to the law of $|B_\delta|$.

4. For all $\alpha \geq 1$ and $c > 0$ there exists a unique continuous non-negative solution ρ of the following SDE

$$d\rho = \frac{c}{(\rho)^\alpha}\, dt + dB, \quad t \geq 0, \quad \rho(0) \geq 0, \tag{10}$$

while for all $0 < \alpha < 1$ and $c \geq 0$ there exists a unique pair (ρ, L) solving the Skorokhod problem:

$$d\rho = \frac{c}{(\rho)^\alpha}\, dt + dL + dB, \quad t \geq 0, \quad \rho(0,x) = x, \; L(0,x) = 0, \tag{11}$$

and $L \neq 0$.

5. The following integration by parts formulae hold for the invariant measure $1_{[0,\infty)}(x)x^{\delta-1}dx$ of ρ_δ:

$$\int_0^\infty f'(x)\, x^{\delta-1}\, dx = -\int_0^\infty f(x)\, \frac{\delta-1}{x}\, x^{\delta-1}\, dx, \quad \delta > 1, \tag{12}$$

$$\int_0^\infty f'(x)\, dx = -f(0), \qquad \forall f \in C_0^\infty([0,\infty)). \tag{13}$$

In particular, in the critical case $\delta = 1$ a boundary term appears, while for $\delta > 1$ only a logarithmic-derivative term appears.

Notice that the exponent in the non-linear term of (8) is equal to -1, i.e. to minus the critical dimension for (10)-(11) and (12)-(13): the same happens for the exponent in the non-linear term of (1), which is equal to -3, i.e. to minus the critical dimension for (6)-(7) and (2)-(5). Moreover, the maps

$$(1,\infty) \ni \delta \mapsto \frac{\delta-1}{2} \in (0,\infty), \quad (3,\infty) \ni \delta \mapsto \frac{(\delta-1)(\delta-3)}{8} \in (0,\infty)$$

are both increasing and bijective.

The paper is organized as follows. In section 2 we prove the integration by parts formula (2). Section 3 is devoted to the study of equation (1). In section 4, we study equations (6) and (7).

We fix some notations: we set $H := L^2(0,1)$ and we denote the canonical scalar product in H by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. We set $K_0 := \{x \in H,\ x \geq 0\}$, $\mathcal{O} := [0, +\infty) \times [0,1]$ and

$$C_0 := C_0(0,1) := \{c : [0,1] \mapsto \mathbb{R} \ \text{ continuous}, \ c(0) = c(1) = 0\},$$

$$A : D(A) \subset H \mapsto H, \quad D(A) := W^{2,2} \cap W_0^{1,2}(0,1), \quad A := \frac{1}{2} \frac{\partial^2}{\partial \xi^2}.$$

By $W = \{W(t,\xi) : (t,\xi) \in \mathcal{O}\}$ we denote a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. W is a Gaussian process with zero mean and covariance function

$$\mathbb{E}\left[W(t,\xi) W(t',\xi')\right] = (t \wedge t')(\xi \wedge \xi'), \qquad (t,\xi), (t',\xi') \in \mathcal{O}.$$

We denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,\xi) : (s,\xi) \in [0,t] \times [0,1]\}$. We also consider a linear Brownian Motion $(B_\tau)_{\tau \in [0,1]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of W, and a $\mathbb{R}^\delta$-valued BM $(B_\delta(t))_{t \geq 0}$, $\mathbb{N} \ni \delta \geq 2$. Then we set:

- π_δ, $\delta > 1$, is the law on $L^2(0,1)$ of the δ-d Bessel Bridge between 0 and 0 on $[0,1]$, i.e. the law of the unique non-negative process $(x_\delta(\tau))_{\tau \in [0,1]}$ solving the SDE:

$$d\, x_\delta = \frac{\delta - 1}{2 x_\delta}\, d\tau \, - \, \frac{x_\delta}{1 - \tau}\, d\tau \, + \, dB, \quad \tau \in [0,1[, \quad x_\delta(0) = 0. \qquad (14)$$

  If $\delta \in \mathbb{N}$, $\delta \geq 2$, then π_δ is equal to the law of $(|B_\delta(\tau)|)_{\tau \in [0,1]}$, conditioned on $\{B_\delta(1) = 0\}$.

- π_3^r, $r \in (0,1)$, is the law on $L^2(0,r)$ of the 3-d Bessel Bridge between 0 and 0 on $[0,r]$, i.e. the law of the unique non-negative process $(x_3^r(\tau))_{\tau \in [0,1]}$ solving the SDE:

$$d\, x_3^r \, = \, \frac{d\tau}{x_3^r} \, - \, \frac{x_3^r}{r - \tau}\, d\tau \, + \, dB, \quad \tau \in [0,r[, \quad x_3^r(0) = 0.$$

  Moreover, π_3^r is the law of $(|B_3(\tau)|)_{\tau \in [0,r]}$, conditioned on $\{B_3(r) = 0\}$.

- Let $r \in (0,1)$. For $y \in C([0,r])$ and $z \in C([0,1-r])$ we set:

$$y \oplus_r z \in H, \quad [y \oplus_r z]\,(\tau) := y(r)\, 1_{[0,r]} + z(\tau - r)\, 1_{(r,1]}.$$

Then we define for all $\varphi \in C_b(H)$:

$$\int \varphi(x)\, \sigma_0(r, dx) \;:=\; \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\, \mathbb{E}\left[\varphi\left(x_3^r \oplus_r \hat{x}_3^{1-r}\right)\right] \tag{15}$$

where x_3^r and $\hat{x}_3^r$, $r \in (0,1)$, are independent and identically distributed.

We introduce the following function spaces:

- $C_b(H)$ is the space of all $\varphi : H \mapsto \mathbb{R}$ being bounded and uniformly continuous in the norm of H. $C_b^1(H)$ is the space of Fréchet differentiable $\varphi \in C_b(H)$ with bounded and continuous gradient $\nabla\varphi : H \mapsto H$.
- $\mathrm{Exp}_A(H)$ is the linear span of $\{1, \cos(\langle\cdot, h\rangle), \sin(\langle\cdot, h\rangle) : h \in D(A)\}$.
- $\mathrm{Lip}(H)$ is the space of all $\varphi : H \mapsto \mathbb{R}$ such that

$$\|\varphi\|_{\mathrm{Lip}} \;:=\; \sup_x |\varphi(x)| \;+\; \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \;<\; \infty.$$

  $\mathrm{Lip}(K_0)$ is the set of $\varphi : K_0 \mapsto \mathbb{R}$ such that $H \ni x \mapsto \varphi(x^+)$ is in $\mathrm{Lip}(H)$, where $x^+(\tau) := \sup\{x(\tau), 0\}$, $\tau \in [0,1]$.
- $C_b^1(K_0)$ is the set of all $\varphi \in \mathrm{Lip}(K_0)$ such that there exists a bounded continuous vector field $\nabla\varphi : H \mapsto H$, which we call the gradient of φ, satisfying:

$$\lim_{t \downarrow 0} \frac{1}{t}(\varphi(x \,+\, t\, h) - \varphi(x)) \;=\; \langle \nabla\varphi(x), h\rangle, \qquad \forall\; h \in K_0.$$

2 IbPF on the δ-d Bessel Bridge

The aim of this section is to prove the following

Theorem 1 *For all $\delta > 3$, $\varphi \in C_b^1(H)$ and $h \in D(A)$, we have*

$$\int_{K_0} \partial_h \varphi \, d\pi_\delta = -\int_{K_0} \varphi(x) \left(\langle x, h''\rangle + \frac{(\delta-1)(\delta-3)}{4} \langle x^{-3}, h\rangle \right) \pi_\delta(dx). \tag{16}$$

We recall the following Theorem, proved in [Za 00]:

Theorem 2 *For all $\varphi \in C_b^1(K_0)$ and $h \in D(A)$, we have*

$$\int_{K_0} \partial_h \varphi \, d\pi_3 \;=\; -\int_{K_0} \varphi(x)\,\langle x, h''\rangle\, d\pi_3 \;-\; \int_0^1 dr\;h(r) \int \varphi(x)\,\sigma_0(r,dx). \quad (17)$$

We define:

$$c(\delta) \;:=\; \frac{\delta-3}{2}, \quad \kappa(\delta) \;:=\; \frac{(\delta-1)(\delta-3)}{4} \qquad \delta > 3.$$

Proof of Theorem 1–We fix $\delta > 3$. The main idea of the proof is the following: regularizing the SDE (14) which defines x_δ, we obtain a monotone family of processes $(z^\epsilon)_{\epsilon>0}$ on $[0,1]$, converging uniformly to x_δ as $\epsilon \downarrow 0$, such that for all $\epsilon > 0$ the law of z^ϵ is absolutely continuous w.r.t. π_3, with an explicit density. Using (17), we write an integration by parts formula w.r.t. the law of z^ϵ: then we prove that, letting $\epsilon \downarrow 0$, the boundary term, i.e. the integral w.r.t. σ_0, converges to 0, and the other terms give (16). Several steps of this proof will be used also during the proof of Theorem 3 below.

We divide the proof of (16) into several steps. We can assume without loss of generality that h has compact support in $(0,1)$ and:

$$\varphi \;\geq\; 0, \qquad h \;\geq\; 0. \quad (18)$$

Step 1. Let $Q_\delta := (x_\delta)^2$. By Itô's formula Q_δ satisfies:

$$\begin{cases} d\,Q_\delta = \delta\, d\tau \;-\; \dfrac{Q_\delta}{1-\tau}\, d\tau \;+\; 2\,\sqrt{Q_\delta}\, dB, \quad \tau \in [0,1[\\ \\ Q_\delta(0) = 0 \end{cases} \quad (19)$$

We set the following SDE on $[0,1[$ for $\epsilon > 0$:

$$\begin{cases} d\,q^\epsilon = \left(3 \;+\; 2c(\delta)\dfrac{\sqrt{|q^\epsilon|}}{\epsilon + \sqrt{|q^\epsilon|}} \;-\; \dfrac{q^\epsilon}{1-\tau}\right) d\tau \;+\; 2\,\sqrt{|q^\epsilon|}\, dB_\tau, \\ \\ q^\epsilon(0) = 0 \end{cases} \quad (20)$$

Equation (20) is a one-dimensional SDE with $\frac{1}{2}$-Hölder-continuous diffusion coefficient and bounded drift on every interval $[0,a[$, $a \in [0,1[$. Then by

Theorem IX.3.5 of [RY 91] pathwise existence and uniqueness holds for (20). Moreover, by comparison with (19) we obtain by Theorem IX.3.7 of [RY 91] that $Q_\delta \geq q^\epsilon \geq Q_3$. In particular $q^\epsilon(\tau) > 0$ for all $\tau \in]0,1[$ and we can apply Itô's formula to $z^\epsilon := \sqrt{q^\epsilon}$, obtaining:

$$\begin{cases} d\, z^\epsilon = \dfrac{1}{z^\epsilon}\, d\tau \,+\, \dfrac{c(\delta)}{\epsilon + z^\epsilon}\, d\tau \,-\, \dfrac{z^\epsilon}{1-\tau}\, d\tau \,+\, dB, \quad \tau \in [0,1[\\ \\ z^\epsilon(0) = 0 \end{cases} \tag{21}$$

Step 2. We list a few properties of $(z^\epsilon)_{\epsilon>0}$:

A. $]0,\infty[\ni \epsilon \mapsto z^\epsilon$ is monotone non-increasing.

B. $]0,\infty[\ni \epsilon \mapsto \epsilon + z^\epsilon$ is monotone non-decreasing.

C. $z^\epsilon \uparrow x_\delta$ and $\epsilon + z^\epsilon \downarrow x_\delta$, uniformly on $[0,1]$, as $\epsilon \downarrow 0$.

Proof

A. Consider $\epsilon_1 \geq \epsilon_2 > 0$, and set $b := (z^{\epsilon_1} - z^{\epsilon_2})^+$: then

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}(b)^2 \;&=\; (z^{\epsilon_1} - z^{\epsilon_2})^+ \left[\left(\frac{1}{z^{\epsilon_1}} - \frac{1}{z^{\epsilon_2}} \right) + \left(\frac{c(\delta)}{\epsilon_1 + z^{\epsilon_1}} - \frac{c(\delta)}{\epsilon_1 + z^{\epsilon_2}} \right) \right] \\ &\quad + b\, c(\delta) \left(\frac{1}{\epsilon_1 + z^{\epsilon_2}} - \frac{1}{\epsilon_2 + z^{\epsilon_2}} \right) - \frac{b^2}{1-\tau} \;\leq\; 0 \end{aligned}$$

which implies $z^{\epsilon_1} \leq z^{\epsilon_2}$.

B. Consider $\epsilon_1 \geq \epsilon_2 > 0$, and set $w^\epsilon := \epsilon + z^\epsilon$, $b := (w^{\epsilon_2} - w^{\epsilon_1})^+$. Then by (A.) and $c(\delta) \geq 0$:

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}(b)^2 \;&=\; (w^{\epsilon_2} - w^{\epsilon_1})^+ \left[\left(\frac{1}{z^{\epsilon_2}} - \frac{1}{z^{\epsilon_1}} \right) \,+\, c(\delta) \left(\frac{1}{w^{\epsilon_2}} - \frac{1}{w^{\epsilon_1}} \right) \right] \\ &\quad - b\, \frac{z^{\epsilon_2} - z^{\epsilon_1}}{1-\tau} \;\leq\; 0 \end{aligned}$$

which implies $w^{\epsilon_1} \geq w^{\epsilon_2}$.

C. By (A.), $]0,\infty[\ni \epsilon \mapsto q^\epsilon = (z^\epsilon)^2$ is monotone non-increasing. By Dominated Convergence Theorem, we find that $q := \lim_{\epsilon\downarrow 0} q^\epsilon$ satisfies (19). Since pathwise uniqueness holds for (19), we find $q = Q_\delta$ and therefore $x_\delta = \lim_{\epsilon\downarrow 0} z^\epsilon$. We conclude by (A.) and (B.).

Step 3. We prove now that for all $\psi \in C_b(H)$:

$$\mathbb{E}[\psi(z^\epsilon)] = \mathbb{E}[\psi(x_3)\,\Gamma^\epsilon], \qquad \text{where :} \tag{22}$$

$$\Gamma^\epsilon := \exp\left(\int_0^1 \frac{c(\delta)}{\epsilon + x_3}\, dB - \frac{1}{2}\int_0^1 \left[\frac{c(\delta)}{\epsilon + x_3}\right]^2 d\tau\right).$$

First notice that $0 \le (\epsilon + x_3)^{-1} \le \epsilon^{-1}$. Then $\mathbb{E}[\Gamma^\epsilon] = 1$, and the right hand side of (22) defines a probability measure on H. If we set:

$$\hat{B}(\tau) := B(\tau) - \int_0^\tau \frac{c(\delta)}{\epsilon + x_3}\, ds, \quad \tau \in [0,1]$$

then by the Girsanov Theorem $\hat{B}$ is a Brownian Motion under $\Gamma^\epsilon d\mathbb{P}$. Therefore x_3 is, under $\Gamma^\epsilon d\mathbb{P}$, a weak solution to (21). By Itô's formula, $Q_3 = (x_3)^2$ is, under $\Gamma^\epsilon d\mathbb{P}$, a weak solution to (19). Since pathwise uniqueness holds for (19), by Yamada-Watanabe's Theorem uniqueness in law holds for (19), i.e. the law of Q_3 under $\Gamma^\epsilon d\mathbb{P}$ is equal to the law of q_ϵ under $\mathbb{P}$: see [YW 71]. Since $C([0,1];[0,\infty)) \ni x \mapsto \sqrt{x} \in C([0,1];[0,\infty))$ is continuous, (22) is proven.

Step 4. We can write Γ^ϵ as a function of x_3 only. Indeed, by (14) with $\delta = 3$:

$$dB = dx_3 - \frac{1}{x_3}\, d\tau + \frac{x_3}{1-\tau}\, d\tau,$$

$$d\,\log(\epsilon + x_3) = \frac{dx_3}{\epsilon + x_3} - \frac{1}{2}\frac{d\tau}{(\epsilon + x_3)^2}, \quad \log(\epsilon + x_3(0)) = \log(\epsilon + x_3(1)),$$

$$\int_0^1 \frac{dB}{\epsilon + x_3} = \int_0^1 \left(\frac{1}{2(\epsilon + x_3)^2} - \frac{1}{x_3(\epsilon + x_3)} + \frac{1}{1-\tau}\frac{x_3}{\epsilon + x_3}\right) d\tau.$$

We obtain that

$$\begin{aligned}\Gamma^\epsilon &= \gamma^\epsilon(x_3) \\ &:= \exp\left(c(\delta)\int_0^1 \left(\frac{1-c(\delta)}{2(\epsilon + x_3)^2} - \frac{1}{x_3(\epsilon + x_3)} + \frac{1}{1-\tau}\frac{x_3}{\epsilon + x_3}\right) d\tau\right),\end{aligned}$$

where $\gamma^\epsilon : K_0 \mapsto \mathbb{R}$ is in $L^1(\pi_3)$ and (22) becomes for all $\psi \in C_b(H)$:

$$\mathbb{E}[\psi(z^\epsilon)] \;=\; \mathbb{E}[\psi(x_3)\gamma^\epsilon(x_3)] \;=\; \int \psi(x)\gamma^\epsilon(x)\,\pi_3(dx).$$

Notice that γ^ϵ is not in $C_b^1(K_0)$. If we set for all $\rho > 0$:

$$K_0 \ni x \mapsto \gamma_\rho^\epsilon(x) := \tag{23}$$

$$\exp\left(c(\delta) \int_0^1 \left(\frac{1-c(\delta)}{2(\epsilon+x)^2} - \frac{1}{(\rho+x)(\epsilon+x)} + \frac{1}{1+\rho-\tau}\frac{x}{\epsilon+x} \right) d\tau \right).$$

then γ_ρ^ϵ is in $C_b^1(K_0)$ and for all $x \in K_0$:

$$\langle \nabla \log \gamma_\rho^\epsilon(x), h\rangle \;:=\; \lim_{t\downarrow 0} \frac{1}{t}\left(\log\gamma_\rho^\epsilon(x+t\,h) - \log\gamma_\rho^\epsilon(x)\right)$$

$$= \int_0^1 c(\delta)\left[\frac{c(\delta)-1}{(\epsilon+x)^3} + \frac{1}{(\rho+x)^2(\epsilon+x)} + \frac{1}{(\rho+x)(\epsilon+x)^2} \right.$$

$$\left. + \frac{\epsilon}{(1+\rho-\tau)(\epsilon+x)^2}\right] h(\tau)\,d\tau.$$

By (17) in Theorem 2, we obtain:

$$\begin{aligned} \int_{K_0} \partial_h\varphi\,\gamma_\rho^\epsilon\,d\pi_3 \;\;&=\;\; -\int_{K_0} \varphi(x)\left[\langle x, h''\rangle \;+\; \langle \nabla\log\gamma_\rho^\epsilon, h\rangle\right]\gamma_\rho^\epsilon\,\pi_3(dx) \\ &\qquad - \int_0^1 dr\;h(r)\int \varphi(x)\,\gamma_\rho^\epsilon(x)\,\sigma_0(r,dx). \end{aligned} \tag{24}$$

Step 5. We want to let $\rho\downarrow 0$ in (24). Notice that:

$$\gamma_\rho^\epsilon(x) \;\le\; \gamma^\epsilon(x)\,\exp\left(\frac{1}{\epsilon}\int_0^1 \frac{d\tau}{x}\right) \qquad \text{for } \pi_3-\text{a.e. } x, \tag{25}$$

$$\int \langle h, x^{-2}\rangle\,\gamma^\epsilon(x)\,\exp\left(\frac{1}{\epsilon}\int_0^1\frac{d\tau}{x}\right)\,\pi_3(dx)$$

$$= \mathbb{E}\left[\langle h, (z^\epsilon)^{-2}\rangle\,\exp\left(\frac{1}{\epsilon}\int_0^1\frac{d\tau}{z^\epsilon}\right)\right] \;\le\; \mathbb{E}\left[\langle h, (x_3)^{-2}\rangle\,\exp\left(\frac{1}{\epsilon}\int_0^1\frac{d\tau}{x_3}\right)\right]$$

$$\le\; \mathbb{E}\left[\langle h, (x_3)^{-2p}\rangle\right]^{1/p}\mathbb{E}\left[\exp\left(\frac{q}{\epsilon}\int_0^1\frac{d\tau}{x_3}\right)\right]^{1/q}$$

for every $p, q > 1$ with $(1/p) + (1/q) = 1$. Recall that by (14):

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{q}{\epsilon}\int_0^1 \frac{d\tau}{x_3}\right)\right] &\leq \mathbb{E}\left[\exp\left(\frac{2q}{\epsilon}\int_0^{1/2} \frac{d\tau}{x_3}\right)\right] \\
&\leq \mathbb{E}\left[\exp\left(\frac{2q}{\epsilon}\left(x_3(1/2) + \int_0^{1/2} \frac{x_3}{1-\tau}d\tau - B(1/2)\right)\right)\right] < \infty
\end{aligned} \tag{26}$$

$$\mathbb{E}\left[\langle h, (x_3)^{-2p}\rangle\right] = \int_0^1 d\tau\, h(\tau) \int_0^\infty dy \, \frac{C}{\sqrt{\tau^3(1-\tau)^3}}\, y^{2(1-p)} e^{-y^2/2\tau(1-\tau)} < \infty$$

for $p < 2$, since h has compact support in $(0,1)$. Then, by Dominated Convergence Theorem, we can let $\rho \downarrow 0$ in (24) and obtain:

$$\begin{aligned}
\int_{K_0} \partial_h \varphi\, \gamma^\epsilon\, d\pi_3 &= -\int_{K_0} \varphi(x) \left[\langle x, h''\rangle + \langle \nabla \log \gamma^\epsilon, h\rangle\right] \gamma^\epsilon(x)\, \pi_3(dx) \\
&\quad - \int_0^1 dr\ h(r) \int \varphi(x)\, \gamma^\epsilon(x)\, \sigma_0(r, dx),
\end{aligned} \tag{27}$$

where for π_3-a.e. x we set:

$$\begin{aligned}
\langle h, \nabla\ \log \gamma^\epsilon(x)\rangle &:= \int_0^1 h(\tau)\, c(\delta) \left[\frac{c(\delta)-1}{(\epsilon+x)^3} + \frac{1}{x^2(\epsilon+x)}\right. \\
&\qquad \left. + \frac{1}{x(\epsilon+x)^2} + \frac{\epsilon}{(1-\tau)(\epsilon+x)^2}\right](\tau)\, d\tau,
\end{aligned}$$

and $\langle h, \nabla \log \gamma^\epsilon\rangle \in L^1(\pi_3)$, since h has compact support in $(0,1)$.

Step 6. Recall that $\gamma^\epsilon\, d\pi_3$ is the law of z^ϵ, so that:

$$\int_{K_0} \varphi(x)\, \langle \nabla \log \gamma^\epsilon(x), h\rangle \gamma^\epsilon(x)\, \pi_3(dx) = \mathbb{E}[\varphi(z^\epsilon)\, \langle \nabla \log \gamma^\epsilon(z^\epsilon), h\rangle].$$

Now we prove that:

$$\lim_{\epsilon \downarrow 0} \mathbb{E}[\varphi(z^\epsilon)\, \langle \nabla \log \gamma^\epsilon(z^\epsilon), h\rangle] = \kappa(\delta)\ \mathbb{E}\left[\varphi(x_\delta)\, \langle h, \frac{1}{(x_\delta)^3}\rangle\right]. \tag{28}$$

We set:

$$\zeta^\epsilon \;:=\; \frac{1}{(z^\epsilon)^2(\epsilon+z^\epsilon)} \;+\; \frac{1}{z^\epsilon(\epsilon+z^\epsilon)^2} \;\geq\; \frac{2}{(\epsilon+z^\epsilon)^3}. \tag{29}$$

Then (28) is implied by (30)-(31)-(32):

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\varphi(z^\epsilon)\ \langle h, \zeta^\epsilon\rangle\right] \;=\; \mathbb{E}\left[\varphi(x_\delta)\langle h, \frac{2}{(x_\delta)^3}\rangle\right] \tag{30}$$

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\varphi(z^\epsilon)\ \langle h, \frac{c(\delta)-1}{(\epsilon+z^\epsilon)^3}\rangle\right] \;=\; \mathbb{E}\left[\varphi(x_\delta)\langle h, \frac{c(\delta)-1}{(x_\delta)^3}\rangle\right] \tag{31}$$

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\varphi(z^\epsilon)\ \langle h, \frac{\epsilon}{(1-\tau)(\epsilon+z^\epsilon)^2}\rangle\right] \;=\; 0. \tag{32}$$

First, (31) and (32) follow by (C.) in step 2 and the Dominated Convergence Theorem: indeed $\epsilon + z^\epsilon \geq x_\delta$, and since $\delta > 3$:

$$\int \langle h, x^{-3}\rangle\, \pi_\delta(dx) \;=\; \int_0^1 d\tau\, h(\tau) \int x^{-3}(\tau)\, \pi_\delta(dx)$$

$$=\; \int_0^1 d\tau\, h(\tau) \int_0^\infty dy\, \frac{1}{y^3}\, \frac{C_\delta\, y^{\delta-1}}{[\tau(1-\tau)]^{\delta/2}} \exp\left\{-\frac{y^2}{2\tau(1-\tau)}\right\} \;<\; \infty,$$

since h has compact support in $(0,1)$. On the other hand, (30) is natural but not immediate: indeed, the map

$$]0,\infty[\, \ni \epsilon \mapsto \frac{1}{(z^\epsilon)^2(\epsilon+z^\epsilon)} \;+\; \frac{1}{z^\epsilon(\epsilon+z^\epsilon)^2}$$

is not monotone a priori; moreover the easy estimate:

$$\frac{1}{(z^\epsilon)^2(\epsilon+z^\epsilon)} \;+\; \frac{1}{z^\epsilon(\epsilon+z^\epsilon)^2} \;\leq\; \frac{2}{(x_3)^3},$$

is not useful for a Dominated Convergence argument, since $\mathbb{E}[\langle h, (x_3)^{-3}\rangle] = +\infty$ if $h \geq 0$ and h is not equal to 0 a.e. In order to prove (30), we proceed in a different way. By (31):

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\left[\varphi(z^\epsilon)\ \langle h, \frac{2}{(\epsilon+z^\epsilon)^3}\rangle\right] \;=\; \mathbb{E}\left[\varphi(x_\delta)\langle h, \frac{2}{(x_\delta)^3}\rangle\right].$$

Now by (18) and (29) we have:

$$\begin{aligned}&\left|\mathbb{E}\Big[\varphi(z^\epsilon)\langle h,\zeta^\epsilon\rangle\Big] \;-\; \mathbb{E}\Big[\varphi(z^\epsilon)\;\langle h,\frac{2}{(\epsilon+z^\epsilon)^3}\rangle\Big]\right| \\ &\leq \|\varphi\|_\infty\,\mathbb{E}\Big[\,\Big|\langle h,\zeta^\epsilon\;-\;\frac{2}{(\epsilon+z^\epsilon)^3}\rangle\Big|\,\Big] \\ &\leq \|\varphi\|_\infty\;\left(\mathbb{E}\left[\langle h,\zeta^\epsilon\rangle\right]-\mathbb{E}\left[\langle h,\frac{2}{(\epsilon+z^\epsilon)^3}\rangle\right]\right).\end{aligned}$$

Therefore, in order to prove (30), it is enough to prove that

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\langle h,\zeta^\epsilon\rangle\;-\;\langle h,\frac{2}{(\epsilon+z^\epsilon)^3}\rangle\right]\;=\;0.$$

By Itô's formula we find:

$$\begin{aligned}0\;=\;\frac{h}{\epsilon+z^\epsilon}\Big|_{\tau=0}^{\tau=1}\;=\;\int_0^1 d\left(\frac{h}{\epsilon+z^\epsilon}\right)\;=\;\int_0^1\frac{h'}{\epsilon+z^\epsilon}\,d\tau \\ +\int_0^1\left[-\frac{h}{(\epsilon+z^\epsilon)^2}\left[\left(\frac{1}{z^\epsilon}-\frac{z^\epsilon}{1-\tau}+\frac{c(\delta)}{\epsilon+z^\epsilon}\right)d\tau+dB\right]+\frac{h}{(\epsilon+z^\epsilon)^3}\,d\tau\right]\end{aligned}$$

so that we can compute:

$$\begin{aligned}\mathbb{E}\left[\int_0^1\frac{h}{z^\epsilon(\epsilon+z^\epsilon)^2}\right]\;&=\;\mathbb{E}\left[\int_0^1\frac{h'}{\epsilon+z^\epsilon}+\frac{h\,z^\epsilon}{(1-\tau)(\epsilon+z^\epsilon)^2}+\frac{(1-c(\delta))h}{(\epsilon+z^\epsilon)^3}\right] \\ &\to\;\mathbb{E}\left[\int_0^1\frac{h'}{x_\delta}+\frac{h}{(1-\tau)x_\delta}+(1-c(\delta))\,\frac{h}{(x_\delta)^3}\right]\end{aligned}$$

where the convergence of each term can be justified by either Dominated or Monotone convergence. Analogously, we obtain for all $\alpha>0$:

$$\begin{aligned}\frac{h}{\alpha+z^\epsilon}\Big|_{\tau=0}^{\tau=1}\;=0\;\;=\;&\int_0^1\left(\frac{h'}{\alpha+z^\epsilon}\;+\;\frac{h}{(\alpha+z^\epsilon)^3}\right)\,d\tau \\ &-\int_0^1\frac{h}{(\alpha+z^\epsilon)^2}\left[\left(\frac{1}{z^\epsilon}-\frac{z^\epsilon}{1-\tau}+\frac{c(\delta)}{\epsilon+z^\epsilon}\right)d\tau+dB\right],\end{aligned}$$

$$\mathbb{E}\left[\int_0^1 \frac{h}{(\alpha+z^\epsilon)^2(\epsilon+z^\epsilon)}\right]$$

$$= \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+z^\epsilon}+\frac{h}{(1-\tau)z^\epsilon}+\frac{h}{(\alpha+z^\epsilon)^3}-\frac{h}{z^\epsilon(\alpha+z^\epsilon)^2}\right]$$

$$\leq \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+z^\epsilon}+\frac{h}{(1-\tau)z^\epsilon}\right],$$

$$\frac{h}{\alpha+x_\delta}\Bigg|_{\tau=0}^{\tau=1} = 0 = \int_0^1\left(\frac{h'}{\alpha+x_\delta}+\frac{h}{(\alpha+x_\delta)^3}\right)d\tau$$

$$-\int_0^1\left[\frac{h}{(\alpha+x_\delta)^2}\left(\frac{c(\delta)+1}{x_\delta}-\frac{x_\delta}{(1-\tau)}\right)d\tau + dB\right].$$

$$\mathbb{E}\left[\int_0^1\left(\frac{(c(\delta)+1)h}{x_\delta(\alpha+x_\delta)^2}-\frac{h}{(\alpha+x_\delta)^3}\right)\right]=\mathbb{E}\left[\int_0^1 \frac{h'}{\alpha+x_\delta}+\frac{h\,x_\delta}{(1-\tau)(\alpha+x_\delta)^2}\right].$$

Letting first $\alpha\downarrow 0$ and then $\epsilon\downarrow 0$ we obtain:

$$\limsup_{\epsilon\downarrow 0}\mathbb{E}\left[\int_0^1 \frac{h}{(z^\epsilon)^2(\epsilon+z^\epsilon)}\right] \leq \frac{1}{c(\delta)}\mathbb{E}\left[\int_0^1 \frac{h'}{x_\delta}+\frac{h}{(1-\tau)x_\delta}\right]$$

$$\mathbb{E}\left[\int_0^1 \frac{h}{(x_\delta)^3}\right] = \frac{1}{c(\delta)}\,\mathbb{E}\left[\int_0^1 \frac{h'}{x_\delta}+\frac{h}{(1-\tau)x_\delta}\right].$$

Therefore:

$$0 \leq \limsup_{\epsilon\downarrow 0}\mathbb{E}\left[\langle h,\zeta^\epsilon\rangle - \langle h, \frac{2}{(\epsilon+z^\epsilon)^3}\rangle\right] \leq 0,$$

and, by (29), (30) is proved.

Step 7. We turn to the last term in (27). Notice that

$$\int \varphi(x)\,\gamma^\epsilon(x)\sigma_0(r,dx) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\,\cdot$$

$$\cdot\int \varphi\left(w\oplus_r z\right)\,\gamma^{\epsilon,r}(w)\,\gamma^{\epsilon,1-r}(z)\mathcal{Z}^{\epsilon,r}(w)\,\pi_3^r(dw)\otimes\pi_3^{1-r}(dz),$$

where $\gamma^{\epsilon,r},\mathcal{Z}^{\epsilon,r}\in L^1(\pi_3^r)$, $r\in(0,1)$, are defined by:

$$\gamma^{\epsilon,r}(w) := \exp\left(c(\delta)\int_0^r\left(\frac{1-c(\delta)}{2(\epsilon+w)^2}-\frac{1}{w(\epsilon+w)}+\frac{1}{r-\tau}\frac{w}{\epsilon+w}\right)d\tau\right),$$

$$\mathcal{Z}^{\epsilon,r}(w) \;:=\; \exp\left(c(\delta)\int_0^r \left(\frac{1}{1-\tau} - \frac{1}{r-\tau}\right)\frac{w}{\epsilon+w}\,d\tau\right) \;\leq\; \exp\left(\frac{c(\delta)}{1-r}\right).$$

Arguing as in steps 3-4, by (14) $\gamma^{\epsilon,r}\,d\pi_3^r$ is the law of $w^{\epsilon,r}$, where:

$$\begin{cases} d\,w^{\epsilon,r} = \dfrac{1}{w^{\epsilon,r}}\,d\tau \,+\, \dfrac{\delta-3}{2(\epsilon+w^{\epsilon,r})}\,d\tau \,-\, \dfrac{w^{\epsilon,r}}{r-\tau}\,d\tau \,+\, dB, \quad \tau\in[0,r[\\ \\ w^{\epsilon,r}(0)=0 \end{cases}$$

and $w^{\epsilon,r}\uparrow x_\delta^r$ as $\epsilon\downarrow 0$. Moreover, since a.s. $w^{1,r}(\tau)>0$ for all $\tau\in(0,1)$, we have $\mathcal{Z}^{\epsilon,r}(w^{\epsilon,r})\to 0$ a.s. as $\epsilon\downarrow 0$ and by Dominated Convergence Theorem we obtain, since h has compact support in $(0,1)$:

$$\lim_{\epsilon\downarrow 0}\int_0^1 dr\;h(r)\int \varphi(x)\,\gamma^\epsilon(x)\,\sigma_0(r,dx)$$

$$= \lim_{\epsilon\downarrow 0}\int_0^1 dr\,\frac{h(r)}{\sqrt{2\pi r^3(1-r)^3}}\,\mathbb{E}\left[\varphi(w^{\epsilon,r}\oplus_r w^{\epsilon,1-r})\,\mathcal{Z}^{\epsilon,r}(w^{\epsilon,r})\right] \;=\; 0$$

and the proof of (16) is complete. □

Corollary 1 *For all* $\psi\in\mathrm{Lip}(H)$ *there exists a field* $\nabla\psi\in L^\infty(K_0,\pi_\delta;H)$ *such that for all* $h\in D(A)$*:*

$$\lim_{t\downarrow 0}\frac{1}{t}(\psi(\cdot+t\,h)-\psi) \;=:\; \partial_h\psi \;=\; \langle\nabla\psi,h\rangle \qquad \text{weakly in } L^2(\pi_\delta).$$

We call $\nabla\psi$ *the gradient of* ψ*. Then, (16) holds for all* $\varphi\in\mathrm{Lip}(H)$*. Moreover, for all* $\psi\in\mathrm{Lip}(H)$ *and* $\varphi\in\mathrm{Exp}_A(H)$*, we have:*

$$\frac{1}{2}\int_{K_0}\langle\nabla\psi,\nabla\varphi\rangle\,d\pi_\delta \;=\; -\int_{K_0}\psi\,L_\delta\varphi\,d\pi_\delta$$

where $L_\delta\varphi\in L^1(\pi_\delta)$ *is defined as:*

$$L_\delta\varphi(x) := \frac{1}{2}\mathrm{Tr}\left[D^2\varphi(x)\right] + \langle x, A\nabla\varphi(x)\rangle + \frac{\kappa(\delta)}{2}\,\langle x^{-3},\nabla\varphi(x)\rangle,\;\; x\in K_0.$$

Proof–The family $\{(\psi(\cdot + t\,h) - \psi))/t\}_{t>0}$ is bounded in $L^2(\pi_\delta)$. For all $\varphi \in \mathrm{Exp}_A(H)$:

$$\lim_{t\downarrow 0} \int_{K_0} \frac{1}{t}(\psi(\cdot + t\,h) - \psi)\,\varphi\, d\pi_\delta$$

$$= - \int_{K_0} \psi\,\langle \nabla\varphi, h\rangle\, d\pi_\delta - \int_{K_0} \varphi\psi(x)\,\big(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\big)\,\pi_\delta(dx). \tag{33}$$

Indeed, (33) holds for all $\psi \in C_b^1(H)$; moreover, the family of functionals

$$C_b^1(H) \ni \psi \mapsto \int_{K_0} \frac{1}{t}(\psi(\cdot + t\,h) - \psi)\,\varphi\, d\pi_\delta, \quad t > 0,$$

is uniformly bounded in the sup-norm, by (16). By the density of $C_b^1(H)$ in $C_b(H)$ in the sup-norm, we obtain (33) for all $\psi \in C_b(H)$. Then, (33) allows to identify all limit points of $(\psi(\cdot + t\,h) - \psi))/t$ in the weak topology of $L^2(\pi_\delta)$ as $t \downarrow 0$. The last formula follows from (16). □

Corollary 2 *For all $h \in D(A)$ and $\varphi \in C_b(H)$ we have:*

$$\lim_{\delta\downarrow 3} \frac{\delta - 3}{2} \int \varphi(x)\,\langle x^{-3}, h\rangle\,\pi_\delta(dx) \;=\; \int_0^1 dr\;h(r) \int \varphi(x)\,\sigma_0(r, dx)$$

3 SPDE generated by the δ-Bessel Bridge, $\delta > 3$

This section is devoted to the proof of the following

Theorem 3 *Let $\delta > 3$.*

(i) For all $x \in K_0 \cap C_0$, there exists a unique random continuous non-negative $u_\delta : [0,\infty) \times [0,1] \mapsto [0,\infty)$, such that $(u_\delta)^{-3} \in L^1_{loc}([0,\infty) \times (0,1))$, solving the SPDE:

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\delta}{\partial \xi^2} + \dfrac{(\delta-1)(\delta-3)}{8\,(u_\delta)^3} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\[2ex] u_\delta(0,\cdot) = x, \;\; u_\delta(t,0) = u_\delta(t,1) = 0, \quad t \geq 0. \end{cases} \tag{34}$$

Moreover, u_δ is $(\mathcal{F}_t)$-adapted. We set $X_\delta(t,x) := u_\delta(t,\cdot) \in H$, $t \geq 0$.

(ii) The process X_δ is symmetric with respect to its unique invariant probability measure π_δ, law of the δ-dimensional Bessel Bridge on $[0,1]$. Moreover, X_δ is Strong Feller in H: indeed, for all bounded and Borel $\varphi : H \mapsto \mathbb{R}$ we have for all $x, y \in K_0$, $t > 0$,

$$|\mathbb{E}[\varphi(X_\delta(t,x))] - \mathbb{E}[\varphi(X_\delta(t,y))]| \le \|\varphi\|_\infty (1 \wedge t)^{-\frac{1}{2}} \|x - y\|. \tag{35}$$

(iii) X_δ is the diffusion associated with the Dirichlet form $(\mathcal{D}^\delta, W^{1,2}(\pi_\delta))$, closure in $L^2(\pi_\delta)$ of the symmetric bilinear form:

$$\mathrm{Lip}(H) \ni \varphi, \psi \mapsto D^\delta(\varphi, \psi) := \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\pi_\delta.$$

(iv) Let $\delta \in \mathbb{N} \cap [4, \infty)$. We set: $\Phi_\delta : H^\delta \mapsto K_0$, $\Phi_\delta(y)(\tau) := |y(\tau)|_{\mathbb{R}^\delta}$, $\tau \in [0,1]$. Then $\mathcal{D}^\delta$ is the image of Λ^δ under the map Φ_δ, i.e.

$$W^{1,2}(\pi_\delta) = \{\varphi \in L^2(\pi_\delta) : \varphi \circ \Phi_\delta \in W^{1,2}(\mu^{\otimes \delta})\},$$

$$\mathcal{D}^\delta(\varphi, \psi) = \Lambda^\delta(\varphi \circ \Phi_\delta, \psi \circ \Phi_\delta), \quad \forall \varphi, \psi \in W^{1,2}(\pi_\delta).$$

Recall that for all $\delta \in \mathbb{N}$, $\delta \ge 3$, we denote by $(\Lambda^\delta, W^{1,2}(\mu^{\otimes \delta}))$ the Dirichlet Form with state space $H^\delta = L^2(0,1;\mathbb{R}^\delta)$:

$$W^{1,2}(\mu^{\otimes \delta}) \ni F, G \mapsto \Lambda^\delta(F,G) := \frac{1}{2} \int_{H^\delta} \langle \overline{\nabla} F, \overline{\nabla} G \rangle_{H^\delta} \, d\mu^{\otimes \delta}$$

where μ is the law on $L^2(0,1)$ of a Brownian Bridge on $[0,1]$, $F, G : H^\delta \mapsto \mathbb{R}$ and $\overline{\nabla} F : H^\delta \mapsto H^\delta$ is the gradient of F in H^δ. Since μ is equal to the Gaussian measure $\mathcal{N}(0, (-2A)^{-1})$, then $(\Lambda^\delta, W^{1,2}(\mu^{\otimes \delta}))$ generates the process Z_δ, solution of the $\mathbb{R}^\delta$-valued linear SPDE (3): see [Za 01] and Chap. 8 of [DPZ 96].

We first recall the definition given by Nualart and Pardoux in [NP 92] of solution of the SPDE with reflection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial \xi^2} - f(\xi, u(t,\xi)) + \dfrac{\partial^2 W}{\partial t \partial \xi} + \eta(t,\xi) \\[2ex] u(0,\cdot) = x, \ u(t,0) = u(t,1) = 0, \quad t \ge 0 \\[2ex] u \ge 0, \ d\eta \ge 0 \ \int_{\mathcal{O}} u \, d\eta = 0. \end{cases} \tag{36}$$

with $x : [0,1] \mapsto [0, +\infty)$ continuous, $x(0) = x(1) = 0$ and $f : [0,1] \times [0,\infty) \mapsto \mathbb{R}$ measurable. We suppose that

(H1) $f(\xi, \cdot)$ is continuously differentiable for all $\xi \in [0,1]$, and for some $c > 0$

$$|f| \leq c, \quad |\partial_y f(\xi, y)| \leq c, \ \forall \xi \in [0,1], \ y \in [0, \infty).$$

(H2) There exists $C \geq 0$ such that for all $\xi \in [0,1]$:

$$\left| \int_0^t f(\xi, u) \, du \right| \leq C, \qquad \forall t \geq 0.$$

Following [NP 92], we set:

Definition 1 *A pair (u, η) is said to be a solution of the SPDE with reflection (36), also called the Nualart-Pardoux equation, and initial value $x \in K_0 \cap C_0(0,1)$, if:*

- *$\{u(t,\xi) : (t,\xi) \in \mathcal{O}\}$ is a continuous and adapted process, i.e. $u(t,\xi)$ is $\mathcal{F}_t$-measurable for all $(t,\xi) \in \mathcal{O}$, and a.s. $u(\cdot,\cdot)$ is continuous on $\mathcal{O}$, $u(t,\cdot) \in K_0 \cap C_0(0,1)$ for all $t \geq 0$, and $u(0,\cdot) = x$.*

- *η is a random positive measure on $\mathcal{O}$ such that $\eta([0,T] \times [\delta, 1-\delta]) < +\infty$ for all $T, \delta > 0$, and η is adapted, i.e. $\eta(B)$ is $\mathcal{F}_t$-measurable for every Borel set $B \subset [0,t] \times [0,1]$.*

- *For all $t \geq 0$ and $h \in D(A)$*

$$\langle u(t, \cdot), h \rangle - \int_0^t \langle u(s, \cdot), Ah \rangle \, ds + \int_0^t \langle f(\cdot, u(s, \cdot)), h \rangle \, ds =$$

$$= \langle x, h \rangle - \int_0^1 h'(\xi) \, W(t,\xi) \, d\xi + \int_0^t \int_0^1 h(\xi) \, \eta(dt, d\xi).$$

- $\int_{\mathcal{O}} u \, d\eta = 0.$

In [NP 92], the following theorem is proved:

Theorem 4 *Assume that f satisfies (H1),(H2). Then for all $x \in K_0 \cap C_0(0,1)$, there exists a unique solution (u,η) of the SPDE with reflection (36).*

We set:

$$F : K_0 \mapsto \mathbb{R}, \quad F(x) := \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds.$$

$$\pi_3^F(dx) := \frac{1}{\pi_3(e^{-2F})} \exp(-2F(x))\, \pi_3(dx)$$

The following Theorem has been proved in [Za 00] and [Za 01].

Theorem 5 *If u is the solution of the Nualart-Pardoux SPDE (36), then the process $x \mapsto u$ is the diffusion associated with the symmetric Dirichlet Form $(\mathcal{E}, W^{1,2}(\pi_3^F))$, closure in $L^2(\pi_3^F)$ of the symmetric bilinear form:*

$$\mathrm{Exp}_A(H) \ni \varphi, \psi \mapsto \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\pi_3^F.$$

In particular, $x \mapsto u$ is symmetric with respect to π_3^F, and π_3^F is the unique invariant probability measure of $x \mapsto u$. Finally, $\mathrm{Lip}(K_0) \subset W^{1,2}(\pi_3^F)$.

Remark 1 Let u_δ be the unique solution to (34), for all $\delta > 3$. For all $(t, \xi) \in \mathcal{O}$, $(3, \infty) \ni \delta \mapsto u_\delta(t, \xi)$ is non-decreasing, and as $\delta \downarrow 3$: $u_\delta \downarrow u$ uniformly on $[0, T] \times [0, 1]$, $T \geq 0$, and

$$\frac{\delta - 3}{4\,(u_\delta)^3}\, dt\, d\xi \;\to\; \eta(dt, d\xi) \qquad \text{distributionally on } \mathcal{O},$$

where (u, η) is the solution of the SPDE with reflection (36), with $f \equiv 0$.

In the proof of Theorem 3 we consider solutions to SPDEs of the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial \xi^2} - g(u) + \dfrac{\partial^2 W}{\partial t \partial \xi} \\[2ex] u(t, 0) = u(t, 1) = b, \quad t \geq 0, \\[2ex] u(0, \cdot) = x(\cdot) + b \in L^2(0, 1), \end{cases} \tag{37}$$

where $g : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ is measurable, $g(u) := g(\cdot, u(\cdot, \cdot))$, $b \in \mathbb{R}$ and $x \in H$.

Lemma 1 *Let $g_\rho : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ be measurable, $\rho > 0$, such that $\mathbb{R} \ni y \mapsto g_\rho(\xi, y)$ is monotone non-decreasing, Lipschitz-continuous uniformly in $\xi \in [0, 1]$, and satisfies:*

$$|g_\rho(\xi, y)| \;\leq\; c(1 + |y|), \qquad \forall\, y \in \mathbb{R}, \;\; \rho > 0,$$

for some $c \geq 0$. For all $b \in \mathbb{R}$, let u_ρ^b be the unique solution of the SPDE (37) with $g := g_\rho$. Then, a.s. we have:

- *If $\rho \mapsto g_\rho(\cdot,\cdot)$ is monotone non-decreasing, then $\rho \mapsto u_\rho^b(\cdot,\cdot)$ is monotone non-increasing for all $b \in \mathbb{R}$.*
- *If $\rho \mapsto g_\rho(\cdot,\cdot)$ is monotone non-increasing, then $\rho \mapsto u_\rho^b(\cdot,\cdot)$ is monotone non-decreasing for all $b \in \mathbb{R}$.*
- *$b \mapsto u_\rho^b(\cdot,\cdot)$ is monotone non-decreasing for all $\rho > 0$.*

Proof–We prove the first assertion: the others follow analogously. Let $\rho_1 \geq \rho_2 > 0$ and set $\phi := (u^b_{\rho_1} - u^b_{\rho_2})^+$. Then by Lemma 6.1, p. 147 in [BL 82]:

$$\begin{aligned}\frac{d}{dt}\|\phi\|^2 &= 2\,\langle \phi, A(u^b_{\rho_1} - u^b_{\rho_2})\rangle - 2\langle \phi, g_{\rho_1}(u^b_{\rho_1}) - g_{\rho_1}(u^b_{\rho_2})\rangle \\ &= -\left\|\frac{\partial \phi}{\partial \xi}\right\|^2 - 2\langle \phi, g_{\rho_1}(u^b_{\rho_1}) - g_{\rho_1}(u^b_{\rho_2})\rangle - 2\langle \phi, g_{\rho_1}(u^b_{\rho_2}) - g_{\rho_2}(u^b_{\rho_2})\rangle \leq 0.\end{aligned}$$

Since $\|\phi(0,\cdot)\| = 0$, by Gronwall's Lemma $\phi \equiv 0$, so that $u^b_{\rho_2} \geq u^b_{\rho_1}$. $\square$

Proof of Theorem 3–We divide the proof into several steps. In steps 1-5 we prove (i) and (ii). The idea is to approximate u_δ from below, by means of solutions v^ϵ, $\epsilon > 0$, of Nualart-Pardoux-type equations. We choose v^ϵ so that its invariant measure is the law of z^ϵ, see (21), which converges to π_δ. In step 6 we prove (iii) and in step 7 we prove (iv). We shall refer to several definitions and technical points of the proof of Theorem 1.

Step 1. Uniqueness of solutions of (34) follows from the dissipativity of the coefficients: indeed, let u^1 and u^2 are two non-negative continuous solutions of (34), and set for $\varepsilon > 0$, $h_\varepsilon(\xi) := [\xi(1-\xi)/\varepsilon] \wedge 1$, $\xi \in [0,1]$, and $\phi := u^1 - u^2$. Notice that, by (34), $h_\varepsilon \cdot u_i^{-3} \in L^1([0,T] \times [0,1])$, for all $T \geq 0$, $i = 1,2$, and by Theorem 6.4, p. 131 in [BL 82], $\phi(t,\cdot) \in C^1([0,1])$ for all $t > 0$ and $\partial\phi/\partial\xi$ is in $L^\infty_{loc}(\mathcal{O})$. Then:

$$\begin{aligned}\frac{d}{dt}\|h_\varepsilon \cdot \phi\|^2 &= -\int_0^1 h_\varepsilon \cdot \left(\frac{\partial \phi}{\partial \xi}\right)^2 d\xi - \langle h'_\varepsilon \phi, \frac{\partial \phi}{\partial \xi}\rangle \\ &\quad + \kappa(\delta)\,\langle h_\varepsilon \cdot (u^1 - u^2), \frac{1}{(u^1)^3} - \frac{1}{(u^2)^3}\rangle \leq -\langle h'_\varepsilon \cdot \phi, \frac{\partial \phi}{\partial \xi}\rangle.\end{aligned}$$

As $\varepsilon \downarrow 0$, $\langle h'_\varepsilon \cdot \phi, \frac{\partial \phi}{\partial \xi} \rangle \to 0$ since $\phi(t,0) = \phi(t,1) = 0$, $t \geq 0$, so that $\phi \equiv 0$.

Step 2. We define for $0 < \rho \leq \epsilon$ and $c > 0$: $f_\rho^\epsilon : [0,1] \times [0,\infty) \mapsto \mathbb{R}$,

$$f_\rho^\epsilon(\xi, a) \;:=\; -\frac{c(\delta)}{2}\left[\frac{c(\delta)-1}{(\epsilon+a)^3} + \frac{1}{(\rho+a)^2(\epsilon+a)} + \frac{1}{(\rho+a)(\epsilon+a)^2} + \frac{\epsilon}{(1+\rho-\xi)(\epsilon+a)^2}\right],$$

$$g_{\rho,c}^\epsilon : [0,1] \times \mathbb{R} \mapsto \mathbb{R}, \quad g_{\rho,c}^\epsilon(\xi,a) := f_\rho^\epsilon(\xi, a^+) - \frac{a^-}{c},$$

where $a^+ = \sup\{a, 0\}$ and $a^- = a^+ - a$. The definition of f_ρ^ϵ is suggested by (23) above. Notice that f_ρ^ϵ and $[0,\infty) \ni a \mapsto -\kappa(\delta)/2(\epsilon+a)^3$ satisfy (H1) and (H2) above. Let $x \in K_0 \cap C_0$. We define for all $0 < \rho \leq \epsilon$ and $c > 0$:

- $v_{\rho,c}^\epsilon$ as the solution of the SPDE (37) with $b = 0$ and $g = g_{\rho,c}^\epsilon$
- u_c^ϵ as the solution of the SPDE (37) with $b = 0$ and

$$g(\xi,a) = -\frac{\kappa(\delta)}{2(\epsilon+a^+)^3} - \frac{a^-}{c} \qquad (\xi,a) \in [0,1] \times \mathbb{R}$$

- $(v_\rho^\epsilon, \eta_\rho^\epsilon)$ as the solution of the SPDE with reflection (36) with $f = f_\rho^\epsilon$
- $(u^\epsilon, \theta^\epsilon)$ as the solution of the SPDE with reflection (36) with

$$f(\xi,a) = -\frac{\kappa(\delta)}{2(\epsilon+a)^3} \qquad (\xi,a) \in [0,1] \times [0,\infty).$$

Notice that:

(a1) $a \mapsto g_{\rho,c}^\epsilon(\xi,a)$ is non-decreasing

(a2) $\rho \mapsto g_{\rho,c}^\epsilon(\xi,a)$ is non-decreasing

(a3) $(0,a) \ni \epsilon \mapsto g_{\rho,c}^\epsilon(\xi, a-\epsilon)$ is non-increasing

(a4) $-f_\rho^\epsilon(\xi,a) \geq \kappa(\delta)[2(\epsilon+a)^3]^{-1}$, $a \geq 0$.

By the proof of Theorem 4 given in [NP 92], we obtain in particular that $v_{\rho,c}^\epsilon \uparrow v_\rho^\epsilon$ and $u_c^\epsilon \uparrow u^\epsilon$ uniformly on compact sets of $\mathcal{O}$, as $c \downarrow 0$. By Lemma 1 and (a1)-a(4) we obtain:

(b1) $\rho \mapsto v^\epsilon_\rho$ is non-increasing

(b2) $\epsilon \mapsto \epsilon + v^\epsilon_\rho$ is non-decreasing

(b3) $\epsilon \mapsto u^\epsilon$ is non-increasing, $\epsilon \mapsto \epsilon + u^\epsilon$ is non-decreasing.

(b4) $v^\epsilon_\rho \geq u^\epsilon$.

We shall prove that $x \mapsto v^\epsilon_\rho$ converges to a process $x \mapsto v$, which is symmetric with respect to π_δ. Then, we shall prove that u^ϵ converges to a continuous process $x \mapsto u_\delta$ satisfying (34), and that $v(t,\cdot) \equiv u_\delta(t,\cdot)$ in $L^2(0,1)$ for all $t \geq 0$. The proof will be based only on monotonicity arguments, on the integration by parts formula w.r.t. π_δ (16) and on the explicit knowledge of the invariant measure of $x \mapsto v^\epsilon_\rho$, given by Theorem 5.

Step 3. We set for all $\epsilon, \rho, c > 0$: $X^\epsilon_{\rho,c}(t,x) := v^\epsilon_{\rho,c}(t,\cdot)$, $X^\epsilon_\rho(t,x) := v^\epsilon_\rho(t,\cdot)$. We have for all $t \geq 0$, $x, x' \in C_0 \cap K_0$:

$$\|X^\epsilon_{\rho,c}(t,x) - X^\epsilon_{\rho,c}(t,x')\|^2 \leq e^{-\pi^2 t} \|x - x'\|^2.$$

By Theorem 5, the process X^ϵ_ρ is symmetric with respect to $\gamma^\epsilon_\rho \, d\pi_3$, defined in (23). By (b1) we obtain that, almost surely, there exists the limit $v^\epsilon := \lim_{\rho \downarrow 0} v^\epsilon_\rho = \sup_{\rho > 0} v^\epsilon_\rho$. We claim that for every $t \geq 0$, $v^\epsilon(t,\cdot) \in L^2(0,1)$ a.s. Indeed, we have:

$$\|X^\epsilon_\rho(t,0)\| \leq \|X^\epsilon_\rho(t,x)\| \leq \|X^\epsilon_\rho(t,0)\| + \|x\|, \tag{38}$$

The first inequality follows from a comparison argument, and the second one from the Lipschitz continuity of $X^\epsilon_\rho(t,\cdot)$. Therefore, we can reduce to the case $x \equiv 0$. Integrating the first inequality in (38) with respect to $(\gamma^\epsilon_\rho \, d\pi_3) \otimes \mathbb{P}$, since $\gamma^\epsilon_\rho \, d\pi_3$ is the invariant measure of X^ϵ_ρ, we obtain:

$$\mathbb{E}[\|X^\epsilon_\rho(t,0)\|] \leq \int \mathbb{E}[\|X^\epsilon_\rho(t,x)\|] \, \gamma^\epsilon_\rho(x) \, \pi_3(dx) = \int \|x\| \gamma^\epsilon_\rho(x) \, \pi_3(dx)$$

By Beppo-Levi's Theorem and (25)-(26), we obtain:

$$\mathbb{E}\left[\left\|\lim_{\rho \downarrow 0} X^\epsilon_\rho(t,0)\right\|\right] \leq \int \|x\| \gamma^\epsilon(x) \, \pi_3(dx) = \mathbb{E}[\|z^\epsilon\|] < \infty,$$

and the claim is proved. We set $X^\epsilon(t,x) := \lim_{\rho \downarrow 0} X^\epsilon_\rho(t,x)$. By Dominated Convergence, we obtain for all $\psi, \varphi \in C_b(H)$:

$$\mathbb{E}[\psi(z^\epsilon) \, \varphi(X^\epsilon(t,z^\epsilon))] = \mathbb{E}[\psi(X^\epsilon(t,z^\epsilon)) \, \varphi(z^\epsilon)]$$

i.e. X^ϵ is symmetric w.r.t. to the law $\gamma^\epsilon\, d\pi_3$ of z^ϵ. Moreover, $X^\epsilon(t,\cdot)$ is 1-Lipschitz. Analogously, by (b2) $\epsilon\mapsto\epsilon+v^\epsilon$ is non-decreasing. Therefore, there exist the limits $v:=\lim_{\epsilon\downarrow 0}v^\epsilon$, $H\ni X_\delta(t,x):=\lim_{\epsilon\downarrow 0}X^\epsilon(t,x)$ for all $t\geq 0$. Be the equicontinuity of $\{X^\epsilon(t,\cdot)\}_{\epsilon>0}$, we have for $\psi,\varphi\in C_b(H)$:

$$\mathbb{E}[\psi(x_\delta)\varphi(X_\delta(t,x_\delta))] \;=\; \lim_{\epsilon\downarrow 0}\mathbb{E}[\psi(z^\epsilon)\varphi(X^\epsilon(t,z^\epsilon))]$$

$$=\;\lim_{\epsilon\downarrow 0}\mathbb{E}[\psi(X^\epsilon(t,z^\epsilon))\varphi(z^\epsilon)] \;=\; \mathbb{E}[\psi(X_\delta(t,x_\delta))\varphi(x_\delta)],$$

i.e. X_δ is symmetric w.r.t. π_δ. Moreover for all $t\geq 0$, $x,x'\in C_0\cap K_0$:

$$\|X_\delta(t,x)-X_\delta(t,x')\|^2 \;\leq\; e^{-\pi^2 t}\,\|x-x'\|^2. \tag{39}$$

Let now m_1 and m_2 be invariant probability measures for X_δ. Then, if q_1 and q_2 are random variable with law, respectively, m_1 and m_2, and independent of W, we have for all $\varphi\in C_b(H)$:

$$|m_1(\varphi)-m_2(\varphi)| \;=\; |\mathbb{E}[\varphi(X_\delta(t,q_1))-\varphi(X_\delta(t,q_2))]| \;\to\; 0 \quad \text{as } t\to\infty.$$

Therefore, X_δ is symmetric w.r.t. its unique invariant probability measure π_δ.

Step 4. Fix $t\geq 0$. By Dominated Convergence, we obtain for $h\geq 0$:

$$\langle X^\epsilon(t,x),h\rangle-\langle x,h\rangle-\frac{1}{2}\int_0^t\langle h'',X^\epsilon(s,x)\rangle\,ds-\langle h,W(t)\rangle \geq \int_0^t\int_0^1 \frac{h\,\kappa(\delta)}{2(\epsilon+X^\epsilon)^3}$$

Since $\epsilon\mapsto\epsilon+v^\epsilon$ is non-decreasing we can let $\epsilon\downarrow 0$, and obtain by Beppo-Levi's Theorem:

$$\langle X_\delta(t,x),h\rangle-\langle x,h\rangle-\frac{1}{2}\int_0^t\langle h'',X_\delta(s,x)\rangle\,ds-\langle h,W(t)\rangle \geq \frac{\kappa(\delta)}{2}\int_0^t\int_0^1 \frac{h}{(X_\delta)^3}$$

Since π_δ is invariant for X_δ, we obtain:

$$\mathbb{E}\left[\langle X_\delta(t,x_\delta),h\rangle-\langle x_\delta,h\rangle-\frac{1}{2}\int_0^t\langle h'',X_\delta(s,x_\delta)\rangle\,ds-\langle h,W(t)\rangle\right]$$

$$=\;-\,\frac{t}{2}\,\mathbb{E}\left[\langle h'',x_\delta\rangle\right]$$

$$\frac{\kappa(\delta)}{2}\,\mathbb{E}\left[\int_0^t\int_0^1 \frac{h(\xi)}{[X_\delta(s,x_\delta)(\xi)]^3}\,d\xi\,ds\right] \;=\; t\,\frac{\kappa(\delta)}{2}\,\mathbb{E}\left[\int_0^1 \frac{h(\xi)}{[x_\delta(\xi)]^3}\,d\xi\right]$$

By (16) with $\varphi \equiv 1$, we obtain that there exists a measurable set $G_t \subseteq H\times\Omega$, with $[\pi_\delta \otimes \mathbb{P}](G_t) = 1$, such that for all $(x,\omega) \in G_t$:

$$\langle X_\delta(t,x),h\rangle - \langle x,h\rangle - \frac{1}{2}\int_0^t \langle h'', X_\delta(s,x)\rangle\,ds - \langle h, W(t)\rangle = \frac{\kappa(\delta)}{2}\int_0^t\int_0^1 \frac{h}{(X_\delta)^3}. \tag{40}$$

Step 5. By (b4) $v^\epsilon \geq u^\epsilon$, and by (b3) there exists the limit in the uniform norm of u^ϵ as $\epsilon \downarrow 0$. Setting $u_\delta := \lim_{\epsilon\downarrow 0} u^\epsilon$, we have that u_δ is a.s. continuous, and $v \geq u_\delta$. Setting $Y(t,x) := u_\delta(t,\cdot)$, we have that $Y(t,\cdot)$ is 1-Lipschitz. Moreover, by Beppo-Levi's Theorem, for $h \geq 0$:

$$\langle u_\delta(t,\cdot),h\rangle \geq \langle x,h\rangle + \frac{1}{2}\int_0^t \left(\langle u_\delta(s,\cdot),h''\rangle + \langle \frac{\kappa(\delta)}{(u_\delta(s,\cdot))^3},h\rangle\right)ds + \langle W(t),h\rangle.$$

If now $h(\xi) := \xi(1-\xi)$, $\xi \in [0,1]$, then $h \in D(A)$, $h > 0$ on $(0,1)$ and $h'' \leq 0$. By (40) for all $t \geq 0$, $(x,\omega) \in G_t$:

$$\begin{aligned}
\langle v(t,\cdot),h\rangle &\geq \langle u_\delta(t,\cdot),h\rangle \\
&\geq \langle x,h\rangle + \frac{1}{2}\int_0^t \left(\langle u_\delta(s,\cdot),h''\rangle + \langle \frac{\kappa(\delta)}{(u_\delta(s,\cdot))^3},h\rangle\right)ds + \langle W(t),h\rangle \\
&\geq \langle x,h\rangle + \frac{1}{2}\int_0^t \left(\langle v(s,\cdot),h''\rangle + \langle \frac{\kappa(\delta)}{(v(s,\cdot))^3},h\rangle\right)ds + \langle W(t),h\rangle \\
&= \langle v(t,\cdot),h\rangle
\end{aligned}$$

so that for all $(x,\omega) \in G_t$, $u_\delta(t,\cdot) = v(t,\cdot)$ in H. By the Fubini-Tonelli Theorem, for $\mathbb{P}$-a.e. ω, we have $\pi_\delta(G_t \cap (K_0 \times \{\omega\})) = 1$. By the continuity of $X_\delta(t,\cdot)$ and $Y(t,\cdot)$, we obtain that $u_\delta(t,\cdot) = v(t,\cdot)$ in H for all $x \in K_0$, a.s. Moreover, for all $t \geq 0$ and $h \in D(A)$:

$$\lim_{\epsilon\downarrow 0}\int_0^t\int_0^1 h\,d\theta^\epsilon \;=\; 0.$$

Therefore, u_δ is a.s. continuous, solves (34) and is symmetric with respect to its unique invariant measure π_δ. Finally, we notice that $v^\epsilon_{\rho,c}$ satisfies a white-noise driven SPDE with dissipative non-linearity of Nemytskii type. By

Proposition 8.3.3 of [Ce 01], we have for all bounded and Borel $\varphi : H \mapsto \mathbb{R}$, $x, y \in H$, $t > 0$:

$$\left|\mathbb{E}[\varphi(X^\epsilon_{\rho,c}(t,x))] - \mathbb{E}[\varphi(X^\epsilon_{\rho,c}(t,y))]\right| \le \|\varphi\|_\infty (1 \wedge t)^{-\frac{1}{2}} \|x-y\|,$$

and (35) follows letting $c, \rho, \epsilon \downarrow 0$.

Step 6. We prove (iii). Let $\delta > 3$. We set for all $\psi \in \mathrm{Lip}(H)$, $\lambda > 0$:

$$R^\epsilon_\rho(\lambda)\psi(x) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[\psi(X^\epsilon_\rho(t,x))\right] dt, \qquad x \in K_0,$$

$$R^\epsilon(\lambda)\psi(x) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[\psi(X^\epsilon(t,x))\right] dt = \lim_{\rho\downarrow 0} R^\epsilon_\rho(\lambda)\psi(x), \qquad x \in K_0,$$

$$R_\delta(\lambda)\psi(x) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[\psi(X_\delta(t,x))\right] dt = \lim_{\epsilon\downarrow 0} R^\epsilon(\lambda)\psi(x), \qquad x \in K_0.$$

By (39), $\{R^\epsilon_\rho(\lambda)\psi, R^\epsilon(\lambda), R_\delta(\lambda)\psi\} \subset \mathrm{Lip}(K_0)$, and by Theorem 5, $R^\epsilon_\rho(\lambda)\psi \in W^{1,2}(\gamma^\epsilon_\rho\, d\pi_3)$. By Theorem 5 and by (17) we have for all $\varphi \in \mathrm{Exp}_A(H)$:

$$\int_{K_0} (\psi - \lambda R^\epsilon_\rho(\lambda)\psi)\, \varphi\, \gamma^\epsilon_\rho\, d\pi_3 = \frac{1}{2}\int_{K_0} \langle \nabla R^\epsilon_\rho(\lambda)\psi, \nabla\varphi\rangle\, \gamma^\epsilon_\rho\, d\pi_3$$
$$= -\int_{K_0} R^\epsilon_\rho(\lambda)\psi \left(\frac{1}{2}\mathrm{Tr}\left[D^2\varphi\right] + \langle x, A\nabla\varphi\rangle + \frac{1}{2}\langle \nabla \log \gamma^\epsilon_\rho, \nabla\varphi\rangle\right) \gamma^\epsilon_\rho\, d\pi_3.$$

Letting $\rho \downarrow 0$ and $\epsilon \downarrow 0$ we obtain by Corollary 1:

$$\int_{K_0} (\psi - \lambda R_\delta(\lambda)\psi)\varphi\, d\pi_\delta = -\int_{K_0} R_\delta(\lambda)\psi\, L_\delta\varphi\, d\pi_\delta = \frac{1}{2}\int_{K_0} \langle \nabla R_\delta(\lambda)\psi, \nabla\varphi\rangle d\pi_\delta.$$

By a standard approximation procedure, for all $\psi \in \mathrm{Lip}(H)$ there exists a net $(\varphi_i)_{i\in I} \subset \mathrm{Exp}_A(H)$ such that

$$\sup_i \|\varphi_i\|_{\mathrm{Lip}} \le \|\psi\|_{\mathrm{Lip}}, \quad \lim_{i\in I} \varphi_i(x) = \psi(x),\ \forall x \in H.$$

By Corollary 1, ψ admits a generalized gradient $\nabla\psi \in L^\infty(K_0, \pi_\delta; H)$. We claim that $(\nabla\varphi_i)_i$ converges to $\nabla\psi$ weakly in $L^2(K_0, \pi_\delta; H)$. Indeed, let $\mathcal{K}$

be a weak limit of $(\nabla\varphi_i)_i$. By Corollary 1, we have for all $\varphi \in \mathrm{Exp}_A(H)$ and $h \in D(A)$:

$$\int_{K_0} \langle \mathcal{K}, h\rangle\, \varphi\, d\pi_\delta \;=\; -\int_{K_0} \psi\, \langle \nabla\varphi, h\rangle\, d\pi_\delta$$

$$-\int_{K_0} \psi\varphi(x)\, \left(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\right) \pi_\delta(dx) \;=\; \int_{K_0} \langle \nabla\psi, h\rangle\, \varphi\, d\pi_\delta$$

and this proves the claim. We obtain for all $\psi_1, \psi_2 \in \mathrm{Lip}(H)$:

$$\int_{K_0} \lambda R_\delta(\lambda)\psi_1\, \psi_2\, d\pi_\delta \;+\; \frac{1}{2}\int_{K_0} \langle \nabla R_\delta(\lambda)\psi_1, \nabla\psi_2\rangle\, d\pi_\delta \;=\; \int_{K_0} \psi_1\, \psi_2\, d\pi_\delta. \quad (41)$$

Therefore, $(D^\delta, \mathrm{Lip}(H))$ is closable in $L^2(\pi_\delta)$, and the unique continuous extension of $(R_\delta(\lambda))_{\lambda>0}$ to $L^2(\pi_\delta)$ is the strongly continuous resolvent associated with the closure $(\mathcal{D}^\delta, W^{1,2}(\pi_\delta))$.

Step 7.–We prove (iv). Let $\delta \in \mathbb{N} \cap [4, \infty)$. Since the image measure of $\mu^{\otimes\delta}$ under Φ_δ is π_δ, there exists a measurable set $\Omega_0 \subseteq H^\delta$ with $\mu^{\otimes\delta}(\Omega_0) = 1$, such that for all $y \in \Omega_0$, $|y| > 0$ on $(0,1)$, so that for all $h \in C_0(0,1)$ the following map is well-defined:

$$\Omega_0 \;\ni\; y \;\mapsto\; h\,\frac{y}{|y|} \;\in\; C\left([0,1]; \mathbb{R}^\delta\right).$$

By Theorem 8.3.2 of [DPZ 96], for all $G \in \mathrm{Lip}(H^\delta)$ there exists a sequence $\{G_n\} \subset C^1_b(H^\delta)$, such that

$$\|G_n\|_{\mathrm{Lip}(H^\delta)} \;\leq\; \|G\|_{\mathrm{Lip}(H^\delta)}, \quad G_n \to G \quad \text{in } W^{1,2}(H^\delta, \mu^{\otimes\delta}).$$

Then, by a density argument, for all $G \in \mathrm{Lip}(H^\delta)$:

$$\lim_{t\downarrow 0} \frac{1}{t}\left[G\left(y \,+\, t\,h\,\frac{y}{|y|}\right) - G(y)\right] \;=\; \langle \overline{\nabla}\, G(y), h\,\frac{y}{|y|}\rangle_{H^\delta} \qquad \text{in } L^2(\mu^{\otimes\delta}).$$

Then, for $h \in C_0(0,1)$ and $G := \varphi \circ \Phi_\delta$ with $\varphi \in \mathrm{Lip}(H)$:

$$\langle \nabla\varphi(|y|), h\rangle \;:=\; \lim_{t\downarrow 0} \frac{1}{t}\left(\varphi(|y| + t\,h) - \varphi(|y|)\right)$$

$$= \lim_{t\downarrow 0} \frac{1}{t}\left[[\varphi\circ\Phi_\delta]\left(y \,+\, t\,h\,\frac{y}{|y|}\right) - [\varphi\circ\Phi_\delta]\,(y)\right]$$

$$= \langle \overline{\nabla}\,[\varphi\circ\Phi_\delta]\,(y), h\,\frac{y}{|y|}\rangle_{H^\delta}, \qquad \text{in } L^2(\mu^{\otimes\delta}).$$

For all $\varphi, \psi \in \mathrm{Lip}(H)$, it follows that:

$$\mathcal{D}^\delta(\varphi, \psi) \;=\; \Lambda^\delta\left(\varphi \circ \Phi_\delta, \psi \circ \Phi_\delta\right), \tag{42}$$

and by the density of $\mathrm{Lip}(H)$ in $W^{1,2}(\pi_\delta)$, (42) holds for all $\varphi, \psi \in W^{1,2}(\pi_\delta)$. It remains to prove that if $\varphi \in L^2(\pi_\delta)$ satisfies $\varphi \circ \Phi_\delta \in W^{1,2}(\mu^{\otimes\delta})$, then $\varphi \in W^{1,2}(\pi_\delta)$. By (42), $\mathcal{Y}_\delta := \{\varphi \circ \Phi_\delta : \varphi \in W^{1,2}(\pi_\delta)\}$ is a closed subspace of $W^{1,2}(\mu^{\otimes\delta})$. Therefore, setting $\Lambda_1^\delta := (\cdot,\cdot)_{L^2(\mu^{\otimes\delta})} + \Lambda^\delta$, for all $G \in W^{1,2}(\mu^{\otimes\delta})$ there exists a unique $\Gamma_\delta G \in W^{1,2}(\pi_\delta)$, such that for all $\varphi \in W^{1,2}(\pi_\delta)$:

$$\Lambda_1^\delta(G, \varphi \circ \Phi_\delta) \;=\; \Lambda_1^\delta([\Gamma_\delta G] \circ \Phi_\delta, \varphi \circ \Phi_\delta) \;=\; \mathcal{D}_1^\delta(\Gamma_\delta G, \varphi),$$

where $\mathcal{D}_1^\delta := (\cdot,\cdot)_{L^2(\pi^\delta)} + \mathcal{D}^\delta$. Moreover, Γ_δ is Markovian, i.e. $G \geq 0$ implies $\Gamma_\delta G \geq 0$ and $\Gamma_\delta 1 = 1$. Therefore:

$$\|\Gamma_\delta G\|_{L^\infty(\pi_\delta)} \;\leq\; \|G\|_{L^\infty(\mu^{\otimes\delta})}, \;\; \forall G \in W^{1,2}(\mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta}).$$

By (i)-(ii)-(iii), $\mathcal{D}^\delta$ is a quasi-regular symmetric Dirichlet form: see [MR 92]. Then, for all $h \in D(A)$, $\varphi \in W^{1,2}(\pi_\delta) \cap L^\infty(\pi_\delta)$:

$$\int_{K_0} \langle \nabla\varphi, h\rangle \, d\pi_\delta = -\int_{K_0} \varphi^*(x)\left(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\right)\pi_\delta(dx),$$

where φ^* is a $\mathcal{D}^\delta$-quasi-continuous π_δ-version of φ. For all $\psi \in \mathrm{Lip}(H)$ we have by (41):

$$\Lambda_1^\delta(G, [R_\delta(1)\psi] \circ \Phi_\delta) \;=\; \mathcal{D}_1^\delta(\Gamma_\delta G, [R_\delta(1)\psi]) \;=\; \int_{K_0} (\Gamma_\delta G)^* \, \psi \, d\pi_\delta$$

for all $G \in W^{1,2}(\mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta})$. Then there exists $C_\psi \geq 0$ such that:

$$|\Lambda_1^\delta(G, [R_\delta(1)\psi] \circ \Phi_\delta)| \;\leq\; C_\psi \, \|G\|_\infty \quad \forall G \in W^{1,2}(\mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta}),$$

and by Theorem 4.2 in [Fu 99], there exists a finite signed measure Σ_ψ on H^δ, charging no Λ^δ-exceptional set, such that for all $G \in W^{1,2}(\mu^{\otimes\delta}) \cap L^\infty(\mu^{\otimes\delta})$:

$$\Lambda^\delta(G, [R_\delta(1)\psi] \circ \Phi_\delta) = -\int_{H^\delta} G^* \, d\Sigma_\psi,$$

where G^* is a Λ^δ-quasi-continuous $\mu^{\otimes\delta}$-version of G, and for all $\varphi \in C_b(H)$:

$$\int_{H^\delta} \varphi \circ \Phi_\delta \, d\Sigma_\psi \;=\; \int_{H^\delta} \varphi \circ \Phi_\delta \,\cdot\, \psi \circ \Phi_\delta \, d\mu^{\otimes\delta}.$$

Now, to complete the proof, it is enough to prove that $\{[R_\delta(1)\psi] \circ \Phi_\delta : \psi \in \mathrm{Lip}(H)\}$ is dense in $\{\varphi \circ \Phi_\delta : \varphi \in L^2(\pi_\delta)\} \cap W^{1,2}(\mu^{\otimes\delta})$ w.r.t. Λ_1^δ. Suppose that $\varphi \in L^2(\pi_\delta)$, $\varphi \circ \Phi_\delta \in W^{1,2}(\mu^{\otimes\delta})$, and:

$$\Lambda_1^\delta(\varphi \circ \Phi_\delta, [R_\delta(1)\psi] \circ \Phi_\delta) \;=\; 0 \qquad \forall \psi \in \mathrm{Lip}(H),$$

We set $G_m := ([\varphi \circ \Phi_\delta]^* \wedge m) \vee (-m)$, $m \in \mathbb{N}$, and

$$G_{n,m}(y) := G_m \circ \Phi_\delta(Z_\delta(1/n, y)), \quad y \in H^\delta.$$

where Z_δ is the solution of (3). Then $(G_{n,m}) \subset C_b^1(H^\delta)$, $|G_{n,m}| \leq m$, $G_{n,m} \to \varphi_m \circ \Phi_\delta$ Λ^δ-quasi everywhere as $n \to \infty$ and in $W^{1,2}(\mu^{\otimes\delta})$. Moreover:

$$\Lambda_1^\delta(G_{n,m}, [R_\delta(1)\psi] \circ \Phi_\delta) \;=\; -\int G_{n,m}\, d\Sigma_\psi,$$

and passing to the limit in $n \to \infty$ and $m \to \infty$, we obtain for all $\psi \in \mathrm{Lip}(H)$:

$$\begin{aligned} 0 \;&=\; \Lambda_1^\delta(\varphi \circ \Phi_\delta, [R_\delta(1)\psi] \circ \Phi_\delta) \;=\; -\int [\varphi \circ \Phi_\delta]^*\, d\Sigma_\psi \\ &=\; -\int_{K_0} [\varphi \circ \Phi_\delta]^* \,\cdot\, \psi \circ \Phi_\delta \, d\mu^{\otimes\delta}, \end{aligned}$$

which implies $\varphi \equiv 0$. $\square$

Corollary 3 *For all $\delta > 3$, $(t,\xi) \in (0,\infty) \times (0,1)$ and $x \in C_0 \cap K_0$, the law of $u_\delta(t,\xi)$ is absolutely continuous w.r.t. $y^{\delta-1}dy$ on $[0,\infty)$.*

Proof–This follows from (ii) in Theorem 3. $\square$

Corollary 4 *For all $\delta > 3$, the Log-Sobolev and the Poincaré inequalities hold for equation (1), i.e. for all $\varphi \in W^{1,2}(\pi_\delta)$:*

$$\int_{K_0} |\varphi - \pi_\delta(\varphi)|^2 \, d\pi_\delta \;\leq\; \frac{1}{2\pi^2} \int_{K_0} \|\nabla\varphi\|^2 \, d\pi_\delta,$$

$$\int_{K_0} \varphi^2 \log(\varphi^2)\, d\pi_\delta \;\leq\; \frac{1}{2\pi^2} \int_{K_0} \|\nabla\varphi\|^2 \, d\pi_\delta \;+\; \|\varphi\|^2_{L^2(\pi_\delta)} \log(\|\varphi\|^2_{L^2(\pi_\delta)}).$$

For the proof, see e.g. [St 93], [DPDG 00] and [DP 01].

4 SPDEs with positive unbounded drift

In this section we apply the results of the previous sections, to prove the following

Theorem 6

- *Let $\alpha \geq 3$, $C > 0$. Then for all $x \in C_0 \cap K_0$, there exists a unique non-negative continuous adapted $\hat{u}_\alpha$ on $\mathcal{O}$, such that $(\hat{u}_\alpha)^{-\alpha} \in L^1_{loc}([0,\infty) \times (0,1))$, solution of*

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \xi^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t \partial \xi} \\[2ex] \hat{u}_\alpha(t,0) = \hat{u}_\alpha(t,1) = 0, \quad t \geq 0 \\[2ex] \hat{u}_\alpha(0,\cdot) = x, \end{cases} \tag{43}$$

- *Let $0 < \alpha < 3$, $C \geq 0$. Then for all $x \in C_0 \cap K_0$, there exists a unique $(\hat{u}_\alpha, \hat{\eta}_\alpha)$, such that $(\hat{u}_\alpha)^{-\alpha} \in L^1_{loc}([0,\infty) \times (0,1))$, solution of the following SPDE of the Nualart-Pardoux type:*

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \xi^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t \partial \xi} + \hat{\eta}_\alpha \\[2ex] \hat{u}_\alpha(0,\cdot) = x, \ \hat{u}_\alpha(t,0) = \hat{u}_\alpha(t,1) = 0, \quad t \geq 0 \\[2ex] \hat{u}_\alpha \geq 0, \ d\hat{\eta}_\alpha \geq 0, \ \int_{\mathcal{O}} \hat{u}_\alpha \, d\hat{\eta}_\alpha = 0 \end{cases} \tag{44}$$

  Moreover $(\hat{\eta}_\alpha)_{x \in C_0 \cap K_0}$ is not identically equal to 0.

Proof—Let $\hat{f} : (0,\infty) \mapsto \mathbb{R}$, smooth and monotone non-decreasing, possibly unbounded in a neighbourhood of 0. We claim that there exists a unique pair $(\hat{u}, \hat{\eta})$, solution of the Nualart-Pardoux equation (36) with $f = \hat{f}$, such that $h \cdot \hat{f}(\hat{u}) \in L^1([0,T] \times [0,1])$ for all $h \in D(A)$ and $T \geq 0$. Indeed, if we set, for $\epsilon > 0$, $(\hat{u}^\epsilon, \hat{\eta}^\epsilon)$ as the solution of the Nualart-Pardoux (36) with $f = \hat{f}(\cdot + \epsilon)$, then, arguing as in step 2 of the proof of Theorem 3, we have that $\epsilon \mapsto \hat{u}^\epsilon$ is monotone non-increasing and $\epsilon \mapsto \epsilon + \hat{u}^\epsilon$ is monotone non-decreasing. Moreover, $\epsilon \mapsto \hat{\eta}^\epsilon$ is monotone non-decreasing. Therefore, $\hat{u}^\epsilon$ converges uniformly con compact subsets of $\mathcal{O}$ to a continuous function $\hat{u}$ and

$\hat\eta^\epsilon$ converges distributionally to a measure $\hat\eta$, and by Beppo-Levi's Theorem, $(\hat u, \hat\eta)$ is the wanted solution. Uniqueness follows from the proof of Theorem 4, given in [NP 92].

Therefore, for all $\alpha \geq 0$ and $C \geq 0$, there exists a unique pair $(\hat u_\alpha, \hat\eta_\alpha)$, solution of the SPDE with reflection (36) with $f(\xi, a) = -C\, a^{-\alpha}$, $a > 0$. If $\alpha = 3$ and $C > 0$, then we proved in Theorem 3 that $\hat\eta_3 \equiv 0$.

Let $\alpha > 3$, $C > 0$ and $x \in C_0 \cap K_0$. Notice that we can write:

$$\frac{1}{a^3} = \frac{1}{a^3} \vee 1 + \frac{1}{a^3} \wedge 1 - 1, \qquad a > 0.$$

Consider for all $\epsilon > 0$, the solution $(\hat v^\epsilon, \hat\theta^\epsilon)$ of the SPDE with reflection (36) with

$$f(\xi, a) = -C\left(\frac{1}{(\epsilon + a^+)^3} \vee 1 - 1\right) + \frac{a^-}{c}, \qquad a \in \mathbb{R}.$$

By Lemma 1, $\hat u^\epsilon \geq \hat v^\epsilon$ and $\hat\eta^\epsilon \leq \hat\theta^\epsilon$, $\epsilon > 0$. Arguing as in steps 2-5 of the proof of Theorem 3, we can prove that, letting $\epsilon \downarrow 0$, $\hat v^\epsilon$ converges, uniformly on compact sets of $\mathcal{O}$, to a continuous $\hat v$, such that for all $h \in D(A)$, $t \geq 0$:

$$\begin{aligned}\langle \hat v(t,\cdot), h\rangle &= \langle x, h\rangle + \frac{1}{2}\int_0^t \langle h'', \hat v(s,\cdot)\rangle\, ds + \langle h, W(t)\rangle \\ &\quad + \frac{C}{2}\int_0^t\int_0^1 h\left(\frac{1}{(\hat v)^3} \vee 1 - 1\right),\end{aligned}$$

$$\lim_{\epsilon\downarrow 0}\int_0^t\int_0^1 h\, d\hat\theta^\epsilon = 0, \quad \text{so that :} \quad \int_0^t\int_0^1 h\, d\hat\eta = \lim_{\epsilon\downarrow 0}\int_0^t\int_0^1 h\, d\hat\eta^\epsilon = 0.$$

Therefore, $\hat\eta_\alpha = 0$ and $\hat u_\alpha$ satisfies (6).

Let $\alpha \in (0,3)$. By Theorem 10 in [Za 00], we have for all $h \in D(A)$ and $\varphi \in C_b(H)$:

$$\begin{aligned}&\int_{K_0} \mathbb{E}\left[\int_0^1 h(\xi)\int_0^\infty e^{-t}\,\hat\eta_\alpha(dt, d\xi)\right] \varphi\, \exp(-2F_\alpha)\, d\pi_3 \\ &= \int_0^1 dr\, h(r)\int_{K_0} \varphi\, e^{-2F_\alpha}\, d\sigma_0(r,\cdot),\end{aligned}$$

$$\text{where :} \qquad F_\alpha(x) = \begin{cases} \dfrac{C}{\alpha-1}\displaystyle\int_0^1 \frac{1}{[x(\xi)]^{\alpha-1}}\,d\xi, & 1<\alpha<3 \\[2ex] C\displaystyle\int_0^1 \log\left[\frac{1}{x(\xi)}\right]\,d\xi, & \alpha=1 \\[2ex] -\dfrac{C}{1-\alpha}\displaystyle\int_0^1 [x(\xi)]^{1-\alpha}\,d\xi, & 0<\alpha<1 \end{cases}$$

For all $\alpha \in (0,3)$, e^{-2F_α} is in $L^1(\pi_3)$ and not identically equal to 0. Therefore $(\hat{\eta}_\alpha)_{x\in C_0\cap K_0}$ is not identically 0. $\square$

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