# Bifurcations and critical phenomena for some infinite-dimensional physical systems II.

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**Abstract**

In this paper we study critical phenomena and bifurcations of solutions of infinite-dimensional system of ordinary differential equations which has applications to some important physical problems.

## 0. Introduction

This paper is the continuation of paper [1] and is devoted to study of bifurcations of solutions of general infinite-dimensional system of ordinary differential equations, describing the motion of infinitely many number of particles $P_{\vec{n}}$ numbered by all $r$-dimensional integer vectors $\vec{n} = (n_1, \ldots, n_2) \in \mathbf{Z}^r$ which are moving in $r$-dimensional real space $\mathbf{R}^r$. Here $r$ is a natural number and $\mathbf{Z}^r$ is the integer lattice in $\mathbf{R}^r$. We suppose that each particle $P_{\vec{n}}$ interacts only with the neighboring particles along each direction in $\mathbf{Z}^r$ and also assume that each particle is under the action of external field of forces along each coordinate axis in $\mathbf{R}^r$. The point $P_{\vec{n}}$ is given by $r$-dimensional vector $x_{\vec{n}} = (x_{\vec{n}}^{(1)}, \ldots, x_{\vec{n}}^{(r)}) \in \mathbf{R}^r$, and each coordinate $x_{\vec{n}}^{(S)}$ satisfies the following infinite-dimensional system of ordinary differential equation ([1]):

$$\frac{dx_{\vec{n}}^{(S)}}{dt} = -f^{(S)}\left(x_{\vec{n}}^{(S)}\right) + \beta^{(S)} + 2\chi\left(-2rx_{\vec{n}}^{(S)} + \sum_{\vec{n}' \in \Gamma_{\vec{n}}} x_{\vec{n}'}^{(S)}\right) \ , \tag{0.1}$$

where $S = 1, \ldots, r$; $\vec{n} = (n_1, \ldots, n_r) \in \mathbf{Z}^r$, $f^{(S)}(x)$ is a given smooth function, $\beta^{(S)}$ and $\chi$ are parameters, $\Gamma_{\vec{n}}$ is the set of all vectors $\vec{n}' = (n_1', \ldots, n_r') \in \mathbf{Z}^r$ such that

$$\sum_{S=1}^{r} |n_S - n_S'| = 1 \ , \tag{0.2}$$

$t$ is an independent variable characterizing the time ([1]).

In this paper we assume that the absolute value of the function $f^{(S)}(x)$ is bounded, that is there exist constants $\gamma'_S$ and $\gamma''_S$ such that

$$\gamma'_S = \min_x f^{(S)}(x) \leq \max_x f^{(S)}(x) = \gamma''_S \ . \tag{0.3}$$

In the paper [1] we obtained rigorous results about qualitative behavior of solutions of the system (0.1) for the one-dimensional case $r = 1$ and in the particular case for which in the equation (0.1) the constant $\chi = \frac{1}{2}$, the parameter $\beta^{(1)} > 0$ and the function $f^{(1)}(x) = b \sin x$, where $a$ constant $b > 0$.

In particular in this case one found bifurcations of spatially homogeneous solutions and the critical value of parameter $\beta^{(1)}$ for which the behavior of a spatially homogeneous solutions changes in a drastic way in the limit $t \to \infty$.

In addition, the proofs of these facts in [1] essentially used the form of the function $f^{(1)}(x) = b \sin x$.

In present paper we study bifurcations and critical phenomena of general system (0.1) for any natural $r$ and for arbitrary smooth function $f^{(S)}(x)$ satisfying the condition (0.3). Here we consider as spatially homogeneous solutions of the system (0.1) (sectioin 1) so the solutions which are not spatially homogeneous (section 2), and we study bifurcations as with respect to changes of parameters $\beta^{(S)}$ (section 1) so with respect to changes of initial data for fixed parameters $\beta^{(S)}$ (section 2). In all cases the qualitative behavior of solutions change fundamentally by passing through bifurcation in the limit $t \to \infty$. These rigorous results allow to give an explanation of the origin of critical phenomena in some well-known and popular physical problems ("sandpile model", "avalanche problem"), which mentioned in [1] about ([2] - [8]).

# 1 Bifurcations of spatially homogeneous solutions under changes of parameters

**Definition 1.1.** A solution $\vec{x}^{(S)} = (\vec{x}_{\vec{n}}^{(S)}(t))$ of the system (0.1) is said to be spatially homogeneous if the function $x_{\vec{n}}^{(S)}(t)$ does not depend on a vector $\vec{n} \in \mathbf{Z}$.

We fix the function $f^{(S)}(x)$ and the parameter $\chi$ in the equation (0.1) and study the dependence of spatially homogeneous solutions on parameters $\beta^{(S)}$.

**Theorem 1.1.** Let $\vec{x}^{(S)} = (x_{\vec{n}}^{(S)}(t))$ be spatially homogeneous solution of the system (0.1). Then two following statements hold:

I. If $\beta^{(S)} > \gamma''_S$, then $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = +\infty$, $\frac{dx_{\vec{n}}^{(S)}}{dt}(t) > 0$, and if $\beta^{(S)} < \gamma'_S$, then $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = -\infty$, $\frac{dx_{\vec{n}}^{(S)}}{dt}(t) < 0$.

II. We assume that at least one of two following conditions hold:

1) $\gamma'_S \leq \beta^{(S)} \leq \gamma''_S$ and the initial data $x_0^{(S)} = x_{\vec{n}}^{(S)}(t_0)$ satisfy the inequalities $f^{(S)}(x_0^{(S)}) \leq \beta^{(S)}$, $x_0^{(S)} \leq \overline{x}^{(S)}$, where $\overline{x}^{(S)}$ is a quantity such that $f^{(S)}(\overline{x}^{(S)}) \geq \beta^{(S)}$;

2) $\gamma'_S \leq \beta^{(S)} \leq \gamma''_S$ and the initial data $x_0^{(S)} = x_{\vec{n}}^{(S)}(t_0)$ satisfy the inequalities $f^{(S)}(x_0^{(S)}) \geq \beta^{(S)}$, $x_0^{(S)} \geq \tilde{x}^{(S)}$, where $\tilde{x}^{(S)}$ is a quantity such that $f^{(S)}(\tilde{x}^{(S)}) \leq \beta^{(S)}$. Then the solution $x_{\vec{n}}^{(S)}(t)$ of the system (0.1) with initial data $x_0^{(S)} = x_{\vec{n}}^{(S)}(t_0)$ has the finite limit $\hat{x} = \lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) < \infty$.

Proof. Let us consider the quation

$$\frac{dx}{dt} = -f^{(S)}(x) + \beta^{(S)} \; . \tag{1.1}$$

We need to prove the following three lemmas about solutions $x(t)$ of the equation (1.1).

**Lemma 1.1** If $\beta^{(S)} > \gamma''_S$, then $\lim_{t\to+\infty} x(t) = +\infty$, $\frac{dx}{dt}(t) > 0$, and if $\beta^{(S)} < \gamma'_S$, then $\lim_{t\to+\infty} x(t) = -\infty$, $\frac{dx}{dt}(t) < 0$.

Proof. By virtue of (0.3) and (1.1) if $\beta^{(S)} > \gamma''_S$, then $\frac{dx}{dt}(t) > \beta^{(S)} - \gamma''_S > 0$, and if $\beta^{(S)} < \gamma'_S$, then $\frac{dx}{dt}(t) < \beta^{(S)} - \gamma'_S < 0$.

These inequalities prove Lemma 1.1.

**Lemma 1.2** We assume that $\gamma'_S \leq \beta^{(S)} \leq \gamma''_S$ and the initial data $x_0 = x(t_0)$ satisfies the inequalities $f^{(S)}(x_0) \leq \beta^{(S)}$, $x_0 \leq \overline{x}$, where $\overline{x}$ is a quantity such that $f^{(S)}(\overline{x}) \geq \beta^{(S)}$. Then the solution of the equation (1.1) with initial data $x_0$ has a finite limit $\hat{x} = \lim_{t\to+\infty} x(t) < \infty$.

Proof. From inequalities $x_0 \leq \overline{x}$, $f^{(S)}(x_0) \leq \beta^{(S)}$, $f^{(S)}(\overline{x}) \geq \beta^{(S)}$ it follows that there exists a number $\overline{\overline{x}}$ satisfying the conditions $x_0 \leq \overline{\overline{x}} \leq \overline{x}$, $f^{(S)}(\overline{\overline{x}}) = \beta^{(S)}$ and such that in the interval $(x_0, \overline{\overline{x}})$ there are no numbers $\xi$ for which $f^{(S)}(\xi) = \beta^{(S)}$. Now we prove that $\lim_{t\to+\infty} x(t) = \overline{\overline{x}}$. According to the inequality $f^{(S)}(x_0) \leq \beta^{(S)}$ and to the choice of a number $\overline{\overline{x}}$ the inequality $\beta^{(S)} - f^{(S)}(x) > 0$ holds in the interval $x_0 < x < \overline{\overline{x}}$, and the solution $x(t)$ of the equation (1.1) will increase, if the variable $t$ increases. Therefore for any $\epsilon > 0$ there exists a value $\overline{\overline{t}} \geq t_0$ such that $\overline{\overline{x}} - \epsilon \leq x(\overline{\overline{t}}) \leq \overline{\overline{x}}$. We represent the function $f^{(S)}(x)$ in the neighborhood of the point $x = \overline{\overline{x}}$ in the form

$$f^{(S)}(x) = \beta^{(S)} + a_\nu (x - \overline{\overline{x}})^\nu + O(|x - \overline{\overline{x}}|^{\nu+1}) \; , \tag{1.2}$$

where $\nu$ is a natural number, $a_\nu$ is a constant and $|O(|x - \overline{\overline{x}}|^{\nu+1})| \leq \text{const}\; |x - \overline{\overline{x}}|^{\nu+1}$. From this representation it follows that for sufficiently small $\epsilon > 0$ the integral

$\int_{\overline{\overline{x}}-\epsilon}^{\overline{\overline{x}}} \frac{dx}{\beta^{(S)} - f^{(S)}(x)} = \infty$. Therefore, by virtue of (1.1) either for any finite number $t \geq t_0$ the solution $x(t)$ of the equation (1.1) satisfies the inequality $x(t) < \overline{\overline{x}}$, or for all $t \geq t_0$ $x(t) \equiv \overline{\overline{x}}$.

Lemma 1.2 is proved.

**Lemma 1.3** We assume that $\gamma'_S \leq \beta^{(S)} \leq \gamma''_S$ and the initial data $x_0 = x(t_0)$ satisfies the inequalities $f^{(S)}(x_0) \geq \beta^{(S)}$, $x_0 \geq \tilde{x}$, where $\tilde{x}$ is a quantity such that $f^{(S)}(\tilde{x}) \leq \beta^{(S)}$. Then the solution of the equation (1.1) with initial data $x_0$ has a finite limit $\hat{x} = \lim_{t\to+\infty} x(t) < \infty$.

Proof. From inequalities $x_0 \geq \tilde{x}$, $f^{(S)}(x_0) \geq \beta^{(S)}$, $f^{(S)}(\tilde{x}) \leq \beta^{(S)}$ it follows that there exists a number $\overline{\overline{x}}$ satisfying the conditions $x_0 \geq \overline{\overline{x}} \geq \tilde{x}$, $f^{(S)}(\overline{\overline{x}}) = \beta^{(S)}$, and such that in the interval $(\overline{\overline{x}}, x_0)$ there are no numbers $\xi$ for which $f^{(S)}(\xi) = \beta^{(S)}$. We prove that $\lim_{t\to+\infty} x(t) = \overline{\overline{x}}$. According to the inequality $f^{(S)}(x_0) \geq \beta^{(S)}$ and to the choice of a number $\overline{\overline{x}}$, the inequality $\beta^{(S)} - f^{(S)}(x) < 0$ holds in the interval $\overline{\overline{x}} < x < x_0$, and the solution $x(t)$ of the equation (1.1) will decrease if the variable $t$ increases. Therefore for any $\epsilon > 0$ there exists a value $\overline{\overline{t}} \geq t_0$ such that $\overline{\overline{x}} \leq x(\overline{\overline{t}}) \leq \overline{\overline{x}} + \epsilon$. Now we use the representation (1.2) of the function $f^{(S)}(x)$ in the neighborhood of the point $x = \overline{\overline{x}}$ and obtain that for sufficiently small $\epsilon > 0$ the integral $\int_{\overline{\overline{x}}}^{\overline{\overline{x}}+\epsilon} \frac{dx}{\beta^{(S)} - f^{(S)}(x)} = -\infty$. Therefore by virtue of (1.1) either for any number $t \geq t_0$ the solution $x(t)$ of the equation (1.1) satisfies the inequality $x(t) > \overline{\overline{x}}$, or for all $t \geq t_0$ $\quad x(t) \equiv \overline{\overline{x}}$.

Lemma 1.3 is proved.

Since $\vec{x}^{(S)} = (x_{\vec{n}}^{(S)}(t))$ is a spatially homogeneous solution of the system (0.1) then for all vectors $\vec{n} \in \mathbf{Z}^r$ the function $x_{\vec{n}}^{(S)}(t)$ is equal to some function $x(t)$, and substituting $x_{\vec{n}}^{(S)}(t) = x(t)$ into equation (0.1), we obtain the equation (1.1) for the function $x(t)$. Now the statement I of Theorem 1.1 follows from Lemma 1.1 and the statement II follows from Lemmas 1.2 and 1.3. Theorem 1.1 is proved.

**Corollary 1.1** Let the function $f^{(S)}(x)$ be periodic and $\vec{x}^{(S)} = (x_{\vec{n}}^{(S)}(t))$ be a spatially homogeneous solution of the system (0.1). Then it is possible one of following three cases:

1) if $\gamma'_S \leq \beta^{(S)} \leq \gamma''_S$, then $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = \hat{x} < \infty$;

2) if $\beta^{(S)} > \gamma''_S$, then $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = +\infty$;

3) if $\beta^{(S)} < \gamma'_S$, then $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = -\infty$.

**Corollary 1.2** If the function $f^{(S)}(x)$ is periodic then values $\beta^{(S)} = \gamma'_S$ and $\beta^{(S)} = \gamma''_S$ are critical for which the spatially homogeneous solution of the system (0.1) changes its behavior fundamentally in the limit $t \to +\infty$.

## 2 Bifurcations of solutions under changes of initial data

In this section we find solution of the system (0.1) which are not spatially homogeneous and study their bifurcation with respect to change of initial data.

**Theorem 2.1** Suppose that the function $f^{(S)}(x)$ and a constant $\beta^{(S)}$ satisfy the equaltiy

$$2\beta^{(S)} \equiv f^{(S)}(x) + f^{(S)}(-x) \ . \tag{2.1}$$

Then for any number $x_0^{(S)}$ there exists a solution $x_{\vec{n}}^{(S)} = x_{\vec{n}}^{(S)}(t)$ of the system (0.1) such that for any two vectors $\vec{n}' = (n_1', \ldots, n_r') \in \mathbf{Z}^r$, $\vec{n}'' = (n_1'', \ldots, n_r'') \in \mathbf{Z}^r$ whose components satisfy the equality

$$\sum_{\nu=1}^{r} |n_\nu' - n_\nu''| = 1 \ , \tag{2.2}$$

the equality

$$x_{\vec{n}'}^{(S)}(t) \equiv -x_{\vec{n}''}^{(S)}(t) \tag{2.3}$$

is valid, at the initial moment of time $t = t_0$ one of two equalities $x_{\vec{n}}^{(S)}(t_0) = x_0^{(S)}$, $x_{\vec{n}}^{(S)}(t_0) = -x_0^{(S)}$ is valid and following assertions hold:

1) if $\chi \le 0$ and $8r\chi x_0^{(S)} < \beta^{(S)} - \gamma_S''$, then for any $\vec{n} \in \mathbf{Z}^r$ $\quad \lim_{t\to+\infty} |x_{\vec{n}}^{(S)}(t)| = +\infty$;

2) if $\chi \ge 0$ and $8r\chi x_0^{(S)} > \beta^{(S)} - \gamma_S'$, then for any $\vec{n} \in \mathbf{Z}^r$ $\quad \lim_{t\to+\infty} |x_{\vec{n}}^{(S)}(t)| = +\infty$;

3) if $\beta^{(S)} - \gamma_S'' \le 8r\chi x_0^{(S)} \le \beta^{(S)} - \gamma_S'$, $\beta^{(S)} - f^{(S)}(x_0^{(S)}) \ge 8r\chi x_0^{(S)}$ and there exists a number $\overline{x}^{(S)} \ge x_0^{(S)}$ such that $\beta^{(S)} - f^{(S)}(\overline{x}^{(S)}) \le 8r\chi x_0^{(S)}$, then for any $\vec{n} \in \mathbf{Z}^r$ $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = \hat{x}^{(S)} < \infty$;

4) if $\beta^{(S)} - \gamma_S'' \le 8r\chi x_0^{(S)} \le \beta^{(S)} - \gamma_S'$, $\beta^{(S)} - f^{(S)}(x_0^{(S)}) \le 8r\chi x_0^{(S)}$ and there exists a number $\tilde{x}^{(S)} \le x_0^{(S)}$ such that $\beta^{(S)} - f^{(S)}(\tilde{x}^{(S)}) \ge 8r\chi\tilde{x}^{(S)}$, then for any $\vec{n} \in \mathbf{Z}^r$ $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = \hat{x}^{(S)} < \infty$.

Proof. Let us consider the equation

$$\frac{dy}{dt} = -f^{(S)}(y) + \beta^{(S)} - 8r\chi y \tag{2.4}$$

and prove four lemmas.

**Lemma 2.1** If $\chi \le 0$ and an initial data $y_0 = y(t_0)$ satisfies the inequality

$$8r\chi y_0 < \beta^{(S)} - \gamma_S'' \ , \tag{2.5}$$

then $\lim_{t\to+\infty} y(t) = +\infty$.

Proof. By virtue of (0.3), (2.5) and the inequality $\chi \le 0$ for all $y \ge y_0$ we have: $-f^{(S)}(y) + \beta^{(S)} - 8r\chi y \ge -\gamma''_S + \beta^{(S)} - 8r\chi y_0 > C' > 0$, where $C'$ is a constant.

Therefore according to (2.4) for all $t \ge t_0$ the inequality $\frac{dy}{dt}(t) \ge C' > 0$ holds, and the assertion of Lemma 2.1 is valid.

**Lemma 2.2** If $\chi \ge 0$ and an initial data $y_0 = y(t_0)$ satisfies the inequality

$$8r\chi y_0 > \beta^{(S)} - \gamma'_S \ , \tag{2.6}$$

then $\lim_{t\to+\infty} y(t) = -\infty$.

Proof. By virtue of (0.3), (2.6) and the inequality $\chi \ge 0$, for all $y \le y_0$ we have:

$$-f^{(S)}(y) + \beta^{(S)} - 8r\chi y \le -\gamma'_S + \beta^{(S)} - 8r\chi y_0 < C'' < 0 \ ,$$

where $C''$ is a constant. Therefore according to (2.4) the inequality $\frac{dy}{dt} \le C'' < 0$ holds, and the assertion of Lemma 2.2 is valid.

**Lemma 2.3** Let $\beta^{(S)} - \gamma''_S \le 8r\chi y_0 \le \beta^{(S)} - \gamma'_S$, $\beta^{(S)} - f^{(S)}(y_0) \ge 8r\chi y_0$ and there exists $\overline{y} \ge y_0$ such that $\beta^{(S)} - f^{(S)}(\overline{y}) \le 8r\chi\overline{y}$. Then the solution $y(t)$ of the equation (2.4) with initial data $y(t_0) = y_0$ has a finite limit $\lim_{t\to+\infty} y(t) = \hat{y} < \infty$.

Proof. From inequalities $y_0 \le \overline{y}$, $\beta^{(S)} - f^{(S)}(y_0) \ge 8r\chi y_0$, $\beta^{(S)} - f^{(S)}(\overline{y}) \le 8r\chi\overline{y}$ it follows that there exists a number $\overline{\overline{y}}$ satisfying the conditions $y_0 \le \overline{\overline{y}} \le \overline{y}$, $\beta^{(S)} - f^{(S)}(\overline{\overline{y}}) = 8r\chi\overline{\overline{y}}$, and such that there are no a number $\eta$ in the interval $(y_0, \overline{\overline{y}})$ for which $\beta^{(S)} - f^{(S)}(\eta) = 8r\chi\eta$. We prove that $\lim_{t\to+\infty} y(t) = \overline{\overline{y}}$. According to the inequality $\beta^{(S)} - f^{(S)}(y_0) \ge 8r\chi y_0$ and to the choice of a number $\overline{\overline{y}}$, the inequality

$$-f^{(S)}(y) + \beta^{(S)} - 8r\chi y > 0$$

holds in the interval $y_0 < y < \overline{\overline{y}}$, and the solution $y(t)$ of the equation (2.4) will increase, if the variable $t$ increases. Therefore for any $\epsilon > 0$ there exists a value $\overline{\overline{t}} \ge t_0$ such that $\overline{\overline{y}} - \epsilon \le y(\overline{\overline{t}}) \le \overline{\overline{y}}$. We represent the function $f^{(S)}(y)$ in the neighborhood of the point $y = \overline{\overline{y}}$ in the form

$$f^{(S)}(y) = \beta^{(S)} + b_\nu (y - \overline{\overline{y}})^\nu + O(|y - \overline{\overline{y}}|^{\nu+1}) \ , \tag{2.7}$$

where $\nu$ is a natural number, $b_\nu$ is a constant and

$$|O(|y - \overline{\overline{y}}|^{\nu+1})| \le \text{const } |y - \overline{\overline{y}}|^{\nu+1} \ .$$

From this representation it follows that for sufficiently small $\epsilon > 0$ the integral

$\int_{\overline{\overline{y}}-\epsilon}^{\overline{\overline{y}}} \frac{dy}{-f^{(S)}(y)+\beta^{(S)}-8r\chi y^{(S)}} = \infty$. Therefore, by virtue of (2.4) either for any finite number $t \geq t_0$ the solution $y(t)$ of the equation (2.4) satisfies the inequality $y(t) < \overline{\overline{y}}$, or for all $t \geq t_0 \quad y(t) \equiv \overline{\overline{y}}$. Lemma 2.3 is proved.

**Lemma 2.4** Let $\beta^{(S)} - \gamma_S'' \leq 8r\chi y_0 \leq \beta^{(S)} - \gamma_S'$, $\beta^{(S)} - f^{(S)}(y_0) \leq 8r\chi y_0$ and there exists $\tilde{y} \leq y_0$ such that $\beta^{(S)} - f^{(S)}(\tilde{y}) \geq 8r\chi y_0$. Then the solution $y(t)$ of the equation (2.4) with the initial data $y(t_0) = y_0$ has a finite limit $\lim_{t\to+\infty} y(t) = \hat{y} < \infty$.

Proof. From the inequalities $y_0 \geq \tilde{y}$, $\beta^{(S)} - f^{(S)}(y_0) \leq 8r\chi y_0$, $\beta^{(S)} - f^{(S)}(\tilde{y}) \geq 8r\chi\tilde{y}$ it follows that there exists a number $\overline{\overline{y}}$ satisfying the conditions $y_0 \geq \overline{\overline{y}} \geq \tilde{y}$, $\beta^{(S)} - f^{(S)}(\overline{\overline{y}}) = 8r\chi\overline{\overline{y}}$ and such that there is no number $\eta$ in the interval $(\overline{\overline{y}}, y_0)$ for which $\beta^{(S)} - f^{(S)}(\eta) = 8r\chi\eta$. We prove that $\lim_{t\to+\infty} y(t) = \overline{\overline{y}}$. According to the inequality $\chi \geq 0$, $\beta^{(S)} - f^{(S)}(y_0) \leq 8r\chi y_0$ and to the choice of a number $\overline{\overline{y}}$, the inequality

$$-f^{(S)}(y) + \beta^{(S)} - 8r\chi y < 0$$

holds in the interval $\overline{\overline{y}} < y < y_0$, and the solution $y(t)$ of the equation (2.4) will decrease, if the varibale $t_0$ increases. Therefore for any $\epsilon > 0$ there exists a value $\overline{\overline{t}} \geq t_0$ such that $\overline{\overline{y}} \leq y(\overline{\overline{t}}) \leq \overline{\overline{y}} + \epsilon$. Now we use the representation (2.7) of the function $f^{(S)}(y)$ in the neighborhood of the point $y = \overline{\overline{y}}$ and obtain that for sufficiently small $\epsilon > 0$ the integral $\int_{\overline{\overline{y}}}^{\overline{\overline{y}}+\epsilon} \frac{dy}{-f^{(S)}(y)+\beta^{(S)}-8r\chi y^{(S)}} = -\infty$. Therefore, either for any $t \geq t_0$ the solution $y(t)$ of the equation (2.4) satisfies the inequality $y(t) > \overline{\overline{y}}$, or for all $t \geq t_0 \quad y(t) \equiv \overline{\overline{y}}$. Lemma 2.4 is proved.

Now we shall look for a solution $x_{\vec{n}}^{(S)}(t)$ of the system (0.1) such that for any vectors $\vec{n}' = (n_1', \ldots, n_r') \in \mathbf{Z}^r$, $\vec{n}'' = (n_1'', \ldots, n_r'') \in \mathbf{Z}^r$, satisfying the equality (2.2), the equality (2.3) would be valid. Using the definition of the set $\Gamma_{\vec{n}}$ in the equation (0.1) and the equality (0.2), we obtain that if $\vec{n}' \in \Gamma_{\vec{n}}$, then functions $x_{\vec{n}'}^{(S)}$ in the equation (0.1) satisfy the equality

$$x_{\vec{n}'}^{(S)}(t) \equiv -x_{\vec{n}}^{(S)}(t) \ , \tag{2.8}$$

and the equality (0.1) for $x_{\vec{n}}^{(S)}(t)$ takes the form of equation

$$\frac{dx_{\vec{n}}^{(S)}}{dt} = -f^{(S)}(x_{\vec{n}}^{(S)}) + \beta^{(S)} - 8r\chi x_{\vec{n}}^{(S)} \ , \tag{2.9}$$

which coincide with equation (2.4) for the function $y(t)$.

Since according to the equality (2.1) the equality $\beta^{(S)} - f^{(S)}(x_{\vec{n}}^{(S)}) \equiv -(\beta^{(S)} - f^{(S)}(-x_{\vec{n}}^{(S)}))$ holds then by virtue of (2.8) the same equality as (2.9) is valid for all vectors $\vec{n}' \in \Gamma_{\vec{n}}$:

$$\frac{dx_{\vec{n}'}^{(S)}}{dt} = -f^{(S)}(x_{\vec{n}'}^{(S)}) + \beta^{(S)} - 8r\chi x_{\vec{n}'}^{(S)} \ .$$

Therefore, if the equation (2.9) is valid for some vector $\vec{n}$, then it is valid for all $\vec{n} \in \mathbf{Z}^r$, and to find solutions $x_{\vec{n}}^{(S)}(t)$ of the system (0.1), satisfying the condition (2.3), it is necessary to find all solutions of the equation (2.4), which are uniquely defined by giving of initial data $y_0 = y(t_0)$ at the initial moment of time $t = t_0$. Now the assertion 1) - 4) of Theorem 2.1 follows respectively from Lemmas 2.1 - 2.4. Theorem 2.1 is proved.

**Corollary 2.1** Let the constant $\beta^{(S)} = 0$, and the function $f^{(S)}(x)$ satisfies the equality $f^{(S)}(x) = -f^{(S)}(-x)$.
Then all statements of Theorem 2.1 hold.

**Example 2.1** $\beta^{(S)} = 0$, $f^{(S)}(x) = b\sin(\omega x)$, where $b$ and $\omega$ are constants.

**Corollary 2.2** If the function $f^{(S)}(x)$ is periodic, the equality (2.1) holds and the inequality $\beta^{(S)} - \gamma_S'' \leq 8r\chi x_0^{(S)} \leq \beta^{(S)} - \gamma_S'$ is valid, then for any $\vec{n} \in \mathbf{Z}^r$ $\lim_{t\to+\infty} x_{\vec{n}}^{(S)}(t) = \hat{x}^{(S)} < \infty$.

**Corollary 2.3** If the function $f^{(S)}(x)$ is periodic, $\vec{n}'$ and $\vec{n}''$ are integer vectors satisfying (2.2) and the equality (2.1) holds then the values $x_0^{(S)} = \frac{\beta^{(S)}-\gamma_S''}{8r\chi}$, $x_0^{(S)} = \frac{\beta^{(S)}-\gamma_S'}{8r\chi}$ are critical for which the solutions $x_{\vec{n}}^{(S)}(t)$ of the system (0.1) with initial data $x_{\vec{n}'}^{(S)}(t_0) = x_0^{(S)}$, $x_{\vec{n}''}^{(S)}(t_0) = -x_0^{(S)}$ satisfying the equality (2.3) change its behavior fundamentally in the limit $t \to +\infty$.

# 3 Summary and conclusions

This article is the continuation of the paper [1] and devoted to study of critical phenomena and bifurcations of solutions of general infinite-dimensional system of ordinary differential equations, describing the motion of infinitely many numbers of particles in many dimensional real space. One consider as spatially homogeneous solutions so and solutions which are not spatially homogeneous, and one study the bifurcations with respect to changes of parameters so with respect to changes of initial data. As consequences we find critical values of parameters and initial data for which the behavior of solutions changes in drastic way.

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